Research Article

Laplace Operator with Caputo-Type Marichev–Saigo–Maeda Fractional Differential Operator of Extended Mittag-Leffler Function

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Received 6 January 2021; Revised 26 February 2021; Accepted 1 April 2021; Published 19 April 2021

Academic Editor: Xiaohua Ding

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In this paper, the Laplace operator is used with Caputo-Type Marichev–Saigo–Maeda (MSM) fractional differentiation of the extended Mittag-Leffler function in terms of the Laplace function. Further in this paper, some corollaries and consequences are shown which are the special cases of our main findings. We apply the Laplace operator on the right-sided MSM fractional differential operator and on the left-sided MSM fractional differential operator. We also apply the Laplace operator on the right-sided MSM fractional differential operator with the Mittag-Leffler function and the left-sided MSM fractional differential operator with the Mittag-Leffler function.

1. Introduction

Fractional calculus is a fast-growing field of mathematics that shows the relations of fractional-order derivatives and integrals. Fractional calculus is an effective subject to study many complex real-world systems. In recent years, many researchers have calculated the properties, applications, and extensions of fractional integral and differential operators involving various special functions.

Integral and differential operators in fractional calculus have become a research subject in recent decades due to the ability to have arbitrary order. Special functions are the functions that have improper integrals or series. Some of the well-known functions are the gamma function, beta function, and hypergeometric function.

Many researchers established compositions of new fractional derivative formulas called Marichev–Saigo–Maeda Caputo-type fractional operators on well-known functions like the Mittag-Leffler function.

Fernandez et al. [1] proposed the definitions for fractional derivatives and integrals, starting from the classical Riemann–Liouville formula and its generalizations and

modifying it by replacing the power function kernel with other kernel functions. They demonstrated, under some assumptions, how all of these modifications can be considered as special cases of a single, unifying model of fractional calculus. They provided a fundamental connection with classical fractional calculus by writing these general fractional operators in terms of the original Riemann–Liouville fractional integral operator.

Fernandez et al. [2] considered an integral transform introduced by Prabhakar, involving generalized multiparameter Mittag-Leffler functions, which can be used to introduce and investigate several different models of fractional calculus. They derived a new series expression for this transform, in terms of classical Riemann–Liouville fractional integrals, and used it to obtain or verify series formulas in various specific cases corresponding to different fractional calculus models.

Srivastava et al. [3] considered the well-known Mittag-Leffler functions of one, two, and three parameters and established some new connections between them using fractional calculus. In particular, they expressed the three-parameter Mittag-Leffler function as a fractional derivative

of the two-parameter Mittag-Leffler function, which is, in turn, a fractional integral of the one-parameter Mittag-Leffler function. Hence, they derived an integral expression for the three-parameter one in terms of the one-parameter one.

Khan et al. [4] studied the fractional-order model of HIV/AIDS involving the Liouville–Caputo and Atangana–Baleanu–Caputo derivatives. The generalised HIV/AIDS model allows and shows that certain infected individuals switch from symptomatic to asymptomatic phases. Special iterative solutions were obtained by the use of Laplace and Sumudu transform.

Khan [5] established the existence of positive solutions (EPS) and the Hyers–Ulam (HU) stability of a general class of nonlinear Atangana–Baleanu–Caputo (ABC) fractional differential equations (FDEs) with singularity and nonlinear p -Laplacian operator in Banach's space.

Khan et al. [6] are interested in using the Atangana–Baleanu fractional differential form to analyse the HIV-TB co-infected model. The model is studied for the existence, uniqueness of solution, Hyers–Ulam (HU) stability, and numerical simulations with the assumption of specific parameters.

Ahmad et al. [7] presented the mathematical model with different compartments for the transmission dynamics of coronavirus-19 disease (COVID-19) under the fractional-order derivative. Some results regarding the existence of at least one solution through fixed point results have been derived. Then, for the concerned approximate solution, the modified Euler method for fractional-order differential equations (FODEs) is utilized.

Shah et al. [8] studied a compartmental mathematical model for the transmission dynamics of the novel coronavirus-19 under Caputo fractional-order derivative. By using the fixed point theory of Schauder and Banach, they established some necessary conditions for the existence of at least one solution to model under investigation and its uniqueness. After the existence, a general numerical algorithm based on the Haar collocation method is established to compute the approximate solution of the model.

Sher et al. [9] studied the novel coronavirus (2019-nCoV or COVID-19) which is a threat to the whole world

nowadays. They considered a fractional-order epidemic model which describes the dynamics of COVID-19 under a non-singular kernel type of fractional derivative. An attempt is made to discuss the existence of the model using the fixed point theorem of Banach and Krasnoselskii.

Manzoor et al. [10] used the beta operator with Caputo (MSM) fractional differentiation of the extended Mittag-Leffler function in terms of beta function. They applied the beta operator on the right-sided MSM fractional differential operator and on the left-sided MSM fractional differential operator. They also applied the beta operator on the right-sided MSM fractional differential operator with the Mittag-Leffler function and the left-sided MSM fractional differential operator with the Mittag-Leffler function.

