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Research Article

Robust Asymptotical Stability and Stabilization of Fractional-Order Complex-Valued Neural Networks with Delay

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The robust asymptotical stability and stabilization for a class of fractional-order complex-valued neural networks (FCNNs) with parametric uncertainties and time delay are considered in this paper. It is worth noting that our system combines complex numbers, uncertain parameters, time delay, and fractional orders, which is universal in practical application. Using the theorem of homeomorphism, the sufficient condition of the existence and uniqueness of the equilibrium point for the system is obtained. Then, the sufficient criteria of robust asymptotical stability and stabilization for the addressed models are established, respectively. Finally, we give two numerical examples to verify the feasibility and effectiveness of the theoretical results.

1. Introduction

In recent decades, fractional-order calculus, which can be regarded as a generalization of traditional integer-order calculus, has attracted the interest of a lot of researchers in various fields of science and engineering. It mainly depends on the fact that the properties of some actual processes modeled by fractional differential equations will be more accurate or more applicative, such as diffusion processes [1], biological modeling [2], and image processing [3]. Furthermore, when it comes to some actual dynamical systems, it is much better to describe them by fractional-order models rather than the integer-order counterpart, which mainly benefits from the properties of memory and heredity of fractional derivatives. Among the behaviors of dynamical systems, stability is extremely vital, and numerous pieces of literature concerning the stability of fractional-order dynamical systems have been widely reported (see [4–8]).

Artificial neural network, as the technical reproduction of biological neural network in a simplified sense, was first proposed by McCulloch and Pitts [9] in the 1940s, in which an algorithm-based computational threshold logic model is created for the neural network models. With the development of electronic technology, the research on neural networks has aroused remarkable attention again after a period of silence and subsequently widely applied in image compression [10], speech [11], and natural language processing [12]. Furthermore, the dynamical behaviors of neural networks have been discussed extensively, such as stability [13–16], stabilization [17, 18], and synchronization [19, 20].

As known to all, the locality of the integer-order derivative operator may lead to the limitation when describing the fractional-order neural networks. On the contrary, fractional-order derivative has the character of nonlocality which is superior to the integer-order one. Hence, some processes of dynamic systems involving the historical dependence can be reflected better. In the past decade, some scholars have incorporated the fractional order into neural networks, thus opening up the research field of fractional neural networks, and amounts of significant results about real-valued fractional-order neural networks have been investigated (see [21–27]). In [21], authors investigated a class of neural networks with simplified connectivity structures (ring or hub structure) which is important in characterizing the dynamics of networks models and analyzed the nonlinear dynamics and chaos for the addressed models. In [23], authors gave the expended second method of Lyapunov in the fractional-order case and a vital Caputo fractional-order inequality to construct more efficient Lyapunov functions, and then the stability and synchronization of fractional Hopfield neural networks were discussed. In [25], a stability theorem about fractional-order Hopfield neural networks with time delay and a comparison theorem for a class of fractional-order systems with time delay were established, and on this basis, the condition for global asymptotical stability was obtained. In [27], the global Mittag-Leffler stability and the global synchronization in finite time of a class of fractional-order neural networks with discontinuous activation were analyzed.

From the perspective of the number domain, the complex-valued fractional-order neural networks, as an extension of the real-valued ones, can figure out more practical problems, such as the complex signal in neural networks [28]. In fact, when compared with real-valued fractional-order neural networks, FCNNs are much more complicated because of the state vectors, connection weights, and activation functions of which are all complex values. Recently, a lot of interesting study reports have been done (see [29–36]). Actually, in [29–32], the studies of complex-valued neural networks were carried out by dividing complex-valued parameters or variables into two real values by the definition of complex numbers whereas in [33] authors extended some lemmas to the complex field and dealt with the quasi-projective synchronization of fractional-order complex-valued recurrent neural networks directly in the complex domain instead of separating complex number. Ground on the above idea, in [34], authors considered the finite-time synchronization of the FCNNs by the designed sign function in the complex field. In [35], authors investigated not only the quasi-projective synchronization but also the complete synchronization of the FCNNs added the time delays. In [36], several sufficient conditions guaranteeing the finite-time synchronization of FCNNs were given by employing the graph-theoretic method under a new controller.

As a matter of fact, when modeling the actual dynamical networks, it always fails in acquiring exact values of parameters of the addressed models. This is mainly because some perturbations, named as uncertain parameters, from the models or the environment, are inevitable. It should be pointed out that the influences of such uncertain parameters which may derail the stability or some other properties should not be overlooked in the investigation of the dynamics of nonlinear systems. For the reasons above, a lot of research achievements on fractional neural networks with parametric uncertainties have been obtained [37–39]. In [37], a fractional memristive system with time-varying feedback weights was established, and some criteria ensuring the global asymptotical stability for the considered models were investigated by using Lyapunov techniques. In [38], by employing the method of the comparison principle, some sufficient conditions were deduced to ensure the robust globally asymptotical synchronization of a class of memristor-based fractional-order complex-valued neural networks with multiple time delays. In [39], authors discussed the robust synchronization of the fractional-order complex-valued neural networks with mixed time-varying delays and impulses by applying the approach of adaptive error feedback control.

To the best of our knowledge, although there have been some results on the stability and stabilization analysis of fractional-order complex-valued neural networks with delay and parameter uncertainties, most of them were disposed by separating the complex-valued neural networks into two real-valued ones, and only a few of which were explored directly in complex field. Inspired by the aforementioned discussion, in this paper, we will study the robust asymptotical stability and stabilization of fractional-order complex-valued delayed neural networks (FCDNNs). Based on several established lemmas, the existence, uniqueness, robust asymptotical stability, and robust asymptotical stabilization of the equilibrium point of the addressed models will be studied.

1.1. Notations. $\mathbb{R}$ and $\mathbb{C}$ are the domain of real-valued number and the domain of complex-valued number, respectively. $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$, respectively, represent spaces composed of all n-dimensional complex vectors and the set of all $m\times n$-dimensional complex-valued matrices. For any $x=(x_1,x_2,\ldots,x_n)^T\in\mathbb{C}$, the norm of x is $\|x\|=(\sum_{j=1}^{n}x_j\overline{x_j})^{(1/2)}$. For any matrix $A=(a_{pq})_{n\times n}\in\mathbb{C}^{n\times n}$, A^T and A^* represent the transpose and the conjugate transpose of matrix A, respectively. $\lambda_{\max}(A)(\lambda_{\min}(A))$ stands for the maximum (minimum) eigenvalue of A, $A\geq 0$ $(A\leq 0)$ means the matrix A is Hermitian and positive semidefinite (negative semidefinite), and $A>0$ $(A<0)$ means the matrix A is Hermitian and positive definite (negative definite). In addition, I represents the appropriate dimension identity matrix.

2. Preliminaries

Definition 1 (see [40]). The Caputo fractional derivative of order α for a function $z(t)\in C^n[[t_0,+\infty),\mathbb{C}]$ is defined as

$$ {}_{t_0}^{C}D_t^{\alpha}z(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{n-\alpha-1}z^{(n)}(s)\mathrm{d}s, \tag{1}$$

where $\alpha>0$, $\Gamma(\alpha)=\int_0^{+\infty}e^{-t}t^{\alpha-1}\mathrm{d}t$, and n is the first integer greater than α, that is, $n-1<\alpha<n$, and in particular, when $0<\alpha<1$, one has

$$ {}_{t_0}^{C}D_t^{\alpha}z(t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}z'(s)\mathrm{d}s. \tag{2}$$

In this paper, we would use the following property and lemmas.

