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Research Article

The Dynamical Analysis of Computer Viruses Model with Age Structure and Delay

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This paper deals with the dynamical behaviors for a computer viruses model with age structure, where the loss of the acquired immunity and delay are incorporated. Through some rigorous analyses, an explicit formula for the basic reproduction number of the model is calculated, and some results about stability and instability of equilibria for the model are established. These findings show that the age structure and delay can produce Hopf bifurcation for the computer viruses model. The numerical examples are executed to validate the theoretical results.

1. Introduction

With the popularization of computers and the rapid development of information network technology, the network has brought great convenience to human work, study, and life. However, the network has brought us a lot of harm as well. The spread of computer viruses in the network is a common phenomenon. Once the computers in the network are infected with the viruses, the normal programs of the computers may not be able to run, the files in the computers may be damaged, the important information in the computer are lost, and so on. Therefore, it is more important issue to better understand the dynamical spread of computer viruses in the network.

Since the spread of computer viruses in the network is very similar to the spread of biological viruses in populations [1], in recent years, many authors have constructed the computer viruses transmission models based on the epidemic model framework, including SIS models [2, 3], SIR models [4], SIRS models [5–7], SEIR models [8], SEIS models [9, 10], and so on [11–14]. In particular, for the different types of the computer viruses models, the term vaccination is introduced to describe the process of installing the newest version antivirus software in the uninfected computer [5, 6, 14, 15]. The process can lead to the uninfected computer acquire temporary immunity.

In fact, after the virus-infected computers are successfully disinfected, the computer's antivirus system will inevitably be upgraded to strengthen defenses, which will cause the recovery computers to obtain short-term immune protection. That is, the recovery computers will stay in the recovery class for a while. For the classic SIRS model, the outflow of the recovery computers is often described by an ordinary equation as follows:

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = -\mu R(t) - \gamma R(t), \tag{1}$$

where $R(t)$ denotes the number of the recovery computers at time t, μ is the rate at which one computer is removed from the network, and γ is the removal rate of the recovery computers, which describes the recovery computers leaving the recovery class and entering the susceptible class again since they lose its immunity. However, the diversity of computer virus leads the recovery computers must stay in the recovery class for some time, and then they lose immune protection and become susceptible ones again. It means that the removal rate γ of the recovery computers depends on the length of the recovery time. To this end, we assume that the removal rate γ in (1) should be replaced with the following piecewise function:

$$\gamma(a) = \begin{cases} \gamma_*, & a > \tau, \\ 0, & a \le \tau, \end{cases} \tag{2}$$

where $\gamma(a) \in L_+^\infty((0,+\infty),\mathbb{R})$, γ_* is a positive constant, and τ is the shortest time for a recovery computer to maintain its acquired immunity. Therefore, the outflow of the recovery computers in (1) can be rewritten by a partial differential equation as

$$\frac{\partial R(t,a)}{\partial t} + \frac{\partial R(t,a)}{\partial a} = -\mu R(t,a) - \gamma(a)R(t,a), \quad t \ge 0, a \ge 0, \tag{3}$$

where $R(t,a)$ denotes the density of the recovery computers with respect to the acquired immunity age a at time t. Obviously, the loss rate $\gamma(a)$ of the acquired immunity of the recovery computers obeys a non-Markovian process.

Let $S(t)$ and $I(t)$ be the number of the susceptible computers and infected computers at time t, respectively, b be the rate at which external computers are connected to the network, p be the vaccination proportion of computers which are connected to the network, β be the transmission rate, α be the rate at which the infected computers recover due to the antivirus treatment, r_2 be the self-recovery rate of the infected computers, and α_1 be the acquired temporary immunity rate of the susceptible computers. Based on the model in [5, 6], we display the flowchart of infection progression in Figure 1.

Following the transmission mechanism and schematic diagram, we propose the computer virus model with age structure and delay in the following:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = (1-p)b - \beta S(t)I(t) - (\alpha_1 + \mu)S(t) \\ \quad + r_2 I(t) + \displaystyle\int_0^{+\infty} \gamma(a)R(t,a)\mathrm{d}a, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta S(t)I(t) - (\alpha + r_2 + \mu)I(t), \\ \dfrac{\partial R(t,a)}{\partial t} + \dfrac{\partial R(t,a)}{\partial a} = -\mu R(t,a) - \gamma(a)R(t,a), \\ \quad t \ge 0,\ a \ge 0, \end{cases} \tag{4}$$

with the boundary condition

$$R(t,0) = pb + \alpha_1 S(t) + \alpha I(t), \quad t \ge 0, \tag{5}$$

and the initial condition

$$S(0) = S_0 > 0, I(0) = I_0 \ge 0, R(0,\cdot) = R_0(\cdot) \in L_+^1(0,+\infty). \tag{6}$$

This paper is organized as follows. Some preliminaries results and the well-posedness of system (4) are presented in section 2. In section 3, we give an explicit expression of basic reproduction number $\mathscr{R}_0(\tau)$ and discuss the existence of all the feasible equilibria. In section 4, we study the global stability of the virus-free equilibrium E_0 when $\mathscr{R}_0(\tau) < 1$ and local stability of the computer virus equilibrium E_* when $\mathscr{R}_0(\tau) > 1$ and $\tau = 0$. In section 5, we study the Hopf bifurcations occurring from the computer virus equilibrium E_* with the increase in τ. In section 6, we present some numerical examples to illustrate our theoretical results and give the conclusions.

2. Preliminaries

In this section, we will mainly discuss the non-negativity and ultimately boundedness of the solutions of system (4) with non-negative initial condition. The existence and uniqueness of the solution of system (4) directly follows from Lemma A. 1 in the Appendix since the structure of model (4) satisfies the assumptions 1 − 5 in Lemma A. 1.

Theorem 1. *If $\phi = (S_0, I_0, R_0(\cdot)) \in \mathbb{R}_+^2 \times L_+^1$, then the solution of system (4) is non-negative for all $t \ge 0$ and the set*

$$\Gamma = \left\{(S,I,R) \in \mathbb{R}_+^2 \times L_+^1((0,+\infty),\mathbb{R}) \colon S + I + \int_0^{+\infty} R(t,a)\mathrm{d}a \le \frac{b}{\mu}\right\}, \tag{7}$$

which is positively invariant with respect to system (4); moreover, it is ultimately bounded for t large enough.

Proof. The second equation of system (4) implies that

$$I(t) = I_0 e^{\int_0^t [\beta S(\theta) - \alpha - r_2 - \mu]\mathrm{d}\theta} \ge 0, \tag{8}$$

which means that, for any initial value $I_0 \in \mathbb{R}_+$, $I(t)$ remains non-negative for $t \ge 0$.

For any $S_0 \in \mathbb{R}_+$, we claim that $S(t)$ remains non-negative for any $t \in \mathbb{R}_+$. Suppose that the claim does not hold and then it follows from the continuity of the solution of system (4) associated with the initial condition that there exists a t_S such that $S(t) \ge 0$ for $t \in [0, t_S)$, $S(t_S) = 0$, and $S'(t_S) < 0$. Then, by using of the third and fourth equation of system (4), we can get

$$R(t_S, a) = \begin{cases} (pb + \alpha_1 S(t_S - a) + \alpha I(t_S - a))e^{-\int_0^a (\mu + \gamma(\theta))\mathrm{d}\theta}, & a \le t_S, \\ R_0(a - t_S)e^{-\int_{a-t_S}^a (\mu + \gamma(\theta))\mathrm{d}\theta}, & a > t_S. \end{cases} \tag{9}$$

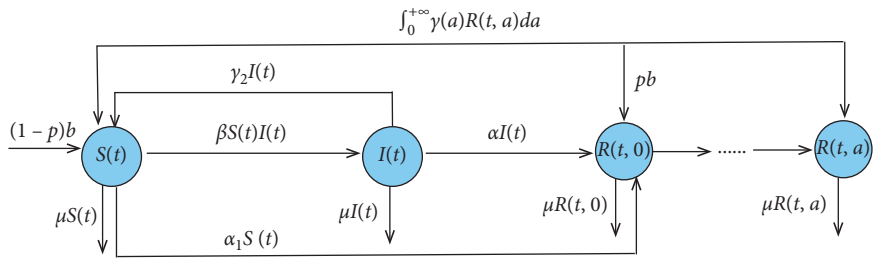

Figure 1: The flowchart of infection progression for the computer viruses based on the SIRS model.

It is clear that $R(t_S,a)\geq 0$ since $S(t_S-a)>0$, $I(t_S-a)\geq 0$, and $R_0(\cdot)\in L_+^1(0,+\infty)$. Plugging such a value into the first equation of (4) leads to

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t}\Big|_{t=t_S}=(1-p)b+r_2I(t_S)+\int_0^{+\infty}\gamma(a)R(t_S,a)\mathrm{d}a>0, \tag{10}$$

which contradicts with $S'(t_S)<0$. Hence, the claim holds and $S(t)$ remains non-negative for any $t\in\mathbb{R}_+$ if $S_0\geq 0$.

