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Research Article

Classical Theory of Linear Multistep Methods for Volterra Functional Differential Equations

Yunfei Li [1] and Shoufu Li [2]

[1] *College of Mathematics and Physics, Hunan Province Cooperative Innovation Center for the Construction and Development of Dongting Lake Ecological Economic Zone, Hunan University of Arts and Science, Changde 415000, Hunan, China*
[2] *Department of Mathematics, Xiangtan University, Xiangtan 411105, China*

Correspondence should be addressed to Yunfei Li; feiyun2cc@163.com

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Based on the linear multistep methods for ordinary differential equations (ODEs) and the canonical interpolation theory that was presented by Shoufu Li who is exactly the second author of this paper, we propose the linear multistep methods for general Volterra functional differential equations (VFDEs) and build the classical stability, consistency, and convergence theories of the methods. The methods and theories presented in this paper are applicable to nonneutral, nonstiff, and nonlinear initial value problems in ODEs, Volterra delay differential equations (VDDEs), Volterra integro-differential equations (VIDEs), Volterra delay integro-differential equations (VDIDEs), etc. At last, some numerical experiments verify the correctness of our theories.

1. Introduction

VFDEs contain many subtypes, such as VDDEs, VIDEs, VDIDEs, etc. Certainly ODEs are also a subtype of them. VFDEs are widely applied in many fields of science and technology (see [1–5] and their references), for which there have been remarkable theoretical and numerical analysis research results. As for VDDEs, please refer to [6–19], as for VIDEs, please refer to [20–23], and as for VDIDEs, please refer to [24–29]. In recent decades, Li [30–35] has carried on systematic research for stiff general VFDEs and the numerical methods for them. In 2014, Li [36] established the classical stability and convergence theories of Runge–Kutta methods for nonstiff, nonlinear general VFDEs. It is well known that in solving ODEs, linear multistep methods have significant advantages in the aspects of format simplification and computation cost, and experts have presented many famous linear multistep methods such as Backward Differentiation Formula (BDF) methods and Adams methods. Based on the linear multistep methods for ODEs and the canonical interpolation theory presented by Li [34], we propose the linear multistep methods for general VFDEs and build the classical stability, consistency, and convergence theories of the methods. The methods and theories presented in this paper are applicable to nonneutral, nonstiff, and nonlinear initial value problems in ODEs, VDDEs, VIDEs, VDIDEs, etc., and are interesting companions to the methods and theories in [36]. Furthermore, the paper may have some value for the study of numerical methods for fractional order VFDEs because many numerical methods for fractional-order differential equations are presented by extending the methods for integer-order differential equations.

This paper is organized as follows. In Section 2, the linear multistep methods for general VFDEs are introduced. The classical stability of the linear multistep methods is discussed in Section 3. The classical consistency and convergence analyses are carried out in Section 4. Some numerical experiments are carried out in Section 5, which verifies the correctness of the theories presented in this paper.

2. Derivation of the Numerical Methods

Let $\mathbf{R}^m$ be the m-dimensional Euclidean space with standard inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. For any given closed interval $\mathbf{I} \subset \mathbf{R}$, let $\mathbf{C}_m(\mathbf{I})$ denote the Banach space consisting of all continuous mappings $x\colon \mathbf{I} \longrightarrow \mathbf{R}^m$ with the maximum norm $\|x\| = \max_{t\in \mathbf{I}} \|x(t)\|$.

Consider VFDEs of the form [30–32, 34, 36]

$$\begin{cases} y'(t) = f(t, y(t), y(\cdot)), & t \in [0, T], \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \quad (1)$$

where $T \in (0, +\infty)$ and $\tau \in [0, +\infty]$ are constants, $\varphi(t) \in \mathbf{C}_m[-\tau, 0]$ is an initial function, and $f\colon [0, T] \times \mathbf{R}^m \times \mathbf{C}_m[-\tau, T] \longrightarrow \mathbf{R}^m$ is a given continuous mapping which satisfies the classical Lipschitz conditions [36]:

