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Research Article

Stability of a Three-Species Cooperative System with Time Delays and Stochastic Perturbations

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Considering the impacts of time delays and different kinds of stochastic perturbations, we propose two three-species delayed cooperative systems with stochastic perturbations in this paper. We establish the sufficient criteria of the asymptotical stability and stability in probability by constructing a neutral stochastic differential equation and some suitable functionals. The impacts of time delays and stochastic perturbations to the system dynamics are revealed by some numerical simulations at the end.

1. Introduction

In biological world, mutualism is an important interaction and many cooperative models have been proposed to describe such biological phenomena (see [1–3] and references cited therein).

In nature, time delays usually appear and bring some important influence to the dynamics of ecosystem models. Kuang [4] says that ignoring time delays means ignoring the reality. It is essential to take the influence of time delays into account in mathematical modelling [5–9]. Furthermore, in view of the complexity of natural world, single-species or two-species ecological models often cannot describe some natural phenomena accurately and many vital behaviours can only be exhibited by systems with three or more species [10–13]. Considering the effect of time delays to three-species cooperative system, we propose the following deterministic model:

$$\begin{cases} dN_1(t) = N_1(t)\left(r_1 - a_{11}N_1(t) + a_{12}N_2(t-\tau_{12}) + a_{13}N_3(t-\tau_{13})\right)dt, \\ dN_2(t) = N_2(t)\left(r_2 + a_{21}N_1(t-\tau_{21}) - a_{22}N_2(t) + a_{23}N_3(t-\tau_{23})\right)dt, \\ dN_3(t) = N_3(t)\left(r_3 + a_{31}N_1(t-\tau_{31}) + a_{32}N_2(t-\tau_{32}) - a_{33}N_3(t)\right)dt. \end{cases} \tag{1}$$

For (1), we define

$$\begin{aligned} \alpha_1 &= \left(a_{11}, -a_{12}, -a_{13}\right)^T, \\ \alpha_2 &= \left(-a_{21}, a_{22}, -a_{23}\right)^T, \\ \alpha_3 &= \left(-a_{31}, -a_{32}, a_{33}\right)^T, \\ R &= \left(r_1, r_2, r_3\right)^T, \\ A &= \det\left(\alpha_1, \alpha_2, \alpha_3\right), \\ A_1 &= \det\left(R, \alpha_2, \alpha_3\right), \\ A_2 &= \det\left(\alpha_1, R, \alpha_3\right), \\ A_3 &= \det\left(\alpha_1, \alpha_2, R\right), \end{aligned} \tag{2}$$

and assume that there exists a unique positive equilibrium point as follows:

$$\begin{aligned} N_1^* &= \frac{A_1}{A}, \\ N_2^* &= \frac{A_2}{A}, \\ N_3^* &= \frac{A_3}{A}. \end{aligned} \tag{3}$$

On the contrary, the growth of populations is often subject to environmental fluctuation, so it is necessary to consider stochastic perturbation in the process of mathematical modelling [14–18]. Usually, there are many kinds of stochastic perturbations. The authors in [19] proposed that the stochastic perturbation of state variable around the steady-state N_1^*, N_2^*, and N_3^* was Brownian white noise, which was proportional to the distance from the equilibrium state. Consequently, we obtain the following three-species stochastic cooperative system with time delays:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\big(r_1 - a_{11}N_1(t) + a_{12}N_2(t-\tau_{12}) \\ \qquad + a_{13}N_3(t-\tau_{13})\big)\mathrm{d}t + \sigma_1\big(N_1(t) - N_1^*\big)\mathrm{d}\omega_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\big(r_2 + a_{21}N_1(t-\tau_{21}) - a_{22}N_2(t) \\ \qquad + a_{23}N_3(t-\tau_{23})\big)\mathrm{d}t + \sigma_2\big(N_2(t) - N_2^*\big)\mathrm{d}\omega_2(t), \\ \mathrm{d}N_3(t) = N_3(t)\big(r_3 + a_{31}N_1(t-\tau_{31}) + a_{32}N_2(t-\tau_{32}) \\ \qquad - a_{33}N_3(t)\big)\mathrm{d}t + \sigma_3\big(N_3(t) - N_3^*\big)\mathrm{d}\omega_3(t), \end{cases} \tag{4}$$

with initial data

$$N_i(\theta) = \phi_i(\theta) > 0, \quad \theta \in [-\tau, 0],\ i = 1, 2, 3, \tag{5}$$

where $N_i(t)$ stands for the population size of the i^{th} species at time t, $r_i > 0$ is the growth rate of $N_i(t)$, $a_{ii} > 0$ represents the intraspecific competitive coefficient of $N_i(t)$, a_{ij} is the interspecific cooperative rate, $\tau_{ij} > 0$ is time delay, $\tau = \max\{\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}\}$, $\phi_i(\theta)$ is positive and continuous function defined on $[-\tau, 0]$, σ_i^2 denotes the intensity of white noise, and $\omega_i(t)_{t>0}$ is the standard independent Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\ge 0}, P)$, $i, j = 1, 2, 3$, $j \ne i$.

Furthermore, Liu [20] proposed that the stochastic perturbation of state variable was proportional to $N_i(N_i - N_i^*)$ $(i = 1, 2, 3)$, and then, we get the following model:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\big(r_1 - a_{11}N_1(t) + a_{12}N_2(t-\tau_{12}) + a_{13}N_3(t-\tau_{13})\big)\mathrm{d}t \\ \qquad + \sigma_1 N_1(t)\big(N_1(t) - N_1^*\big)\mathrm{d}\omega_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\big(r_2 + a_{21}N_1(t-\tau_{21}) - a_{22}N_2(t) + a_{23}N_3(t-\tau_{23})\big)\mathrm{d}t \\ \qquad + \sigma_2 N_2(t)\big(N_2(t) - N_2^*\big)\mathrm{d}\omega_2(t), \\ \mathrm{d}N_3(t) = N_3(t)\big(r_3 + a_{31}N_1(t-\tau_{31}) + a_{32}N_2(t-\tau_{32}) - a_{33}N_3(t)\big)\mathrm{d}t \\ \qquad + \sigma_3 N_3(t)\big(N_3(t) - N_3^*\big)\mathrm{d}\omega_3(t). \end{cases} \tag{6}$$

For the deterministic system, whether there exists a positive equilibrium state is an important topic, which attracts many attentions of researchers, while stochastic model cannot tend to a positive fixed point; i.e., there exists no traditional positive equilibrium state, so it is popular to study the dynamics of stochastic system around the equilibrium state of its corresponding deterministic system. Furthermore, some kinds of delays are considered in biological systems, but many obtained results have no relation with delays and the effects of time delays are not revealed clearly [9, 11]. Time delays are the sources of instability in population dynamics, and they can cause population fluctuations, so it is interesting to study the effects of delays to the system dynamics. Motivated by these, we propose two three-species delayed cooperative systems with stochastic perturbation and aim to investigate how time delays affect the stability in probability around the equilibrium state and then by numerical examples to validate our theoretical results. To the best of our knowledge, this paper is the first attempt to investigate the influence of time delays on the stability in probability of a stochastic three-species cooperative system.

The rest work of this paper is structured as follows. Section 2 begins with definitions and some important lemmas and notations. Section 3 focuses on the asymptotical stability and stability in probability of (4) and (6), respectively. Some numerical simulations are given in Section 4 to validate our theoretical results. Finally, a brief conclusion and future direction are given in Section 5 to conclude the paper.

2. Preliminaries

Let $\{\Omega, \sigma, P\}$ be a probability space and $\{f_t, t \ge 0\}$ be a family of σ-algebras. We consider the following neutral stochastic differential equation:

$$d\big(x(t) - F(t, x_t)\big) = a(t, x_t)\mathrm{d}t + b(t, x_t)\mathrm{d}\omega(t), \quad x_0 = \phi \in H, \tag{7}$$

where H represents the space with all f_0-adapted functions $\phi(s) \in R^n$ and $x_t(s) = x(t+s), s \leq 0, \omega(t)$ denotes the m-dimensional f_t-adapted Brownian process, $a(t,\phi)$ and $b(t,\phi)$ are n-dimensional vector and $n \times m$-dimensional matrix, respectively. Define

$$\|\phi\|_0 = \sup_{s\leq 0} |\phi(s)|,$$
$$\|\phi\|_1 = \sup_{s\leq 0} \mathbb{E}\left\{|\phi(s)|^2\right\}, \quad (8)$$

where $\mathbb{E}$ denotes the mathematical expectation.

Definition 1 (see [19]). The zero solution of (7) is said to be stable in probability if for any $\varepsilon, \varepsilon > 0$, there exists a number $\delta > 0$ such that the solution $x(t) = x(t,\phi)$ satisfies $P\{|x(t,\phi)| > \varepsilon\} < \varepsilon$ for any initial function $\phi \in H$ such that $P\{\|\phi_0\| \leq \delta\} = 1$, in which P is the probability of an event.