Based on the above analysis, we directly integrate the third equation in system (4) along the characteristic line yields that

$$R(t,a)=\begin{cases}(pb+\alpha_1S(t-a)+\alpha I(t-a))e^{-\int_0^a(\mu+\gamma(\theta))\mathrm{d}\theta}, & a\leq t,\\ R_0(a-t)e^{-\int_{a-t}^a(\mu+\gamma(\theta))\mathrm{d}\theta}, & a>t.\end{cases} \tag{11}$$

The non-negativity of S and I together with $R_0(\cdot)\in L_+^1(0,+\infty)$ ensures that $R(t,\cdot)$ remains non-negative for all $t\geq 0$.

In the following, we proceed with the ultimate boundedness of the solutions of system (4). Let $\overline{R}(t)=\int_0^{+\infty}R(t,a)\mathrm{d}a$, which represents the total number of the recovery individuals at time t. Then, adding those equations in system (4) yields

$$\begin{aligned}\frac{\mathrm{d}(S(t)+I(t)+\overline{R}(t))}{\mathrm{d}t}&=(1-p)b-(\mu+\alpha_1)S(t)+\int_0^{+\infty}\gamma(a)R(t,a)\mathrm{d}a-(\alpha+\mu)I(t)-0\int_0^{+\infty}\left[\frac{\partial R(t,a)}{\partial a}-(\mu+\gamma(a))R(t,a)\right]\mathrm{d}a\\&=(1-p)b-\alpha_1S(t)+\alpha I(t)-\mu(S(t)+I(t)+\overline{R}(t))-\int_0^{+\infty}\frac{\partial R(t,a)}{\partial a}\mathrm{d}a\\&=(1-p)b-\alpha_1S(t)+\alpha I(t)-\mu(S(t)+I(t)+\overline{R}(t))-\lim_{a\longrightarrow+\infty}R(t,a)+R(t,0).\end{aligned} \tag{12}$$

It is reasonable to assume that $\lim_{a\longrightarrow+\infty}R(t,a)=0$ according to the biological significance. Noting that $R(t,0)=pb+\alpha_1S(t)+\alpha I(t)$, we obtain

$$\frac{\mathrm{d}(S(t)+I(t)+\overline{R}(t))}{\mathrm{d}t}=b-\mu(S(t)+I(t)+\overline{R}(t)). \tag{13}$$

Therefore,

$$\lim_{t\longrightarrow+\infty}(S(t)+I(t)+\overline{R}(t))=\frac{b}{\mu}. \tag{14}$$

It is not hard to see that the set

$$\Gamma=\left\{(S,I,R)\in\mathbb{R}_+^2\times L_+^1((0,+\infty),\mathbb{R}):\ S+I+\int_0^{+\infty}R(t,a)\mathrm{d}a\leq\frac{b}{\mu}\right\}, \tag{15}$$

which is positively invariant with respect to system (4). Consequently, system (4) is ultimately bounded.

For the sake of convenience, we account for the dynamics of system (4) taken the initial values from Γ. □

3. The Existence of the Equilibria

In this section, we focus on the existence of the virus-free equilibrium and the computer virus equilibrium of system (4). To this end, we need to solve the following equation:

$$\begin{cases} (1-p)b - \beta SI - (\alpha_1 + \mu)S + r_2 I + \int_0^{+\infty} \gamma(a)R(a)\mathrm{d}a = 0, \\ \beta \mathrm{SI} - (\alpha + r_2 + \mu)I = 0, \\ \dfrac{\mathrm{d}R(a)}{\mathrm{d}a} = -(\mu + \gamma(a))R(a), \\ R(0) = pb + \alpha_1 S + \alpha I. \end{cases} \tag{16}$$

The virus-free equilibrium E_0 means that there is no virus-infected computers in the entire network; therefore, in order to get the virus-free equilibrium E_0 of system (4), we assume $I = 0$. Then, equation (16) can be rewritten as the following equations:

$$\begin{cases} (1-p)b - (\alpha_1 + \mu) + \int_0^{+\infty} \gamma(a)R(a)\mathrm{d}a = 0, \\ \dfrac{\mathrm{d}R(a)}{\mathrm{d}a} = -(\mu + \gamma(a))R(a), \\ R(0) = pb + \alpha_1 S. \end{cases} \tag{17}$$

The second and third equations in (17) yields

$$R(a) = (pb + \alpha_1 S)e^{-\int_0^a (\mu+\gamma(\theta))\mathrm{d}\theta}. \tag{18}$$

Taking $R(a) = (pb + \alpha_1 S)e^{-\int_0^a (\mu+\gamma(\theta))\mathrm{d}\theta}$ into the first equation in (17), we obtain

$$S = \frac{(1-p)b + pbM(\tau)}{\mu + \alpha_1 - \alpha_1 M(\tau)}, \tag{19}$$

with

$$M(\tau) = \int_0^{+\infty} \gamma(a)e^{-\int_0^a (\mu+\gamma(\theta))\mathrm{d}\theta}\mathrm{d}a = \frac{\gamma_* e^{-\mu\tau}}{\mu + \gamma_*}. \tag{20}$$

Then, we know that system (4) always has the virus-free equilibrium $E_0 = (S^0, 0, R^0(a))$, where

$$S^0 = \frac{(1-p)b + pbM(\tau)}{\mu + \alpha_1 - \alpha_1 M(\tau)}, R^0(a) = \left(pb + \alpha_1 S^0\right)e^{-\int_0^a (\mu+\gamma(\theta))\mathrm{d}\theta}. \tag{21}$$

The computer virus equilibrium E_* means that there are always virus-infected computers in the entire network; therefore, in order to get the computer virus equilibrium E_* of system (4), we assume $I > 0$. Then, equation (16) can be rewritten as the following equations:

$$\begin{cases} (1-p)b - \beta SI - (\alpha_1 + \mu)S + r_2 I + \int_0^{+\infty} \gamma(a)R(a)\mathrm{d}a = 0, \\ \beta S - (\alpha + r_2 + \mu) = 0, \\ \dfrac{\mathrm{d}R(a)}{\mathrm{d}a} = -(\mu + \gamma(a))R(a), \\ R(0) = pb + \alpha_1 S + \alpha I. \end{cases} \tag{22}$$

The second equation in (22) states that $S = \alpha + r_2 + \mu/\beta$. Furthermore,

$$R(a) = \left(pb + \frac{\alpha_1(\alpha + r_2 + \mu)}{\beta} + \alpha I\right)e^{-\int_0^a (\mu+\gamma(\theta))\mathrm{d}\theta}. \tag{23}$$

Substituting $S = \alpha + r_2 + \mu/\beta$ and $R(a) = (pb + \alpha_1(\alpha + r_2 + \mu)/\beta + \alpha I)e^{-\int_0^a \mu+\gamma(\theta)\mathrm{d}\theta}$ into the first equation in (22), we can get

$$I = \frac{(\alpha + r_2 + \mu)(\alpha_1 + \mu - \alpha_1 M(\tau))}{\beta(\mu + \alpha_1 - \alpha M(\tau))}(\mathscr{R}_0(\tau) - 1), \tag{24}$$

with

$$\mathscr{R}_0(\tau) = \beta \times \frac{1}{\alpha + r_2 + \mu} \times \frac{(1-p)b + pb\gamma_* e^{-\mu\tau}/\mu + \gamma_*}{\mu + \alpha_1 - \alpha_1\gamma_* e^{-\mu\tau}/\mu + \gamma_*}. \tag{25}$$

Obviously, inequalities $\mathscr{R}_0(\tau) > 1$ guarantee $I > 0$. That is, when $\mathscr{R}_0(\tau) > 1$, system (4) has the computer virus equilibrium $E_* = (S_*, I_*, R_*(a))$, where

$$\begin{aligned} S_* &= \frac{\alpha + r_2 + \mu}{\beta}, R_*(a) = \left(pb + \frac{\alpha_1(\alpha + r_2 + \mu)}{\beta} + \alpha I_*\right)e^{-\int_0^a \mu+\gamma(\theta)\mathrm{d}\theta}, \\ I_* &= \frac{(\alpha + r_2 + \mu)(\mu + \alpha_1 - \alpha_1 M(\tau))}{\beta(\mu + \alpha - \alpha M(\tau))}(\mathscr{R}_0(\tau) - 1). \end{aligned} \tag{26}$$

Therefore, the following theorem gives the existence of the equilibria of system (4).