$$\begin{cases} \left\| f(t, u_1, \psi(\cdot)) - f(t, u_2, \psi(\cdot)) \right\| \leq L_1 \left\| u_1 - u_2 \right\|, \\ \forall t \in [0, T], u_1, u_2 \in \mathbf{R}^m, \psi \in \mathbf{C}_m[-\tau, T], \\ \left\| f(t, u, \psi_1(\cdot)) - f(t, u, \psi_2(\cdot)) \right\| \leq L_2 \max\limits_{\xi \in [-\tau, t]} \left\| \psi_1(\xi) - \psi_2(\xi) \right\|, \\ \forall t \in [0, T], u \in \mathbf{R}^m, \psi_1, \psi_2 \in \mathbf{C}_m[-\tau, T], \end{cases} \quad (2)$$

where L_1 and L_2 are classical Lipschitz constants. Equations (1) has a unique solution $y(t)$ [37], and we further make the same assumption as [36] that the system $y'(t) = f(t, y(t), y(\cdot))(t \in [0, T])$ in (1) is stable for the initial function $\varphi(t) \in \mathbf{C}_m[-\tau, 0]$, which is to say there exists a function $C(t) > 0 (t \in [0, T])$ which satisfies $\max_{t \in [0, T]} C(t)$ has only moderate size such that

$$\| y(t) - z(t) \| \leq C(t) \max_{\xi \in [-\tau, 0]} \| \varphi(\xi) - \chi(\xi) \|, \quad \forall t \in [0, T], \quad (3)$$

where $z(t)$ is the solution of the perturbed problem:

$$\begin{cases} z'(t) = f(t, z(t), z(\cdot)), & t \in [0, T], \\ z(t) = \chi(t), & t \in [-\tau, 0]. \end{cases} \quad (4)$$

We use $\mathbb{D}(L_1, L_2)$ to denote the problem class consisting of all the equations of form (1) which satisfy condition (2) and all the other assumption conditions made above. The problem class $\mathbb{D}(L_1, L_2)$ contains nonneutral initial value problems in ODEs, DDEs, IDEs, DIDEs, etc. [36].

Combine the linear k-step method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f(t_{n+i}, y_{n+i}), \quad (5)$$

for ODEs with piecewise Lagrangian interpolation operators Π_{ni}^h which are constructed based on canonical interpolation theory [34, 36], and we get the linear k-step method:

$$\begin{cases} y_{ni}^h(t) = \Pi_{ni}^h(t; \psi, y_1, y_2, \ldots, y_{n+k}), & -\tau \leq t \leq t_{n+i}, i = 0, 1, \ldots, k, \\ \sum\limits_{i=0}^{k} \alpha_i y_{n+i} = h \sum\limits_{i=0}^{k} \beta_i f\left(t_{n+i}, y_{n+i}, y_{ni}^h(\cdot)\right), \end{cases} \quad (6)$$

for the VFDEs (1), where $k \geq 1$, all α_i and β_i are real constants, $\alpha_k \neq 0$, $|\alpha_0| + |\beta_0| > 0$, $\rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i$ is called the characteristic polynomial of the method (6), $h = (T/N)$ is the stepsize, N is a given appropriate positive integer, $t_n = nh (n = 0, 1, \ldots, N)$ are grid points, $\psi \in \mathbf{C}_m[-\tau, 0]$ is an approximation to the initial function $\varphi(t)$, $y_0 := \psi(0)$, $y_n \in \mathbf{R}^m$ is an approximation to the true value $y(t_n)$, and the piecewise interpolation operators $\Pi_{ni}^h\colon \mathbf{R} \times \mathbf{C}_m[-\tau, 0] \times \mathbf{R}^{m(n+k)} \longrightarrow \mathbf{C}_m[-\tau, t_{n+i}]$ based on p-degree piecewise Lagrangian interpolation polynomials are defined as follows: if $p = 0$,

$$\Pi_{ni}^h(t; \psi, y_1, y_2, \ldots, y_{n+k}) = \begin{cases} \psi(t), & t \in [-\tau, 0], \\ y_j, & t \in (t_{j-1}, t_j], j = 1, 2, \ldots, n+i, \end{cases} \quad (7)$$