Lemma 1. *Systems (4) and (6) have a unique globally positive solution on $t > -\tau$ for any initial data given above, respectively.*

Remark 1. The proof is standard. For more details, refer to [20, 21].

Lemma 2 (see [19, 22]). *If there exists a functional $V(t,x)$ such that*

$$\lambda_1 |x(t)|^2 \leq V(t,x_t) \leq \lambda_2 \|x_t\|_0^2, \quad \lambda_1, \lambda_2 > 0, LV(t,x_t) \leq 0, \quad (9)$$

for any function $\phi \in H$ such that $P\{\|\phi\|_0 \leq \delta\} = 1$, $\delta > 0$, is sufficiently small positive constant, and then, the zero solution of (7) is stable in probability.

For convenience in the later, if there is no special mention, we always denote $f(t), t \in R$, by f and give the following notations to end this section:

$$\begin{aligned}
\xi_1 &= a_{21}(a_{21} + a_{23}),\\
\eta_1 &= a_{31}(a_{31} + a_{32}),\\
\xi_2 &= a_{12}(a_{12} + a_{13}),\\
\eta_2 &= a_{32}(a_{31} + a_{32}),\\
\xi_3 &= a_{13}(a_{12} + a_{13}),\\
\eta_3 &= a_{23}(a_{21} + a_{23}),\\
\Delta_1 &= 2a_{11} - a_{12} - a_{13} - \frac{\sigma_1^2}{N_1^*},\\
\Delta_2 &= 2a_{22} - a_{21} - a_{23} - \frac{\sigma_2^2}{N_2^*},\\
\Delta_3 &= 2a_{33} - a_{31} - a_{32} - \frac{\sigma_1^2}{N_3^*},\\
\tilde{\Delta}_1 &= 2a_{11} - a_{12} - a_{13} - \sigma_1^2 N_1^*,\\
\tilde{\Delta}_2 &= 2a_{22} - a_{21} - a_{23} - \sigma_2^2 N_2^*,\\
\tilde{\Delta}_3 &= 2a_{33} - a_{31} - a_{32} - \sigma_1^2 N_3^*.
\end{aligned} \quad (10)$$

3. Stochastic Stability

In this section, we investigate the asymptotical stability and stable in probability of (4) and (6) around the equilibrium state of (1), respectively. Let $x_1 = N_1 - N_1^*$, $x_2 = N_2 - N_2^*$, and $x_3 = N_3 - N_3^*$, then system (4) is transformed to the following equivalent system:

$$\begin{cases}
\mathrm{d}x_1(t) = \left(-a_{11}N_1^* x_1(t) + a_{12}N_1^* x_2(t-\tau_{12}) + a_{13}N_1^* x_3(t-\tau_{13}) - a_{11}x_1^2 \right.\\
\qquad \left. + a_{12}x_1x_2(t-\tau_{12}) + a_{13}x_1x_3(t-\tau_{13})\right)\mathrm{d}t + \sigma_1 x_1 \mathrm{d}\omega_1(t),\\
\mathrm{d}x_2(t) = \left(a_{21}N_2^* x_1(t-\tau_{21}) - a_{22}N_2^* x_2(t) + a_{23}N_2^* x_3(t-\tau_{23}) - a_{22}x_2^2 \right.\\
\qquad \left. + a_{21}x_2x_1(t-\tau_{21}) + a_{23}x_2x_3(t-\tau_{23})\right)\mathrm{d}t + \sigma_2 x_2 \mathrm{d}\omega_2(t),\\
\mathrm{d}x_3(t) = \left(-a_{33}N_3^* x_3(t) + a_{31}N_3^* x_3(t-\tau_{31}) + a_{31}N_3^* x_3(t-\tau_{32}) - a_{33}x_3^2 \right.\\
\qquad \left. + a_{31}x_3x_1(t-\tau_{31}) + a_{32}x_3x_2(t-\tau_{32})\right)\mathrm{d}t + \sigma_3 x_3 \mathrm{d}\omega_3(t).
\end{cases} \quad (11)$$

By the equivalent property of above transformation, the stability of (4) around the equilibrate state of (1) is equivalent to the stability of zero solution of corresponding equivalent system (11). Consequently, we only need to study the stability of zero solution of system (11).

By use of the definition of derivative, system (11) is equivalent to the following neutral system:

$$\begin{cases} d\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)\mathrm{d}s + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)\mathrm{d}s\right) \\ = \left(-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3 - a_{11}x_1^2 + a_{12}x_1x_2(t-\tau_{12}) \right. \\ \quad \left. + a_{13}x_1x_3(t-\tau_{13})\right)\mathrm{d}t + \sigma_1 x_1 \mathrm{d}\omega_1(t), \\ d\left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)\mathrm{d}s + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)\mathrm{d}s\right) \\ = \left(a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3 - a_{22}x_2^2 + a_{21}x_2x_1(t-\tau_{21}) \right. \\ \quad \left. + a_{23}x_2x_3(t-\tau_{23})\right)\mathrm{d}t + \sigma_2 x_2 \mathrm{d}\omega_2(t), \\ d\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)\mathrm{d}s + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)\mathrm{d}s\right) \\ = \left(a_{31}N_3^* x_1 + a_{32}N_3^* x_2 - a_{33}N_3^* x_3 - a_{33}x_3^2 + a_{31}x_3x_1(t-\tau_{31}) \right. \\ \quad \left. + a_{32}x_3x_2(t-\tau_{32})\right)\mathrm{d}t + \sigma_3 x_3 \mathrm{d}\omega_3(t). \end{cases} \tag{12}$$

The linear case of (12) is as follows:

$$\begin{cases} d\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)\mathrm{d}s + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)\mathrm{d}s\right) \\ = \left(-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3\right)\mathrm{d}t + \sigma_1 x_1 \mathrm{d}\omega_1(t), \\ d\left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)\mathrm{d}s + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)\mathrm{d}s\right) \\ = \left(a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3\right)\mathrm{d}t + \sigma_2 x_2 \mathrm{d}\omega_2(t), \\ d\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)\mathrm{d}s + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)\mathrm{d}s\right) \\ = \left(a_{31}N_3^* x_1 + a_{32}N_3^* x_2 - a_{33}N_3^* x_3\right)\mathrm{d}t + \sigma_3 x_3 \mathrm{d}\omega_3(t). \end{cases} \tag{13}$$

For the linear and neutral system (13), the following statement is true.

Theorem 1. *If the following condition holds,*

$$(H_1)\begin{pmatrix} \Delta_1 & -a_{21} & -a_{31} \\ -a_{12} & \Delta_2 & -a_{32} \\ -a_{13} & -a_{23} & \Delta_3 \end{pmatrix}\begin{pmatrix} N_1^* \\ N_2^* \\ N_3^* \end{pmatrix} > 2\tau \begin{pmatrix} 0 & \xi_1 & \eta_1 \\ \xi_2 & 0 & \eta_2 \\ \xi_3 & \eta_3 & 0 \end{pmatrix}\begin{pmatrix} (N_1^*)^2 \\ (N_2^*)^2 \\ (N_3^*)^2 \end{pmatrix}, \tag{14}$$

then the zero solution of (13) is globally asymptotically stable, almost surely, i.e., $\lim_{t\longrightarrow\infty} x_i(t) = 0, i = 1, 2, 3.$

Proof. First, by the characteristics of neutral functional differential system (13), we define

$$\begin{aligned} V_1 &= \left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)\mathrm{d}s + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)\mathrm{d}s\right)^2, \\ V_2 &= \left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)\mathrm{d}s + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)\mathrm{d}s\right)^2, \\ V_3 &= \left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)\mathrm{d}s + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)\mathrm{d}s\right)^2. \end{aligned} \tag{15}$$

Applying Itô formula to V_1, we have

$$\begin{aligned} LV_1 &= 2\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)\mathrm{d}s + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)\mathrm{d}s\right) \\ &\quad \times \left(-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3\right) + \sigma_1^2 x_1^2 \\ &= -2a_{11}N_1^* x_1^2 + 2a_{12}N_1^* x_1x_2 + 2a_{13}N_1^* x_1x_3 + \sigma_1^2 x_1^2 - 2a_{11}a_{22}(N_1^*)^2 \int_{t-\tau_{12}}^{t} x_1(t)x_2(s)\mathrm{d}s \\ &\quad + 2a_{12}^2 (N_1^*)^2 \int_{t-\tau_{12}}^{t} x_2(t)x_2(s)\mathrm{d}s + 2a_{12}a_{13}(N_1^*)^2 \int_{t-\tau_{12}}^{t} x_3(t)x_2(s)\mathrm{d}s - 2a_{11}a_{13}(N_1^*)^2 \\ &\quad \times \int_{t-\tau_{13}}^{t} x_1(t)x_3(s)\mathrm{d}s + 2a_{12}a_{13}(N_1^*)^2 \int_{t-\tau_{13}}^{t} x_2(t)x_3(s)\mathrm{d}s + 2a_{13}^2 (N_1^*)^2 \int_{t-\tau_{13}}^{t} x_3(t)x_3(s)\mathrm{d}s \\ &\le -2a_{11}N_1^* x_1^2 + a_{12}N_1^*\left(x_1^2 + x_2^2\right) + a_{13}N_1^*\left(x_1^2 + x_3^2\right) + \sigma_1^2 x_1^2 \\ &\quad + a_{12}(a_{12} + a_{13})(N_1^*)^2 x_2^2 \tau + a_{13}(a_{13} + a_{12})(N_1^*)^2 x_3^2 \tau \\ &\quad + a_{12}(a_{12} + a_{13})(N_1^*)^2 \int_{t-\tau_{12}}^{t} x_2^2(s)\mathrm{d}s + a_{13}(a_{12} + a_{13})(N_1^*)^2 \int_{t-\tau_{13}}^{t} x_3^2(s)\mathrm{d}s. \end{aligned} \tag{16}$$

