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Research Article

Sharp Bound of the Number of Zeros for a Liénard System with a Heteroclinic Loop

Junning Cai ,[1] Minzhi Wei ,[1] and Guoping Pang[2]

[1]*Department of Applied Mathematics, Guangxi University of Finance and Economics, Nanning, Guangxi 530003, China*
[2]*School of Mathematics and Statistics, Yulin Normal University, Yulin 537000, China*

Correspondence should be addressed to Minzhi Wei; xiaoyanxiong123@163.com

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In the presented paper, the Abelian integral $I(h)$ of a Liénard system is investigated, with a heteroclinic loop passing through a nilpotent saddle. By using a new algebraic criterion, we try to find the least upper bound of the number of limit cycles bifurcating from periodic annulus.

1. Introduction

A well-known analytic system with planar polynomial differential equation of degree n is of the form:

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y). \tag{1}$$

In 1977, Arnold [1] proposed weak Hilbert's 16th problem and studied the number of zeros of the Abelian integral:

$$I(h, \delta) = \oint_{\Gamma_h} q\mathrm{d}x - p\mathrm{d}y, \quad h \in J, \tag{2}$$

where p and q are the polynomials of degree $n \geq 2$ and Γ_h are some closed ovals of corresponding Hamiltonian. More precisely, $H(x, y)$ is the Hamiltonian function of special form of (1):

$$\begin{aligned} \dot{x} &= H_y + \varepsilon p(x, y, \delta), \\ \dot{y} &= -H_x + \varepsilon q(x, y, \delta), \end{aligned} \tag{3}$$

where $H(x, y)$, $p(x, y)$, and $q(x, y)$ are the polynomials of x and y, their degrees satisfy $\max\{\deg p, \deg q\} = n$, $\deg(H) = n + 1$, and ε is a positive and sufficiently small parameter.

More precisely, the following Liénard system of type (m, n) attracted more and more attentions from mathematicians [2–15]:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= f(x) + \varepsilon g(x)y, \end{aligned} \tag{4}$$

where $f(x)$ and $g(x)$ are the polynomials of degrees m and n, respectively. For example, Wang and Xiao [16] concluded that the number of limit cycles in the system bifurcating from period annulus is at most three. Qi and Zhao [17] considered the Liénard system of type (5, 3), [18], Asheghi and Zangeneh [19] considered the Liénard system of type (5, 4), and Sun [20] studied the limit cycles of type (7, 6) with a heteroclinic loop connecting two nilpotent saddles. In this paper, we intend to study on a following Liénard system that is a small perturbation of the Hamiltonian vector field:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x+1)^3\left(x + \frac{1}{3}\right) + \varepsilon\left(a_1 x + a_2 x^3 + a_3 x^5 + x^7\right)y, \end{aligned} \tag{5}$$

with $0 < \varepsilon \ll 1$, and a_1, a_2, a_3 are the constants. Equation (5) holds the hyperelliptic Hamiltonian function:

$$\overline{H}(x,y)=\frac{1}{2}y^2-\frac{1}{6}x^6-\frac{2}{3}x^5-x^4-\frac{2}{3}x^3-\frac{1}{6}x^2=\frac{1}{2}y^2+A(x). \tag{6}$$

The level sets (i.e., $\overline{H}(x,y)=h$) of the Hamiltonian function (6) are sketched in Figure 1. $\overline{H}(x,y)=h$ defines a family of closed orbits of system (5)$|\varepsilon=0$, denoted by $\{\Gamma_h\}$. Γ_0 is the corresponding orbit to $h=0$, it incloses an elementary center $(-(1/3),0)$, and $\Gamma_{-(8/2187)}$ defines two heteroclinic orbits, connecting a nilpotent saddle $(-1,0)$ and a hyperbolic saddle $(0,0)$. The Melnikov function on Γ_h is

$$\begin{aligned}I(h,\delta)&=\oint_{\Gamma h}\left(a_1x+a_2x^3+a_3x^5+x^7\right)y\mathrm{d}x\\&\equiv a_1I_1(h)+a_2I_2(h)+a_3I_3(h)+I_4(h),\end{aligned} \tag{7}$$

for $h\in(-(8/2187),0)$, where $\delta=(a_1,a_1,a_3,1)$ and $I_i(h)=\oint_{\Gamma h}x^{2i-1}y\mathrm{d}x$, $i=1,2,3,4$. Our main work is to provide a complete description of the number of limit cycles for perturbed system in the whole plane.