Property 1 (see [40]). For any constants u_1 and u_2, the linearity of Caputo fractional-order derivative is described by

$$ {}_{t_0}^{C}D_t^{\alpha}(u_1x(t)+u_2y(t))=u_1{}_{t_0}^{C}D_t^{\alpha}x(t)+u_2{}_{t_0}^{C}D_t^{\alpha}y(t). \tag{3}$$

Lemma 1 (see [33]). *Let* $x(t)\in\mathbb{C}$ *be a continuous and analytic function, for any order* $0<\alpha<1$ *and any time instant* $t\geq t_0$.

$$ {}_{t_0}^{C}D_t^{\alpha} x(t)\overline{x(t)} \le x(t){}_{t_0}^{C}D_t^{\alpha}\overline{x(t)} + \overline{x(t)}{}_{t_0}^{C}D_t^{\alpha} x(t). \tag{4} $$

Lemma 2 (see [41]). *Let $X(t) \in \mathbb{R}^n$. Then, for any order $0 < \alpha < 1$ and any time instant $t \ge t_0$,*

$$ \frac{1}{2}{}_{t_0}^{C}D_t^{\alpha} X^T(t)X(t) \le X^T(t){}_{t_0}^{C}D_t^{\alpha} X(t). \tag{5} $$

Lemma 3. *For a differentiable vector $X(t) \in \mathbb{C}^n$ and a Hermitian positive matrix Q, the following inequality holds:*

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} X^*(t)QX(t) &\le X^T(t)Q^T{}_{t_0}^{C}D_t^{\alpha}\overline{X(t)} \\ &\quad + X^*(t)Q{}_{t_0}^{C}D_t^{\alpha} X(t), \quad t \ge t_0. \end{aligned} \tag{6} $$

Proof. Because $Q > 0$, there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$, such that $Q = P^*P$. Let $Z(t) = PX(t) = Z_R(t) + iZ_I(t) \in \mathbb{C}^n$, where $Z_R(t)$ and $Z_I(t) \in \mathbb{R}^n$; then, we can easily obtain

$$ \begin{aligned} X^*(t)QX(t) &= X^*(t)P^*PX(t) = (PX(t))^*PX(t) \\ &= Z^*(t)Z(t) = Z_R^T(t)Z_R(t) + Z_I^T(t)Z_I(t). \end{aligned} \tag{7} $$

Using Lemma 2, the following inequality holds:

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} X^*(t)QX(t) &= {}_{t_0}^{C}D_t^{\alpha} Z_R^T(t)Z_R(t) + {}_{t_0}^{C}D_t^{\alpha} Z_I^T(t)Z_I(t) \\ &\le 2\left(Z_R^T(t){}_{t_0}^{C}D_t^{\alpha} Z_R(t) + Z_I^T(t){}_{t_0}^{C}D_t^{\alpha} Z_I(t)\right). \end{aligned} \tag{8} $$

It is easily found that

$$ \begin{aligned} Z^T(t){}_{t_0}^{C}D_t^{\alpha}\overline{Z(t)} + Z^*(t){}_{t_0}^{C}D_t^{\alpha} Z(t) &= \left(Z_R(t) + iZ_I(t)\right)^T{}_{t_0}^{C}D_t^{\alpha}\overline{\left(Z_R(t) + iZ_I(t)\right)} + \left(Z_R(t) + iZ_I(t)\right)^*{}_{t_0}^{C}D_t^{\alpha}\left(Z_R(t) + iZ_I(t)\right) \\ &= \left(Z_R^T(t) + iZ_I^T(t)\right){}_{t_0}^{C}D_t^{\alpha}\left(Z_R(t) - iZ_I(t)\right) + \left(Z_R^T(t) - iZ_I^T(t)\right){}_{t_0}^{C}D_t^{\alpha}\left(Z_R(t) + iZ_I(t)\right) \\ &= 2\left(Z_R^T(t){}_{t_0}^{C}D_t^{\alpha} Z_R(t) + Z_I^T(t){}_{t_0}^{C}D_t^{\alpha} Z_I(t)\right). \end{aligned} \tag{9} $$

Submitting (9) into (8), we obtain

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha} X^*(t)QX(t) &\le 2\left(Z_R^T(t){}_{t_0}^{C}D_t^{\alpha} Z_R(t) + Z_I^T(t){}_{t_0}^{C}D_t^{\alpha} Z_I(t)\right) \\ &= Z^T(t){}_{t_0}^{C}D_t^{\alpha}\overline{Z(t)} + Z^*(t){}_{t_0}^{C}D_t^{\alpha} Z(t) \\ &= (PX(t))^T{}_{t_0}^{C}D_t^{\alpha}\overline{(PX(t))} + (PX(t))^*{}_{t_0}^{C}D_t^{\alpha} PX(t) \\ &= X^T(t)P^T\overline{P}{}_{t_0}^{C}D_t^{\alpha}\overline{X(t)} + X^*(t)P^*P{}_{t_0}^{C}D_t^{\alpha} X(t) \\ &= X^T(t)Q^T{}_{t_0}^{C}D_t^{\alpha}\overline{X(t)} + X^*(t)Q{}_{t_0}^{C}D_t^{\alpha} X(t). \end{aligned} \tag{10} $$

Thus, inequality (6) holds. This ends the proof.

Based on the results in [42], the following lemma can be easily obtained. □

Lemma 4. *Let $H \in \mathbb{C}^{n\times k}$ and $M \in \mathbb{C}^{l\times n}$ be constant matrices and $F(t) \in \mathbb{C}^{k\times l}$ be a time-varying matrix. If $F^*(t)F(t) \le I$, then for $\eta > 0$, the following inequality holds:*

$$ \pm\left(HF(t)M + M^*F^*(t)H^*\right) \le \eta HH^* + \eta^{-1}M^*M. \tag{11} $$

Proof. Note that $F^*(t)F(t) \le I$, and it holds that

$$ \begin{aligned} \left[\left(\sqrt{\eta}HF(t)\right)^* \mp \frac{1}{\sqrt{\eta}}M\right]^*\left[\left(\sqrt{\eta}HF(t)\right)^* \mp \frac{1}{\sqrt{\eta}}M\right] &= \left[\sqrt{\eta}HF(t) \mp \frac{1}{\sqrt{\eta}}M^*\right]\left[\sqrt{\eta}F^*(t)H^* \mp \frac{1}{\sqrt{\eta}}M\right] \\ &= \eta HF(t)F^*(t)H^* \mp HF(t)M \mp M^*F^*(t)H^* + \frac{1}{\eta}M^*M \\ &\le \eta HH^* + \frac{1}{\eta}M^*M \mp HF(t)M \mp M^*F^*(t)H^*. \end{aligned} \tag{12} $$

Obviously,

$$\left[\left(\sqrt{\eta}HF(t)\right)^{*}\mp\frac{1}{\sqrt{\eta}}M\right]^{*}\left[\left(\sqrt{\eta}HF(t)\right)^{*}\mp\frac{1}{\sqrt{\eta}}M\right]\geq 0. \tag{13}$$

Consequently, (11) holds, and the proof is completed. □

Lemma 5. *Suppose that $X(t)$ and $Y(t)\in\mathbb{C}^n$ are two continuous and differentiable vector functions, then for any constant $\epsilon>0$, the following inequality holds:*

$$\pm\left(X^{*}(t)Y(t)+Y^{*}(t)X(t)\right)\leq \epsilon X^{*}(t)X(t)+\epsilon^{-1}Y^{*}(t)Y(t). \tag{14}$$

Lemma 6 (see [43]). *$F(x)$ is a homeomorphism on $\mathbb{C}^n$ if $F(x)\in\mathbb{C}^n$ satisfies the following condition:*

(1) $F(x)$ is injective on $\mathbb{C}^n$

(2) $\|F(x)\|\longrightarrow\infty$ as $\|x\|\longrightarrow\infty$

Lemma 7 (see [44]). *Let $V(x(t))\in\mathbb{C}$ be a continuously differentiable function of $x(t)\in\mathbb{C}^n$ and satisfies*

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}V(x(t))\leq -aV(x(t))+\sum_{j=1}^{n}b_jV\left(x\left(t-\tau_j\right)\right), & 1\leq j\leq n,\\ V(x(t))=h(t)\geq 0, & t\in[-\tau,0], \end{cases} \tag{15}$$

where $0<\alpha<1$. If $a>\sqrt{2}\sum_{j=1}^{n}b_j$ and $b_j>0(j=1,2,\ldots,n)$, then

$$\lim_{t\longrightarrow+\infty}V(t)=0, \tag{16}$$

with $h(t)\geq 0$, $\tau_j>0(j=1,2,\ldots,n)$.