Kilbas et al. [11] worked on the composition of Riemann–Liouville fractional integration and differential operators. Rao et al. [12] introduced the result that fractional integration and fractional differentiation are interchanged. Agarwal and Jain [13] developed fractional calculus formula of polynomial using the series expansion method. Choi and Agarwal [14] aimed to find confident integral transforms and fractional integral formula for the generalized hypergeometric function. Agarwal and Choi [15] proved certain image formulas of various fractional integral operators involving Gauss hypergeometric function. Further, it is expressed in terms of Hadamard product.

Nadir and Khan [16] applied Caputo-type MSM fractional differentiation on the Mittag-Leffler function. Nadir and Khan [17, 18] used fractional integral operator associated with the extended Mittag-Leffler function.

Nadir et al. [19] studied the extended versions of the generalized Mittag-Leffler function. Nadir and Khan [20] applied Weyl fractional calculus operators on the extended Mittag-Leffler function. Mondal and Nisar [21] applied the Marichev–Saigo–Maeda operator on the Bessel function. Nadir and Khan [22] applied the Marichev–Saigo–Maeda differential operator and generalized incomplete hypergeometric functions. Maitama and Zhao [23] worked on a new integral transform called Shehu transform, a generalization of Sumudu and Laplace transform, for solving differential equations.

Srivastava et al. [24] defined a function

$$\Theta(\{k_n\}_{n \in N_0}; x) := \left\{ \begin{array}{l} \sum_{n=0}^{\infty} k_n \frac{z^n}{n!} \quad \left(\begin{array}{l} |x| < \Re \\ 0 < \Re < \infty \\ k_0 := 1 \end{array} \right) \\ m_0 x \omega \exp(x) \left[1 + O\left(\frac{1}{2}\right) \right] \quad \left(\begin{array}{l} \Re(x) \rightarrow \infty \\ m_0 > 0; \omega \in \mathbb{C} \end{array} \right) \end{array} \right\}, \quad (1)$$

where $\Theta(\{k_n\}_{n \in N_0}; x)$ is considered to be analytical with $|x| < \Re$, $0 < \Re < \infty$, $\{k_n\}_{n \in N_0}$ is a sequence of Taylor–Maclaurin coefficient, and m_0 and ω are constants and depend upon the bounded sequence $\{k_n\}_{n \in N_0}$.

The series

$$E_{\varepsilon, \mu}^{(\{k_n\}_{n \in N_0}; \gamma)}(x, p) = \sum_{k=0}^{\infty} \frac{B_{\rho}^{(\{k_n\}_{n \in N_0}; \gamma)}(\gamma + k, 1 - \gamma; \rho)}{B(\gamma, 1 - \gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)}, \quad (2)$$

where $\left(\begin{array}{l} x, \mu, \gamma \in C; \Re(\varepsilon) > 0 \\ \Re(\mu) > 0, \Re(\gamma) > 1; \rho \geq 0 \end{array} \right)$ is known as the extension of the Mittag-Leffler function. It was defined by Parmar [25].

Mittag-Leffler functions with special cases are given below.

- (i) When $K_n = ((\rho)_n / (\sigma)_n)$, then the extended form of (2) takes the form

$$E_{\varepsilon, \mu}^{(\rho, \sigma); \gamma}(x; \rho) = \sum_{k=0}^{\infty} \frac{B_{\rho}^{(\rho, \sigma)}(\gamma + k, 1 - \gamma; \rho)}{B(\gamma, 1 - \gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)}, \quad (3)$$

under the condition

$$\left(\begin{array}{l} x, \mu, \gamma \in C; \Re(\varepsilon) > 0, \Re(\sigma) > 0 \\ \Re(\varepsilon) > 0, \Re(\mu) > 0, \Re(\gamma) > 1; \rho \geq 0 \end{array} \right). \quad (4)$$

- (ii) If we select a bounded sequence $K_n = 1$, then (2) reduces to the definition of Özarslan and Yilmaz [26].

$$E_{\varepsilon, \mu}^{\gamma}(x; \rho) = \sum_{k=0}^{\infty} \frac{B(\gamma + k, 1 - \gamma; \rho)}{B(\gamma, 1 - \gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)}, \quad (5)$$

$$\cdot \left(\begin{array}{l} x, \mu, \gamma \in C; \Re(\varepsilon) > 0 \\ \Re(\mu) > 0, \Re(\gamma) > 1; \rho \geq 0 \end{array} \right).$$

- (iii) Another special case of (2) is when $K_n = 1$ and $\rho = 0$; then, (2) reduces to Prabhakar's function [27] of three parameters.