By the same way, we have

$$
\begin{aligned}
LV_2 &= 2\left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)\mathrm{d}s + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)\mathrm{d}s\right) \\
&\quad \times \left(a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3\right) + \sigma_2^2 x_2^2 \\
&= -2a_{22}N_2^* x_2^2 + 2a_{21}N_2^* x_1 x_2 + 2a_{23}N_2^* x_2 x_3 + \sigma_2^2 x_2^2 + 2a_{21}^2 \left(N_2^*\right)^2 \int_{t-\tau_{21}}^{t} x_1(t) x_1(s)\mathrm{d}s \\
&\quad - 2a_{21}a_{22}\left(N_2^*\right)^2 \int_{t-\tau_{21}}^{t} x_2(t) x_1(s)\mathrm{d}s + 2a_{21}a_{23}\left(N_2^*\right)^2 \int_{t-\tau_{21}}^{t} x_3(t) x_1(s)\mathrm{d}s + 2a_{21}a_{23}\left(N_2^*\right)^2 \\
&\quad \times \int_{t-\tau_{23}}^{t} x_1(t) x_3(s)\mathrm{d}s - 2a_{22}a_{23}\left(N_2^*\right)^2 \int_{t-\tau_{23}}^{t} x_2(t) x_3(s)\mathrm{d}s + 2a_{23}^2\left(N_2^*\right)^2 \int_{t-\tau_{13}}^{t} x_3(t) x_3(s)\mathrm{d}s \\
&\le -2a_{22}N_2^* x_2^2 + a_{21}N_2^*\left(x_1^2 + x_2^2\right) + a_{23}N_2^*\left(x_2^2 + x_3^2\right) + \sigma_2^2 x_2^2 \\
&\quad + a_{21}\left(a_{21} + a_{23}\right) 2\left(N_2^*\right)^2 x_1^2 \tau + a_{23}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 x_3^2 \tau \\
&\quad + a_{21}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 \int_{t-\tau_{21}}^{t} x_1^2(s)\mathrm{d}s + a_{23}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 \int_{t-\tau_{23}}^{t} x_3^2(s)\mathrm{d}s, \\
LV_3 &= 2\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)\mathrm{d}s + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)\mathrm{d}s\right) \\
&\quad \times \left(a_{31}N_3^* x_1 + a_{32}N_3^* x_2 - a_{33}N_3^* x_3\right) + \sigma_3^2 x_3^2 \\
&= -2a_{33}N_3^* x_3^2 + 2a_{31}N_3^* x_1 x_3 + 2a_{32}N_3^* x_2 x_3 + \sigma_3^2 x_3^2 + 2a_{31}^2\left(N_3^*\right)^2 \int_{t-\tau_{31}}^{t} x_1(t) x_1(s)\mathrm{d}s \\
&\quad + 2a_{31}a_{32}\left(N_3^*\right)^2 \int_{t-\tau_{31}}^{t} x_2(t) x_1(s)\mathrm{d}s - 2a_{31}a_{33}\left(N_3^*\right)^2 \int_{t-\tau_{31}}^{t} x_3(t) x_1(s)\mathrm{d}s + 2a_{32}a_{31}\left(N_3^*\right)^2 \\
&\quad \times \int_{t-\tau_{32}}^{t} x_1(t) x_2(s)\mathrm{d}s + 2a_{32}^2\left(N_3^*\right)^2 \int_{t-\tau_{32}}^{t} x_2(t) x_2(s)\mathrm{d}s - 2a_{32}a_{33}\left(N_3^*\right)^2 \int_{t-\tau_{32}}^{t} x_3(t) x_2(s)\mathrm{d}s \\
&\le -2a_{33}N_3^* x_3^2 + a_{31}N_3^*\left(x_1^2 + x_3^2\right) + a_{32}N_3^*\left(x_2^2 + x_3^2\right) + \sigma_3^2 x_3^2 \\
&\quad + a_{32}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 x_2^2 \tau + a_{31}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 x_1^2 \tau \\
&\quad + a_{31}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 \int_{t-\tau_{31}}^{t} x_1^2(s)\mathrm{d}s + a_{32}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 \int_{t-\tau_{32}}^{t} x_2^2(s)\mathrm{d}s.
\end{aligned}
\tag{17}
$$

Adding both sides of LV_1, LV_2, and LV_3 yields

$$
\begin{aligned}
L\left(V_1 + V_2 + V_3\right) \le & \left[-\left(2a_{11} - a_{12} - a_{13}\right)N_1^* + \sigma_1^2 + a_{21}N_2^* + a_{31}N_3^*\right] x_1^2 \\
& + \tau\left[a_{21}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 + a_{31}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2\right] x_1^2 \\
& + a_{31}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 \int_{t-\tau_{31}}^{t} x_1^2(s)\mathrm{d}s + a_{21}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 \int_{t-\tau_{21}}^{t} x_1^2(s)\mathrm{d}s \\
& \left[-\left(2a_{22} - a_{21} - a_{23}\right)N_2^* + \sigma_2^2 + a_{12}N_1^* + a_{32}N_3^*\right] x_2^2 \\
& + \tau\left[a_{12}\left(a_{12} + a_{13}\right)\left(N_1^*\right)^2 + a_{32}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2\right] x_2^2 \\
& + a_{12}\left(a_{12} + a_{13}\right)\left(N_1^*\right)^2 \int_{t-\tau_{12}}^{t} x_2^2(s)\mathrm{d}s + a_{32}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 \int_{t-\tau_{32}}^{t} x_2^2(s)\mathrm{d}s \\
& \left[-\left(2a_{33} - a_{31} - a_{32}\right)N_3^* + \sigma_1^2 + a_{13}N_1^* + a_{23}N_2^*\right] x_3^2 \\
& + \tau\left[a_{23}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 + a_{13}\left(a_{13} + a_{12}\right)\left(N_1^*\right)^2\right] x_3^2 \\
& + a_{23}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 \int_{t-\tau_{23}}^{t} x_3^2(s)\mathrm{d}s + a_{13}\left(a_{12} + a_{13}\right)\left(N_1^*\right)^2 \int_{t-\tau_{13}}^{t} x_3^2(s)\mathrm{d}s.
\end{aligned}
\tag{18}
$$

Define

$$\begin{aligned} V_4 &= a_{21}(a_{21}+a_{23})(N_2^*)^2 \int_{t-\tau_{21}}^{t} (\theta - t + \tau_{21}) x_1^2(\theta)\mathrm{d}\theta \\ &\quad + a_{31}(a_{31}+a_{32})(N_3^*)^2 \int_{t-\tau_{31}}^{t} (\theta - t + \tau_{31}) x_1^2(\theta)\mathrm{d}\theta, \\ V_5 &= a_{12}(a_{12}+a_{13})(N_1^*)^2 \int_{t-\tau_{12}}^{t} (\theta - t + \tau_{12}) x_2^2(\theta)\mathrm{d}\theta \\ &\quad + a_{32}(a_{31}+a_{32})(N_3^*)^2 \int_{t-\tau_{32}}^{t} (\theta - t + \tau_{32}) x_2^2(\theta)\mathrm{d}\theta, \\ V_6 &= a_{13}(a_{12}+a_{13})(N_1^*)^2 \int_{t-\tau_{13}}^{t} (\theta - t + \tau_{13}) x_3^2(\theta)\mathrm{d}\theta \\ &\quad + a_{23}(a_{21}+a_{23})(N_2^*)^2 \int_{t-\tau_{23}}^{t} (\theta - t + \tau_{23}) x_3^2(\theta)\mathrm{d}\theta. \end{aligned} \tag{19}$$