Theorem 2. *System (4) has always the virus-free equilibrium $E_0 = (S^0, 0, R^0(a))$; In addition, when $\mathscr{R}_0(\tau) > 1$, system (4) also has a unique computer virus equilibrium $E_* = (S_*, I_*, R_*(a))$.*

In fact, $\mathscr{R}_0(\tau)$ in (25) is the basic reproduction number of system (4); that is, it represents the total number of the newly infected cases by an infectious individual in the entire infection period. Here, β is the infection rate by an infected computer, $1/(\alpha + r_2 + \mu)$ is the average infection period of the infected computers, and $(1-p)b + pb(\gamma_* e^{-\mu\tau}/\mu + \gamma_*)/\mu + \alpha_1 - \alpha_1(\gamma_* e^{-\mu\tau}/\mu + \gamma_*)$ denotes the total number of susceptible individuals.

4. The Stability of the Equilibria

In this section, we study the global stability of the virus-free equilibrium E_0 by the fluctuation lemma and the local stability of the computer virus equilibrium E_* by analyzing the linearizing system (4).

4.1. The Stability of the Virus-Free Equilibrium E_0

Theorem 3. *If $\mathscr{R}_0(\tau) < 1$, then the virus-free equilibrium $E_0 = (S^0, 0, R^0(a))$ of system (4) is locally asymptotically stable while if $\mathscr{R}_0(\tau) > 1$, E_0 is unstable.*

Proof. Linearizing system (4) at the virus-free equilibrium E_0 and considering the exponential forms of that linear system, we obtain the characteristic equation of system (4) at E_0 as follows:

$$\left(\lambda + \alpha + r_2 + \mu - \beta S^0\right)\left(\lambda + \mu + \alpha_1 - \alpha_1 \int_0^{+\infty} \gamma(a) e^{-\int_0^a \lambda+\mu+\gamma(\theta)\mathrm{d}\theta} \mathrm{d}a\right) = 0. \tag{27}$$

It is clear that $\lambda_1 = \beta S^0 - (\alpha + r_2 + \mu) = (\alpha + r_2 + \mu)(\mathscr{R}_0(\tau) - 1)$ is the root of the characteristic (27). The characteristic root $\lambda_1 < 0$ when $\mathscr{R}_0(\tau) < 1$ and $\lambda_1 > 0$ when $\mathscr{R}_0(\tau) > 1$.

In addition, the remaining roots of (27) satisfy the following equation:

$$\Lambda(\lambda) = \lambda + \mu + \alpha_1 - \alpha_1 \int_0^{+\infty} \gamma(a) e^{-\int_0^a \lambda+\mu+\gamma(\theta)\mathrm{d}\theta} \mathrm{d}a = 0. \tag{28}$$

If $\lambda \in \mathbb{R}$, then $\Lambda(\lambda)$ is a continuous real function strictly increasing and satisfies that

$$\Lambda(0) = \mu + \alpha_1 - \alpha_1 M(\tau) > 0, \quad \lim_{\lambda \longrightarrow +\infty} \Lambda(\lambda) = +\infty. \tag{29}$$

It implies that $\Lambda(\lambda) = 0$ has no positive real root. Suppose that $\lambda = u + iv$ is an arbitrary complex root of $\Lambda(\lambda) = 0$ and it satisfies $u > 0$, then we have

$$\Lambda(u + iv) = u + iv + \mu + \alpha_1 - \alpha_1 \int_0^{+\infty} \gamma(a) e^{-\int_0^a u+iv+\mu+\gamma(\theta)\mathrm{d}\theta} \mathrm{d}a = 0, \tag{30}$$

which implies that the real part $\mathrm{Re}(\Lambda(u + iv)) = 0$ and the imaginary part $\mathrm{Im}(\Lambda(u + iv)) = 0$. Separating the real and imaginary part of $\Lambda(u + iv)$, we obtain

$$\begin{aligned}
\mathrm{Re}(\Lambda(u + iv)) &= u + \mu + \alpha_1 - \alpha_1 \int_0^{+\infty} \gamma(a) e^{-ua} \cos va e^{-\int_0^a \mu+\gamma(\theta)\mathrm{d}\theta} \mathrm{d}a \\
&= u + \mu + \alpha_1\left(1 - \gamma_* e^{\gamma_* \tau} \int_\tau^{+\infty} e^{-(u+\mu+\gamma_*)a} \cos va \mathrm{d}a\right) \\
&= u + \mu + \alpha_1\left(1 - \frac{\gamma_*}{u + \mu + \gamma_*} e^{-(u+\mu)\tau} \frac{\cos v\tau - v/u + \mu + \gamma_* \sin v\tau}{1 + v^2/(u + \mu + \gamma_*)^2}\right) \\
&= u + \mu + \alpha_1\left(1 - \frac{\gamma_*}{u + \mu + \gamma_*} e^{-(u+\mu)\tau} \frac{\cos\left(\arccos 1/\sqrt{1 + v^2/(u + \mu + \gamma_*)^2} + v\tau\right)}{\sqrt{1 + v^2/(u + \mu + \gamma_*)^2}}\right).
\end{aligned} \tag{31}$$

It is obvious that $\mathrm{Re}(\Lambda(u + iv)) > 0$ since $\sqrt{1 + v^2/(u + \mu + \gamma_*)^2} > 1$ and $-1 \leq \cos(\arccos 1/\sqrt{1 + v^2/(u + \mu + \gamma_*)^2} + v\tau) \leq 1$. Based on the above analysis, we know $\lambda = u + iv\,(u > 0)$ cannot be the root of $\Lambda(\lambda) = 0$. That is, $\Lambda(\lambda) = 0$ has no complex root with positive real part.

Summarizing the above analysis, we can see that the virus-free equilibrium E_0 is locally asymptotically stable when $\mathscr{R}_0(\tau) < 1$ and the virus-free equilibrium E_0 is unstable when $\mathscr{R}_0(\tau) > 1$.

In the following, we discuss the global stability of E_0 by employing the fluctuation lemma. Let

$$\psi_\infty = \liminf_{t \longrightarrow \infty} \psi(t), \text{ and } \psi^\infty = \limsup_{t \longrightarrow \infty} \psi(t), \tag{32}$$

and the fluctuation lemma is given as follows. □

Lemma 1 (fluctuation lemma [16]). *Let ψ: $\mathbb{R}_+ \longrightarrow \mathbb{R}$ be a bounded and continuously differentiable function. Then, there exist sequences $\{q_n\}$ and $\{t_n\}$ such that $q_n \longrightarrow \infty$, $t_n \longrightarrow \infty$, $\psi(q_n) \longrightarrow \psi_\infty$, $\psi'(q_n) \longrightarrow 0$, $\psi(t_n) \longrightarrow \psi^\infty$, and $\psi(t_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Lemma 2 (see [17]). *Suppose g: $\mathbb{R}_+ \longrightarrow \mathbb{R}$ is a bounded function. Then,*

$$\liminf_{t\longrightarrow\infty}\int_0^t k(\theta)g(t-\theta)\mathrm{d}\theta \geq g_\infty\|k\|_1, \tag{33}$$

where $\|k\|_1 = \int_0^\infty k(s)ds$.

Theorem 4. *If* $\mathscr{R}_0(\tau) < 1$, *then the virus-free equilibrium* E_0 *of system (4) is globally asymptotically stable for any* $\phi \in \Gamma$.

Proof. In order to establish the globally asymptotical stability of the virus-free equilibrium E_0, according to Theorem 3, it is sufficient to prove that E_0 is attractive in Γ. Let $(S(t), I(t), R(t,a))$ be a solution of system (4) with $\phi \in \Gamma$. Integrating the third equation of system (4) with the boundary condition yields

$$R(t,a) = \begin{cases} (pb + \alpha_1 S(t-a) + \alpha I(t-a))e^{-\int_0^a (\mu+\gamma(\theta))\mathrm{d}\theta}, & a \leq t, \\ R_0(a-t)e^{-\int_{a-t}^a (\mu+\gamma(\theta))\mathrm{d}\theta}, & a > t. \end{cases} \tag{34}$$

With the assistance of the fluctuation lemma, it is easy to get

$$S^\infty \leq \frac{(1-p)b + pbM(\tau)}{\mu + \alpha_1 - \alpha_1 M(\tau)}. \tag{35}$$

Furthermore, it follows from the second equations of system (4) that

$$\begin{aligned} \frac{\mathrm{d}I(t)}{\mathrm{d}t} &\leq \beta\frac{(1-p)b + pbM(\tau)}{\mu + \alpha_1 - \alpha_1 M(\tau)}I(t) - (\alpha + r_2 + \mu)I(t) \\ &= (\alpha + r_2 + \mu)(\mathscr{R}_0(\tau) - 1)I(t). \end{aligned} \tag{36}$$

It is clear that $I^\infty \longrightarrow 0$ under the condition $\mathscr{R}_0(\tau) < 1$.