$$E_{\varepsilon, \mu}^{\gamma}(x; \rho) = \sum_{k=0}^{\infty} \frac{(\gamma_k) x^k}{\Gamma(\varepsilon k + \mu) k!}, \quad (6)$$

$$(\varepsilon, \mu, \gamma \in C; \Re(\varepsilon) > 0, \Re(\mu) > 0).$$

- (iv) If we set $\varepsilon = \mu = 1$, then our expression for $E_{\varepsilon, \mu}^{(\{k_n\}_{n \in N_0}; \gamma)}$, $E_{\varepsilon, \mu}^{(\rho, \sigma); \gamma}$, and $E_{\varepsilon, \mu}^{\gamma}$ reduces to the extended confluent hypergeometric functions:

$$E_{\varepsilon, \mu}^{(\{k_n\}_{n \in N_0}; \gamma)}(x; p) = \Phi_p^{(\{k_n\}_{n \in N_0})}(\gamma; 1; x), \quad (7)$$

$$E_{\varepsilon, \mu}^{((p, q); \gamma)}(x; p) = \Phi_p^{(p, q)}(\gamma; 1; x),$$

$$E_{1, 1}^{\gamma}(x; p) = \Phi_p(\gamma; 1; x).$$

2. The Confluent Hypergeometric Function

The confluent hypergeometric function by Rainville is defined as ${}_2F_1(a, b; c; x)$ which is represented by hypergeometric series.

$${}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}, \quad (8)$$

3. The Hadamard Product of the Power Series

As indicated in Pohlen [28], let $g(z) = \sum_{m=0}^{\infty} x_m z^m$ and $h(z) = \sum_{m=0}^{\infty} y_m z^m$ be two power series; then, the Hadamard product of power series is defined as

$$(g * h)(z) = \sum_{m=0}^{\infty} x_m y_m z^m = (h \cdot g)(z), \quad (|z| < R), \quad (9)$$

where

$$R = \lim_{m \rightarrow \infty} \left| \frac{x_m y_m}{x_{m+1} y_{m+1}} \right| \left(\lim_{k \rightarrow \infty} \left| \frac{x_m}{x_{m+1}} \right| \right) \left(\lim_{m \rightarrow \infty} \left| \frac{y_m}{y_{m+1}} \right| \right) = R_g \cdot R_h, \quad (10)$$

where R_g and R_h are radii of convergence of the above series $g(z)$ and $h(z)$, respectively. Therefore, in general, it is to be noted that if one power series is an analytical function, then the series of Hadamard products is also the same like an analytical function.

4. Laplace Transform

The Laplace transform of the function $f(x)$ on an interval $[0, \infty)$ by Sneddon [29] is defined as

$$\mathcal{L}[f(x); l] = \int_0^{\infty} e^{-lx} f(x) dx, \quad (11)$$

where $l \in C$ and $x \geq 0$.

5. Appell Function

Appell function by Rainville [30] of first kind F_3 is basically two-variable hypergeometric function defined as

$$F_3(\omega, \omega', v, v', \eta; x, y) = \sum_{m, n} \frac{(\omega)_m (\omega')_n (v)_m (v')_n}{(\eta)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{(\omega)_m (v)_m}{(\eta)_m} {}_2F_1 \left[\begin{array}{c} \omega', v' \\ \eta + m \end{array}; y \right] \frac{x^m}{m!}. \quad (12)$$

6. The Left-Sided MSM Fractional Differential Operator

The left-sided MSM fractional differential operator containing Appell F_3 function in their kernel by Saigo and Maeda [31] is defined as follows: let $\alpha, \alpha_1, \omega, \omega_1, \mu, \rho \in C$ and $x > 0$; then,

$$\begin{aligned} (D_{0+}^{\alpha, \alpha_1, \omega, \omega_1, \mu} f)(x) &= (I_{0+}^{-\alpha, -\alpha_1, -\omega, -\omega_1, -\mu} f)(x) \\ &= \frac{d^n}{dx^n} (I_{0+}^{\alpha, -\alpha_1, -\omega, -\omega_1, -\mu+n, +\mu+n} f)(x), \end{aligned} \quad (13)$$

where $\Re(\mu) > 0$ and $n = [\Re(-\mu) + 1]$.

7. The Right-Sided MSM Fractional Differential Operator

The right-sided MSM fractional differential operator containing Appell function F_3 in their kernel by Saigo and Maeda [31] is defined as follows.

Let $\alpha, \alpha_1, \omega, \omega_1, \mu, \rho \in \mathbb{C}$ and $x > 0$. Then,

$$\begin{aligned} (D_{0-}^{\alpha, \alpha_1, \omega, \omega_1, \mu} f)(x) &= (I_{0-}^{-\alpha, -\alpha_1, -\omega, -\omega_1, -\mu} f)(x) \\ &= (-1)^n \frac{d^n}{dx^n} (I_{x, \infty}^{\alpha, -\alpha_1, -\omega+n, -\omega_1, -\mu+n} f)(x), \end{aligned} \quad (14)$$

where $\Re(\mu) > 0$ and $n = [\Re(-\mu) + 1]$.