and if $p \geq 1$,

$$\Pi_{ni}^h(t; \psi, y_1, y_2, \ldots, y_{n+k}) = \begin{cases} \psi(t), & t \in [-\tau, 0], \\ L_{j,p}(t), & t \in (t_{j-1}, t_j], j = 1, 2, \ldots, n+i, \end{cases} \quad (8)$$

with

$$\begin{cases} L_{j,p}(t) = \sum_{v=0}^{p} l_{q+v}(t) y_{q+v}, & q = j - \bar{j}, j = 1, 2, \ldots, n+i, \\ l_{q+v}(t) = \prod_{s=0, s\neq v}^{p} \dfrac{t - t_{q+s}}{t_{q+v} - t_{q+s}}, & q = j - \bar{j}, v = 0, 1, \ldots, p, j = 1, 2, \ldots, n+i, \end{cases} \tag{9}$$

where the integer $\bar{j}$ can be freely determined under the conditions

$$1 \le \bar{j} \le j, \quad j + p - n - k \le \bar{j} \le p. \tag{10}$$

According to [34, 36], we know a piecewise interpolation operator Π_{ni}^{h} defined by (7) or by (8) and (9) which satisfies the canonical condition

$$\begin{cases} \max_{-\tau \le t \le t_{n+i}} \left\| \Pi_{ni}^{h}(t; \psi, y_1, y_2, \ldots, y_{n+k}) - \Pi_{ni}^{h}(t; \chi, z_1, z_2, \ldots, z_{n+k}) \right\| \le c_\pi \max\left\{ \max_{1\le j \le n+k} \left\| y_j - z_j \right\|, \max_{-\tau \le t \le 0} \|\psi(t) - \chi(t)\| \right\}, \\ \forall \psi, \chi \in \mathbf{C}_m[-\tau, 0], y_j, z_j \in \mathbf{R}^m, j = 1, 2, \ldots, n+k, \end{cases} \tag{11}$$

where the constant $c_\pi \ge 1$ only depends on the polynomial order p and does not depend on n, h, y_j, z_j, ψ and χ. In this paper, we always assume that c_π is of moderate size. In fact, Li [34, 36] has given the computational formula of c_π and found c_π, respectively, equals 1, 1, 1.250, and 1.631 when p, respectively, equals 0, 1, 2, and 3. We call method (5) as the mother method of method (6).

If $\beta_k = 0$ in method (6), in order to reduce the amount of calculation, we define

$$y_{ni}^{h}(t) = \Pi_{ni}^{h}(t; \psi, y_1, y_2, \ldots, y_{n+k-1}), \quad -\tau \le t \le t_{n+i}, i = 0, 1, \ldots, k-1, \tag{12}$$

and now the method is an explicit linear k-step method which is a special case of (6).

In particular, we point out that when the linear k-step method (6) for VFDEs starts, on the one hand, like method (5) for ODEs, it needs k starting values, and on the other hand, it may need several grid points as interpolation nodes to construct interpolation operator Π_{ni}^{h}. Based on the above statements, in this paper, we always assume the $\breve{k} - 1$ $(\breve{k} \ge tk)$ values $y_1, y_2, \ldots, y_{\breve{k}-1}$, which are called additional starting values, should be provided in advance by other ways, and method (6) starts with $n = \breve{k} - k \ge 0$. By the Banach contraction mapping principle, we conclude that, for any given starting data $\psi \in \mathbf{C}_m[-\tau, 0], y_1, y_2, \ldots, y_{\breve{k}-1} \in \mathbf{R}^m$, method (6) can uniquely determine the sequence $\left\{ y_n, n = \breve{k}, \breve{k}+1, \ldots, N \right\}$ when $h \le h_\varphi$, where h_φ can be any number under the condition

$$0 < h_\varphi < \frac{|\alpha_k|}{|\beta_k| (L_1 + c_\pi L_2)}. \tag{13}$$

3. Stability Analysis

Firstly, we list some concepts and elementary facts about scalar linear difference equations as follows [35, 38, 39].