According to Lemma 1, we can select a sequence $\{t_n\}$ such that $t_n \longrightarrow \infty$, $S(t_n) \longrightarrow S_\infty$, and $S'(t_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore,

$$\begin{aligned} \frac{\mathrm{d}S(t_n)}{\mathrm{d}t} &= (1-p)b - \beta S(t_n)I(t_n) + \int_0^{t_n}\gamma(a)(pb + \alpha_1 S(t_n - a) + \alpha I(t_n - a))e^{-\int_0^a \mu+\gamma(\theta)\mathrm{d}\theta}\mathrm{d}a \\ &\quad + \int_{t_n}^{+\infty}\gamma(a)R_0(a-t_n)e^{-\int_{a-t_n}^a \mu+\gamma(\theta)\mathrm{d}\theta}\mathrm{d}a - (\mu + \alpha_1)S(t_n) + r_2 I(t_n). \end{aligned} \tag{37}$$

Employing Lemma 2, one admits that

$$\begin{aligned} &\lim_{n\longrightarrow\infty}\int_0^{t_n}\gamma(a)(pb + \alpha_1 S(t_n - a) + \alpha I(t_n - a))e^{-\int_0^a \mu+\gamma(\theta)\mathrm{d}\theta}\mathrm{d}a \geq (pb + \alpha_1 S_\infty + \alpha I_\infty)M(\tau), \\ &\lim_{n\longrightarrow\infty}\int_{t_n}^{+\infty}\gamma(a)R_0(a-t_n)e^{-\int_{a-t_n}^a \mu+\gamma(\theta)\mathrm{d}\theta}\mathrm{d}a = 0. \end{aligned} \tag{38}$$

Let $n \longrightarrow \infty$. Equation (37) indicates that

$$\begin{aligned} 0 &\geq (1-p)b - \beta S_\infty I^\infty - (\mu + \alpha_1)S_\infty + r_2 I_\infty \\ &\quad + (pb + \alpha_1 S_\infty + \alpha I_\infty)M(\tau). \end{aligned} \tag{39}$$

Note that $\lim_{n\longrightarrow\infty} I(t_n) = 0$, then $I_\infty \leq I^\infty \longrightarrow 0$. And then, we immediately admit that

$$\frac{(1-p)b + pbM(\tau)}{\mu + \alpha_1 - \alpha_1 M(\tau)} \leq S_\infty \leq S^\infty \leq \frac{(1-p)b + pbM(\tau)}{\mu + \alpha_1 - \alpha_1 M(\tau)}. \tag{40}$$

Therefore, $\lim_{t\longrightarrow\infty} S(t) = (1-p)b + pbM(\tau)/\mu + \alpha_1 - \alpha_1 M(\tau)$. It follows from (34) that

$$\lim_{t\longrightarrow\infty} R(t,a) = \left(pb + \alpha_1 \frac{(1-p)b + pbM(\tau)}{\mu + \alpha_1 - \alpha_1 M(\tau)}\right) e^{-\int_0^a \mu+\gamma(\theta)\mathrm{d}\theta}. \tag{41}$$

Consequently, if $\mathscr{R}_0(\tau) < 1$, then $S(t), I(t), R(t,\cdot) \longrightarrow E_0$ for $\phi \in \Gamma$. This completes the proof. □

4.2. The Stability of the Computer Virus Equilibrium E_*. In this subsection, we will discuss the stability of the computer virus equilibrium E_* of system (4) when $\mathscr{R}_0(\tau) > 1$. Then, similar to Theorem 3, linearizing system (4) around the computer virus equilibrium E_* and accouting for exponential forms of solution for that linear system and taking (2) into the linear system, we obtain the characteristic equation of system (4) at the computer virus equilibrium E_* as follows:

$$\begin{aligned} f(\lambda,\tau) &= \lambda^3 + m_2(\tau)\lambda^2 + m_1(\tau)\lambda + m_0(\tau) \\ &\quad + \left(n_1(\tau)\lambda + n_0(\tau)\right)e^{-\lambda\tau} = 0, \end{aligned} \tag{42}$$

where

$$\begin{aligned} m_2(\tau) &= \mu + \beta I_* + \mu + \alpha_1 + \gamma_*, \\ m_1(\tau) &= \beta I_*(\mu + \alpha + \gamma_*) + (\mu + \alpha_1)(\mu + \gamma_*), \\ m_0(\tau) &= \beta I_* \alpha(\mu + \gamma_*), \\ n_1(\tau) &= -\alpha_1 \gamma_* e^{-\mu\tau}, \\ n_0(\tau) &= -\beta I_* \alpha \gamma_* e^{-\mu\tau}. \end{aligned} \tag{43}$$

It is clear that $m_i(\tau) > 0$, $i = 0, 1, 2$, and $n_i(\tau) < 0$, $i = 0, 1$. In the case where $\tau = 0$, the following result holds.

Theorem 5. *If $\mathscr{R}_0(\tau) > 1$, then the computer virus equilibrium E_* of system (4) is locally asymptotically stable for $\tau = 0$.*

Proof. When $\tau = 0$, by the direct computation, the coefficients of (42) become

$$\begin{aligned} m_1(0) + n_1(0) &= \frac{(\alpha + r_2 + \mu)(\mu + \alpha_1 - \alpha_1\gamma_*/\mu + \gamma_*)}{(\mu + \alpha - \alpha\gamma_*/\mu + \gamma_*)}(\mathscr{R}_0(0) - 1)(\mu + \alpha + \gamma_*) \\ &\quad + (\mu + \alpha_1)(\mu + \gamma_*) - \alpha_1\gamma_* > 0, \\ m_0(0) + n_0(0) &= \frac{(\alpha + r_2 + \mu)(\mu + \alpha_1 - \alpha_1\gamma_*/\mu + \gamma_*)}{(\mu + \alpha - \alpha\gamma_*/\mu + \gamma_*)}(\mathscr{R}_0(0) - 1)\alpha\mu > 0, \\ (m_1(0) + n_1(0))m_2(0) - (m_0(0) + n_0(0)) &= \frac{(\alpha + r_2 + \mu)(\mu + \alpha_1 - \alpha_1\gamma_*/\mu + \gamma_*)}{(\mu + \alpha - \alpha\gamma_*/\mu + \gamma_*)}(\mathscr{R}_0(0) - 1) \\ &\quad \cdot \left[\left(2\mu + \frac{(\alpha + r_2 + \mu)(\mu + \alpha_1 - \alpha_1\gamma_*/\mu + \gamma_*)}{(\mu + \alpha - \alpha\gamma_*/\mu + \gamma_*)}(\mathscr{R}_0(0) - 1) + \alpha_1 + \gamma_*\right)(\mu + \alpha + \gamma_*) - \alpha\mu\right] \\ &\quad + \left(2\mu + \frac{(\alpha + r_2 + \mu)(\mu + \alpha_1 - \alpha_1\gamma_*/\mu + \gamma_*)}{(\mu + \alpha - \alpha\gamma_*/\mu + \gamma_*)}(\mathscr{R}_0(0) - 1) + \alpha_1 + \gamma_*\right) \\ &\quad \cdot \left[(\mu + \alpha_1)(\mu + \gamma_*) - \alpha_1\gamma_*\right]. \end{aligned} \tag{44}$$

Therefore, the Routh–Hurwitz criterion ensures that all the roots of $f(\lambda, 0)$ have negative real parts. Namely, E_* is locally asymptotically stable for $\mathscr{R}_0(\tau) > 1$ and $\tau = 0$.