3. System Description

In this paper, we consider a class of fractional-order complex-valued neural networks with the following vector form:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}x(t) &= -(D+\Delta D(t))x(t)+(A+\Delta A(t))f(x(t))\\ &\quad +(B+\Delta B(t))f(x(t-\tau))+U, \end{aligned} \tag{17}$$

where $\alpha\in(0,1)$, $t\geq t_0$. $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T\in\mathbb{C}^n$; $x_p(t)$ $(p=1,2,\ldots,n)$ is the state variable of the p-th neuron at time t; $D=\operatorname{diag}(d_1,d_2,\ldots,d_n)\in\mathbb{C}^{n\times n}>0$ denotes the self-connection weight; $A=(a_{pq})_{n\times n}\in\mathbb{C}^{n\times n}$ $(p=1,2,\ldots,n, q=1,2,\ldots,n)$ and $B=(b_{pq})_{n\times n}\in\mathbb{C}^{n\times n}$ $(p=1,2,\ldots,n,\ q=1,2,\ldots,n)$ are the interconnection weight matrix and delayed connection weight matrix, respectively; $f(x(t))=(f_1(x_1(t)),f_2(x_2(t)),\ldots,f_n(x_n(t)))^T\in\mathbb{C}^n$ represents the neuron activation function; $\tau>0$ corresponds to the time transimission delay at time t; $U=(u_1,u_2,\ldots,u_n)^T\in\mathbb{C}^n$ is the external input vector. In addition, the initial condition is of the following form:

$$x(s)=w(s),\quad s\in[-\tau,0], \tag{18}$$

where $w(s)$ is continuous on $[-\tau,0]$.

The following assumptions hold throughout this paper.

(H1) The activation function vector of neurons $f(t)$ is Lipschitz continuous with a Lipschitz constant $l>0$ and $x,y\in\mathbb{C}^n$, that is,

$$\|f(x)-f(y)\|\leq l\|x-y\|. \tag{19}$$

(H2) Suppose that $M_i\in\mathbb{C}^{n\times k}, N_i\in\mathbb{C}^{l\times n}$ are known constant matrices, $F_i(t)\in\mathbb{C}^{k\times l}(i=1,2,3)$ are unknown time-varying matrices, and $F_i^{*}(t)F_i(t)\leq I(i=1,2,3)$. Then, the uncertain parameters $\Delta D(t)$, $\Delta A(t)$, and $\Delta B(t)$ are of the following forms:

$$\begin{aligned} \Delta D(t)&=M_1F_1(t)N_1,\\ \Delta A(t)&=M_2F_2(t)N_2,\\ \Delta B(t)&=M_3F_3(t)N_3. \end{aligned} \tag{20}$$

4. Existence and Uniqueness of the Equilibrium Point

Theorem 1. *Assume that (H1) and (H2) hold, if there exist a Hermitian positive matrix Q and positive constants l and ϵ such that the following condition holds:*

$$\begin{aligned} \Psi &= -D^{*}Q^{*}-Q\,D+\epsilon Q(\Omega+\Theta)Q^{*}\\ &\quad +\epsilon^{-1}N_1^{*}N_1+\epsilon^{-1}l^2\left(2+\lambda_{\max}\left(N_2^{*}N_2\right)\right.\\ &\quad \left.+\lambda_{\max}\left(N_3^{*}N_3\right)\right)I<0, \end{aligned} \tag{21}$$

in which $\Omega=M_1M_1^{*}+M_2M_2^{*}+M_3M_3^{*}$ and $\Theta=AA^{*}+BB^{*}$, then system (17) with parametric uncertainties has a unique equilibrium point.

Proof. Suppose the equilibrium point of system (17) be $\tilde{x}=(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n)^T$, then

$$\begin{aligned} 0 &= -(D+\Delta D(t))\tilde{x}+(A+\Delta A(t))f(\tilde{x})\\ &\quad +(B+\Delta B(t))f(\tilde{x})+U. \end{aligned} \tag{22}$$

Define a map $\mathfrak{F}: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ in the form as follows:

$$\begin{aligned}\mathfrak{F}(\vartheta) = &-(D + \Delta D(t))\vartheta + (A + \Delta A(t))f(\vartheta)\\ &+ (B + \Delta B(t))f(\vartheta) + U.\end{aligned} \tag{23}$$

In the following, we will prove that $\mathfrak{F}(\vartheta)$ is a homeomorphism which indicates that system (17) has a unique equilibrium point. The proving procedure is divided into two steps. □

Step 1. We prove that $\mathfrak{F}$ is an injective map on $\mathbb{C}^n$.

Suppose that there exist two different complex-valued vectors $\vartheta, \varphi \in \mathbb{C}^n$ such that $\mathfrak{F}(\vartheta) = \mathfrak{F}(\varphi)$, then we have

$$\begin{aligned}&-(D + \Delta D(t))(\vartheta - \varphi) + (A + \Delta A(t))(f(\vartheta) - f(\varphi))\\ &+ (B + \Delta B(t))(f(\vartheta) - f(\varphi)) = 0.\end{aligned} \tag{24}$$

Multiplying both sides of the above equation by $2(\vartheta - \varphi)^* Q$, one has

$$\begin{aligned}&-2(\vartheta - \varphi)^* Q(D + \Delta D(t))(\vartheta - \varphi)\\ &+ 2(\vartheta - \varphi)^* Q(A + \Delta A(t))(f(\vartheta) - f(\varphi))\\ &+ 2(\vartheta - \varphi)^* Q(B + \Delta B(t))(f(\vartheta) - f(\varphi)) = 0,\end{aligned} \tag{25}$$

that is,

$$\begin{aligned}&-2(\vartheta - \varphi)^* Q D(\vartheta - \varphi) - 2(\vartheta - \varphi)^* Q M_1 F_1(t) N_1(\vartheta - \varphi)\\ &+ 2(\vartheta - \varphi)^* Q(A + B)(f(\vartheta) - f(\varphi))\\ &+ 2(\vartheta - \varphi)^* Q\left(M_2 F_2(t) N_2 + M_3 F_3(t) N_3\right)(f(\vartheta) - f(\varphi)) = 0.\end{aligned} \tag{26}$$

Note that $F_i^*(t) F_i(t) \leq I$, and then it follows from Lemma 5 and assumption (H1) that

$$-2(\vartheta - \varphi)^* Q D(\vartheta - \varphi) = -(\vartheta - \varphi)^* \left(D^* Q^* + Q D\right)(\vartheta - \varphi), \tag{27}$$

$$\begin{aligned}&-2(\vartheta - \varphi)^* Q M_1 F_1(t) N_1(\vartheta - \varphi)\\ &\leq \epsilon(\vartheta - \varphi)^* Q M_1 M_1^* Q^*(\vartheta - \varphi) + \epsilon^{-1}(\vartheta - \varphi)^* N_1^* F_1^*(t) F_1(t) N_1(\vartheta - \varphi)\\ &\leq (\vartheta - \varphi)^* \left(\epsilon Q M_1 M_1^* Q^* + \epsilon^{-1} N_1^* N_1\right)(\vartheta - \varphi),\end{aligned} \tag{28}$$