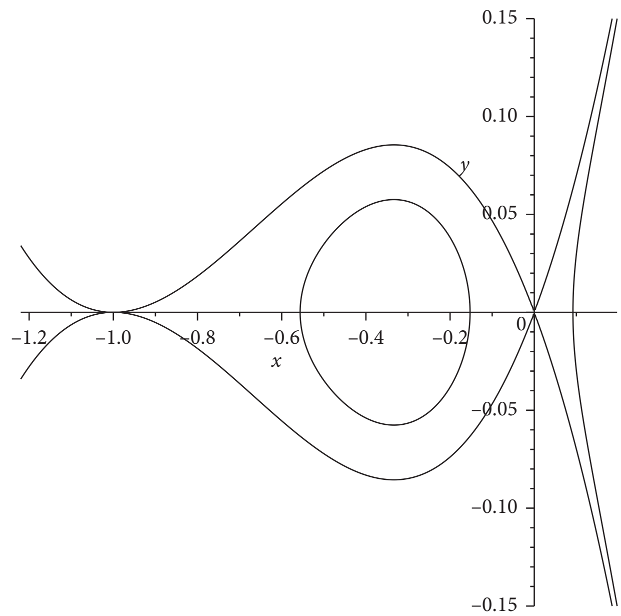

Figure 1: The level set of $H(x,y)$.

2. Some Preliminaries

For system (3), some related definitions and significative results are introduced, it can be seen in [21–23] in detail.

Definition 1. Assume that $f_0,f_1,f_2,\ldots,f_{n-1}$ are analytic functions on a real open interval J.

(i) The family of sets $\{f_0,f_1,f_2,\ldots,f_{n-1}\}$ is called a Chebyshev system (T-system for short) provided that any nontrivial linear combination $k_0f_0(x)+k_1f_1(x)+\cdots+k_{n-1}f_{n-1}(x)$ has at most $n-1$ isolated zeros on J.

(ii) An ordered set of n functions $\{f_0,f_1,f_2,\ldots,f_{n-1}\}$ is called a complete Chebyshev system (CT-system for short) provided any nontrivial linear combination $k_0f_0(x)+k_1f_1(x)+\cdots+k_{n-1}f_{n-1}(x)$ has at most $i-1$ zeros for all $i=1,2,\ldots,n$. Moreover, it is called an extended complete Chebyshev system (ECT-system for short) if the multiplicities of zeros are taken into account.

(iii) The continuous Wronskian of $\{f_0,f_1,f_2,\ldots,f_{n-1}\}$ at $x\in R$ is

$$W[f_0,f_1,f_2,\ldots,f_{k-1}]=\det\left(f_i^j\right)_{0\le i,j\le k-1}=\begin{vmatrix}f_0(x) & f_1(x) & \ldots & f_{k-1}(x)\\ f_0'(x) & f_1'(x) & \ldots & f_{k-1}'(x)\\ \vdots & \ldots & \ldots & \vdots\\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \ldots & f_{k-1}^{(k-1)}(x)\end{vmatrix}, \tag{8}$$

where $f'(x)$ is the first-order derivative of $f(x)$ and $f_i^j(x)$ is the jth order derivative of $f_i(x)$, $i\ge 2$. The definitions imply that the function tuple $\{f_0,f_1,f_2,\ldots,f_{n-1}\}$ is an ECT-system on J; therefore, it is an ECT-system on J and then a T-system on J; however, the inverse implications are not true.

Let $H(x,y)=(1/2)y^2+A(x)$ in (5) be an analytic function. The set of ovals $\Gamma_h=H(x,y)=h$ inside periodic annulus is defined by $h\in(h_1,h_2)=J$. Supposed that P is a punctured neighborhood of the origin foliated by ovals Γ_h, then the projection of P on the x-axis is an interval (x_l,x_r) with $x_l<0<x_r$. It is easy to know that $xA\prime(x)>0$, $\forall x\in(x_l,x_r)\backslash\{0\}$, such that $A(x)$ has a zero of even multiplicity at $x=0$, and there exists an analytic involution $z(x)$, which is defined by $A(x)=A(z(x))$.