In fact, in the case $\mathscr{R}_0(\tau) > 1$, the characteristic roots have continuous dependence on τ which implies that Theorem 4 is still valid for $\tau > 0$ sufficiently small. However, some roots of (34) may cross the imaginary axis to the right part as τ increases. Therefore, we will further insight into the stability of E_* when $\mathscr{R}_0(\tau) > 1$ and $\tau > 0$ in the next section. □

5. Stability and Hopf Bifurcation When $R_0(\tau) > 1$ and $\tau > 0$

When $\tau > 0$, we rewrite the characteristic equation $f(\lambda, \tau) = 0$ as a transcendental equation as follows:

$$P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0, \tag{45}$$

where

$$P(\lambda,\tau)=\lambda^3+m_2(\tau)\lambda^2+m_1(\tau)\lambda+m_0(\tau),$$
$$Q(\lambda,\tau)=n_1(\tau)\lambda+n_0(\tau). \tag{46}$$

It is clear that $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are both analytic function with respect to λ and differentiable with respect to τ. Following section 2 in [18], we need to justify the following hypotheses:

(i) $P(0,\tau)+Q(0,\tau)\neq 0$

(ii) $P(i\omega,\tau)+Q(i\omega,\tau)\neq 0$

(iii) $\limsup\{|Q/P|: |\lambda|\longrightarrow\infty, \mathrm{Re}\lambda\geq 0\}<1$, $\forall\tau\in R_+$

(iv) $F(\omega,\tau)=|P(i\omega,\tau)|^2-|Q(i\omega,\tau)|^2$

(v) Each positive roots $\omega(\tau)$ of $F(\omega,\tau)=0$ is continuous and differentiable in τ whenever it exists

Through a tedious manipulation, we derive

$$\begin{aligned}
P(0,\tau)+Q(0,\tau)&=m_0(\tau)+n_0(\tau)=\beta I_*\alpha\left(\mu+\gamma_*-\gamma_*e^{-\mu\tau}\right)>0,\\
P(i\omega,\tau)+Q(i\omega,\tau)&=-\left(2\mu+\beta I_*+\alpha_1+\gamma_*\right)+\beta I_*\alpha\left(\mu+\gamma_*-\gamma_*e^{-\mu\tau}\right)\\
&\quad+i\omega\left[\beta I_*\left(\mu+\alpha+\gamma_*\right)+\left(\mu+\alpha_1\right)\left(\mu+\gamma_*\right)-\alpha_1\gamma_*e^{-\mu\tau}-\omega^2\right]\neq 0,\\
\lim_{|\lambda|\longrightarrow\infty}\left|\frac{Q(\lambda,\tau)}{P(\lambda,\tau)}\right|&=\lim_{|\lambda|\longrightarrow\infty}\left|\frac{n_1(\tau)\lambda+n_0(\tau)}{\lambda^3+d_2(\tau)\lambda^2+d_1(\tau)\lambda+d_0(\tau)}\right|=0,
\end{aligned} \tag{47}$$

which implies that conditions (i), (ii), and (iii) are satisfied.

Noting that

$$\begin{aligned}
|P(i\omega,\tau)|^2&=\omega^6+\left(m_2^2(\tau)-2m_1(\tau)\right)\omega^4\\
&\quad+\left(m_1^2(\tau)-2m_2(\tau)m_0(\tau)\right)\omega^2+m_0^2(\tau),\\
|Q(i\omega,\tau)|^2&=n_1^2(\tau)\omega^2+n_0^2(\tau),
\end{aligned} \tag{48}$$

which admits

$$\begin{aligned}
F(\omega,\tau)&=\omega^6+\left(m_2^2(\tau)-2m_1(\tau)\right)\omega^4\\
&\quad+\left(m_1^2(\tau)-2m_2(\tau)m_0(\tau)-n_1^2(\tau)\right)\omega^2\\
&\quad+m_0^2(\tau)-n_0^2(\tau).
\end{aligned} \tag{49}$$

Obviously, condition (iv) readily holds, and the implicit function theorem ensures that condition (v) is also satisfied.

Let $\lambda=i\omega$, $\omega>0$, be one purely imaginary root of $f(\lambda,\tau)=0$. Then, we calculate that

$$\begin{aligned}
n_1(\tau)\omega\sin\omega\tau+n_0(\tau)\cos\omega\tau&=m_2(\tau)\omega^2-m_0(\tau),\\
n_0(\tau)\sin\omega\tau-n_1(\tau)\omega\cos\omega\tau&=m_1(\tau)\omega-\omega^3.
\end{aligned} \tag{50}$$

Setting $\Theta=\omega^2$, then (49) can be rewrittten as

$$Q(\Theta):=\Theta^3+q_2(\tau)\Theta^2+q_1(\tau)\Theta+q_0(\tau), \tag{51}$$

where $q_2(\tau)=m_2^2(\tau)-2m_1(\tau)$, $q_1(\tau)=m_1^2(\tau)-2m_2(\tau)m_0(\tau)-n_1^2(\tau)$, and $q_0(\tau)=m_0^2(\tau)-n_0^2(\tau)$. It is easy to verify $q_0(\tau)>0$.

Let $Q'(\Theta)=3\Theta^2+2q_2(\tau)\Theta+q_1(\tau)$. If $q_2^2(\tau)-3q_1(\tau)<0$, then $Q'(\Theta)=0$ has no real roots. While if $q_2^2(\tau)-3q_1(\tau)\geq 0$, then $Q'(\Theta)=0$ has two real roots, which are $\Theta_1=-q_2(\tau)-\sqrt{q_2^2(\tau)-3q_1(\tau)}/3$ and $\Theta_2=-q_2(\tau)-\sqrt{q_2^2(\tau)-3q_1(\tau)}/3$, respectively. The following lemma gives the results on the positive root of the equation $Q(\Theta)=0$.

Lemma 3.

(i) If $q_2^2(\tau)<3q_1(\tau)$, then $Q(\Theta)=0$ has no positive root

(ii) If $q_2^2(\tau)\geq 3q_1(\tau)$ and $\Theta_2\leq 0$, then $Q(\Theta)=0$ has no positive root

(iii) If $q_2^2(\tau)\geq 3q_1(\tau)$, $\Theta_2>0$, and $Q(\Theta_2)>0$, then $Q(\Theta)=0$ has no positive root

(iv) If $q_2^2(\tau)\geq 3q_1(\tau)$, $\Theta_2>0$, and $Q(\Theta_2)\leq 0$, then $Q(\Theta)=0$ has the positive roots

If (51) does not have any positive root, then the stability of E_* will not change as τ increases. Therefore, we have the following result.

Theorem 6. *Suppose that $\mathscr{R}_0(\tau)>1$ and $\tau>0$.*

(i) If $q_2^2(\tau)<3q_1(\tau)$, then the computer virus equilibrium E_* of system (4) is locally asymptotically stable

(ii) If $q_2^2(\tau)\geq 3q_1(\tau)$ and $\Theta_2\leq 0$, then the computer virus equilibrium E_* of system (4) is locally asymptotically stable

(iii) If $q_2^2(\tau)\geq 3q_1(\tau)$, $\Theta_2>0$, and $Q(\Theta_2)>0$, then the computer virus equilibrium E_* of system (4) is locally asymptotically stable

In what follows, we assume that (51) has one positive root. It implies that the stability of the computer virus equilibrium E_* may change once τ passes through some specific values. Let Θ_* be the root of $Q(\Theta)=0$. Namely, $\omega(\tau_*)=\sqrt{\Theta_*}$ is the unique positive real root of $F(\omega,\tau)=0$. For the sake of convenience, let us define a set by

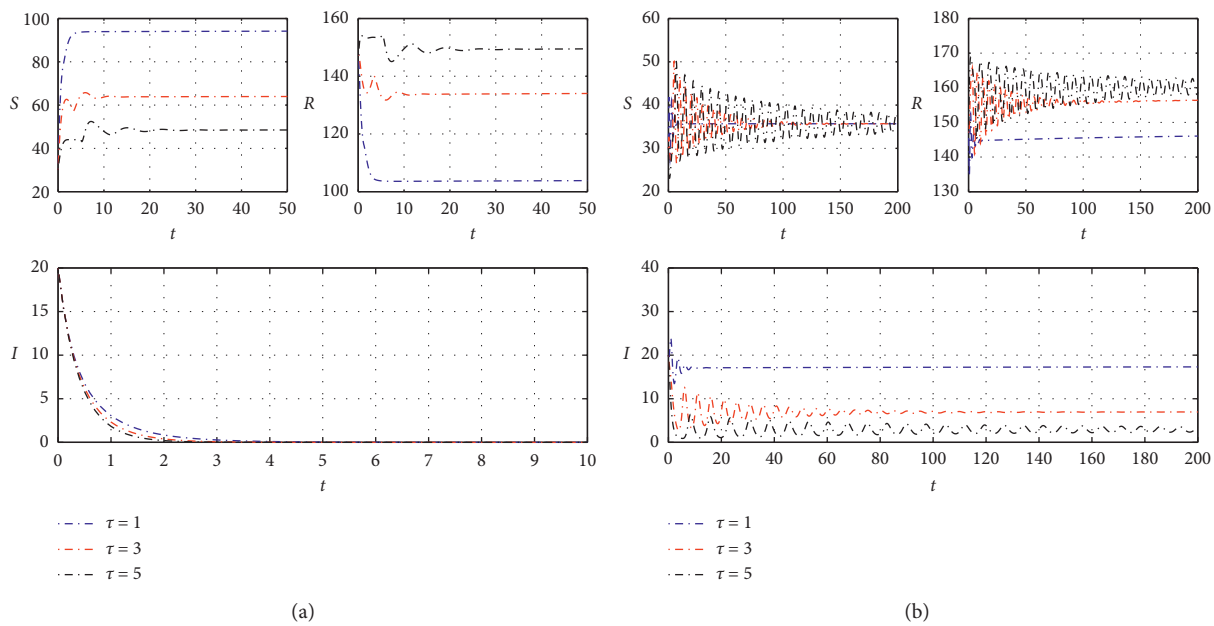

Figure 2: The impact of τ on the number of the susceptible computers $S(t)$, the infected computers $I(t)$, and the recovery computers $R(t)$: (a) $\mathscr{R}_0(\tau) < 1$; (b) $\mathscr{R}_0(\tau) > 1$.