Consider the kth order scalar linear difference equation with constant coefficients:

$$\sum_{j=0}^{k} a_j \bar{y}_{m+j} = b_m, \quad m = q_0, q_0 + 1, \ldots, \tag{14}$$

here and later, $q_0 \ge 0$ is an integer constant, and where $k \ge 1$ is an integer constant, every a_j is a complex constant, $a_k \ne 0$, any b_m is a complex number depending on m. If a complex number sequence $\bar{y}_{q_0}, \bar{y}_{q_0+1}, \ldots$, satisfies equation (14), it is denoted by $\{\bar{y}_m\}$, in this paper, and is called a solution of (14). It is easily known that equation (14) has a unique solution when given k initial values $\bar{y}_{q_0}, \bar{y}_{q_0+1}, \ldots, \bar{y}_{q_0+k-1}$. If every b_m is zero, equation (14) is a so-called homogeneous equation:

$$\sum_{j=0}^{k} a_j \bar{y}_{m+j} = 0, \quad m = q_0, q_0 + 1, \ldots. \tag{15}$$

A system of k linearly independent solutions $\{\bar{y}_m^{(1)}\}, \{\bar{y}_m^{(2)}\}, \ldots, \{\bar{y}_m^{(k)}\}$ of equation (15) is called a fundamental system of (15), and the general solution $\{\bar{y}_m\}$ of equation (15) can be written by

$$\overline{y}_m = \sum_{i=1}^{k} d_i \overline{y}_m^{(i)}, \quad m = q_0, q_0 + 1, \ldots, \tag{16}$$

where each d_i is an arbitrary complex constant.

Call the algebraic equation

$$\sum_{j=0}^{k} a_j \xi^j = 0, \tag{17}$$

which corresponds to (14) or (15) the characteristic equation.

Proposition 1 (see [35, 38]). *Suppose $\xi_1, \xi_2, \ldots, \xi_{k_0}$ $(k_0 \le k)$ are all different roots of the characteristic equation (17), and let r_j denote the multiple number of ξ_j. Then, when $a_0 \ne 0$, the $k = \sum_{j=1}^{k_0} r_j$ sequences with the elements $\overline{y}_m$ which can be equal to*

$$\xi_j^{m-q_0}, (m - q_0)\xi_j^{m-q_0}, \ldots, (m - q_0)(m - q_0 - 1)\ldots(m - q_0 - r_j + 2)\xi_j^{m-q_0}, \quad j = 1, 2, \ldots, k_0, \tag{18}$$

form a fundamental system of equation (15), and when $a_0 = 0$, which means equation (17) has zero root and we further suppose $\xi_1 = 0$ is a r_1-tuple root of the characteristic equation (17), the $k = \sum_{j=1}^{k_0} r_j$ sequences with the elements $\overline{y}_m$ which can be equal to

$$\delta_{(m-q_0)q}, \quad q = 0, 1, \ldots, r_1 - 1, \tag{19}$$

$$\xi_j^{m-q_0}, (m - q_0)\xi_j^{m-q_0}, \ldots, (m - q_0)(m - q_0 - 1)\ldots(m - q_0 - r_j + 2)\xi_j^{m-q_0}, \quad j = 2, 3, \ldots, k_0, \tag{20}$$

form a fundamental system of equation (15), where δ_{ij} is the Kronecker function:

$$\delta_{ij} = \begin{cases} 0, & \text{when } i \ne j, \\ 1, & \text{when } i = j. \end{cases} \tag{21}$$

Remark 1. Some explanations about the proof of Proposition 1 are as follows:

(i) When $a_0 \ne 0$ and $q_0 = 0$, P. Henrici has given the proof in the famous book [38].

(ii) When $a_0 \ne 0$ and $q_0 \ne 0$, from the conclusion in the case that $a_0 \ne 0$ and $q_0 = 0$, the proof is easily completed.