Lemma 1 (see [22]). *On (x_r,x_l), supposed that an analytic function $f_i(x)$ satisfies*

$$I_i(h)=\oint_{\Gamma_h}f_i(x)y^{2s-1}\mathrm{d}x,\quad \text{for } i=0,1,2,\ldots,n-1, \tag{9}$$

where $h\in(h_1,h_2)$, $s\in N$, and I_h is the oval surrounding the origin inside the level curve $\{A(x)+(1/2)y^{2m}=h\}$. Setting

$$l_i(x):=\frac{f_i(x)}{A'(x)}-\frac{f_iz((x))}{A'(x)}. \tag{10}$$

If the following assumptions are satisfied

(i) $W[l_0, l_1, \ldots, l_i]$ *is nonvanishing on* (x_l, x_r) *for* $i = 0, 1, \ldots, n-2$

(ii) $W[l_0, l_1, \ldots, l_{n-1}]$ *has k zeros on* (x_l, x_r) *counting with multiplicities*

(iii) $s > n + k - 2$

then for all nontrivial linear combination of $\{I_0, I_1, \ldots, I_{n-1}\}$ *has at most* $n + k - 1$ *zeros on* (h_1, h_2) *counting the multiplicities. Meantime,* $\{I_0, I_1, \ldots, I_{n-1}\}$ *is called a T-system with accuracy k on* (h_1, h_2)*, where* $W[l_0, l_1, \ldots, l_i]$ *is Wronskian of* $\{l_0, l_1, \ldots, l_{n-1}\}$.

However, the third condition above always not been satisfied, so we usually apply the next lemma to increase the power of y in I_i.

Lemma 2 (see [22]). *Let* Γ_h *be an oval inside the level curve* $\{A(x) + (1/2)y^{2m} = h\}$*,* $F(x)$ *be a function which satisfies* $(F(x)/A\prime(x))$ *which is analytic at* $x = 0$*. Hence,*

$$\oint_{\Gamma_h} F(x) y^{k-2} \mathrm{d}x = \oint_{\Gamma_h} G(x) y^k \mathrm{d}x, \quad \forall k \in N, \tag{11}$$

where $G(x) = (1/k)((F(x))/(A'(x)))'(x)$.

3. The Least Upper Bound of Number of Zeros of $I(h, \delta)$

Multiply $I_i(h)$ by $((2A(x) + y^2)/2h) = 1$, and the following is obtained:

$$\begin{aligned} I_i(h) &= \frac{1}{2h} \oint_{\Gamma h} \left(2A(x) + y^2\right) x^{2i-1} y \mathrm{d}x \\ &= \frac{1}{2h} \left(\oint_{\Gamma h} 2A(x) x^{2i-1} y \mathrm{d}x + \oint_{\Gamma h} x^{2i-1} y^3 \mathrm{d}x \right), \quad i = 0, 1, 3. \end{aligned} \tag{12}$$

Setting $k = 3$ and $F(x) = 2x^{2i-1}A(x)$ and quoting Lemma 2 to $\oint_{\Gamma h} 2x^{2i-1}A(x) y \mathrm{d}x$ yield

$$\oint_{\Gamma h} 2x^{2i-1} A(x) y \mathrm{d}x = \oint_{\Gamma h} G_i(x) y^3 \mathrm{d}x, \tag{13}$$

where $G_i(x) = (1/3)((2x^{2i-1}A(x))/A(x))'(x) = ((2x^{2i-1}(3ix^2 + 4ix - x + i))/(3(3x+1)^2))$.

By substituting (13) into (12) and multiplying $((2A(x) + y^2)/2h) = 1$ again, it changes to

$$\begin{aligned} I_i(h) &= \frac{1}{2h} \oint_{\Gamma h} \left(G_i(x) + x^{2i-1}\right) y^3 \mathrm{d}x \\ &= \frac{1}{4h^2} \oint_{\Gamma h} \left(2A(x) + y^2\right)\left(G_i(x) + x^{2i-1}\right) y^3 \mathrm{d}\phi, \\ &= \frac{1}{4h^2} \oint_{\Gamma h} 2A(x)\left(x^{2i-1} + G_i(x)\right) y^3 \mathrm{d}x \\ &\quad + \frac{1}{4h^2} \oint_{\Gamma h} \left(x^{2i-1} + G_i(x)\right) y^5 \mathrm{d}x. \end{aligned} \tag{14}$$

Quoting Lemma 2, setting $k = 5$ and $F(x) = 2A(x)(x^{2i-1} + G_i(x))$, the following is obtained:

$$\oint_{\Gamma h} 2A(x)\left(x^{2i-1} + G_i(x)\right) y^3 \mathrm{d}x = \oint_{\Gamma h} E_i(x) y^5 \mathrm{d}x, \tag{15}$$

where

$$\begin{aligned} E_i(x) &= \frac{1}{5} \left(\frac{2A(x)(x^{2i-1} + G_i(x))}{A(x)} \right)'(x) = \frac{2x^{2i-1} r_i(x)}{15(3x+1)^4}, \\ r_i(x) &= 81ix^4 + 18x^4 i^2 - 24x^3 + 48i^2 x^3 + 144ix^3 \\ &\quad + 44i^2 x^2 + 84ix^2 - 14x^2 - 4x + 16i^2 x + 24ix \\ &\quad + 3i + 2i^2. \end{aligned} \tag{16}$$