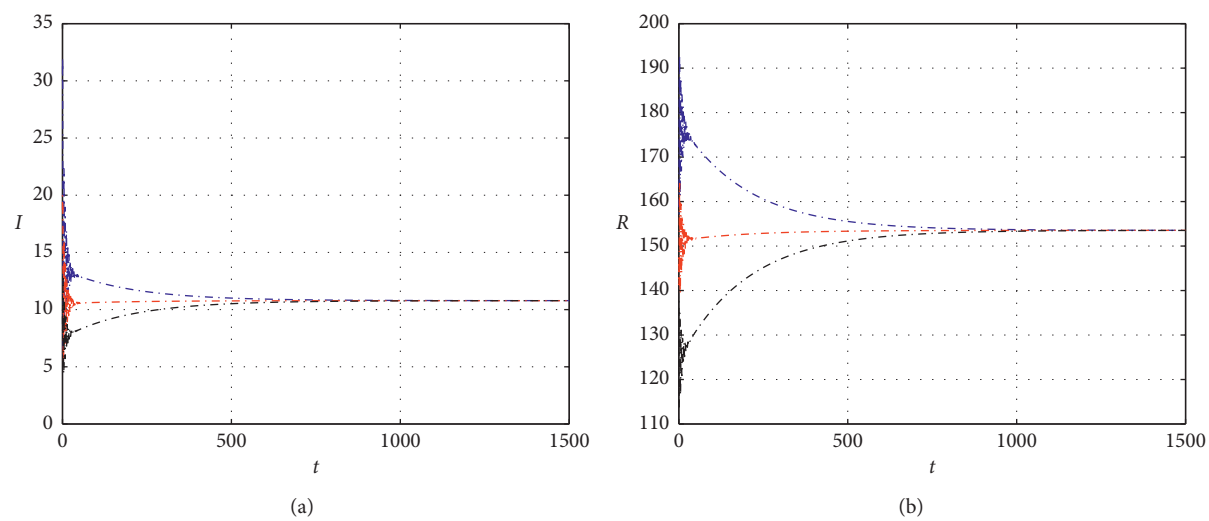

Figure 3: The stability of the computer virus equilibrium E_* of system (4) when $q_2^2(\tau) < 3q_1(\tau)$ and $\mathscr{R}_0(\tau) > 1$: (a) the infectious computer; (b) the recovered computer.

$$\Sigma = \left\{\tau > 0\colon q_2^2(\tau) \geq 3q_1(\tau), \Theta_2 > 0, Q(\Theta_2) \leq 0\right\}. \tag{52}$$

That is, for $\tau \in \Sigma$, there exists $\omega = \omega(\tau) > 0$ such that $F(\omega, \tau) = 0$.

Let $\theta(\tau) \in 0, 2\pi\tau \in \Sigma$ be a solution of the following equations:

$$\begin{aligned} \cos\theta(\tau) &= \frac{n_1(\tau)\omega^4 + (m_2(\tau)n_0(\tau) - m_1(\tau)n_1(\tau))\omega^2 - m_0(\tau)n_0(\tau)}{n_0^2(\tau) + n_1^2(\tau)\omega^2}, \\ \sin\theta(\tau) &= \frac{m_2(\tau)n_1(\tau) - n_0(\tau)\omega^3 + (m_1(\tau)n_0(\tau) - m_0(\tau)n_1(\tau))\omega}{n_0^2(\tau) + n_1^2(\tau)\omega^2}. \end{aligned} \tag{53}$$

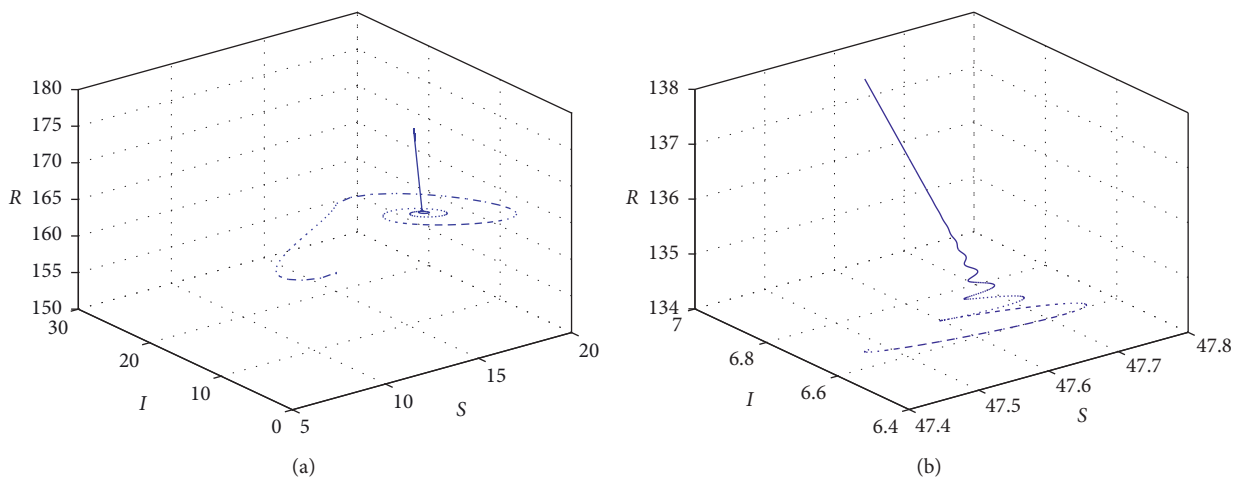

Figure 4: The stability of the computer virus equilibrium E_* of system (4) when $\mathscr{R}_0(\tau) > 1$ and $q_2^2(\tau) \geq 3q_1(\tau)$: (a) $\Theta 2 = -1.6804$; (b) $\Theta 2 = 0.4363$ and $Q(\Theta 2) = 0.0576$.

Then, we conclude that $\omega(\tau)\tau = \theta(\tau) + 2n\pi$. Hence, $\omega_* = \omega(\tau_*)(\omega_* > 0)$ is a purely imaginary root of (45) if and only if τ_* is a zero of $C_n(\tau)$ for some $n \in \mathbb{N}$, which is defined by

$$C_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad \tau \in \Sigma, n \in \mathbb{N}. \tag{54}$$

Theorem 2.2 in [18] implies that the following lemma is true.

Lemma 4 (see [18]). *Assume that $\omega(\tau)$ is a positive real root of $F(\omega, \tau) = 0$ for $\tau \in \Sigma$, and at some $\tau_* \in \Sigma$,*

$$C_n(\tau_*) = 0, \quad \text{for some } n \in \mathbb{N}. \tag{55}$$

Then, a pair of simple conjugate pure imaginary roots $\lambda_+(\tau_*) = i\omega(\tau_*)$ and $\lambda_-(\tau_*) = -i\omega(\tau_*)$ of the characteristic (45) exists at $\tau = \tau_*$ which crosses the imaginary axis from left to right if $H(\tau_*) > 0$ and crosses the imaginary axis from right to left if $H(\tau_*) < 0$, where

$$\begin{aligned} H(\tau_*) &= \operatorname{sign}\left\{\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}\Big|_{\lambda=i\omega(\tau_*)}\right\} \\ &= \operatorname{sign}\{F_\omega'(\omega(\tau_*), \tau_*)\}\operatorname{sign}\left\{\frac{\mathrm{d}C_n(\tau)}{\mathrm{d}\tau}\Big|_{\tau=\tau_*}\right\}. \end{aligned} \tag{56}$$

The relationship between $F(\omega, \lambda)$ and $Q(\Theta)$ leads to

$$\begin{aligned} H(\tau_*) &= \operatorname{sign}\left\{\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}\Big|_{\lambda=i\omega(\tau_*)}\right\} \\ &= \operatorname{sign}\{Q'(\Theta_*)\}\operatorname{sign}\left\{\frac{\mathrm{d}C_n(\tau)}{\mathrm{d}\tau}\Big|_{\tau=\tau_*}\right\}, \end{aligned} \tag{57}$$

which implies that the transversality condition holds and a Hopf bifurcation occurs at $\tau = \tau_*$ when $Q'(\Theta_*) \neq 0$. According to the Hopf bifurcation theorem for functional differential equations [19], we have the following result.