(iii) When $a_0 = 0$ and $q_0 = 0$, the corresponding part of Proposition 1 is based on the proof [38] in the case that $a_0 \ne 0$ and $q_0 = 0$ mentioned in the above (i) and some conclusions in the case that $a_0 = 0$ and $q_0 = 0$ in [35], and we offer the explanations about the proof as follows. Firstly, $\xi_1 = 0$ is a r_1-tuple root of the characteristic equation (17) which means that $a_0 = a_1 = \cdots = a_{r_1-1} = 0$, and it is easily further known as the r_1 sequences with the elements $\overline{y}_m = \delta_{(m-q_0)q}$ $(q = 0, 1, \ldots, r_1 - 1)$ are the solutions of equation (15). Secondly, based on the proof presented by P. Henrici mentioned in the above (i), we can know the sequences with the elements shown in (20) are solutions of equation (15) and that the solutions with the elements shown in (19) and (20) are linearly independent.

(iv) When $a_0 = 0$ and $q_0 \ne 0$, from the conclusion in the case that $a_0 = 0$ and $q_0 = 0$, the proof is easily completed.

Proposition 2 (see [35]). *When the initial values $\overline{y}_{q_0} = \overline{y}_{q_0+1} = \cdots = \overline{y}_{q_0+k-1} = 0$, the solution $\{\overline{y}_m\}$ of equation (14) is given by*

$$\overline{y}_m = \sum_{s=q_0}^{m-k} b_s y_{m-s-1}^*, \quad m = q_0, q_0 + 1, \ldots, \tag{22}$$

where each y_m^* satisfies the homogeneous equation:

$$\left\{ \sum_{j=0}^{k} a_j y_{m+j}^* = 0, \quad m = 0, 1, \ldots, y_0^* = y_1^* = \cdots = y_{k-2}^* = 0, \quad y_{k-1}^* = \frac{1}{a_k}. \right. \tag{23}$$

Remark 2. In this paper, when $q < p$, we define $\sum_{i=p}^{q} Q_i \equiv 0$ and $\max_{p \le i \le q} Q_i \equiv 0$, where each Q_i is any a given expression.

Remark 3. By Theorem 5.2 in [38], Proposition 2 is easily shown.

Proposition 3 (see [35]). *Suppose* $\{\overline{y}_m^{(i)}\}$ $(i = 1, 2, \ldots, k)$ *is a fundamental system of equation (15) and* $\{y_m^*\}$ *is the solution of the homogeneous equation (23). Then, the general solution* $\{\overline{y}_m\}$ *of equation (14) can be expressed by*

$$\overline{y}_m = \sum_{i=1}^{k} c_i \overline{y}_m^{(i)} + \sum_{s=q_0}^{m-k} b_s y_{m-s-1}^*, \quad m = q_0, q_0 + 1, \ldots, \tag{24}$$

where each c_i *is an arbitrary constant.*

Proposition 4 (discrete Bellman inequality). *Let* $\mu_1, \mu_2 \geq 0, h > 0, \eta_{q_0}, \eta_{q_0+1}, \ldots, \eta_N$ *be a sequence of nonnegative real numbers which satisfy*

$$\eta_m \leq \mu_2 + \mu_1 h \sum_{j=q_0}^{m-1} \eta_j, \quad m = q_0, q_0 + 1, \ldots, N. \tag{25}$$

Then,

$$\eta_m \leq \mu_2 e^{\mu_1 (m-q_0)h}, \quad m = q_0, q_0 + 1, \ldots, N. \tag{26}$$

Definition 1. We say method (6) is zero-stable if the sequence $y_{\breve{k}}, y_{\breve{k}+1}, \ldots, y_N$ determined by method (6) applied to equation (1) $\in \mathbb{D}(L_1, L_2)$ with starting data $\psi \in \mathbf{C}_m[-\tau, 0], y_0 := \psi(0), y_1, y_2, \ldots, y_{\breve{k}-1} \in \mathbf{R}^m$ (the method (6) starts with $n = \breve{k} - k$) and the sequence $z_{\breve{k}}, z_{\breve{k}+1}, \ldots, z_N$ determined by the perturbation equations