Lemma 1. Let $\omega, \lambda, \beta, \rho \in \mathbb{C}$, $x > 0$ such that $\Re(\omega) > 0$; then,

$$(D_{+}^{\omega, \lambda, \beta} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(\rho + \beta + \omega + \lambda)}{\Gamma(\rho + \beta)\Gamma(\rho + \lambda)} x^{\rho + \lambda + 1}, \quad (15)$$

where

$$(\Re(\rho) > -\min\{0, \Re(\omega + \lambda + \beta)\}). \quad (16)$$

Lemma 2. Let $\omega, \lambda, \beta, \rho \in \mathbb{C}$, $x > 0$ such that $\Re(\omega) > 0$; then,

$$(D_{-}^{\omega, \lambda, \beta} t^{\rho-1})(x) = \frac{\Gamma(1 - \rho - \lambda)\Gamma(1 - \rho + \omega + \beta)}{\Gamma(1 - \rho + \beta - \lambda)\Gamma(1 - \rho)} x^{\rho + \lambda + 1}, \quad (17)$$

where

$$(\Re(\rho - \sigma k) < 1 + \min\{\Re(-\lambda - n), \Re(\beta + \omega)\} \text{ and } [\Re(\omega)] + 1). \quad (18)$$

Lemma 3. Let $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in \mathbb{C}$ and $m = [\Re(\eta)] + 1$, $\Re(\rho) - m > \max\left\{0, \Re(-\beta + \varepsilon), \Re(-\beta - \beta' - \varepsilon' + \eta)\right\}$; then, the image will be

$$\begin{aligned} (D_{+}^{\beta, \beta', \varepsilon, \varepsilon', \eta} t^{\rho-1})(x) \\ = \frac{\Gamma(\rho)\Gamma(\rho - \varepsilon + \beta)\Gamma(\beta + \beta' + \varepsilon' - \eta + \rho)}{\Gamma(-\varepsilon + \rho)\Gamma(\beta + \varepsilon' - \eta + \rho)\Gamma(\beta + \beta' - \eta + \rho)} x^{\beta + \beta' - \eta + \rho - 1}. \end{aligned} \quad (19)$$

Lemma 4. Let $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in \mathbb{C}$ and $m = [\Re(\eta)] + 1$, $\Re(\rho) + m > \max\left\{\Re(-\varepsilon'), \Re(\beta' + \varepsilon - \eta), \Re(\beta + \beta' - \varepsilon' - \eta) + m\right\}$; then,

$$\begin{aligned} (D_{-}^{\beta, \beta', \varepsilon, \varepsilon', \eta} t^{\rho-1})(x) \\ = \frac{\Gamma(1 + \varepsilon' - \rho)\Gamma(1 - \beta - \beta' + \eta - \rho)\Gamma(1 - \beta' - \varepsilon + \eta - \rho)}{\Gamma(1 - \rho)\Gamma(1 - \beta' + \varepsilon' - \rho)\Gamma(1 - \beta - \beta' - \varepsilon + \eta - \rho)} x^{\beta + \beta' - \eta + \rho - 1}. \end{aligned} \quad (20)$$

8. The Left-Sided MSM Fractional Differential Operator with Mittag-Leffler Function

Theorem 1. Let $\omega, \lambda, \beta, \rho \in \mathbb{C}$, $\Re(\rho) > 0$ be such that $(\Re(\rho + \sigma k) > -\min\{0, \Re(\omega + \lambda + \beta)\})$; then, the following result holds.

$$\begin{aligned} L \left\{ u^{l-1} \left(D_{+}^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (z) \right\} \\ = \frac{Z^{\rho + \lambda - 1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} (x(z/s)^\sigma) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(z/s)^\sigma) \right], \end{aligned} \quad (21)$$

where

$$\begin{aligned}\Delta &= \{l, \sigma, (\rho, \sigma), (\rho + \beta + \omega + \lambda, \sigma)(1, \sigma)\}, \\ \Delta' &= \{(\rho + \beta, \sigma), (\rho + \lambda, \sigma)\}.\end{aligned}\quad (22)$$

Proof

$$\begin{aligned}L \left\{ u^{l-1} \left(D_+^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (z) \right\} \\ = \int_0^\infty e^{-su} u^{l-1} \left\{ \left(D_+^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (z) \right\} du.\end{aligned}\quad (23)$$

By using definition (11) and Lemma (15) and changing ρ by $\rho + \sigma k$, we get

$$\begin{aligned}L \left\{ u^{l-1} \left(D_+^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (z) \right\} \\ = z^{\rho + \sigma k + \lambda - 1} \sum_{k=0}^\infty \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)} \\ \times \frac{\Gamma(\rho + \sigma k) \Gamma(\rho + \sigma k + \beta + \omega + \lambda)}{\Gamma(\rho + \sigma k + \beta) \Gamma(\rho + \sigma k + \lambda)} \int_0^\infty e^{-su} u^{l-1 + \sigma k} du, \\ = \frac{z^{\rho + \sigma k + \lambda - 1}}{s^{l + \sigma k}} \sum_{k=0}^\infty \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)} \\ \times \frac{\Gamma(l + \sigma k) \Gamma(\rho + \sigma k) \Gamma(\rho + \sigma k + \beta + \omega + \lambda)}{\Gamma(\rho + \sigma k + \beta) \Gamma(\rho + \sigma k + \lambda)}.\end{aligned}\quad (24)$$

By using Hadamard product which is given in (9), we get

$$\begin{aligned}L \left\{ u^{l-1} \left(D_+^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (z) \right\} \\ = \frac{Z^{\rho + \lambda - 1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} (x(z/s)^\sigma) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(z/s)^\sigma) \right].\end{aligned}\quad (25)$$