Theorem 7. *Suppose that $\mathscr{R}_0(\tau) > 1$ and $Q'(\Theta_*) \neq 0$. If $q_2^2(\tau) \geq 3q_1(\tau)$, $\Theta_2 > 0$, and $Q(\Theta_2) \leq 0$, then the computer virus equilibrium E_* of system (4) is locally asymptotically stable for $\tau \in 0, \tau_*$, and system (4) undergoes a Hopf bifurcation at the computer virus equilibrium E_* when $\tau = \tau_*$.*

6. The Numerical Simulations

In the following, we will proceed with Matlab to validate the oscillation behaviors of system (4). Let the maximum acquired immunity age be 100, $p = 0.2$, $b = 1$, $\mu = 0.005$, $\alpha = 2.8$, $\gamma_* = 0.9$, $r_2 = 0.05$, and $R(t) = \int_0^{+\infty} R(t, a)\mathrm{d}a$. Then, we first illustrate the impact of τ on the number of the susceptible computers $S(t)$, the infected computers $I(t)$, and the recovery computers $R(t)$. Taking $\tau = 1, 3, 5$, then $\mathscr{R}_0(\tau) < 1$ when $\beta = 0.02$ and $\mathscr{R}_0(\tau) > 1$ when $\beta = 0.08$. Figure 2 displays that the number of the susceptible computers $S(t)$ and the infected computers $I(t)$ decreases as the delay τ increases and the number of the recovery computers $R(t)$ increases as the delay τ increases.

Next, we show that the stability of the computer virus equilibrium E_* and the Hopf bifurcation happens around the computer virus equilibrium E_* under different conditions. Choosing $\beta = 0.08$, $\tau = 2$, and $\alpha_1 = 0.5$, we can obtain $\mathscr{R}_0(2) = 2.1978 > 1$ and $q_2^2(\tau) - 3q_1(\tau) = -5.9250 < 0$. Figure 3 exhibits the solution of system (4) with different initial values which will tend to E_* as t tends to infinity.

If we take $\beta = 0.2$, $\tau = 2$, and $\alpha_1 = 2.5$, then $\mathscr{R}_0(2) = 1.6036 > 1$, $q_2^2(\tau) - 3q_1(\tau) = 12.6772 > 0$, and $\Theta_2 = -1.6804 < 0$. Figure 4(a) shows that the computer virus equilibrium E_* is a stable focus. While if we take $\beta = 0.06$, $\tau = 2$, and $\alpha_1 = 0.5$, then $\mathscr{R}_0(2) = 1.6484 > 1$, $q_2^2(\tau) - 3q_1(\tau) = 0.1442 > 0$, $\Theta_2 = 0.4363 > 0$, and $Q(\Theta_2) = 0.0576 > 0$. Accordingly, Figure 4(b) shows that the computer virus equilibrium E_* is a stable focus again.

Finally, setting $\beta = 0.05$, $\alpha_1 = 0.1$, and $\tau = 2$, we have $\mathscr{R}_0(2) = 2.6688 > 1$, $q_2^2(\tau) - 3q_1(\tau) = 2.7028 > 0$, $\Theta_2 = 1.2942 > 0$, and $Q(\Theta_2) = -0.5037 < 0$. Figure 5 displays that system (4) exists the periodic solutions.

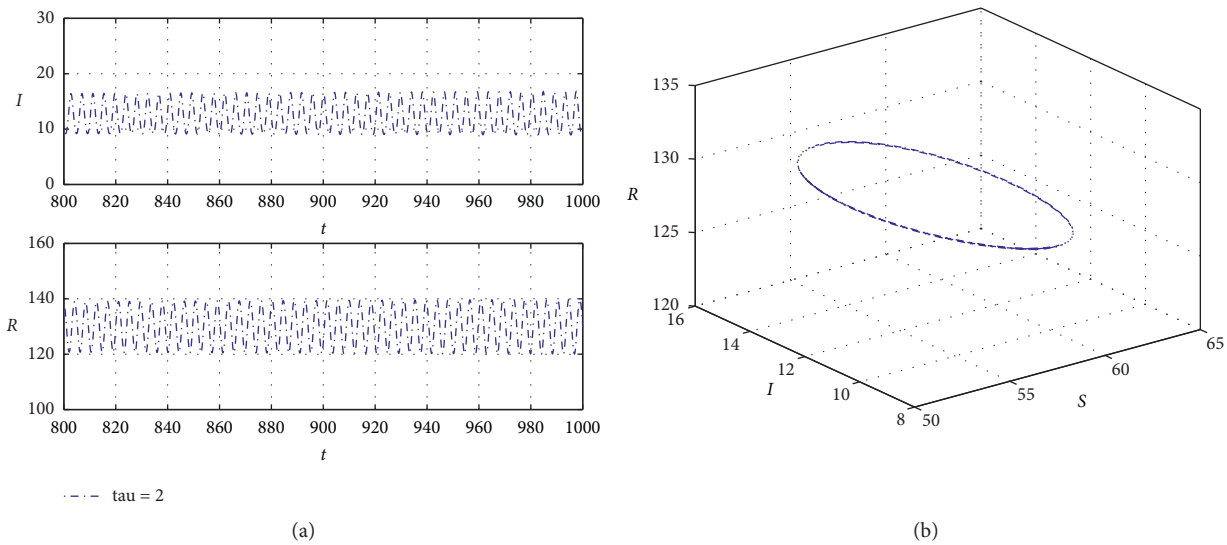

Figure 5: The hopf bifurcation around the computer virus equilibrium E_* when $\mathscr{R}_0(\tau) > 1$: (a) the periodic solution; (b) the phase diagram.

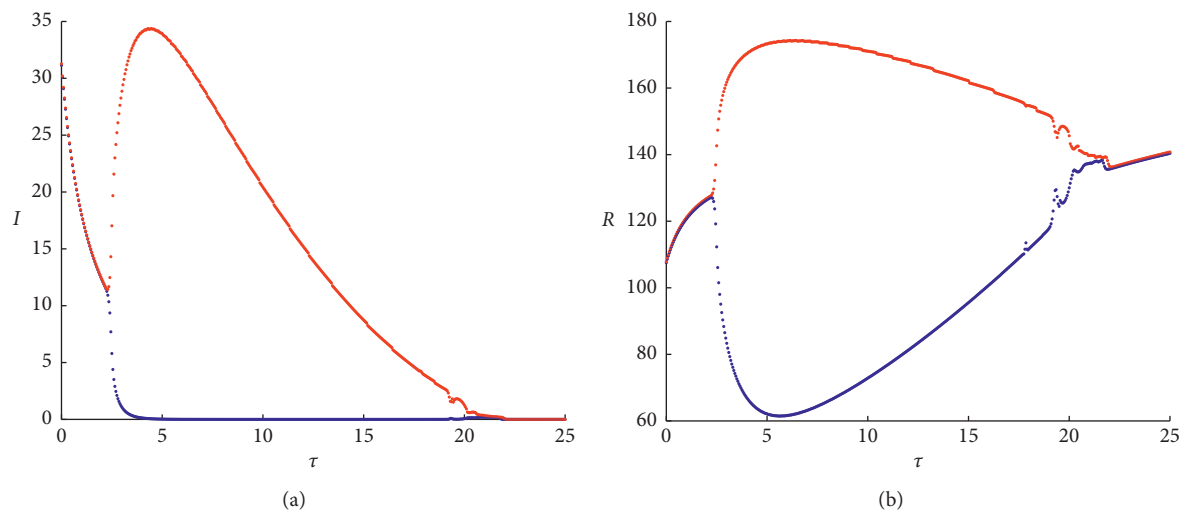

Figure 6: The dynamic behaviors of system (4) as the delay τ increases: (a) the virus-infected computers; (b) the recovery computers.

In addition, choosing the same parameter values as these of Figure 5 and using τ as the bifurcation parameter, we can display the complete dynamic behavior of system (4) as the delay τ increases in Figure 6. It illustrates that system (4) exists the periodic solutions for $\tau \in 2, 23$. When $\tau \in 0, 2$, we obtain $\mathscr{R}_0(\tau) > 1$, namely, system (4) has a unique computer virus equilibrium E_*, which is stable. When $\tau \geq 23$, we obtain $\mathscr{R}_0(\tau) < 1$, namely, system (4) has a unique virus-free equilibrium E_0, which is stable.

7. Conclusion and Discussion

In this paper, we have proposed and analyzed a computer virus model by using of the classic SIRS model with the age structure and delay. The age structure and delay are combined to describe the phenomenon that the recovery computers stay in the recovery class for a short time and eventually become the susceptible computers again since the loss of immune protection. The purpose of this paper is to explore the impact of the age structure and delay on the transmission of the computer virus.