$$\begin{cases} z_{ni}^h(t) = \prod_{ni}^{h}(t; \chi, z_1, z_2, \ldots, z_{n+k}) + \gamma_{ni}(t), & -\tau \leq t \leq t_{n+i}, i = 0, 1, \ldots, k, \\ \sum_{i=0}^{k} \alpha_i z_{n+i} = h\left(\left(\sum_{i=0}^{k} \beta_i f(t_{n+i}, z_{n+i}, z_{ni}^h(\cdot))\right) + \varepsilon_{n+k}\right), & n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k, \end{cases} \tag{27}$$

with starting data $\chi \in \mathbf{C}_m[-\tau, 0], z_0 := \chi(0), z_1, z_2, \ldots, z_{\breve{k}-1} \in \mathbf{R}^m$, satisfy

$$\max_{\breve{k} \leq n \leq N} \|z_n - y_n\| \leq C\left(\max_{1 \leq j \leq \breve{k}-1} \|z_j - y_j\| + \max_{t \in [-\tau, 0]} \|\chi(t) - \psi(t)\| + \max_{\breve{k}-k \leq s \leq N-k} \max_{0 \leq i \leq k} \max_{t \in [-\tau, t_{s+i}]} \|\gamma_{si}(t)\| + \max_{\breve{k} \leq j \leq N} \|\varepsilon_j\|\right), \tag{28}$$

where $\gamma_{ni}(t) \in \mathbf{C}_m[-\tau, t_{n+i}]$ and $\varepsilon_{n+k} \in \mathbf{R}^m$ are both perturbations, $0 < h \leq h_0$, the constant $h_0 > 0$ depends on method (6), L_1 and L_2, and the constant C is independent of h.

Definition 2. We say method (6) satisfies the root condition if for all roots of the characteristic polynomial $\rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i$, the moduli of them are all no larger than 1, and among them any root whose modulus equals 1 is simple root.

Theorem 1. *Method (6) is zero-stable if and only if it satisfies the root condition.*

Proof. Firstly, we prove method (6) satisfies the root condition if it is zero-stable. Consider the ODEs:

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [0, T], \\ y(t) = \eta, & t = 0, \end{cases} \tag{29}$$

which are a special case of (1), where $T \in (0, +\infty)$, $\eta \in \mathbf{R}^m$, and the continuous mapping $f: [0, T] \times \mathbf{R}^m \longrightarrow \mathbf{R}^m$ satisfies the classical Lipschitz condition:

$$\|f(t, u_1) - f(t, u_2)\| \leq L_1 \|u_1 - u_2\|, \quad \forall t \in [0, T], u_1, u_2 \in \mathbf{R}^m, \tag{30}$$

where L_1 is a classical Lipschitz constant. For the ODEs (29), method (6) degrades into its mother method (5). Method (6) is zero-stable implies that its mother method (5) is zero-stable, which means method (6) should satisfy the root condition if it is zero-stable [35, 40].

We assume that method (6) satisfies the root condition, and now we prove it is zero-stable.

Denote $\delta_n = z_n - y_n, n = 0, 1, \ldots, N$. From (6) and (27), we obtain

$$\sum_{i=0}^{k} \alpha_i \delta_{n+i} = \overline{b}_n, \quad n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k, \tag{31}$$

where

$$\overline{b}_n = h\left(\left(\sum_{i=0}^{k}\beta_i\left(f\left(t_{n+i}, z_{n+i}, z_{ni}^{h}(\,\cdot\,)\right) - f\left(t_{n+i}, y_{n+i}, y_{ni}^{h}(\,\cdot\,)\right)\right)\right) + \varepsilon_{n+k}\right). \tag{32}$$

Using Proposition 1 and considering that all α_i are real constants, we can find a real fundamental system [38]$\{\omega_n^{(i)}\}$ $(i = 1, 2, \ldots, k)$ of the following scalar homogeneous equation:

$$\sum_{j=0}^{k}\alpha_j\omega_{n+j} = 0, \quad n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k, \tag{33}$$

and by applying Proposition 3 to (31), we obtain

$$\delta_n = \sum_{i=1}^{k} c_i\omega_n^{(i)} + \sum_{s=\breve{k}-k}^{n-k}\overline{b}_s y_{n-s-1}^{*}, n = \breve{k} - k, \breve{k} - k + 1, \ldots, N, \tag{34}$$

where each $c_i \in \mathbf{R}^m$ is a constant vector and each y_n^* satisfies the scalar homogeneous equation:

$$\left\{\sum_{j=0}^{k}\alpha_j y_{n+j}^{*} = 0, \quad n = 0, 1, \ldots, y_0^* = y_1^* = \cdots = y_{k-2}^* = 0, \quad y_{k-1}^* = \frac{1}{\alpha_k}.\right. \tag{35}$$