Corollary 1. Let $\omega, \lambda, \beta, \rho \in C\mathfrak{R} > 0$ such that $(\Re(\rho + \sigma k) > -\min\{0, \Re(\omega + \lambda + \beta)\})$; under the stated conditions, the right-sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned}L \left\{ u^{l-1} \left(D_+^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^\gamma \right) (x(tu)^\sigma) \right) (z) \right\} \\ = \frac{Z^{\rho + \beta - 1}}{s^l} E_{\varepsilon, \mu}^\gamma (x(z/s)^\sigma) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(z/s)^\sigma) \right],\end{aligned}\quad (26)$$

where

$$\begin{aligned}\Delta &= \{l, \sigma, (\rho, \sigma), (\rho + \beta + \omega + \lambda, \sigma)(1, \sigma)\}, \\ \Delta' &= \{(\rho + \beta, \sigma), (\rho + \lambda, \sigma)\}.\end{aligned}\quad (27)$$

Select a bounded sequence $k_n = 1$ in equation (25) and then proceed (26).

Corollary 2. Let $\omega, \lambda, \beta, \rho \in C\mathfrak{R} > 0$ be such that $(\Re(\rho + \sigma k) > -\min\{0, \Re(\omega + \lambda + \beta)\})$; under the stated conditions, the right-sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned}L \left\{ u^{l-1} \left(D_+^{\omega, \lambda, \beta} \left(t^{\rho-1} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)}(\gamma; 1; x(tu)^\sigma) \right) (z) \right\} \\ = \frac{Z^{\rho + \lambda - 1}}{s^l} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)}(\gamma; 1; x(z/s)^\sigma) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(z/s)^\sigma) \right],\end{aligned}\quad (28)$$

where

$$\begin{aligned}\Delta &= \{l, \sigma, (\rho, \sigma), (\rho + \beta + \omega + \lambda, \sigma)(1, \sigma)\}, \\ \Delta' &= \{(\rho + \beta, \sigma), (\rho + \lambda, \sigma)\}.\end{aligned}\quad (29)$$

If we select $\xi = \mu = 1$, then extension of Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.

Theorem 2. Let

$$\begin{aligned} \beta, \beta', \varepsilon, \varepsilon', \eta, \rho &\in C, \\ m &= [\Re(\eta)] + 1, \\ \Re(\rho) - m &> \max \left\{ \begin{array}{l} \alpha, \Re(-\beta + \varepsilon) \\ \Re(-\beta - \beta' - \varepsilon' + \eta) \end{array} \right\}. \end{aligned} \quad (30)$$

Then,

$$\begin{aligned} &L \left\{ u^{l-1} \left(D_+^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (u) \right\} \\ &= \frac{Z^{\beta+\beta'-\eta+\rho-1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} (x(z/s)^\sigma) * {}_5\Psi_3 \left[\begin{array}{c} \Delta \\ \Delta' \end{array}; (x(z/s)^\sigma) \right], \end{aligned} \quad (31)$$

where

$$\begin{aligned} \Delta &= \{(l, \sigma), (\rho, \sigma)(\beta - \varepsilon + \rho, \sigma), (\beta + \beta' + \varepsilon' - \eta + \rho, \sigma), (1, \sigma)\}, \\ \Delta' &= \{(-\varepsilon + \rho, \sigma), (\beta + \beta' - \eta + \rho, \sigma), (\beta + \varepsilon' - \eta + \rho, \sigma)\}. \end{aligned} \quad (32)$$

Proof

$$L \left\{ u^{l-1} \left(D_+^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (u) \right\}. \quad (33)$$

By using definition of (11) and Lemma (19) and changing ρ by $\rho + \sigma k$, we get

$$\begin{aligned} &L \left\{ u^{l-1} \left(D_+^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (u) \right\} \\ &= Z^{\beta+\beta'-\eta+\rho+\sigma k-1} \sum_{k=0}^{\infty} \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)} \\ &\quad \times \frac{\Gamma(\rho + \sigma k) \Gamma(\rho - \varepsilon + \beta + \sigma k) \Gamma(\beta + \beta' + \varepsilon' - \eta + \rho + \sigma k)}{\Gamma(-\varepsilon + \rho + \sigma k) \Gamma(\beta + \beta' - \eta + \rho + \sigma k) \Gamma(\beta + \varepsilon' - \eta + \rho + \sigma k)} \\ &\quad \times \int_0^\infty e^{-su} u^{l-1+\sigma k} du, \\ &= \frac{Z^{\beta+\beta'-\eta+\rho+\sigma k-1}}{s^{l+\sigma k}} \sum_{k=0}^{\infty} \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\sqrt{(\varepsilon k + \mu)}} \\ &\quad \times \frac{\Gamma(l + \sigma k) \Gamma(\rho + \sigma k) (\Gamma\rho + \sigma k - \varepsilon + \beta) \Gamma(\beta + \beta' + \varepsilon' - \eta + \rho + \sigma k)}{\Gamma(-\varepsilon + \rho + \sigma k) \Gamma(\beta + \varepsilon' - \eta + \rho + \sigma k) \Gamma(\beta + \beta' - \eta + \rho + \sigma k)}. \end{aligned} \quad (34)$$