On the dynamic behavior analysis of system (4), we showed that the non-negativity of the solutions of system (4) and the boundedness of system (4) gave the basic reproduction number $\mathscr{R}_0(\tau)$ and proved that $\mathscr{R}_0(\tau) = 1$ is the threshold that determines whether the epidemic persists or not by studying the stability of both virus-free equilibrium E_0 and the computer virus equilibrium E_*. The virus-free equilibrium E_0 is globally asymptotically stable if $\mathscr{R}_0(\tau) < 1$ and is unstable if $\mathscr{R}_0(\tau) > 1$. Moreover, the computer virus persists in the later case, in the sense that infected computers survive above a certain number for any initial infection numbers. We also proved the existence of Hopf bifurcation around the computer virus equilibrium E_* when E_* is unstable.

The existence of Hopf bifurcation of system (4) means that, if the age structure and delay are introduced together into the computer virus SIRS model, the simple threshold dynamic behavior will be destroyed. We speculate that the essential reason for the changes in the dynamic behavior of system (4) is that the acquired immunity age is subject to a non-Markovian process.

The limitation of this paper is that there is no discussion about the global stability of the computer virus equilibrium E_* when $\mathscr{R}_0(\tau) > 1$ and $\tau = 0$. In addition, the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions from E_* have not been resolved in this paper. We will continue to discuss these aspects in the future.

Appendix

In this section, we show Theorem 1 of [20] as the following Lemma A. 1. Some definitions and notations are first listed as follows.

Let X be a Banach space with norm $\|\cdot\|_X$ and $A > 0$ be the maximal age. We set $X = R^n \times Y$ where $Y = (L^1(0, A))^m$. Let $v \in C([0, T], R^n)$ and $u \in C([0, T], Y)$. Let $D \geq 0$ be a $R^m \times R^m$ diagonal matrix of functions which may not be bounded on $[0, A)$; that is, we may have $\lim_{a \longrightarrow A} d_j(a) = \infty$ for some or all j and if A is finite. Consider the system

$$\begin{aligned} v' &= G(t; v, u), \\ u_t + u_a &= F(t, a; v, u) - Du\ , \\ v(0) &= v_0, \\ u(t, 0) &= B(t; v, u), \\ u(0, a) &= u_0(a), \end{aligned} \tag{A.1}$$

where G, B: $0, T \times 0, A \times X \longrightarrow R^n$, and F: $0, T \times 0, A \times X \longrightarrow Y$ are generally nonlinear. Let X_+ denote the positive cone in X. Let $(v_0, u_0) \in X_+$, $\Omega \subset X_+$ be a bounded invariant set for system (A.1).

Lemma 5. *Assume the following:*

(1) G, B, and F are Lipschitz continuous; that is, for all $v, \overline{v} \in R^n$ and all $u, \overline{u} \in Y$ (or $(v, u) \in \Omega$, $(\overline{v}, \overline{u}) \in \Omega$), and all $t \in [0, T]$, we have

$$\begin{aligned} &|G(t; v, u) - G(t; \overline{v}, \overline{u})|_{R^n} \leq K_G\left(|v - \overline{v}|_{R^n} + \|u - \overline{u}\|_Y\right), \\ &|B(t; v, u) - B(t; \overline{v}, \overline{u})|_{R^m} \leq K_B\left(|v - \overline{v}|_{R^n} + \|u - \overline{u}\|_Y\right), \\ &\|F(t; v, u) - F(t; \overline{v}, \overline{u})\|_Y \leq K_F\left(|v - \overline{v}|_{R^n} + \|u - \overline{u}\|_Y\right). \end{aligned} \tag{A.2}$$

(2) $F(t, a; v, u) = F_1(t, a; v, u) - F_2(t, a; v, u)u$ where F_1, $F_2 \geq 0$ for all $t \in [0, T]$, $a \in 0, A$, $v \in R^n_+$, and $u \in Y_+$ and $|F_2(t, a; v, u)|_{R^m} \leq K_2$ for all $t \in [0, T]$, $a \in 0, A$, $v \in R^n_+$, and $u \in Y_+$ or all $(v, u) \in \Omega$.

(3) $G(t; v, u) = G_1(t; v, u) - G_2(t; v, u)v$ where G_1, $G_2 \geq 0$ for all $t \in [0, T]$, $v \in R^n_+$, and $u \in Y_+$ and $|G_2(t; v, u)|_{R^n} \leq K_2$ for all $t \in [0, T]$, $v \in R^n_+$, and $u \in Y_+$ or all $(v, u) \in \Omega$.

(4) $B(t; v, u) \geq 0$ for all $t \in [0, T]$, $v \in R^n_+$, and $u \in Y_+$.

(5) $|G(t; 0, 0)|_{R^n}$, $\|F(t; 0, 0)\|_Y$, and $B(t; 0, 0)0$ are bounded. Then, for every $(v_0, u_0) \in X_+$ or in Ω, system (A.1) has a unique non-negative solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] F. Cohen, "Computer viruses," *Computers & Security*, vol. 6, no. 1, pp. 22–35, 1987.

[2] J. C. Wierman and D. J. Marchette, "Modeling computer virus prevalence with a susceptible-infected-susceptible model with reintroduction," *Computational Statistics & Data Analysis*, vol. 45, no. 1, pp. 3–23, 2004.

[3] X. Liang, Y. Pei, and Y. Lv, "Modeling the state dependent impulse control for computer virus propagation under media coverage," *Physica A: Statistical Mechanics and Its Applications*, vol. 491, pp. 516–527, 2018.

[4] J. Ren, X. Yang, Q. Zhu, L.-X. Yang, and C. Zhang, "A novel computer virus model and its dynamics," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 1, pp. 376–384, 2012.

[5] C. Gan, X. Yang, W. Liu, Q. Zhu, and X. Zhang, "An epidemic model of computer viruses with vaccination and generalized nonlinear incidence rate," *Applied Mathematics and Computation*, vol. 222, pp. 265–274, 2013.

[6] C. Gan, X. Yang, W. Liu, and Q. Zhu, "A propagation model of computer virus with nonlinear vaccination probability," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 1, pp. 92–100, 2014.

[7] J. Amador and J. R. Artalejo, "Stochastic modeling of computer virus spreading with warning signals," *Journal of the Franklin Institute*, vol. 350, no. 5, pp. 1112–1138, 2013.

[8] U. Fatima, M. Ali, N. Ahmed, and M. Rafiq, "Numerical modeling of susceptible latent breaking-out quarantine computer virus epidemic dynamics," *Heliyon*, vol. 4, p. e00631, 2018.

[9] L.-X. Yang and X. Yang, "A new epidemic model of computer viruses," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 6, pp. 1935–1944, 2014.

[10] L. Chen, K. Hattaf, and J. Sun, "Optimal control of a delayed SLBS computer virus model," *Physica A: Statistical Mechanics and Its Applications*, vol. 427, pp. 244–250, 2015.

[11] J. R. C. Piqueira, A. A. de Vasconcelos, C. E. C. J. Gabriel, and V. O. Araujo, "Dynamic models for computer viruses," *Computers & Security*, vol. 27, no. 7-8, pp. 355–359, 2008.
[12] Z. Hu, H. Wang, F. Liao, and W. Ma, "Stability analysis of a computer virus model in latent period," *Chaos, Solitons & Fractals*, vol. 75, pp. 20–28, 2015.
[13] J. Singh, D. Kumar, Z. Hammouch, and A. Atangana, "A fractional epidemiological model for computer viruses pertaining to a new fractional derivative," *Applied Mathematics and Computation*, vol. 316, pp. 504–515, 2018.
[14] Q. A. Dang and M. T. Hoang, "Positivity and global stability preserving NSFD schemes for a mixing propagation model of computer viruses," *Journal of Computational and Applied Mathematics*, vol. 374, Article ID 112753, 2020.
[15] J. Ren and Y. Xu, "A compartmental model for computer virus propagation with kill signals," *Physica A: Statistical Mechanics and Its Applications*, vol. 486, pp. 446–454, 2017.
[16] W. M. Hirsch, H. Hanisch, and J.-P. Gabriel, "Differential equation models of some parasitic infections: methods for the study of asymptotic behavior," *Communications on Pure and Applied Mathematics*, vol. 38, no. 6, pp. 733–753, 1985.
[17] M. Iannelli, "Mathematical theory of age-structured population dynamics," in *Applied Mathematics Monographs*Vol. 7, Giardini editori e stampatori, Pisa, Italy, 1995.
[18] E. Beretta and Y. Kuang, "Geometric stability switch criteria in delay differential systems with delay dependent parameters," *SIAM Journal on Mathematical Analysis*, vol. 33, no. 5, pp. 1144–1165, 2002.
[19] J. K. Hale, *Theory of Function Differential Equations*, Springer, Heidelberg, Germany, 1977.
[20] X. Li, J. Yang, and M. Martcheva, *Age Structured Epidemic Modelling*, Springer, Geneva, Switzerland, 2020.