Let $V = \sum_{i=1}^{6} V_i$, then

$$\begin{aligned} LV \le & \left\{\left[-\left(2a_{11} - a_{12} - a_{13} - \frac{\sigma_1^2}{N_1^*}\right)N_1^* + a_{21}N_2^* + a_{31}N_3^*\right] \right. \\ & \left. + 2\tau a_{21}(a_{21}+a_{23})(N_2^*)^2 + 2\tau a_{31}(a_{31}+a_{32})(N_3^*)^2\right\} x_1^2(t) \\ & + \left\{\left[-\left(2a_{22} - a_{21} - a_{23} - \frac{\sigma_2^2}{N_2^*}\right)N_2^* + a_{12}N_1^* + a_{32}N_3^*\right] \right. \\ & \left. + 2\tau a_{12}(a_{12}+a_{13})(N_1^*)^2 + 2\tau a_{32}(a_{31}+a_{32})(N_3^*)^2\right\} x_2^2 \\ & + \left\{\left[-\left(2a_{33} - a_{31} - a_{32} - \frac{\sigma_1^2}{N_3^*}\right)N_3^* + a_{13}N_1^* + a_{23}N_2^*\right] \right. \\ & \left. + 2\tau a_{23}(a_{21}+a_{23})(N_2^*)^2 + 2\tau a_{13}(a_{12}+a_{13})(N_1^*)^2\right\} x_3^2. \end{aligned} \tag{20}$$

By the hypothesis (H_1), it is easy to get $LV < 0$ along all trajectories in R_+^3 except N_i^*. Hence, by the stability theory of stochastic functional differential equations [23], the zero solution of (13) is globally asymptotically stable. This completes the proof. □

Remark 2. The globally asymptotical stability of zero solution of (13) means the globally asymptotical stability of the solution of (4) around the equilibrate state of (1), that is, $\lim_{t\longrightarrow\infty} N_i(t) = N_i^*, i = 1, 2, 3$.

Remark 3. The proof method is motivated in [19]. We apply the theory of neutral functional differential equation and define some suitable functionals V_1, V_2, and V_3 to obtain the sufficient conditions assuring the globally asymptotical stability of (13) around the equilibrium state N_i^*, which are much different from those of [20].

Theorem 2. *If (H_1) holds, then the zero solution of system (12) is stable in probability; that is, system (4) around the equilibrium state of (1) is stable in probability.*

Proof. For system (12), define V_1, V_2, and V_3 as before and assume that there exists a number $\delta > 0$ such that $\sup_{t\geq\tau}|x_i(s)| < \delta, i = 1, 2, 3$. By the Itô formula, we calculate LV_1, LV_2, and LV_3 along system (12), respectively, then

$$\begin{aligned}
LV_1 &= 2\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)\mathrm{d}s + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)\mathrm{d}s\right) \\
&\quad \cdot \left(-a_{11}N_1^*x_1 + a_{12}N_1^*x_2 + a_{13}N_1^*x_3 - a_{11}x_1^2 + a_{12}x_1x_2(t-\tau_{12}) + a_{13}x_1x_3(t-\tau_{13})\right) + \sigma_1^2x_1^2 \\
&\leq -2a_{11}N_1^*x_1^2 + a_{12}N_1^*\left(x_1^2 + x_2^2\right) + a_{13}N_1^*\left(x_1^2 + x_3^2\right) \\
&\quad + \sigma_1^2x_1^2 + a_{12}(a_{12}+a_{13})(N_1^*)^2x_2^2\tau + a_{13}(a_{13}+a_{12})(N_1^*)^2x_3^2\tau \\
&\quad + a_{12}(a_{12}+a_{13})(N_1^*)^2 \int_{t-\tau_{12}}^{t} x_2^2(s)\mathrm{d}s + a_{13}(a_{12}+a_{13})(N_1^*)^2 \int_{t-\tau_{13}}^{t} x_3^2(s)\mathrm{d}s \\
&\quad + 2\delta(a_{12}+a_{13})x_1^2 + (a_{12}+a_{13})^2N_1^*\tau\delta x_1^2 + \delta a_{12}^2N_1^* \int_{t-\tau_{12}}^{t} x_2^2(s)\mathrm{d}s + \delta a_{13}^2N_1^* \int_{t-\tau_{13}}^{t} x_3^2(s)\mathrm{d}s, \\
LV_2 &= 2\left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)\mathrm{d}s + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)\mathrm{d}s\right) \\
&\quad \cdot \left(-a_{22}N_2^*x_2 + a_{21}N_2^*x_1 + a_{23}N_2^*x_3 + a_{21}x_2x_1(t-\tau_{21}) - a_{22}x_2^2 + a_{23}x_2x_3(t-\tau_{23})\right) + \sigma_2^2x_2^2 \\
&\leq -2a_{22}N_2^*x_2^2 + a_{21}N_2^*\left(x_1^2 + x_2^2\right) + a_{23}N_2^*\left(x_2^2 + x_3^2\right) + \sigma_2^2x_2^2 \\
&\quad + a_{21}(a_{21}+a_{23})(N_2^*)^2x_1^2\tau + a_{23}(a_{21}+a_{23})(N_2^*)^2x_3^2\tau \\
&\quad + a_{21}(a_{21}+a_{23})(N_2^*)^2 \int_{t-\tau_{21}}^{t} x_1^2(s)\mathrm{d}s + a_{23}(a_{21}+a_{23})(N_2^*)^2 \int_{t-\tau_{23}}^{t} x_3^2(s)\mathrm{d}s + 2\delta(a_{21}+a_{23})x_2^2 \\
&\quad + (a_{21}+a_{23})^2N_2^*\tau\delta x_2^2 + \delta a_{21}^2N_2^* \int_{t-\tau_{23}}^{t} x_1^2(s)\mathrm{d}s + \delta a_{23}^2N_2^* \int_{t-\tau_{23}}^{t} x_3^2(s)\mathrm{d}s, \\
LV_3 &= 2\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)\mathrm{d}s + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)\mathrm{d}s\right) \\
&\quad \cdot \left(a_{31}N_3^*x_1 + a_{32}N_3^*x_2 - a_{33}N_3^*x_3 + a_{31}x_3x_1(t-\tau_{31}) + a_{32}x_3x_2(t-\tau_{32}) - a_{33}x_3^2\right) + \sigma_3^2x_3^2 \\
&\leq -2a_{33}N_3^*x_3^2 + a_{31}N_3^*\left(x_1^2 + x_3^2\right) + a_{32}N_3^*\left(x_2^2 + x_3^2\right) + \sigma_3^2x_3^2 \\
&\quad + a_{32}(a_{31}+a_{32})(N_3^*)^2x_2^2\tau + a_{31}(a_{31}+a_{32})(N_3^*)^2x_1^2\tau \\
&\quad + a_{31}(a_{31}+a_{32})(N_3^*)^2 \int_{t-\tau_{31}}^{t} x_1^2(s)\mathrm{d}s \\
&\quad + a_{32}(a_{31}+a_{32})(N_3^*)^2 \int_{t-\tau_{32}}^{t} x_2^2(s)\mathrm{d}s + 2\delta(a_{31}+a_{32})x_3^2 + (a_{31}+a_{32})^2N_3^*\tau\delta x_3^2 \\
&\quad + \delta a_{31}^2N_3^* \int_{t-\tau_{31}}^{t} x_1^2(s)\mathrm{d}s + \delta a_{32}^2N_3^* \int_{t-\tau_{32}}^{t} x_2^2(s)\mathrm{d}s.
\end{aligned} \tag{21}$$

Define $\widetilde{V}_4, \widetilde{V}_5$, and $\widetilde{V}_6$ as follows:

$$\begin{aligned}\widetilde{V}_4 &= \left[a_{21}(a_{21}+a_{23})(N_2^*)^2 + \delta a_{21}^2 N_2^*\right]\int_{t-\tau_{21}}^{t}(\theta - t + \tau_{21})x_1^2(\theta)\mathrm{d}\theta \\ &+ \left[a_{31}(a_{31}+a_{32})(N_3^*)^2 + \delta a_{31}^2 N_3^*\right]\int_{t-\tau_{31}}^{t}(\theta - t + \tau_{31})x_1^2(\theta)\mathrm{d}\theta, \\ \widetilde{V}_5 &= \left[a_{12}(a_{12}+a_{13})(N_1^*)^2 + \delta a_{12}^2 N_1^*\right]\int_{t-\tau_{12}}^{t}(\theta - t + \tau_{12})x_2^2(\theta)\mathrm{d}\theta \\ &+ \left[a_{32}(a_{31}+a_{32})(N_3^*)^2 + \delta a_{32}^2 N_3^*\right]\int_{t-\tau_{32}}^{t}(\theta - t + \tau_{32})x_2^2(\theta)\mathrm{d}\theta, \\ \widetilde{V}_6 &= \left[a_{13}(a_{12}+a_{13})(N_1^*)^2 + \delta a_{13}^2 N_1^*\right]\int_{t-\tau_{13}}^{t}(\theta - t + \tau_{13})x_3^2(\theta)\mathrm{d}\theta \\ &+ \left[a_{23}(a_{21}+a_{23})(N_2^*)^2 + \delta a_{23}^2 N_2^*\right]\int_{t-\tau_{23}}^{t}(\theta - t + \tau_{23})x_3^2(\theta)\mathrm{d}\theta.\end{aligned} \tag{22}$$