By using Hadamard product which is given in (9), we get

$$\begin{aligned} &L \left\{ u^{l-1} \left(D_+^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (u) \right\} \\ &= \frac{Z^{\beta+\beta'-\eta+\rho-1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} (x(z/s)^\sigma) * {}_5\Psi_3 \left[\begin{array}{c} \Delta \\ \Delta' \end{array}; (x(z/s)^\sigma) \right]. \end{aligned} \quad (35)$$

Corollary 3. Let the parameters $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ and $m = [\Re(\eta)] + 1$, and under the stated conditions, the left-sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned} &L \left\{ u^{l-1} \left(D_+^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(tu)^\sigma) \right) (u) \right\} \\ &= \frac{Z^{\beta+\beta'-\eta+\rho-1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} (x(z/s)^\sigma) * {}_5\Psi_3 \left[\begin{array}{c} \Delta \\ \Delta' \end{array}; (x(z/s)^\sigma) \right], \end{aligned} \quad (36)$$

where

$$\begin{aligned} \Delta &= \{(l, \sigma), (\rho, \sigma)(\beta - \varepsilon + \rho, \sigma), (\beta + \beta' + \varepsilon' - \eta + \rho, \sigma), (1, \sigma)\}, \\ \Delta' &= \{(-\varepsilon + \rho, \sigma), (\beta + \beta' - \eta + \rho, \sigma), (\beta + \varepsilon' - \eta + \rho, \sigma)\}. \end{aligned} \quad (37)$$

Select a bounded sequence $k_n = 1$ in equation (35) and then proceed (36).

Corollary 4. Let the parameters $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in \mathbb{C}$ and $m = [\Re(\eta)] + 1$, and under the stated conditions, the left-

sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned} L \left\{ u^{l-1} \left(D_+^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)} (\gamma; 1; x(tu)^\sigma) \right) \right) (u) \right\} \\ = \frac{Z^{\beta+\beta'-\eta+\rho-1}}{s^l} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)} (\gamma; 1; x(z/s)^\sigma) * {}_5\Psi_3 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(z/s)^\sigma) \right], \end{aligned} \quad (38)$$

where

$$\begin{aligned} \Delta &= \{(l, \sigma), (\rho, \sigma)(\beta - \varepsilon + \rho - m, \sigma), \\ &\quad \cdot (\beta + \beta' + \varepsilon' - \eta + \rho - m, \sigma), (1, \sigma)\}, \\ \Delta' &= \{(-\varepsilon + \rho - m, \sigma), (\beta + \beta' - \eta + \rho, \sigma), \\ &\quad \cdot (\beta + \varepsilon' - \eta + \rho - m, \sigma)\}. \end{aligned} \quad (39)$$

If we select $\xi = \mu = 1$, then extension of the Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.

9. The Right-Sided MSM Fractional Differential Operator with Mittag-Leffler Function

Theorem 3. $\omega, \lambda, \beta, \rho \in \mathbb{C}, \Re > 0$ where $(\Re(\rho - \sigma k) < 1 + \min\{\Re(-\lambda - n), \Re(\beta + \omega)\})$ and $n = [\Re(\omega)] + 1$; then, the following result holds true:

$$\begin{aligned} L \left\{ u^{l-1} \left(D_-^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} \\ = \frac{Z^{\rho+\lambda-1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} (x(zs)^\sigma) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(zs)^{-\sigma}) \right], \end{aligned} \quad (40)$$

where

$$\begin{aligned} \Delta &= \{(l, \sigma), (1 - \rho - \lambda, \sigma), (1 - \rho + \omega + \beta, \sigma)(1, \sigma)\}, \\ \Delta' &= \{(1 - \rho + \beta - \lambda, \sigma), (1 - \rho)\}. \end{aligned} \quad (41)$$

Proof.

$$\begin{aligned} L \left\{ u^{l-1} \left(D_-^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} \\ = \int_0^\infty e^{-su} u^{l-1} \left\{ \left(D_-^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} dz. \end{aligned} \quad (42)$$

By using definition of (11) and Lemma (17) and changing ρ by $\rho - \sigma k$, we get

$$\begin{aligned} L \left\{ u^{l-1} \left(D_-^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0; \gamma} \right)} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} \\ = z^{\rho - \sigma k + \lambda - 1} \sum_{k=0}^\infty \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)} (r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)} \\ \times \frac{\Gamma(1 - \rho + \sigma k - \lambda) \Gamma(1 - \rho + \sigma k + \omega + \beta)}{\Gamma(1 - \rho + \sigma k + \beta - \lambda) \Gamma(1 - \rho + \sigma k)} \times \int_0^\infty e^{-su} u^{l-1+\sigma k} du, \\ = \frac{z^{\rho - \sigma k + \lambda - 1}}{s^{l+\sigma k}} \sum_{k=0}^\infty \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)} (r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)} \\ \times \frac{\Gamma(l + \sigma k) \Gamma(1 - \rho + \sigma k - \lambda) \Gamma(1 - \rho + \sigma k + \omega + \beta)}{\Gamma(1 - \rho + \sigma k + \beta - \lambda) \Gamma(1 - \rho + \sigma k)}. \end{aligned} \quad (43)$$