By substituting (15) into (14) and multiplying $((2A(x) + y^2)/2h) = 1$ again, the following is obtained:

$$\begin{aligned} I_i(h) &= \frac{1}{4h^2} \oint_{\Gamma h} \left(E_i(x) + G_i(x) + x^{2i-1}\right) y^5 \mathrm{d}x \\ &= \frac{1}{8h^3} \oint_{\Gamma h} \left(2A(x) + y^2\right)\left(E_i(x) + G_i(x) + x^{2i-1}\right) y^5 d, \\ &= \frac{1}{8h^3} \oint_{\Gamma h} 2A(x)\left(E_i(x) + G_i(x) + x^{2i-1}\right) y^5 \mathrm{d}x \\ &\quad + \frac{1}{8h^3} \oint_{\Gamma h} \left(E_i(x) + G_i(x) + x^{2i-1}\right) y^7 \mathrm{d}x. \end{aligned} \tag{17}$$

Quoting Lemma 2 once more, then

$$\oint_{\Gamma h} 2A(x)\left(E_i(x) + G_i(x) + x^{2i-1}\right) y^5 \mathrm{d}x = \oint_{\Gamma h} D_i(x) y^7 \mathrm{d}x, \tag{18}$$

where

$$\begin{aligned} D_i(x) &= \frac{1}{7} \left(\frac{2A(x)(x^{2i-1} + G_i(x) + E_i(x))}{A(x)} \right)'(x) \\ &= \frac{2x^{2i-1} \overline{g_i}(x)}{105(3x+1)^6}, \\ \overline{g_i}(x) &= 15i - 24x - 1008x^5 - 1218x^4 - 672x^3 - 178x^2 \\ &\quad + 184ix + 3645ix^6 + 8496ix^5 + 7761ix^4 \\ &\quad + 3848ix^3 + 1107ix^2 + 3120i^2 x^3 + 1072i^2 x^2 \\ &\quad + 200i^2 x + 16i^2 + 1296x^6 i^2 + 4104x^5 i^2 \\ &\quad + 5040x^4 i^2 + 544i^3 x^3 + 228i^3 x^2 + 48i^3 x \\ &\quad + 108x^6 i^3 + 432x^5 i^3 + 684x^4 i^3 + 4i^3. \end{aligned} \tag{19}$$

From the above computation, the following result can be obtained easily.

Lemma 3

$$8h^3 I_i(h) = \oint_{\Gamma h} f_i(x) y^7 \mathrm{d}x \equiv \widetilde{I}_i(h), \tag{20}$$

where $f_i(x) = x^{2i-1} + G_i(x) + E_i(x) + D_i(x)$. *Therefore,* $\{I_1(h), I_2(h), I_3(h), I_4(h)\}$ *is an ECT-system if and only if* $\left\{\widetilde{I}_1(h)t, n\widetilde{I}_2 q(h)h, \underset{I_3 x}{\sim}(h)7, C\widetilde{I}_4;(h)\right\}$ *is as well.*

Take the following function

$$l_i(x) = \left(\frac{f_i}{A'}\right)(x) - \left(\frac{f_i}{A'}\right)(z(x)), \tag{21}$$

where $z(x)$ *is an analytic involution, defined by* $A(x) = A(z(x))$ *on* $(-1, -(1/3))$. *Factoring* $A(x) - A(z(x))$ *yields*

$$-\frac{1}{6}(x-z)q(x,z), \tag{22}$$

where

$$\begin{aligned} q(x,z) = {} & z + 6z^3 + 4z^2 + 4z^2x^2 + 6zx^2 + x + 6z^2x \\ & + 4xz^3 + x^5 + 4x^4 + 6x^3 + 4x^2 + 4xz + 4zx^3 \\ & + 4z^4 + x^2z^3 + x^4z + xz^4 + z^2x^3 + z^5, \end{aligned} \tag{23}$$

which defined $z(x)$ *on* $(-1, 0)$. *Hence,*

$$\frac{\mathrm{d}}{\mathrm{d}x} l_i(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f_i}{A\prime}\right)(x) - \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f_i}{A\prime}\right)(z(x))\right]\frac{\mathrm{d}z}{\mathrm{d}x}, \tag{24}$$

with $(dz/dx) = -(((\partial q(x,z))/\partial x)/((\partial q(x,z))/\partial z))$. *Suppose that* $x \in (-1, -(1/3))$, *then* $z(x) \in (-(1/3), 0)$; *in other words,*

$$-1 < x < -\frac{1}{3} < z < 0. \tag{25}$$

Lemma 4. *The function tuple* $\{l_1(x), l_2(x), l_3(x), l_4(x)\}$ *is an ECT-system for* $x \in (-1, -(1/3))$.