Let $V = \sum_{i=1}^{3} V_i + \sum_{j=4}^{6}\widetilde{V}_j$. Computing the derivatives of $\widetilde{V}_j\,(j = 4,5,6)$ and adding both sides of $LV_i\,(i = 1,2,3)$ and $L\widetilde{V}_j\,(j = 4,5,6)$ reads

$$\begin{aligned}LV \le &\left\{\left[-\left(2a_{11} - a_{12} - a_{13} - \frac{\sigma_1^2}{N_1^*}\right)N_1^* + a_{21}N_2^* + a_{31}N_3^*\right] + 2\tau a_{21}(a_{21}+a_{23})(N_2^*)^2\right. \\ &\left.+2\tau a_{31}(a_{31}+a_{32})(N_3^*)^2 + \tau\delta\left(a_{12}^2 N_1^* + a_{32}^2 N_3^*\right)\right\}x_1^2(t) \\ &+\left\{\left[-\left(2a_{22} - a_{21} - a_{23} - \frac{\sigma_2^2}{N_2^*}\right)N_2^* + a_{12}N_1^* + a_{32}N_3^*\right] + 2\tau a_{12}(a_{12}+a_{13})(N_1^*)^2\right. \\ &\left.+2\tau a_{32}(a_{31}+a_{32})(N_3^*)^2 + \tau\delta\left(a_{13}^2 N_1^* + a_{23}^2 N_2^*\right)\right\}x_2^2 \\ &+\left\{\left[-\left(2a_{33} - a_{31} - a_{32} - \frac{\sigma_1^2}{N_3^*}\right)N_3^* + a_{13}N_1^* + a_{23}N_2^*\right] + 2\tau a_{23}(a_{21}+a_{23})(N_2^*)^2\right. \\ &\left.+2\tau a_{13}(a_{12}+a_{13})(N_1^*)^2 + \tau\delta\left(a_{21}^2 N_2^* + a_{31}^2 N_3^*\right)\right\}x_3^2.\end{aligned} \tag{23}$$

By choosing sufficiently small $\delta > 0$ such that (H_1) holds, then we have $LV < 0$. Thus, it follows from Lemma 2 that the zero solution of (12) is stable in probability. The proof is completed.

Remark 4. In the process of our proof, the same method applied in linear case (Theorem 1) is generalized to nonlinear case. We define $\widetilde{V}_j\,(j = 4,5,6)$ and compute their derivatives so as to eliminate the integral items appeared in $LV_i\,(i = 1,2,3)$. By Lemma 2, we obtain the sufficient conditions assuring the stability in probability of (4) around the equilibrium state N_i^*, which is relatively new in some sense.

Next, we consider system (6). By setting $x_1 = N_1 - N_1^*$, $x_2 = N_2 - N_2^*$, and $x_3 = N_3 - N_3^*$, and then, (6) is transformed to the following equivalent system:

$$\begin{cases}\mathrm{d}x_1(t) = \left(-a_{11}N_1^* x_1(t) + a_{12}N_1^* x_2(t-\tau_{12}) + a_{13}N_1^* x_3(t-\tau_{13}) - a_{11}x_1^2\right. \\ \qquad\left. + a_{12}x_1x_2(t-\tau_{12}) + a_{13}x_1x_3(t-\tau_{13})\right)\mathrm{d}t + \sigma_1 x_1(x_1 + N_1^*)\mathrm{d}\omega_1(t), \\ \mathrm{d}x_2(t) = \left(-a_{22}N_2^* x_2(t) + a_{21}N_2^* x_1(t-\tau_{21}) + a_{23}N_2^* x_3(t-\tau_{23}) - a_{22}x_2^2\right. \\ \qquad\left. + a_{21}x_2x_1(t-\tau_{21}) + a_{23}x_2x_3(t-\tau_{23})\right)\mathrm{d}t + \sigma_2 x_2(x_2 + N_2^*)\mathrm{d}\omega_2(t), \\ \mathrm{d}x_3(t) = \left(-a_{33}N_3^* x_3(t) + a_{31}N_3^* x_3(t-\tau_{31}) + a_{31}N_3^* x_3(t-\tau_{32}) - a_{33}x_3^2\right. \\ \qquad\left. + a_{31}x_3x_1(t-\tau_{31}) + a_{32}x_3x_2(t-\tau_{32})\right)\mathrm{d}t + \sigma_3 x_3(x_3 + N_3^*)\mathrm{d}\omega_3(t).\end{cases} \tag{24}$$

Similarly, the stability of (6) around the equilibrate state of (1) is equivalent to the stability of zero solution of (24). Consequently, we only need to study the stability of zero solution of (24). By use of transformation, system (24) is equivalent to the following system:

$$\begin{cases} d\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)ds + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds\right) \\ = \left(-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3 - a_{11}x_1^2 \right. \\ \left. + a_{12}x_1 x_2(t-\tau_{12}) + a_{13}x_1 x_3(t-\tau_{13})\right)dt + \sigma_1 x_1 (x_1 + N_1^*) d\omega_1(t), \\ d\left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)ds + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)ds\right) \\ = \left(a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3 - a_{22}x_2^2 \right. \\ \left. + a_{21}x_2 x_1(t-\tau_{21}) + a_{23}x_2 x_3(t-\tau_{23})\right)dt + \sigma_2 x_2 (x_2 + N_2^*) d\omega_2(t), \\ d\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)ds + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)ds\right) \\ = \left(a_{31}N_3^* x_1 + a_{32}N_3^* x_2 - a_{33}N_3^* x_3 - a_{33}x_3^2 \right. \\ \left. + a_{31}x_3 x_1(t-\tau_{31}) + a_{32}x_3 x_2(t-\tau_{32})\right)dt + \sigma_3 x_3 (x_3 + N_3^*) d\omega_3(t). \end{cases} \tag{25}$$

Consider the following linear case of (25):

$$\begin{cases} d\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)ds + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds\right) \\ = \left(-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3\right)dt + \sigma_1 x_1 N_1^* d\omega_1(t), \\ d\left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)ds + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)ds\right) \\ = \left(a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3\right)dt + \sigma_2 x_2 N_2^* d\omega_2(t), \\ d\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)ds + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)ds\right) \\ = \left(a_{31}N_3^* x_1 + a_{32}N_3^* x_2 - a_{33}N_3^* x_3\right)dt + \sigma_3 x_3 N_3^* d\omega_3(t). \end{cases} \tag{26}$$

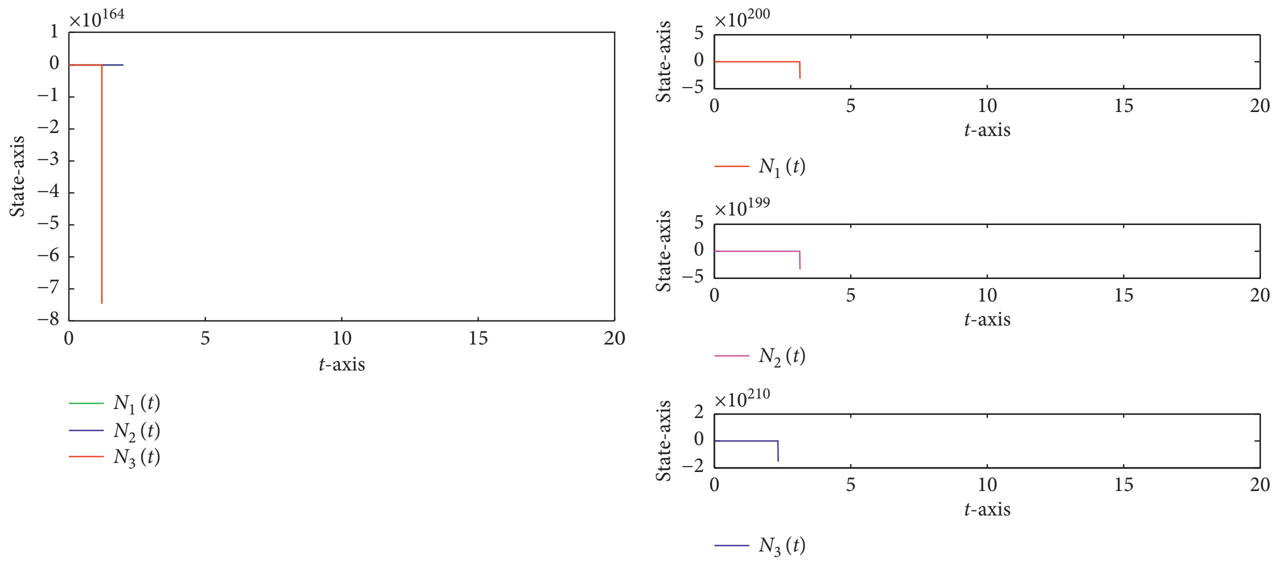
For system (25), the following result holds.