By using Hadamard product which is given in (9), we get

$$\begin{aligned} & L \left\{ u^{l-1} \left(D_-^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} \\ &= \frac{Z^{\rho+\lambda-1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} (x(zs)^{-\sigma}) \\ & \quad * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix} ; (x(zs)^{-\sigma}) \right]. \end{aligned} \quad (44)$$

□

Corollary 5. Let $\omega, \lambda, \beta, \rho \in C\mathfrak{R} > 0$ where $(\Re(\rho - \sigma k) < 1 + \min\{\Re(-\lambda - n), \Re(\beta + \omega)\})$ and $[\Re(\omega)] + 1$; under the stated conditions, the left-sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned} & L \left\{ u^{l-1} \left(D_-^{\omega, \lambda, \beta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\gamma} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} \\ &= \frac{Z^{\rho+\lambda-1}}{s^l} E_{\varepsilon, \mu}^{\gamma} (x(zs)^{-\sigma}) \\ & \quad * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix} ; (x(zs)^{-\sigma}) \right], \end{aligned} \quad (45)$$

where

$$\begin{aligned} \Delta &= \{l, \sigma, (1 - \rho + \sigma - \lambda), (1 - \rho + \omega + \beta, \sigma)(1, \sigma)\}, \\ \Delta' &= \{(1 - \rho + \beta - \lambda, \sigma), (1 - \rho)\}. \end{aligned} \quad (46)$$

Select a bounded sequence $k_n = 1$ in equation (44) and then proceed (45).

Corollary 6. Let $\omega, \lambda, \beta, \rho \in C\mathfrak{R} > 0$ where $(\Re(\rho - \sigma k) < 1 + \min\{\Re(-\lambda - n), \Re(\beta + \omega)\})$ and $[\Re(\omega)] + 1$; under the stated conditions, the left-sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned} & L \left\{ u^{l-1} \left(D_-^{\omega, \lambda, \beta} \left(t^{\rho-1} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)} (\gamma; 1; x(t/u)^{-\sigma}) \right) \right) (z) \right\} \\ &= \frac{Z^{\rho+\lambda+1}}{s^l} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)} (\gamma; 1; x(zs)^{-\sigma}) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix} ; (x(zs)^{-\sigma}) \right], \end{aligned} \quad (47)$$

where

$$\begin{aligned} \Delta &= \{l, \sigma, (1 - \rho - \lambda, \sigma), (1 - \rho + \omega + \beta, \sigma)(1, \sigma)\}, \\ \Delta' &= \{(1 - \rho + \beta - \lambda, \sigma), (1 - \rho)\}. \end{aligned} \quad (48)$$

If we select $\xi = \mu = 1$, then extension of the Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.

Theorem 4. Let $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in C$ and $m = [\Re(\eta)] + 1$, $\Re(\rho) + m > \max \left\{ \Re(-\varepsilon'), \Re(\omega' + \varepsilon - \eta) \right\}$.

$$\begin{aligned} & L \left\{ u^{l-1} \left(D_-^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} \right) (x(t/u)^{-\sigma}) \right) (u) \right\} \\ &= \frac{Z^{\rho+\lambda-1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} (x(zs)^{-\sigma}) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix} ; (x(zs)^{-\sigma}) \right], \end{aligned} \quad (49)$$

where

$$\begin{aligned} \Delta &= \{l, \sigma, (1 - \beta' - \varepsilon + \eta - \rho, \sigma), (1 + \varepsilon' - \rho, \sigma), \\ & \quad \cdot (1 - \beta - \beta' + \eta - \rho, \sigma)(1, \sigma)\}, \\ \Delta' &= \{(1 - \rho, \sigma), (1 - \beta' + \varepsilon' - \rho, \sigma), \\ & \quad \cdot (1 - \beta - \beta' - \varepsilon + \eta - \rho, \sigma)\}. \end{aligned} \quad (50)$$

Proof.

$$\begin{aligned} & L \left\{ u^{l-1} \left(D_-^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} \right) (x(t/u)^{-\sigma}) \right) (u) \right\} \\ &= \int_0^\infty e^{-su} u^{l-1} \left\{ \left(D_-^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} du. \end{aligned} \quad (51)$$

By using definition (11) and Lemma (17) and changing ρ by $\rho - \sigma k$, we get

$$\begin{aligned}
 & L \left\{ u^{l-1} \left(D_-^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} \right) (x(t/u)^{-\sigma}) \right) (u) \right\} \\
 &= z^{\beta+\beta'-\eta+\rho-\sigma k-1} \sum_{k=0}^{\infty} \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)} \\
 &\quad \times \frac{\Gamma(1+\varepsilon'-\rho+\sigma k) \Gamma(1-\beta-\beta'+\eta-\rho+\sigma k) \Gamma(1-\beta'-\varepsilon+\eta-\rho+\sigma k)}{\Gamma(1-\rho+\sigma k) \Gamma(1-\beta'+\varepsilon'-\rho+\sigma k) \Gamma(1-\beta-\beta'-\varepsilon+\eta-\rho+\sigma k)} \\
 &\int_0^{\infty} e^{-su} u^{l-1+\sigma k} dz, \\
 &= \frac{z^{\beta+\beta'-\eta+\rho-\sigma k-1}}{s^{l+\sigma k}} \sum_{k=0}^{\infty} \frac{B_p^{\left(\{k_n\}_{n \in N_0} \right)}(r+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{x^k}{\Gamma(\varepsilon k + \mu)} \\
 &\quad \times \frac{\Gamma(l+\sigma k) \Gamma(1+\varepsilon'-\rho+\sigma k) \Gamma(1-\beta-\beta'+\eta-\rho+\sigma k) \Gamma(1-\beta'-\varepsilon+\eta-\rho+\sigma k)}{\Gamma(1-\rho+\sigma k) \Gamma(1-\beta'+\varepsilon'-\rho+\sigma k) \Gamma(1-\beta-\beta'-\varepsilon+\eta-\rho+\sigma k)}. \tag{52}
 \end{aligned}$$