$$\begin{aligned}&2(\vartheta - \varphi)^* Q(A + B)(f(\vartheta) - f(\varphi))\\ &\quad = 2(\vartheta - \varphi)^* Q A(f(\vartheta) - f(\varphi)) + 2(\vartheta - \varphi)^* Q B(f(\vartheta) - f(\varphi))\\ &\quad \leq \epsilon(\vartheta - \varphi)^* Q A A^* Q^*(\vartheta - \varphi) + \epsilon^{-1}(f(\vartheta) - f(\varphi))^*(f(\vartheta) - f(\varphi))\\ &\qquad + \epsilon(\vartheta - \varphi)^* Q B B^* Q^*(\vartheta - \varphi) + \epsilon^{-1}(f(\vartheta) - f(\varphi))^*(f(\vartheta) - f(\varphi))\\ &\quad \leq (\vartheta - \varphi)^*\left(\epsilon Q A A^* Q^* + \epsilon Q B B^* Q^*\right)(\vartheta - \varphi) + 2\epsilon^{-1} l^2 (\vartheta - \varphi)^*(\vartheta - \varphi)\\ &\quad = (\vartheta - \varphi)^*\left(\epsilon Q A A^* Q^* + \epsilon Q B B^* Q^* + 2\epsilon^{-1} l^2 I\right)(\vartheta - \varphi),\end{aligned} \tag{29}$$

$$\begin{aligned}&2(\vartheta - \varphi)^* Q\left(M_2 F_2(t) N_2 + M_3 F_3(t) N_3\right)(f(\vartheta) - f(\varphi))\\ &\quad = 2(\vartheta - \varphi)^* Q M_2 F_2(t) N_2(f(\vartheta) - f(\varphi)) + 2(\vartheta - \varphi)^* Q M_3 F_3(t) N_3(f(\vartheta) - f(\varphi))\\ &\quad \leq \epsilon(\vartheta - \varphi)^* Q M_2 M_2^* Q^*(\vartheta - \varphi) + \epsilon^{-1}(f(\vartheta) - f(\varphi))^* N_2^* F_2^*(t) F_2(t) N_2(f(\vartheta) - f(\varphi))\\ &\qquad + \epsilon(\vartheta - \varphi)^* Q M_3 M_3^* Q^*(\vartheta - \varphi) + \epsilon^{-1}(f(\vartheta) - f(\varphi))^* N_3^* F_3^*(t) F_3(t) N_3(f(\vartheta) - f(\varphi))\\ &\quad \leq (\vartheta - \varphi)^*\left(\epsilon Q M_2 M_2^* Q^* + \epsilon Q M_3 M_3^* Q^*\right)(\vartheta - \varphi)\\ &\qquad + \epsilon^{-1}(f(\vartheta) - f(\varphi))^* N_2^* N_2(f(\vartheta) - f(\varphi)) + \epsilon^{-1}(f(\vartheta) - f(\varphi))^* N_3^* N_3(f(\vartheta) - f(\varphi))\\ &\quad \leq (\vartheta - \varphi)^*\left(\epsilon Q M_2 M_2^* Q^* + \epsilon Q M_3 M_3^* Q^*\right)(\vartheta - \varphi)\\ &\qquad + \epsilon^{-1}\lambda_{\max}(N_2^* N_2)(f(\vartheta) - f(\varphi))^*(f(\vartheta) - f(\varphi)) + \epsilon^{-1}\lambda_{\max}(N_3^* N_3)(f(\vartheta) - f(\varphi))^*(f(\vartheta) - f(\varphi))\\ &\quad \leq (\vartheta - \varphi)^*\left(\epsilon Q M_2 M_2^* Q^* + \epsilon Q M_3 M_3^* Q^* + \epsilon^{-1}\lambda_{\max}(N_2^* N_2) l^2 I + \epsilon^{-1}\lambda_{\max}(N_3^* N_3) l^2 I\right)(\vartheta - \varphi).\end{aligned} \tag{30}$$

Thereupon, submitting (27)–(30) to (26), it yields that

$$\begin{aligned}0 \leq (\vartheta - \varphi)^* [&-D^* Q^* - Q D + \epsilon Q\left(M_1 M_1^* + M_2 M_2^* + M_3 M_3^* + A A^* + B B^*\right) Q^*\\ &+ \epsilon^{-1} N_1^* N_1 + \epsilon^{-1} l^2 (2 + \lambda_{\max}(N_2^* N_2) + \lambda_{\max}(N_3^* N_3)) I](\vartheta - \varphi),\end{aligned} \tag{31}$$

that is,

$$(\vartheta-\varphi)^*\Psi(\vartheta-\varphi)\geq 0. \tag{32}$$

However, in consequence of ϑ and φ which are two different complex-valued vectors and $\Psi<0$, we have

$$(\vartheta-\varphi)^*\Psi(\vartheta-\varphi)<0, \tag{33}$$

which is contradicted with (32). Therefore, $\vartheta=\varphi$, which implies that the map $\mathfrak{F}$ is injective on $\mathbb{C}^n$.

Step 2. . We prove that $\|\mathfrak{F}(\vartheta)\|\longrightarrow\infty$ as $\|\vartheta\|\longrightarrow\infty$. Let

$$\begin{aligned}\breve{\mathfrak{F}}(\vartheta)&=-(D+\Delta\ D(t))\vartheta+(A+\Delta A(t))(f(\vartheta)-f(0))\\&\quad+(B+\Delta B(t))(f(\vartheta)-f(0)).\end{aligned} \tag{34}$$

Multiplying both sides of (34) by $2\vartheta^*Q$ gives

$$\begin{aligned}2\vartheta^*Q\breve{\mathfrak{F}}(\vartheta)&=-2\vartheta^*Q(D+\Delta\ D(t))\vartheta+2\vartheta^*Q(A+\Delta A(t))(f(\vartheta)-f(0))\\&\quad+2\vartheta^*Q(B+\Delta B(t))(f(\vartheta)-f(0))\\&=-2(\vartheta-0)^*Q\ D(\vartheta-0)-2(\vartheta-0)^*QM_1F_1(t)N_1(\vartheta-0)+2(\vartheta-0)^*Q(A+B)(f(\vartheta)-f(0))\\&\quad+2(\vartheta-0)^*Q\big(M_2F_2(t)N_2+M_3F_3(t)N_3\big)(f(\vartheta)-f(0)).\end{aligned} \tag{35}$$

Making $\varphi=0$ in (27)–(30), respectively, we obtain

$$-2(\vartheta-0)^*Q\ D(\vartheta-0)=-\vartheta^*\left(D^*Q^*+Q\ D\right)\vartheta, \tag{36}$$

$$-2(\vartheta-0)^*QM_1F_1(t)N_1(\vartheta-0)\leq\vartheta^*\left(\epsilon QM_1M_1^*Q^*+\epsilon^{-1}N_1^*N_1\right)\vartheta, \tag{37}$$

$$2(\vartheta-0)^*Q(A+B)(f(\vartheta)-f(0))\leq\vartheta^*\left(\epsilon QAA^*Q^*+\epsilon QBB^*Q^*+2\epsilon^{-1}l^2I\right)\vartheta, \tag{38}$$

$$\begin{aligned}&2(\vartheta-0)^*Q\big(M_2F_2(t)N_2+M_3F_3(t)N_3\big)(f(\vartheta)-f(0))\\&\quad\leq\vartheta^*\left(\epsilon QM_2M_2^*Q^*+\epsilon QM_3M_3^*Q^*+\epsilon^{-1}\lambda_{\max}(N_2^*N_2)l^2I+\epsilon^{-1}\lambda_{\max}(N_3^*N_3)l^2I\right)\vartheta,\end{aligned} \tag{39}$$

and then submitting (36)–(39) to (34), we get

$$2\vartheta^*Q\breve{\mathfrak{F}}(\vartheta)\leq\vartheta^*\Psi\vartheta\leq-\lambda_{\min}(-\Psi)\vartheta^*\vartheta. \tag{40}$$

By applying the Schwartz inequality, it holds that

$$\lambda_{\min}(-\Psi)\|\vartheta\|^2\leq 2\|\vartheta\|\cdot\|Q\|\cdot\|\breve{\mathfrak{F}}(\vartheta)\|, \tag{41}$$

and when $\|\vartheta\|\neq 0$, we get

$$\frac{\lambda_{\min}(-\Psi)}{2\|Q\|}\|\vartheta\|\leq\|\breve{\mathfrak{F}}(\vartheta)\|. \tag{42}$$

Thus, $\|\breve{\mathfrak{F}}(\vartheta)\|\longrightarrow\infty$ as $\|\vartheta\|\longrightarrow\infty$, which implies $\|\mathfrak{F}(\vartheta)\|\longrightarrow\infty$ as $\|\vartheta\|\longrightarrow\infty$. This proof is completed.