Proof. Taking (24) into consideration, with the aid of Maple 16, we can obtain the 4 following Wronskians:

$$\begin{aligned} W[l_1(x)] &= l_1(x) = \frac{3(x-z)w_1(x,z)}{35(3x+1)^7(x+1)^3(3z+1)^7(z+1)^3}, \\ W[l_1(x), l_2(x)] &= \frac{18(x-z)^3 w_2(x,z)}{125(z+1)^6(3z+1)^{13}(x+1)^6(3x+1)^{13}p(x,z)}, \\ W[l_1(x), l_2(x), l_3(x)] &= -\frac{108(x-z)^6 w_3(x,z)}{42875(z+1)^9(3z+1)^{15}z^4(x+1)^9(3x+1)^{15}x^4p^3(x,z)}, \\ W[l_1(x), l_2(x), l_3(x), l_4(x)] &= -\frac{62208(x-z)^{10} w_4(x,z)}{300125(z+1)^{12}(3z+1)^{22}z^5(x+1)^{12}(3x+1)^{22}x^5p^6(x,z)}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} p(x,z) = {} & x^4 + 4x^3 + 2x^3z + 6x^2 + 8x^2z + 3x^2z^2 + 4x \\ & + 12xz + 12xz^2 + 4xz^3 + 1 + 8z + 18z^2 + 16z^3 \\ & + 5z^4, \end{aligned} \tag{27}$$

and $w_1(x,z)$, $w_2(x,z)$, $w_3(x,z)$, and $w_4(x,z)$ are polynomials in $\{x, z\}$ of degrees 15, 32, 53, and 73, respectively. In the following, calculating the resultant with respect to z between $q(x,z)$ and $p(x,z)$ gives

$$R(q,p,z) = 16x^6(3x+4)\left(27x^3 + 54x^2 + 27x - 4\right)(x+1)^{10}. \tag{28}$$

From Sturm's Theorem, we know that $R(q,p,z)$ has no root on $(-1, -(1/3))$, then $p(x,z)$ and $q(x,z)$ have no common root on $(-1, -(1/3))$. In the following, we will check whether $w_i(x,z)$ and $q(x,z)$ have common root under the condition (25).

(i) Calculating the resultant with respect to z between $q(x,z)$ and $w_1(x,z)$, that is, eliminating from $q(x,z) = 0$ and $w_1(x,z) = 0$ gives $R(q, w_1, z) = 15552(x+1)^6(3x+1)^6\varphi_1(x)$, where

$\varphi_1(x)$ is a polynomial of degree 62 in x. Applying Sturm's theorem to $\varphi_1(x)$, there is not any x such that $\varphi_1(x) = 0$, so we conclude that $W_1[l_1(x)] \neq 0$ on $(-1, -(1/3))$.

(ii) Calculating the resultant with respect to z between $q(x,z)$ and $w_2(x,z)$, that is, eliminating from $q(x,z) = 0$ and $w_2(x,z) = 0$ gives $R(q, w_2, z) = 123834728448x^4(3x+1)^{10}(x+1)^{22}\ \varphi_2(x)$, where $\varphi_2(x)$ is a polynomial of degree 122 in x. Applying Sturm's theorem to $\varphi_2(x)$, there are three points, denoted by x_1, x_2, and x_3, such that $\varphi_2(x) = 0$, which $x_1 \approx -0.5231817697$, $x_2 \approx -0.5050882970$, and $x_3 \approx -0.39862246670$.

(iii) Thus we will check if $q(x)$ and $w_2(x,z)$ have any common roots on $(-1, -(1/3))$ by using the program with Maple 16 to find all the possible intervals.

```
>with(RegularChains);
>with(ChainTools);
>with(SemiAlgebraicSetTools);
>sys: = [w2(x,z), q(x,z)];
>R2: = PolynomialRing([x,z]);
>dec: = Triangularize(sys, R);
[regular_chain] > L: = map(Equations, dec, R);
```

$$[[x+1, z], [x, z+1], [x+1, z+1], [3x+1, 3z+1], [\varphi_1^*(x,z), \varphi_2^*(x,z)], [\eta_1^*(x,z), \eta_2^*(x,z)]], \tag{29}$$