By using Hadamard product which is given in (9), we get

$$\begin{aligned}
 & L \left\{ u^{l-1} \left(D_-^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} \right) (x(t/u)^{-\sigma}) \right) (u) \right\} \\
 &= \frac{Z^{\rho+\lambda-1}}{s^l} E_{\varepsilon, \mu}^{\left(\{k_n\}_{n \in N_0}; \gamma \right)} (x(zs)^{-\sigma}) * {}_4\Psi_2 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(zs)^{-\sigma}) \right]. \tag{53}
 \end{aligned}$$

Corollary 7. Let the parameters $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in \mathbb{C}$ and $m = [\Re(\eta)] + 1$, and under the stated conditions, the right-

sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned}
 & L \left\{ u^{l-1} \left(D_-^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} E_{\varepsilon, \mu}^{\gamma} \right) (x(t/u)^{-\sigma}) \right) (z) \right\} \\
 &= \frac{Z^{\beta+\beta'-\eta+\rho-1}}{s^l} E_{\varepsilon, \mu}^{\gamma} (x(z/s)^{\sigma}) * {}_5\Psi_3 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(zs)^{-\sigma}) \right], \tag{54}
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta &= \{(l, \sigma), (1-\beta'-\varepsilon+\eta-\rho, \sigma), (1+\varepsilon'-\rho, \sigma), ((1-\beta-\beta'+\eta-\rho, \sigma)(1, \sigma))\}, \\
 \Delta' &= \{(1-\rho, \sigma), (1-\beta'+\varepsilon'-\rho, \sigma), (1-\beta-\beta'-\varepsilon+\eta-\rho, \sigma)\}. \tag{55}
 \end{aligned}$$

Select a bounded sequence $k_n = 1$ in equation (53) and then proceed (54).

Corollary 8. Let the parameters $\beta, \beta', \varepsilon, \varepsilon', \eta, \rho \in \mathbb{C}$ and $m = [\Re(\eta)] + 1$, and under the stated conditions, the right-sided Caputo fractional differential operator of the extended Mittag-Leffler function is defined by

$$\begin{aligned}
 & L \left\{ u^{l-1} \left(D_-^{\beta, \beta', \varepsilon, \varepsilon', \eta} \left(t^{\rho-1} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)}(\gamma; 1; x(t/u)^{-\sigma}) \right) (z) \right\} \\
 &= \frac{Z^{\beta+\beta'-\eta+\rho-1}}{s^l} \Phi_p^{\left(\{k_n\}_{n \in N_0} \right)}(\gamma; 1; x(sz)^{-\sigma}) * {}_5\Psi_3 \left[\begin{matrix} \Delta \\ \Delta' \end{matrix}; (x(zs)^{-\sigma}) \right], \tag{56}
 \end{aligned}$$

where

$$\begin{aligned}\Delta &= \{(l, \sigma), (1 - \beta - \beta' + \eta - \rho, \sigma), (1 + \varepsilon' - \rho, \sigma), (1 - \beta' - \varepsilon + \eta - \rho, \sigma), (1, \sigma)\}, \\ \Delta' &= \{(1 - \rho, \sigma), (1 - \beta' + \varepsilon' - \rho, \sigma), (1 - \beta - \beta' - \varepsilon + \eta - \rho, \sigma)\}.\end{aligned}\quad (57)$$

If we select $\xi = \mu = 1$, then extension of the Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions.

Remark 1. In this paper, Laplace transform is applied on Caputo MSM fractional differentiation of the extended Mittag-Leffler function. New results and some corollaries had been demonstrated. Above corollaries can easily be derived if we select $\xi = \mu = 1$ and then the above results reduce for classical confluent hypergeometric functions.

10. Conclusions

The outcomes obtained here by the Laplace transform and beta transform with Caputo MSM fractional differentiation of extended Mittag-Leffler function complete our current consideration. It should be noted that the results of our analysis will be sufficiently important, most general in nature, and capable of differential transform techniques with various special functions by suitable selections using arbitrary parameters that will be elaborated in these outcomes. As a result, the findings of our research will be used to guide some future applications in fields such as computational, physical, observable, and design sciences. Differential operators are very useful for solving problems in many fields of applied sciences, especially in extended form of functions like the beta function, gamma function, Gauss hypergeometric function, confluent hypergeometric function, and Mittag-Leffler function.

In the solution of fractional-order integral equations and examinations of the fractional generalisation of the kinetic equation, the Mittag-Leffler function rises clearly.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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