5. Robust Asymptotical Stability

We first transform the equilibrium point of (17) to the origin by the transformation $z(t)=x(t)-\tilde{x}$, and it yields that

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}z(t)&=-(D+\Delta\ D(t))z(t)+(A+\Delta A(t))g(z(t))\\&\quad+(B+\Delta B(t))g(z(t-\tau)),\end{aligned} \tag{43}$$

where $g(z(t))=f(z(t)+\tilde{x})-f(\tilde{x})$ and $g(z(t-\tau))=f(z(t-\tau)+\tilde{x})-f(\tilde{x})$, and the coefficients are the same as ones of system (17). Next, we will analyze the robust asymptotical stability of system (43).

Theorem 2. . *Under the conditions of Theorem 1, if there exist a Hermitian positive matrix Q and positive constants l, η, and ϵ such that the following condition holds:*

$$\frac{\lambda_{\min}(\Pi)}{\lambda_{\max}(Q)}>\frac{\sqrt{2}\lambda_{\max}(\Xi)}{\lambda_{\min}(Q)}, \tag{44}$$

where

$$\begin{aligned}\Pi&=D^*Q+Q\ D-\epsilon l^2I-\eta Q\Omega Q-\epsilon^{-1}Q\Theta Q\\&\quad-\eta^{-1}\left(N_1^*N_1+\lambda_{\max}(N_2^*N_2)l^2I\right),\\\Xi&=l^2\left(\epsilon+\eta^{-1}\lambda_{\max}(N_3^*N_3)\right)I,\end{aligned} \tag{45}$$

in which $\Omega=M_1M_1^*+M_2M_2^*+M_3M_3^*$ and $\Theta=AA^*+BB^*$, then the unique equilibrium point of system (17) is robust asymptotically stable.

Proof. Consider the following Lyapunov function candidate:

$$V(t)=z^*(t)Qz(t). \tag{46}$$

By calculating the α-order Caputo derivatives of $V(t)$ along the trajectories of system (43), we can obtain from Lemma 3 that

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V(t) &\le z^T(t)Q^T{}_{t_0}^{C}D_t^{\alpha}\overline{z(t)} + z^*(t)Q{}_{t_0}^{C}D_t^{\alpha}z(t)\\
&= z^T(t)Q^T\left[-\left(\overline{D} + \overline{M_1F_1(t)N_1}\right)\overline{z(t)} + \left(\overline{A} + \overline{M_2F_2(t)N_2}\right)\overline{g(z(t))} + \left(\overline{B} + \overline{M_3F_3(t)N_3}\right)\overline{g(z(t-\tau))}\right]\\
&\quad + z^*(t)Q\left[-\left(D + M_1F_1(t)N_1\right)z(t) + \left(A + M_2F_2(t)N_2\right)g(z(t)) + \left(B + M_3F_3(t)N_3\right)g(z(t-\tau))\right]\\
&= -z^T(t)Q^T\overline{Dz(t)} - z^*(t)Q\,Dz(t) + z^T(t)Q^T\overline{Ag(z(t))} + z^*(t)QAg(z(t))\\
&\quad + z^T(t)Q^T\overline{Bg(z(t-\tau))} + z^*(t)QBg(z(t-\tau))\\
&\quad - z^T(t)Q^T\overline{M_1F_1(t)N_1z(t)} - z^*(t)QM_1F_1(t)N_1z(t)\\
&\quad + z^T(t)Q^T\overline{M_2F_2(t)N_2g(z(t))} + z^*(t)QM_2F_2(t)N_2g(z(t))\\
&\quad + z^T(t)Q^T\overline{M_3F_3(t)N_3g(z(t-\tau))} + z^*(t)QM_3F_3(t)N_3g(z(t-\tau))\\
&= -z^*(t)D^*Qz(t) - z^*(t)Q\,Dz(t) + g^*(z(t))A^*Qz(t) + z^*(t)QAg(z(t))\\
&\quad + g^*(z(t-\tau))B^*Qz(t) + z^*(t)QBg(z(t-\tau))\\
&\quad - z^*(t)N_1^*F_1^*(t)M_1^*Qz(t) - z^*(t)QM_1F_1(t)N_1z(t)\\
&\quad + g^*(z(t))N_2^*F_2^*(t)M_2^*Qz(t) + z^*(t)QM_2F(t)N_2g(z(t))\\
&\quad + g^*(z(t-\tau))N_3^*F_3^*(t)M_3^*Qz(t) + z^*(t)QM_3F_3(t)N_3g(z(t-\tau)).
\end{aligned} \tag{47}$$

Based on (H1), utilizing Lemma 4, we obtain that

$$\begin{aligned}
&- z^*(t)N_1^*F_1^*(t)M_1^*Qz(t) - z^*(t)QM_1F_1(t)N_1z(t)\\
&\le \eta z^*(t)QM_1M_1^*Qz(t) + \eta^{-1}z^*(t)N_1^*N_1z(t),
\end{aligned} \tag{48}$$

$$\begin{aligned}
&g^*(z(t))N_2^*F_2^*(t)M_2^*Qz(t) + z^*(t)QM_2F_2(t)N_2g(z(t))\\
&\le \eta z^*(t)QM_2M_2^*Qz(t) + \eta^{-1}g^*(z(t))N_2^*N_2g(z(t))\\
&\le \eta z^*(t)QM_2M_2^*Qz(t) + \eta^{-1}\lambda_{\max}(N_2^*N_2)g^*(z(t))g(z(t))\\
&\le \eta z^*(t)QM_2M_2^*Qz(t) + \eta^{-1}\lambda_{\max}(N_2^*N_2)l^2z^*(t)z(t),
\end{aligned} \tag{49}$$

$$\begin{aligned}
&g^*(z(t-\tau))N_3^*F_3^*(t)M_3^*Qz(t) + z^*(t)QM_3F_3(t)N_3g(z(t-\tau))\\
&\le \eta z^*(t)QM_3M_3^*Qz(t) + \eta^{-1}g^*(z(t-\tau))N_3^*N_3g(z(t-\tau))\\
&\le \eta z^*(t)QM_3M_3^*Qz(t) + \eta^{-1}\lambda_{\max}(N_3^*N_3)g^*(z(t-\tau))g(z(t-\tau))\\
&\le \eta z^*(t)QM_3M_3^*Qz(t) + \eta^{-1}\lambda_{\max}(N_3^*N_3)l^2z^*(t-\tau)z(t-\tau).
\end{aligned} \tag{50}$$

Meanwhile, based on Lemma 5 and (H1), we have

$$\begin{aligned}
&g^*(z(t))A^*Qz(t) + z^*(t)QAg(z(t))\\
&\le \epsilon g^*(z(t))g(z(t)) + \epsilon^{-1}z^*(t)QAA^*Qz(t)\\
&\le \epsilon l^2z^*(t)z(t) + \epsilon^{-1}z^*(t)QAA^*Qz(t),
\end{aligned} \tag{51}$$

$$\begin{aligned}
&g^*(z(t-\tau))B^*Qz(t) + z^*(t)QBg(z(t-\tau))\\
&\le \epsilon g^*(z(t-\tau))g(z(t-\tau)) + \epsilon^{-1}z^*(t)QBB^*Qz(t)\\
&\le \epsilon l^2z^*(t-\tau)z(t-\tau) + \epsilon^{-1}z^*(t)QBB^*Qz(t).
\end{aligned} \tag{52}$$