where $\varphi_1^*(x,z) = \varphi_{11}^*(z)x + \varphi_{12}^*(z)$, $\eta_1^*(x,z) = \eta_{11}^*(z)x + \eta_{12}^*(z)$, and φ_{11}^*, φ_{12}^*, and η_2^* are polynomials in z of degree 41, 41, and 42, η_{11}^*, η_{12}^*, and η_2^* are polynomials in z of degree 79, 79, and 80, respectively. It is obvious that, all the roots of the regular chains $[x+1, z]$, $[x, z+1]$, $[x+1, z+1]$, and $[3x+1, 3z+1]$ do not satisfy (14), the regular chains $[\varphi_1^*(x,z), \varphi_2^*(x,z)]$ and $[\eta_1^*(x,z), \eta_2^*(z)]$ are square-free and zero-dimensional (because the number of variables equals the number of polynomials). $L[5][1]$ and $L[5][2]$ represent $\varphi_1^*(x,z)$ and $\varphi_2^*(z)$, and $L[6][1]$ and $L[6][2]$ represent $\eta_1^*(x,z)$ and $\eta_2^*(z)$ in Maple, we use the following program to check their common roots.

```
>C: = Chain([L[5][2], L[5][1]], Empty(R2), R2);
[regular_chain] > RL: = RealRootIsolate
(C, R2, 'abserr' = (1/10^5));
[box, box] > map(BoxValues, RL, R2);
```

$$\begin{aligned}
&\left[x = \left[-\frac{68575}{131072}, -\frac{34287}{65536}\right], z = \left[-\frac{59409777128373836560319652339976029315}{340282366920938463463374607431768211456},\right.\right.\\
&\qquad\left.\left.-\frac{118819554256747673120639304679952058629}{680564733841876926926749214863536422912}\right]\right],\\
&\left[x = \left[-\frac{174237}{131072}, -\frac{43559}{32768}\right], z = \left[-\frac{16398377152993610030696991219693126780993606400880552798989 0579}{411376139330301510538742295639337626245683966408394965837152256},\right.\right.\\
&\qquad\left.\left.-\frac{6559350861197444012278796487877250712397442560352221119595 62315}{1645504557321206042154969182557350504982735865633579863348609024}\right]\right],\\
&\left[x = \left[-\frac{5721}{32768}, -\frac{22883}{131072}\right], z = \left[-\frac{95578876636133688791068498973648060853858107987}{182687704666362864775460604089535377456991567872},\right.\right.\\
&\qquad\left.\left.-\frac{38231550654453475516427399589459224341543243194 7}{730750818665451459101842416358141509827966271488}\right]\right],\\
&x = \left[-\frac{52249}{131072}, -\frac{6531}{16384}\right],\\
&z = \left[-\frac{34511036383350895369252333113760 09}{25961484292674138142652481646100 48}, -\frac{6902207276670179073850466622752017}{5192296858534827628530496329220096}\right],
\end{aligned} \tag{30}$$

```
>C: = Chain([L[6][2], L[6][1]], Empty(R), R2);
[regular_chain] > RL: = RealRootIsolate
(C, R,'abserr' = (1/10^5));
[box, box] > map(BoxValues, RL, R);
```

$$\left[\left[x=\left[\frac{13355}{131072},\frac{3339}{32768}\right],z=-\frac{60472309605497270024023052157479321857023890133197 47}{119726214130147567059245861496117904970213993920593 92},\right.\right.$$

$$-\frac{2418892384219890800960922086299172874280955605327898 7}{4789048565205902682369834459844716198808559756823756 8},\Bigg]\Bigg] \tag{31}$$

$$\left[x=\left[-\frac{66203}{131072},-\frac{33101}{65536}\right],z=\left[\frac{7002027595}{68719476736},\frac{14004055191}{137438953472}\right]\right].$$

It means that there are 6 pairs of common roots of $w_2(x,z)$ and $q(x,z)$ in the listed intervals, respectively. However, there is not any pair of them satisfies the condition (25). It is said that $w_2(x,z)\neq 0$ as $-1<x<-(1/3)$; therefore, $W[l_1(x),l_2(x)]\neq 0$.