Theorem 3. *If the following* (H_2) *holds,*

$$(H_2)\begin{pmatrix} \widetilde{\Delta}_1 & -a_{21} & -a_{31} \\ -a_{12} & \widetilde{\Delta}_2 & -a_{32} \\ -a_{13} & -a_{23} & \widetilde{\Delta}_3 \end{pmatrix}\begin{pmatrix} N_1^* \\ N_2^* \\ N_3^* \end{pmatrix} > 2\tau \begin{pmatrix} 0 & \xi_1 & \eta_1 \\ \xi_2 & 0 & \eta_2 \\ \xi_3 & \eta_3 & 0 \end{pmatrix}\begin{pmatrix} (N_1^*)^2 \\ (N_2^*)^2 \\ (N_3^*)^2 \end{pmatrix}, \tag{27}$$

then system (25) around the equilibrium state N_i^* *is stable in probability; i.e., system (6) around the equilibrium state* N_i^* *is stable in probability.*

Proof. Applying the same manner as before, consider the linear system (26) and define V_1, V_2, and V_3 as before. By using the Itô formula and computing the derivatives of V_1, V_2, and V_3 along solutions of (26), then



(a) (b)

FIGURE 1: Unstable cases with large time delay, where $\sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_3 = 0.3$, and $\tau = 0.8$. Simulations reveal that too large time delay destroys the stability of (4) and (6), respectively. (a) The state graph of system (4); (b) the state graph of system (6).

$$\begin{aligned}
LV_1 &= 2\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)\mathrm{d}s + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)\mathrm{d}s \right) \\
&\quad \times \left(-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3\right) + \sigma_1^2 \left(N_1^*\right)^2 x_1^2 \\
&\le -\left(2a_{11} - a_{12} - a_{13} - \sigma_1^2 N_1^*\right) N_1^* x_1^2 + a_{12}N_1^* x_2^2 + a_{13}N_1^* x_3^2 \\
&\quad + a_{12}\left(a_{12} + a_{13}\right)\left(N_1^*\right)^2 x_2^2 \tau + a_{13}\left(a_{13} + a_{12}\right)\left(N_1^*\right)^2 x_3^2 \tau \\
&\quad + a_{12}\left(a_{12} + a_{13}\right)\left(N_1^*\right)^2 \int_{t-\tau_{12}}^{t} x_2^2(s)\mathrm{d}s + a_{13}\left(a_{12} + a_{13}\right)\left(N_1^*\right)^2 \int_{t-\tau_{13}}^{t} x_3^2(s)\mathrm{d}s, \\
LV_2 &= 2\left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)\mathrm{d}s + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)\mathrm{d}s \right) \\
&\quad \times \left(a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3\right) + \sigma_2 \left(N_2^*\right)^2 x_2 \\
&\le -\left(2a_{22} - a_{21} - a_{23} - \sigma_2^2 N_2^*\right) N_2^* x_2^2 + a_{21}N_2^* x_2^2 + a_{23}N_2^* x_3^2 \\
&\quad + a_{21}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 x_1^2 \tau + a_{23}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 x_3^2 \tau \\
&\quad + a_{21}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 \int_{t-\tau_{21}}^{t} x_1^2(s)\mathrm{d}s + a_{23}\left(a_{21} + a_{23}\right)\left(N_2^*\right)^2 \int_{t-\tau_{23}}^{t} x_3^2(s)\mathrm{d}s, \\
LV_3 &= 2\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)\mathrm{d}s + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)\mathrm{d}s \right) \\
&\quad \times \left(a_{31}N_3^* x_1 + a_{32}N_3^* x_2 - a_{33}N_3^* x_3\right) + \sigma_3^2 \left(N_3^*\right)^2 x_3^2 \\
&\le -\left(2a_{33} - a_{31} - a_{32} - \sigma_3^2 N_3^*\right) N_3^* x_3^2 + a_{31}N_3^* x_3^2 + a_{32}N_3^* x_3^2 \\
&\quad + a_{32}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 x_2^2 \tau + a_{31}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 x_1^2 \tau \\
&\quad + a_{31}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 \int_{t-\tau_{31}}^{t} x_1^2(s)\mathrm{d}s + a_{32}\left(a_{31} + a_{32}\right)\left(N_3^*\right)^2 \int_{t-\tau_{32}}^{t} x_2^2(s)\mathrm{d}s.
\end{aligned} \tag{28}$$

The rest proof is similar to proofs of Theorems 1 and 2, so we omit it. The proof is completed. □

Remark 5. Theorems 1–3 show that both time delays ($\tau_{ij}, i, j = 1, 2, 3$) and stochastic disturbances ($\sigma_i, i = 1, 2, 3$)

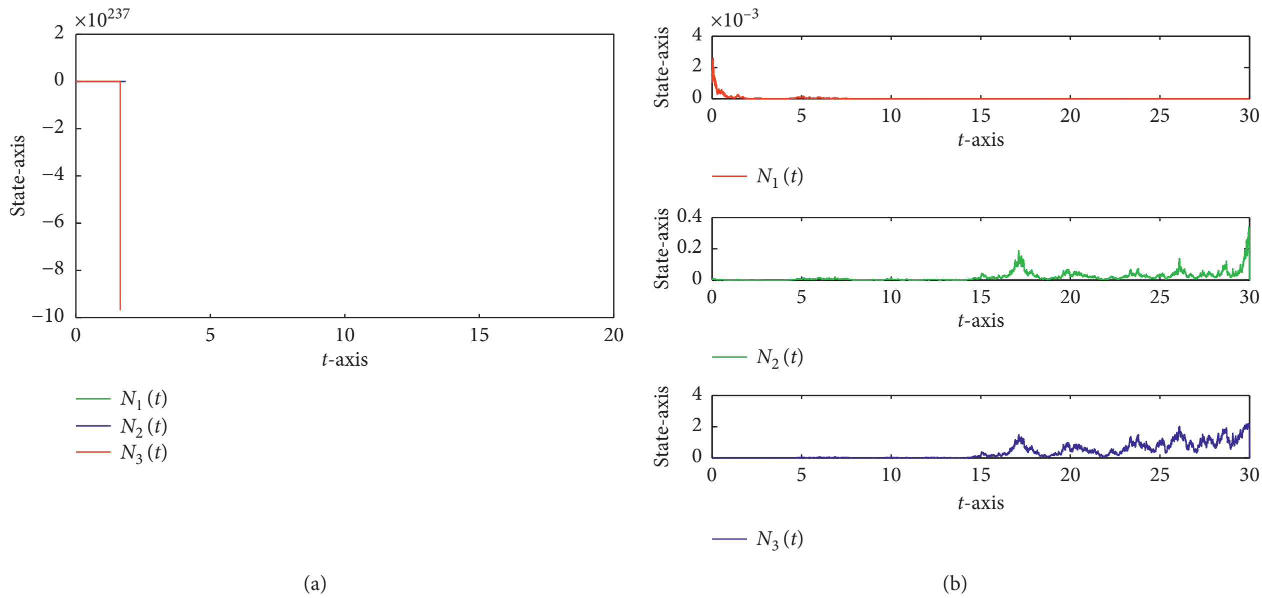


Figure 2: Unstable cases for (4) and (6) with large stochastic perturbations, respectively, which show that too large stochastic disturbances destroy the stability of above systems. (a) For system (4), with $\sigma_1 = 2, \sigma_2 = 2, \sigma_3 = 2$, and $\tau = 0.5$; (b) for system (6), with $\sigma_1 = 0.9, \sigma_2 = 0.9, \sigma_3 = 0.9$, and $\tau = 0.5$.

have important impacts on the globally asymptotical stability and stability in probability around the positive equilibrium state. Conditions (H_1) and (H_2) imply that too large time delays or white noises may destroy the stability of above systems, which are also verified by numerical simulations (see Figures 1 and 2 in the next section).

Remark 6. Theorems 1–3 highlight the effects of time delays and stochastic disturbances on the stability around the positive equilibrium state, where conditions (H_1) and (H_2) seem complexity, but they are not difficult to be verified by computation software like Matlab.

4. Numerical Simulations

In this section, we present some numerical simulations to verify the theoretical result. After giving values of all parameters, by applying the method [24] and writing some suitable Matlab code, we can obtain the numerical results as follows. Let $a_{11} = 1, a_{12} = 0.2, a_{13} = 0.3, a_{21} = 0.3, a_{22} = 1$, $a_{23} = 0.2, a_{31} = 0.2, a_{32} = 0.3, a_{33} = 1$, $r_1 = 1, r_2 = 1.2$, and $r_3 = 1.5$. Then, an easy computation yields that $A = 0.785, A_1 = 1.783, A_2 = 1.928, A_3 = 2.098$ and $N_1^* = 2.2713$, $N_2^* = 2.4561, N_3^* = 2.6726$.