Thereupon, together with (47)–(52), we can get

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}V(t) &\le z^*(t)\big[-D^*Q - Q\,D + \epsilon l^2 I + \eta Q(M_1M_1^* + M_2M_2^* + M_3M_3^*)Q \\ &+ \epsilon^{-1}Q(AA^* + BB^*)Q + \eta^{-1}\big(N_1^*N_1 + \lambda_{\max}(N_2^*N_2)l^2 I\big)z(t) \\ &+ z^*(t-\tau)l^2\big(\epsilon + \eta^{-1}\lambda_{\max}(N_3^*N_3)\big)Iz(t-\tau) \\ &= -z^*(t)\Pi z(t) + z^*(t-\tau)\Xi z(t-\tau).\end{aligned} \quad (53)$$

Note that $\lambda_{\min}(Q)z^*(t)z(t) \le z^*(t)Qz(t) \le \lambda_{\max}(Q)z^*(t)z(t)$, we can get from (53) that

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}V(t) &\le -\lambda_{\min}(\Pi)z^*(t)z(t) + \lambda_{\max}(\Xi)z^*(t-\tau)z(t-\tau) \\ &\le -\frac{\lambda_{\min}(\Pi)}{\lambda_{\max}(Q)}z^*(t)Qz(t) + \frac{\lambda_{\max}(\Xi)}{\lambda_{\min}(Q)}z^*(t-\tau)Qz(t-\tau) \\ &= -\frac{\lambda_{\min}(\Pi)}{\lambda_{\max}(Q)}V(t) + \frac{\lambda_{\max}(\Xi)}{\lambda_{\min}(Q)}V(t-\tau).\end{aligned} \quad (54)$$

Then, we can obtain from (54) and Lemma 7 that $\lim_{t\longrightarrow+\infty}V(t) = 0$; besides, $V(t) = z^*(t)Qz(t) \ge \lambda_{\min}(Q)\|z(t)\|$, which indicates that $z(t) \longrightarrow 0$ as $t \longrightarrow \infty$. That is to say, the origin of system (43) is robust asymptotically stable, which also implies that the equilibrium point of system (17) is robust asymptotically stable. This ends the proof. □

6. Robust Asymptotical Stabilization

Now, we will stabilize the states of system (17) to the equilibrium point $\tilde{x}$, which is equivalent to stabilize the states of system (43) to the origin. For this purpose, we design the following controller:

$$v(t) = -(J + \Delta J)z(t) - (K + \Delta K)z(t-\tau), \quad (55)$$

where J and K are positive gain coefficient matrices to be determined and $\Delta J(t)$ and $\Delta K(t)$ are uncertain parameters. Then, the controlled system can be described as follows:

$$\begin{aligned}{}_{t_0}^{C}D_t^{\alpha}z(t) &= -(D + \Delta\, D(t))z(t) + (A + \Delta A(t))g(z(t)) \\ &\quad +(B + \Delta B(t))(t)g(z(t-\tau)) \\ &\quad -(J + \Delta J)z(t) - (K + \Delta K)z(t-\tau).\end{aligned} \quad (56)$$

In order to proceed further analysis, it is necessary to give the following assumption:

(H3) Suppose that $M_i \in \mathbb{C}^{n\times k}$ and $N_i \in \mathbb{C}^{l\times n}$ are known constant matrices, $F_i(t) \in \mathbb{C}^{k\times l}(i = 4, 5)$ are unknown time-varying matrices, and $F_i^*(t)F_i(t) \le I(i = 4, 5)$. Then, the uncertain parameters $\Delta J(t)$ and $\Delta K(t)$ are of the following forms:

$$\begin{aligned}\Delta J(t) &= M_4F_4(t)N_4, \\ \Delta K(t) &= M_5F(t)_5N_5.\end{aligned} \quad (57)$$

Theorem 3. *Under the conditions of Theorem 1 and (H3), if there exist a Hermitian positive Q and positive constants l, η, and ϵ such that the following condition holds:*

$$\frac{\lambda_{\min}(\Phi)}{\lambda_{\max}(Q)} > \frac{\sqrt{2}\lambda_{\max}(\Upsilon)}{\lambda_{\min}(Q)}, \quad (58)$$

where

$$\begin{aligned}\Phi &= \Pi + J^*Q + QJ - \epsilon QQ - \eta^{-1}N_4^*N_4 \\ &\quad - \eta QM_4M_4^*Q - \eta QM_5M_5^*Q, \\ \Upsilon &= \Xi + \epsilon^{-1}K^*K + \eta^{-1}N_5^*N_5,\end{aligned} \quad (59)$$

then system (43) can achieve robust asymptotical stabilization under the controller (55).

Proof. We still consider the Lyapunov function (46), calculating the derivative of $V(t)$ along the trajectories of system (56), similar to Theorem 2, and we have

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V(t) \le & -z^*(t)D^*Qz(t) - z^*(t)Q\,Dz(t) + g^*(z(t))A^*Qz(t) + z^*(t)QAg(z(t)) \\
& + g^*(z(t-\tau))B^*Qz(t) + z^*(t)QBg(z(t-\tau)) \\
& - z^*(t)N_1^*F_1^*(t)M_1^*Qz(t) - z^*(t)QM_1F_1(t)N_1z(t) \\
& + g^*(z(t))N_2^*F_2^*(t)M_2^*Qz(t) + z^*(t)QM_2F_2(t)N_2g(z(t)) \\
& + g^*(z(t-\tau))N_3^*F_3^*(t)M_3^*Qz(t) + z^*(t)QM_3F_3(t)N_3g(z(t-\tau)) \\
& - z^*(t)J^*Qz(t) - z^*(t)QJz(t) - z^*(t-\tau)K^*Qz(t) - z^*(t)QKz(t-\tau) \\
& - z^*(t)N_4^*F_4^*(t)M_4^*Qz(t) - z^*(t)QM_4F_4(t)N_4z(t) \\
& - z(t-\tau)N_5^*F_5^*(t)M_5^*Qz^*(t) - z^*(t)QM_5F_5(t)N_5z(t-\tau).
\end{aligned}
\tag{60}
$$

According to Lemmas 4 and 5, we can derive that

$$
\begin{aligned}
& - z^*(t)N_4^*F_4^*(t)M_4^*Qz(t) - z^*(t)QM_4F_4(t)N_4z(t) \\
& \le \eta z^*(t)QM_4M_4^*Qz(t) + \eta^{-1}z^*(t)N_4^*N_4z(t),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
& - z(t-\tau)N_5^*F_5^*(t)M_5^*Qz^*(t) - z^*(t)QM_5F_5(t)N_5z(t-\tau) \\
& \le \eta z^*(t)QM_5M_5^*Qz(t) + \eta^{-1}z^*(t-\tau)N_5^*N_5z(t-\tau),
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
& - z^*(t-\tau)K^*Qz(t) - z^*(t)QKz(t-\tau) \\
& \le \epsilon z^*(t)QQz(t) + \epsilon^{-1}z^*(t-\tau)K^*Kz(t-\tau).
\end{aligned}
\tag{63}
$$

Let $\Phi = \Pi + J^*Q + QJ - \epsilon QQ - \eta^{-1}N_4^*N_4 - \eta QM_4 M_4^*Q - \eta QM_5M_5^*Q$ and $\Upsilon = \Xi + \epsilon^{-1}K^*K + \eta^{-1}N_5^*N_5$. Then, together with (48)–(52) and (60)–(63), one gets