(iii) Similarly, we use the same program as (i) and (ii) to find all the possible intervals, which may hold the common roots of $w_3(x,z)$ and $q(x,z)$ and then obtain the following regular chains:

$$\begin{aligned}&[[x,z],[x,z+1],[x+1,z],[x+1,z+1],\\&[3x+1,3z+1],[u_1^*(x,z),u_2^*(x,z)],\\&[\rho_1^*(x,z),\rho_2^*(x,z)]],\end{aligned} \tag{32}$$

where $u_1^*(x,z)=u_{11}^*(z)x+u_{12}^*(z)$, $\rho_1^*(x,z)=\rho_{11}^*(z)x+\rho_{12}^*(z)$, and u_{11}^*,u_{12}^*, and u_2^* are polynomials in z of degree 57, 57, and 58, respectively, $\rho_{11}^*,\rho_{12}^*(z)$, and ρ_2^* are polynomials in z of degree 111, 111, and 112, respectively. Isolating the fifth and sixth regular chains yields

$$\left[x=\left[-\frac{54979}{131072},-\frac{27489}{65536}\right],z=\left[-\frac{856271130202849555762339288322376111382316862455404994678519855401}{336999333339382997433337688587745383420464305281757156013795128115 2},\right.\right.$$

$$\left.\left.-\frac{107033891275356194470292411040297013922789607806925624334814981925}{421249166674228746791672110734681729275580381602196445017243910144}\right]\right],$$

$$\left[x=\left[-\frac{174741}{131072},-\frac{43685}{32768}\right],\right.$$

$$z=\left[-\frac{3030298039091615678970663582957756689217473784749385379759227090702937762986412587479826606535809219}{8749002899132047697490008908470485461412677723572849745703082425639811996797503692894052708092215296},\right.$$

$$\left.\left.-\frac{6060596078183231357941327165915513378434947569498770759518454181405875525972825174959653213071618437}{17498005798264095394980017816940970922825355447145699491406164851279623993595007385788105416184430592}\right]\right],$$

$$x=\left[-\frac{4163}{16384},-\frac{33303}{131072}\right],z=-\frac{361868533714530451234631234914240547837472188608858683748897796324263}{862718293348820473429344482784628181556388621521298319395315527974912},$$

$$-\frac{144747413485812180493852493965696219134988875443543473499559118529705 1}{345087317339528189371737793113851272622555448608519327758126211189964 8},$$

$$x=\left[-\frac{42589}{32768},-\frac{170355}{131072}\right],z=-\frac{480531930410139348156667048862506606461576771541323121412001025602246951271}{904625697166532776746648320380374280103671755200316906558262375061821325312},$$

$$-\frac{961063860820278696313334097725013212923153543082646242824002051204493902541}{1809251394333065553493296640760748560207343510400633813116524750123642650624}\Bigg],$$

$$x = \left[-\frac{69625}{131072}, -\frac{8703}{16384}\right], z = \left[-\frac{2968006133719558087077106477110092074248769964 7}{2283596308329535809693257551119192218212394598 4},\right.$$
$$\left.-\frac{1484003066859779043538553238555046037124384982 3}{1141798154164767904846628775559596109106197299 2}\right]\Bigg],$$
$$x = \left[-\frac{22699}{65536}, -\frac{45397}{131072}\right], z = \left[-\frac{24939827437209994434974131321565448086471946064 8955}{18707220957835557353007165858768422651595936550 0928},\right.$$
$$\left.-\frac{1246991371860499721748706566078272404323597303 24477}{9353610478917778676503582929384211325797968275 0464}\right]\Bigg]\Bigg]$$
$$\Bigg[x = \left[\frac{15487}{131072}, \frac{121}{1024}\right],$$
$$z = \left[-\frac{671186327417864448164206670006314426784341428019480859893071190393381893648895245}{189713759006418854581978701838234268226797542876185500122247305638564871602071142 4},\right.$$
$$\left.-\frac{1677965818544661120410516675015786066960853570048702149732677975983454734122238 11}{4742843975160471364549467545955856705669938571904637503056182640964121790051778 56}\right]\Bigg],$$
$$\Bigg[x = \left[-\frac{11593}{32768}, -\frac{46371}{131072}\right], z = \left[\frac{272465990628347833}{2305843009213693952}, \frac{4359455850053565329}{36893488147419103232}\right]\Bigg]\Bigg]. \quad (33)$$

It means that there are 8 pairs of common roots of $w_3(x,z)$ and $q(x,z)$ in the listed intervals, respectively. However, there is not any pair of them satisfies the condition (25). It is said that $w_3(x,z) \neq 0$ as $-1 < x < -(1/3)$; therefore, $W[l_1(x), l_2(x), l_3(x)] \neq 0$.