If $\sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_3 = 0.3$, and $\tau = 0.5$, with the help of Matlab, by computation, we have

$$\begin{aligned}\Delta_1 &= 1.4824,\\ \Delta_2 &= 1.4959,\\ \Delta_3 &= 1.4663,\\ \widetilde{\Delta}_1 &= 1.4091,\\ \widetilde{\Delta}_2 &= 1.4754,\\ \widetilde{\Delta}_3 &= 1.2592,\end{aligned} \tag{29}$$

and hence,

$$\begin{gathered}\begin{pmatrix} \Delta_1 & -a_{21} & -a_{31} \\ -a_{12} & \Delta_2 & -a_{32} \\ -a_{13} & -a_{23} & \Delta_3 \end{pmatrix}\begin{pmatrix} N_1^* \\ N_2^* \\ N_3^* \end{pmatrix} = \begin{pmatrix} 2.0956 \\ 2.4180 \\ 2.7462 \end{pmatrix},\\ \begin{pmatrix} \widetilde{\Delta}_1 & -a_{21} & -a_{31} \\ -a_{12} & \widetilde{\Delta}_2 & -a_{32} \\ -a_{13} & -a_{23} & \widetilde{\Delta}_3 \end{pmatrix}\begin{pmatrix} N_1^* \\ N_2^* \\ N_3^* \end{pmatrix} = \begin{pmatrix} 1.9291 \\ 2.3677 \\ 2.1927 \end{pmatrix},\\ 2\tau\begin{pmatrix} 0 & \xi_1 & \eta_1 \\ \xi_2 & 0 & \eta_2 \\ \xi_3 & \eta_3 & 0 \end{pmatrix}\begin{pmatrix} (N_1^*)^2 \\ (N_2^*)^2 \\ (N_3^*)^2 \end{pmatrix} = \begin{pmatrix} 1.6191 \\ 1.5873 \\ 1.37719 \end{pmatrix}.\end{gathered} \tag{30}$$

By verification, both (H_1) and (H_2) hold, and then, Theorems 2 and 3 hold. Theorem 2 implies system (4) is stable around the equilibrium of (1). The state graphs of each population are illustrated in Figure 3. Furthermore,

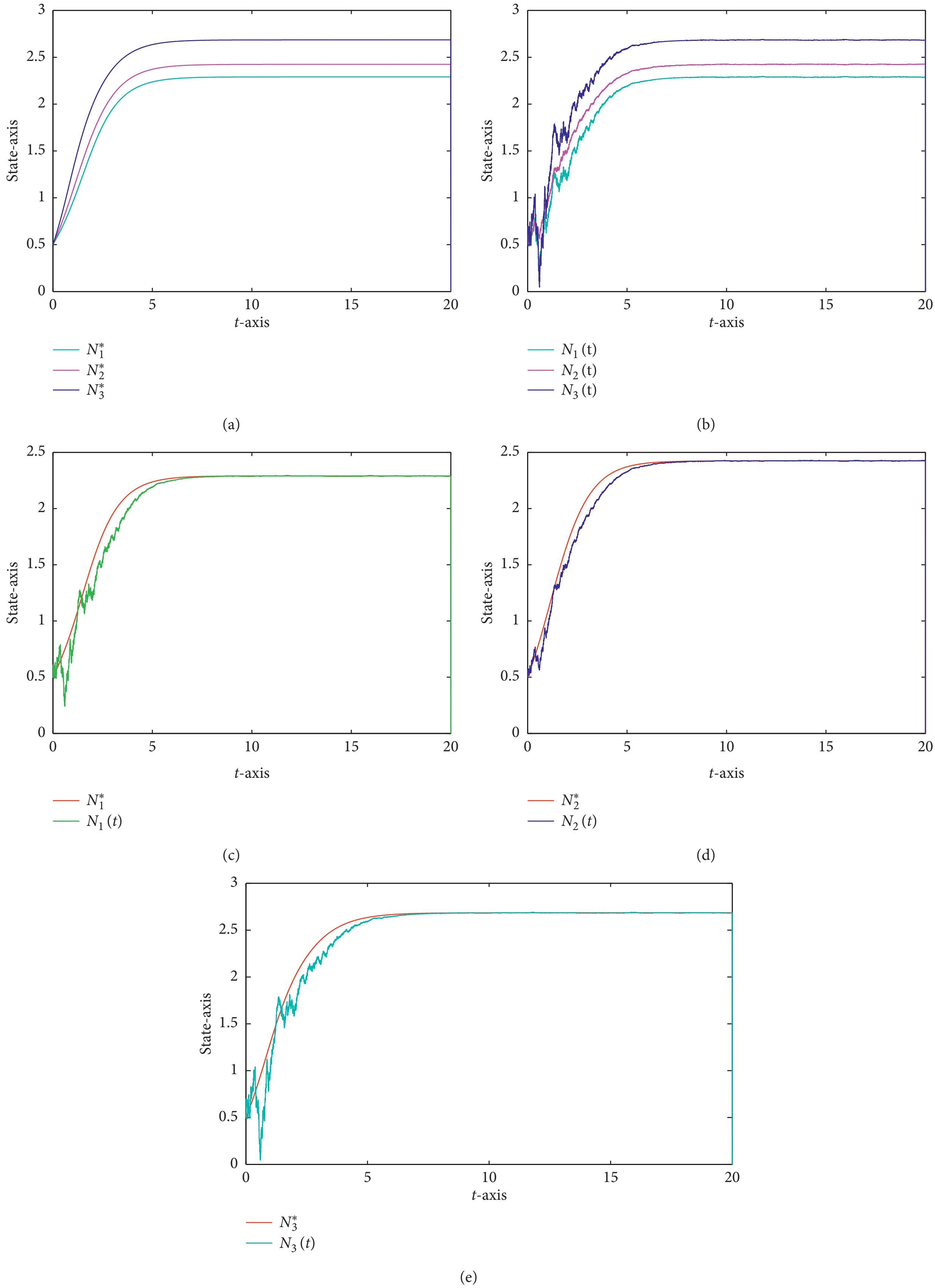


Figure 3: Stable case of system (4) with $\sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_3 = 0.3$, and $\tau = 0.5$: (a) equilibrium of (1) with $N_1^* = 2.2713$, $N_2^* = 2.4561$, and $N_3^* = 2.6726$; (b) the state graph of three species of (4); (c), (d), and (e) are state graphs of populations N_1, N_2, and N_3 of (4) around their corresponding equilibrium points N_1^*, N_2^*, and N_3^*, respectively.

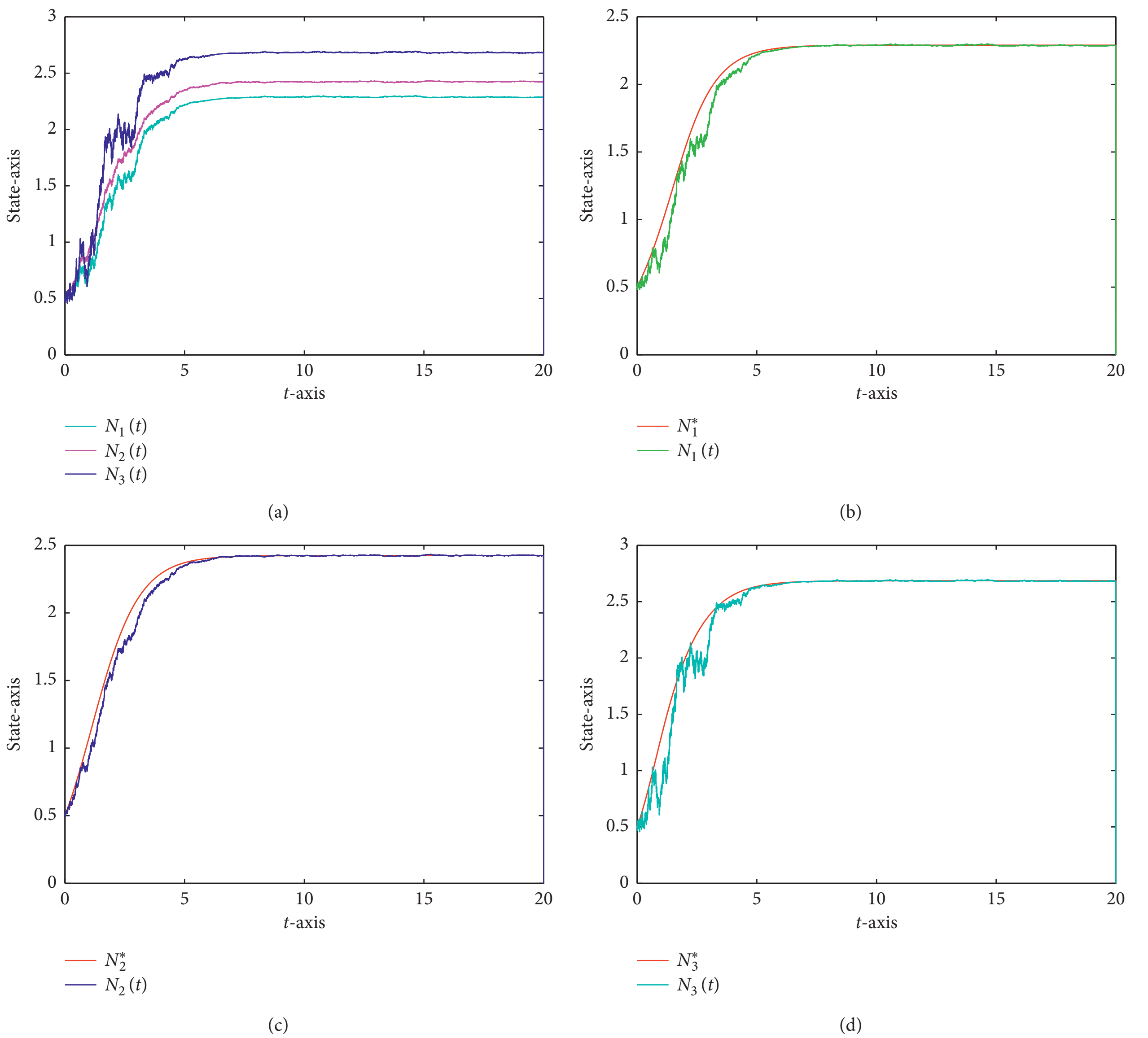


Figure 4: Stable case of system (6) with $\sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_3 = 0.3$, and $\tau = 0.5$: (a) the state graph of three species; (b), (c), and (d) are the state graphs of the populations N_1, N_2, and N_3 around their corresponding equilibrium points N_1^*, N_2^*, and N_3^*, respectively.