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V(t) \le & -z^*(t)\left(\Pi + J^*Q + QJ - \epsilon QQ - \eta^{-1}N_4^*N_4 - \eta QM_4M_4^*Q - \eta QM_5M_5^*Q\right)z(t) \\
& + z^*(t-\tau)\left(\Xi + \epsilon^{-1}K^*K + \eta^{-1}N_5^*N_5\right)z(t-\tau) \\
= & -z^*(t)\Phi z(t) + z^*(t-\tau)\Upsilon z(t-\tau) \\
\le & -\frac{\lambda_{\min}(\Phi)}{\lambda_{\max}(Q)}V(t) + \frac{\lambda_{\max}(\Upsilon)}{\lambda_{\min}(Q)}V(t-\tau).
\end{aligned}
\tag{64}
$$

In the same way, we can obtain from (58), (64), and Lemma 7 that $\lim_{t\longrightarrow+\infty}V(t) = 0$; besides, $V(t) = z^*(t)Qz(t) \ge \lambda_{\min}(Q)\|z(t)\|$, which indicates that $z(t) \longrightarrow 0$ as $t \longrightarrow \infty$. That is to say, system (43) can achieve robust asymptotical stabilization under the controller (55). The proof is completed. □

Remark 1. . In most articles on FCNNs, the method of dealing with complex-valued systems is separating the real and imaginary parts and then treating them separately in the real field to achieve their objects; for instance, see [19–32, 38, 39]. In [33, 34, 45], authors investigated the various dynamical behaviors of the FCNNs in algebraic form without decomposing the complex-valued networks whereas in this paper we treat the concerned complex-valued models as an entirety and use several related inequalities with complex terms to obtain results; besides, our entire process is in the form of vectors.

7. Numerical Simulations

In this section, two numerical simulation examples are chosen to demonstrate the effectiveness of the theoretical results in this paper.

Example 1. Consider the two-dimensional complex-valued fractional neural networks with time delay and uncertain parameters described by the following form:

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}x(t) = &-(D + \Delta D(t))x(t) + (A + \Delta A(t))f(x(t)) \\
&+ (B + \Delta B(t))f(x(t-\tau)) + U,
\end{aligned}
\tag{65}
$$

where $\alpha = 0.95$, $\tau = 0.6$, $x(t) = (x_1(t), x_2(t))^T \in \mathbb{C}^2$, $f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T$, and

$$
\begin{aligned}
D &= \begin{bmatrix} 6+6i & 0 \\ 0 & 6+6i \end{bmatrix}, \\
A &= \begin{bmatrix} 1.25-2i & -0.4+0.8i \\ 1-0.6i & 1.64-1.3i \end{bmatrix}, \\
B &= \begin{bmatrix} 2-0.5i & -1.2+2.1i \\ -1.8-0.2i & 0.4i \end{bmatrix}, \\
U &= \begin{bmatrix} 0 & 0 \end{bmatrix}^T,
\end{aligned}
\tag{66}
$$

and the uncertain parameters $\Delta D(t)$, $\Delta A(t)$, and $\Delta B(t)$ are given as

$$
\begin{aligned}
M_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F_1(t) &= \begin{bmatrix} 0.06 - i0.05\sin t & 0.08 + i0.04\cos t \\ 0.04 + i0.06\cos t & -0.05 - i0.02\sin t \end{bmatrix}, \\
N_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
M_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F_2(t) &= \begin{bmatrix} 0.04 - i0.08\sin t & 0.03 - i0.05\cos t \\ -0.02 + i0.05\cos t & -0.08 - i0.06\sin t \end{bmatrix}, \\
N_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
M_3 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F_3(t) &= \begin{bmatrix} 0.05 - i0.07\sin t & 0.02 - i0.09\cos t \\ -0.04 + i0.08\cos t & -0.01 - i0.06\sin t \end{bmatrix}, \\
N_3 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
\tag{67}
$$

Apparently, the Lipschitz constants are satisfied with $l_j = 1\,(i = 1, 2)$. By calculation, it is easy to find that the conditions of Theorems 1 and 2 are hold. Then, it is deduced from Theorems 1 and 2 that the unique equilibrium point of system (65) is robust asymptotically stable. Figures 1 and 2 illustrate the time responses of the states of system (65) with initial values as follows: $x_1 = -4.5 - 3.5i$ and $x_2 = 1.5 - 6i$.

Example 2. Consider the following two-dimensional complex-valued fractional neural networks with time delay and uncertain parameters described by the form as follows:

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}x(t) = &-(D + \Delta D(t))x(t) + (A + \Delta A(t))f(x(t)) \\
&+ (B + \Delta B(t))f(x(t-\tau)) + U,
\end{aligned}
\tag{68}
$$

where $\alpha = 0.98$, $\tau = 0.5$, $x(t) = (x_1(t), x_2(t))^T \in \mathbb{C}^2$, $f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T$, and

$$
\begin{aligned}
D &= \begin{bmatrix} 6+6i & 0 \\ 0 & 6+6i \end{bmatrix}, \\
A &= \begin{bmatrix} 2+3i & 3-i \\ 4-2i & 1+2i \end{bmatrix}, \\
B &= \begin{bmatrix} -1+2i & 2+i \\ 3-4i & -3+2i \end{bmatrix}, \\
U &= \begin{bmatrix} 0 & 0 \end{bmatrix}^T,
\end{aligned}
\tag{69}
$$

and uncertain parameters $\Delta D(t)$, $\Delta A(t)$, and $\Delta B(t)$ are given as

$$
\begin{aligned}
M_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F_1(t) &= \begin{bmatrix} 0.02\cos t + i0.02\cos t & 0 \\ 0 & 0.02\cos t + i0.02\cos t \end{bmatrix}, \\
N_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
M_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F_2(t) &= \begin{bmatrix} 0.02\cos t + i0.01\sin t & 0.05\cos t + i0.01\sin t \\ 0.01\sin t + i0.02\cos t & 0.02\cos t - i0.01\cos t \end{bmatrix}, \\
N_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
M_3 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
F_3(t) &= \begin{bmatrix} 0.01\sin t + i0.02\sin t & 0.02\cos t - i0.05\sin t \\ 0.03\cos t - i0.01\cos t & 0.01\sin t + i0.03\sin t \end{bmatrix}, \\
N_3 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
\tag{70}
$$

The controller for stabilization is described as

$$
v(t) = -(J + \Delta J)x(t) - (K + \Delta K)x(t-\tau), \tag{71}
$$

where the control gain matrices are

$$
\begin{aligned}
J &= \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}, \\
K &= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix},
\end{aligned}
\tag{72}
$$

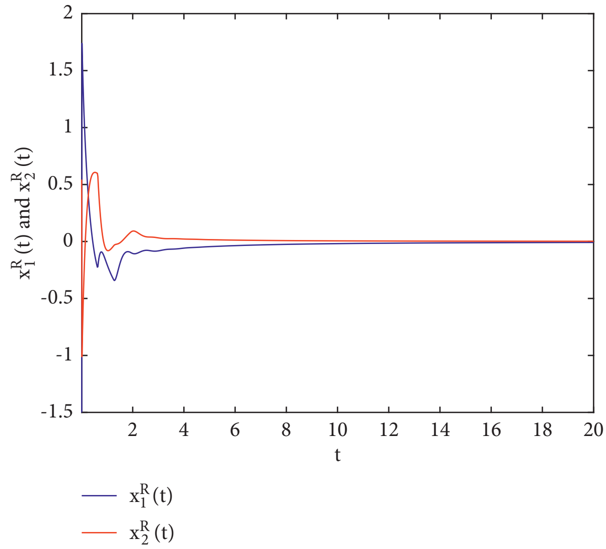

Figure 1: Real parts of the transient states of system (65).