(iv) Similarly, we use the same program as (ii) and (iii) to find all the possible intervals, which may hold the common roots of $w_4(x,z)$ and $q(x,z)$ and then obtain the following regular chains:

$$[[x,z], [x+1,z], [x,z+1], [x+1,z+1], [3x+1,3z+1], [v_1^*(x,z), v_2^*(x,z)], [\omega_1^*(x,z), \omega_2^*(x,z)]], \quad (34)$$

where $v_1^*(x,z) = v_{11}^*(z)x + v_{12}^*(z)$, $\omega_1^*(x,z) = \omega_{11}^*(z)x + \omega_{12}^*(z)$, and v_{11}^*, v_{12}^*, and v_2^* are polynomials in z of degree 75, 75, and 76, respectively, ω_{11}^*, ω_{12}^*, and ω_2^* are polynomials in z of degree 145, 145, and 146, respectively. Isolating the fifth and sixth regular chains yields

```
>C: = Chain([L[6][2], L[6][1]], Empty(R), R2);
[regular_chain] > RL: =
RealRootIsolate(C, R,'abserr' = (1/ 10^5));
[box, box, box, box, box, box,
box, box, box, box, box, box, box, box] > map
(BoxValues, RL, R);
>evalf(%);
```

$$\begin{aligned}
&[[x = [-1.113883972, -1.113876343], z = [-0.01488446846, -0.01488446846]],\\
&[x = [-1.277610779, -1.277603149], z = [-0.1301213260, -0.1301213260]],\\
&[x = [-0.5849761963, -0.5849685669], z = [-0.1345142627, -0.1345142627]],\\
&[x = [-1.329246521, -1.329238892], z = [-0.2715335562, -0.2715335562]],\\
&[x = [-0.3662719727, -0.3662643433], z = [-0.3014530812, -0.3014530812]],\\
&[x = [-1.333335876, -1.333328247], z = [-0.3333242893, -0.3333242893]],\\
&[x = [-0.3014602661, -0.3014526367], z = [-0.3662645453, -0.3662645453]],\\
&[x = [-0.1345214844, -0.1345138550], z = [-0.5849732646, -0.5849732646]],\\
&[x = [-1.078308105, -1.078300476], z = [-0.9149922188, -0.9149922188]],\\
&[x = [-0.9149932861, -0.9149856567], z = [-1.078306231, -1.078306231]],\\
&[x = [-0.01488494873, -0.01487731934], z = [-1.113876799, -1.113876799]],\\
&[x = [-0.1301269531, -0.1301193237], z = [-1.277609231, -1.277609231]],\\
&[x = [-0.2715377808, -0.2715301514], z = [-1.329244729, -1.329244729]],\\
&[x = [-0.3333282471, -0.3333206177], z = [-1.333333333, -1.333333333]]],
\end{aligned} \quad (35)$$

```
>C: = Chain([L[7][2], L[7][1]], Empty(R), R2);
[regular_chain] > RL: = RealRootIsolate(C,
R,'abserr' = (1/10^5));
[box, box, box, box, box, box] > map(BoxValues,
RL, R);
```

```
>evalf(%);
```

$$\begin{aligned}
&[x = [0.1184310913, 0.1184387207], z = [-0.3332116462, -0.3332116462]],\\
&[x = [0.1063232422, 0.1063308716], z = [-0.4784850961, -0.4784850961]],\\
&[x = [0.5298614502, 0.5299377441], z = [-0.7129253486, -0.7129253486]],\\
&[x = [-0.3332138062, -0.3332061768], z = [0.1184337894, 0.1184337894]],\\
&[x = [-0.4784851074, -0.4784774780], z = [0.1063252093, 0.1063252093]],\\
&[x = [-0.7129287720, -0.7129211426], z = [0.5298903953, 0.5298903953]].
\end{aligned} \tag{36}$$

It means that there are 20 pairs of common roots of $w_4(x,z)$ and $q(x,z)$ in the listed intervals, respectively. However, there is not any pair of them satisfies the condition (25). It is said that $w_4(x,z) \neq 0$ as $-1 < x < -(1/3)$; therefore, $W[l_1(x), l_2(x), l_3(x), l_4(x)] \neq 0$.

Based on Lemma 1 and Lemma 4, we obtain the following proposition: □

Proposition 1. $\left\{\widetilde{I_1}(h)t, n\widetilde{I_2}q(h)h, \underset{I_3 x}{\sim}(h)7, C\widetilde{I_4};(h)\right\}$ *is an ECT-system and* $\{I_1(h), I_2(h), I_3(h), I_4(h)\}$ *is as well. Therefore there are at most 3 zeros for* $I(h,\delta)$ *on* $h \in (-(8/2187), 0)$.

4. Conclusion

In, this work, we study the Poincaré bifurcation of the Lie nard system with the form (5), and we prove 4 is the least upper bound of the number of limit cycles by the Poincare bifurcation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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