Theorem 3 indicates system (6) around the equilibrium N_i^* is also stable. The state graphs of each population are illustrated in Figure 4.

If $\tau = 0.8$, other parameters are same as before, then

$$2\tau\begin{pmatrix} 0 & \xi_1 & \eta_1 \\ \xi_2 & 0 & \eta_2 \\ \xi_3 & \eta_3 & 0 \end{pmatrix}\begin{pmatrix} (N_1^*)^2 \\ (N_2^*)^2 \\ (N_3^*)^2 \end{pmatrix} = \begin{pmatrix} 2.5906 \\ 2.5397 \\ 2.2033 \end{pmatrix}. \quad (31)$$

An easy computation yields that neither (H_1) nor (H_2) holds; hence, (4) and (6) around the equilibrium point may be unstable (see Figures 1(a) and 1(b), respectively). The state graphs of each population in Figure 1 show that too large time delays destroy the stability of (4) and (6).

Finally, if $\sigma_1 = 2, \sigma_2 = 2, \sigma_3 = 2, \tau = 0.5$, then $\Delta_1 = -0.2611, \Delta_2 = -0.1286, \Delta_3 = 0.0434$. Hence, (H_1) does not hold and system (4) may be unstable, which is shown in Figure 2(a). If $\sigma_1 = 0.9, \sigma_2 = 0.9, \sigma_3 = 0.9, \tau = 0.5$, then $\tilde{\Delta}_1 = -0.3398, \tilde{\Delta}_2 = -0.4894, \tilde{\Delta}_3 = -0.6648$. Hence, (H_2) does not hold and system (6) may be unstable too (see Figure 2(b)). The state graphs of each population in Figure 2 show that too large stochastic disturbances destroy the stability of above systems.

5. Conclusion and Future Direction

In this paper, we investigate the stability of two three-species cooperative systems with time delays and stochastic disturbances. Theorem 1 gives the sufficient conditions assuring the globally asymptotical stability around the equilibrium state of the corresponding deterministic system. Theorems 2 and 3 imply that systems (4) and (6) are stable in probability around the equilibrium states under some conditions, respectively. By giving Remarks 3–5, we clarify the difference of our proof from some existing methods and, particularly, show that time delays and stochastic perturbations bring some significant impacts on the globally asymptotical stability and stability in probability.

In this paper, we assume time delays are constants, whereas in practice, time delays may be time-varying (see [25–27]), so it is necessary to study the dynamics with time-varying delays. Furthermore, for multispecies predator-prey system or multispecies competitive system, whether some similar results can be obtained? These are interesting and left for our future work.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have read and approved the final manuscript.

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References

[1] M. Liu and K. Wang, "Stochastic Lotka-Volterra systems with Lévy noise," *Journal of Mathematical Analysis and Applications*, vol. 410, no. 2, pp. 750–763, 2014.

[2] M. Liu, K. Wang, and K. Wang, "Population dynamical behavior of Lotka-Volterra cooperative systems with random perturbations," *Discrete & Continuous Dynamical Systems-A*, vol. 33, no. 6, pp. 2495–2522, 2013.

[3] W. Zuo, D. Jiang, X. Sun, T. Hayat, and A. Alsaedi, "Long-time behaviors of a stochastic cooperative Lotka-Volterra system with distributed delay," *Physica A: Statistical Mechanics and Its Applications*, vol. 506, pp. 542–559, 2018.

[4] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Cambridge, MA, USA, 1993.

[5] K. Tran and G. Yin, "Stochastic competitive Lotka-Volterra ecosystems under partial observation: feedback controls for permanence and extinction," *Journal of the Franklin Institute*, vol. 351, no. 8, pp. 4039–4064, 2014.

[6] X. Li, X. Mao, and X. Mao, "Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation," *Discrete & Continuous Dynamical Systems-A*, vol. 24, no. 2, pp. 523–545, 2009.

[7] L. Chen and F. Chen, "Dynamic behaviors of the periodic predator-prey system with distributed time delays and impulsive effect," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 4, pp. 2467–2473, 2011.

[8] M. Liu and K. Wang, "Global asymptotic stability of a stochastic Lotka-Volterra model with infinite delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 8, pp. 3115–3123, 2012.

[9] M. Liu and C. Bai, "Global asymptotic stability of a stochastic delayed predator-prey model with Beddington-DeAngelis functional response," *Applied Mathematics and Computation*, vol. 226, pp. 581–588, 2014.

[10] A. Hastings and T. Powell, "Chaos in a three-species food chain," *Ecology*, vol. 72, no. 3, pp. 896–903, 1991.

[11] M. Liu, C. Bai, and Y. Jin, "Population dynamical behavior of a two-predator one-prey stochastic model with time delay," *Discrete & Continuous Dynamical Systems-A*, vol. 37, no. 5, pp. 2513–2538, 2017.

[12] Z. Wang, Y. Shao, X. Fang, and X. Ma, "An impulsive three-species model with square root functional response and mutual interference of predator," *Discrete Dynamics in Nature and Society*, vol. 2016, Article ID 3897234, 14 pages, 2016.

[13] J. Wu, "Stability of a three-species stochastic delay predator-prey system with Lévy noise," *Physica A: Statistical Mechanics and Its Applications*, vol. 502, pp. 492–505, 2018.

[14] J. R. Beddington and R. M. May, "Harvesting natural populations in a randomly fluctuating environment," *Science*, vol. 197, no. 4302, pp. 463–465, 1977.

[15] R. Rudnicki and K. Pichór, "Influence of stochastic perturbation on prey-predator systems," *Mathematical Biosciences*, vol. 206, no. 1, pp. 108–119, 2007.

[16] N. H. Du and V. H. Sam, "Dynamics of a stochastic Lotka-Volterra model perturbed by white noise," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 82–97, 2006.

[17] Y. Zhao and S. Yuan, "Stability in distribution of a stochastic hybrid competitive Lotka-Volterra model with Lévy jumps," *Chaos, Solitons & Fractals*, vol. 85, pp. 98–109, 2016.

[18] L. Tan, W. Jin, and Y. Suo, "Stability in distribution of neutral stochastic functional differential equations," *Statistics & Probability Letters*, vol. 107, pp. 27–36, 2015.

[19] S. Abbas, D. Bahuguna, and M. Banerjee, "Effect of stochastic perturbation on a two species competitive model," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 3, pp. 195–206, 2009.

[20] Q. Liu, "The effects of time-dependent delays on global stability of stochastic Lotka-Volterra competitive model," *Physica A: Statistical Mechanics and Its Applications*, vol. 420, pp. 108–115, 2015.

[21] Y. Hu, F. Wu, and C. Huang, "Stochastic Lotka-Volterra models with multiple delays," *Journal of Mathematical Analysis and Applications*, vol. 375, no. 1, pp. 42–57, 2011.

[22] V. B. Kolmanovskii and A. D. Myshkis, *Applied Theory of Functional Diffurential Equations*, Springer Science, Berlin, Germany, 1992.

[23] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Publishing, Chichester, England, 1997.

[24] D. J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," *SIAM Review*, vol. 43, no. 3, pp. 525–546, 2001.

[25] Y. Shao, "Exponential stability of periodic neural networks with impulsive effects and time-varying delays," *Applied Mathematics and Computation*, vol. 217, no. 16, pp. 6893–6899, 2011.

[26] P. Chanthorn, G. Rajchakit, J. Thipcha et al., "Robust stability of complex-valued stochastic neural networks with time-varying delays and parameter uncertainties," *Mathematics*, vol. 742, pp. 1–19, 2020.

[27] G. Rajchakit and R. Sriraman, "Robust passivity and stability analysis of uncertain complex-valued impulsive neural networks with time-varying delays," *Neural Processing Letter*, vol. 53, no. 13, 2021.