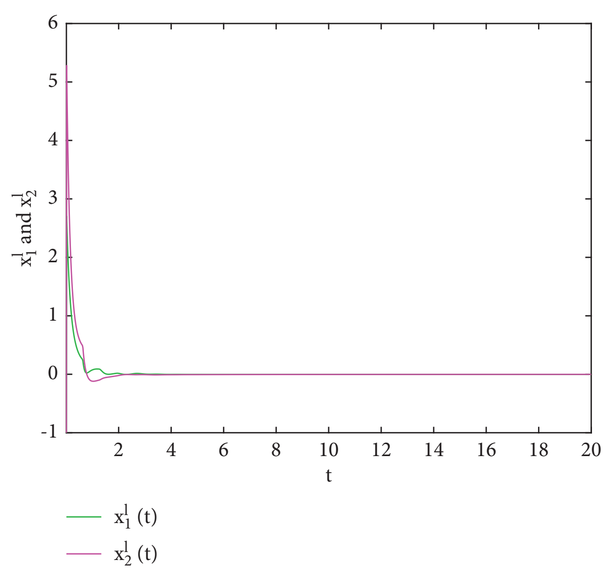

Figure 2: Imaginary parts of the transient states of system (65).

and the uncertain parameters $\Delta J(t)$ and $\Delta K(t)$ are selected as

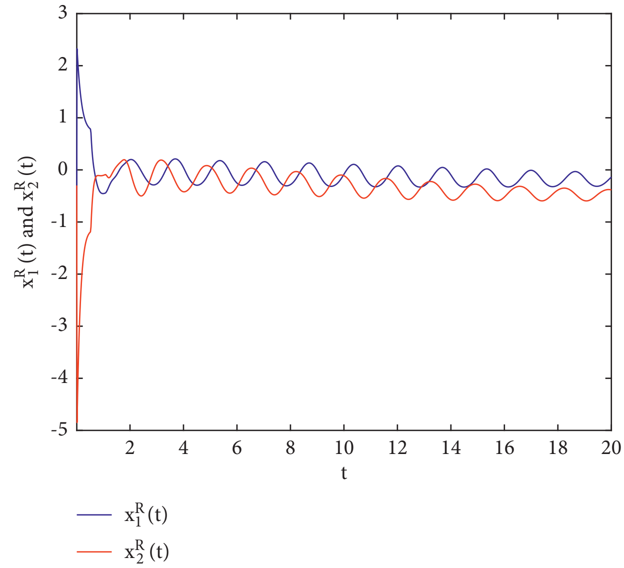

Figure 3: Real parts of the transient states of system (68) without control.

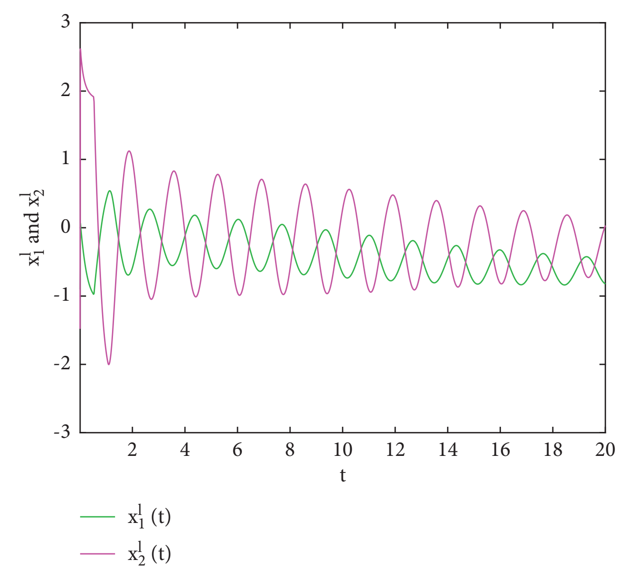

Figure 4: Imaginary parts of the transient states of system (68) without control.

$$M_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_4(t) = \begin{bmatrix} 0.02\ \cos t + i0.02\ \cos t & 0 \\ 0 & 0.02\ \cos t + i0.02\ \cos t \end{bmatrix},$$

$$N_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_5(t) = \begin{bmatrix} 0.02\ \cos t + i0.02\ \cos t & 0 \\ 0 & 0.02\ \cos t + i0.02\ \cos t \end{bmatrix},$$

$$N_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{73}$$

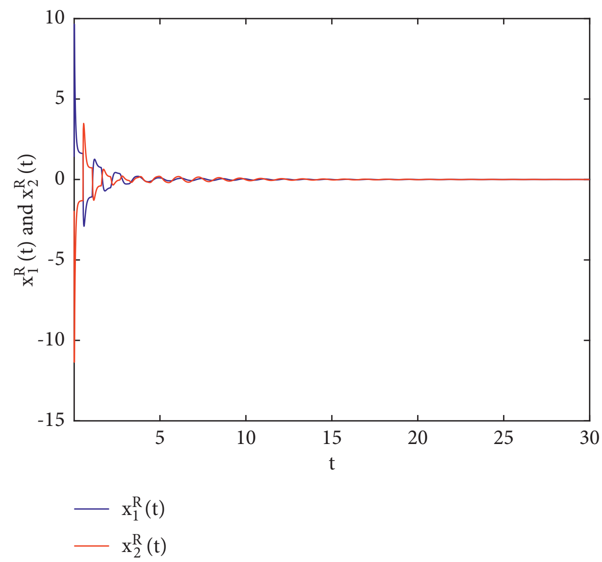

FIGURE 5: Real parts of the transient states of system (68) under controller (71).

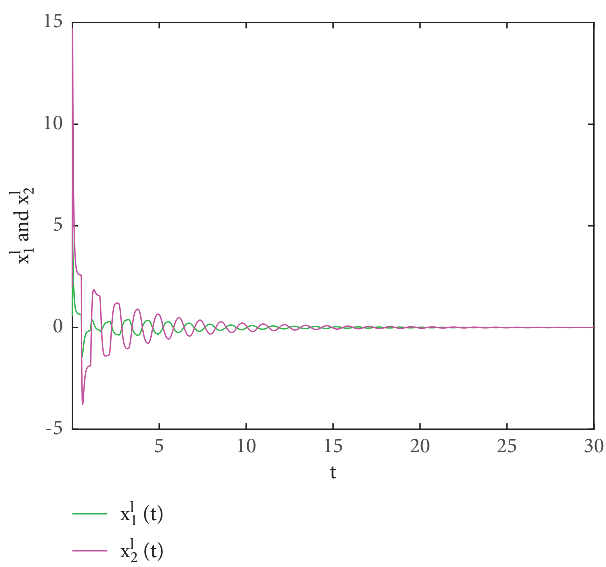

FIGURE 6: Imaginary parts of the transient states of system (68) under controller (71).

Apparently, the Lipschitz constants are satisfied with $l_j = 1\,(i = 1, 2)$. By calculation, it is easy to find that the conditions of Theorems 1 and 3 hold. Then, it is deduced from Theorems 1 and 3 that the unique equilibrium point of system (68) can achieve robust asymptotical stabilization under controller (71). Figures 3–6 illustrate the time responses of the states of system (68) with initial values as follows: $x_1 = -2.5 + 2.5i$; $x_2 = -1.5 - 4i$.

8. Conclusions

In this paper, the robust asymptotical stability and stabilization have been investigated for a class of fractional-order complex-valued neural networks (FCNNs) with parametric uncertainties and time delay. It should be noted that the obtained results have been dealt with directly in the complex domain. By applying the theory of homeomorphism and some inequality techniques, several sufficient conditions ensuring the existence and uniqueness, robust stability, and robust stabilization of the equilibrium point for the concerned network models have been derived. Furthermore, two numerical examples have been designed to verify the feasibility and effectiveness of theoretical results.

The method to prove the stability of the FCNNs in this paper can be generalized in parallel to complex networks or other more complex systems. The distributed delay and time-varying delay, which are more realistic, can be considered in our model. In addition, with different control requirements, various control strategies need to be established to improve system performance. There have been many types of stabilization criteria, such as finite-time stabilization, exponential stabilization, and square stabilization. Among them, finite-time stability has the beneficial characteristics of fast convergence and robustness to uncertainty. In future research, we will focus on the issues above.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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