From the assumption that method (6) satisfies the root condition, we can conclude for all $\omega_n^{(i)}$ and y_{n-s-1}^* in (34), there exists a constant M which is only dependent on the mother method (5) such that

$$\left|\omega_n^{(i)}\right| \le M, \left|y_{n-s-1}^*\right| \le M. \tag{36}$$

It follows from (34) and (36) that

$$\left\|\delta_n\right\| \le M\left(\sum_{i=1}^{k}\left\|c_i\right\| + \sum_{s=\breve{k}-k}^{n-k}\left\|\overline{b}_s\right\|\right), \quad n = \breve{k} - k, \breve{k} - k + 1, \ldots, N. \tag{37}$$

Setting $n = \breve{k} - k, \breve{k} - k + 1, \ldots, \breve{k} - 1$ in (34), we thus find

$$c_i = \sum_{j=1}^{k}\left(W^{-1}\right)_{ij}\delta_{\breve{k}-k-1+j}, \quad i = 1, 2, \ldots, k, \tag{38}$$

where the matrix

$$W = \begin{bmatrix} \omega_{\breve{k}-k}^{(1)} & \omega_{\breve{k}-k}^{(2)} & \cdots & \omega_{\breve{k}-k}^{(k)} \\ \omega_{\breve{k}-k+1}^{(1)} & \omega_{\breve{k}-k+1}^{(2)} & \cdots & \omega_{\breve{k}-k+1}^{(k)} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{\breve{k}-1}^{(1)} & \omega_{\breve{k}-1}^{(2)} & \cdots & \omega_{\breve{k}-1}^{(k)} \end{bmatrix}. \tag{39}$$

By (38), we know

$$\sum_{i=1}^{k}\left\|c_i\right\| \le \left\|\left|W^{-1}\right|\right\| \max_{\breve{k}-k\le j\le \breve{k}-1}\left\|\delta_j\right\|, \tag{40}$$

where $\||W^{-1}|\| = \sum_{i,j=1}^{k}|(W^{-1})_{ij}|$. Combining (37) with (40), we have

$$\left\|\delta_n\right\| \le M\left\|\left|W^{-1}\right|\right\| \max_{\breve{k}-k\le j\le \breve{k}-1}\left\|\delta_j\right\| + M\sum_{s=\breve{k}-k}^{n-k}\left\|\overline{b}_s\right\|, \quad n = \breve{k} - k, \breve{k} - k + 1, \ldots, N. \tag{41}$$

It is known from (2), (6), (11), (27), and (32) that

$$\begin{aligned}
\|\overline{b}_s\| &\leq h\sum_{i=0}^{k}\Big(|\beta_i|\Big(\Big\|f(t_{s+i},z_{s+i},z_{si}^h(\cdot))-f(t_{s+i},y_{s+i},y_{si}^h(\cdot))\Big\|\Big)\Big)+h\|\varepsilon_{s+k}\| \\
&\leq h\sum_{i=0}^{k}\Bigg(|\beta_i|\Bigg(\Big\|f(t_{s+i},z_{s+i},z_{si}^h(\cdot))-f(t_{s+i},y_{s+i},z_{si}^h(\cdot)) \\
&\quad +f(t_{s+i},y_{s+i},z_{si}^h(\cdot))-f(t_{s+i},y_{s+i},y_{si}^h(\cdot))\Big\|\Bigg)\Bigg)+h\|\varepsilon_{s+k}\| \\
&\leq h\sum_{i=0}^{k}\Bigg(|\beta_i|\Bigg(\Big\|f(t_{s+i},z_{s+i},z_{si}^h(\cdot))-f(t_{s+i},y_{s+i},z_{si}^h(\cdot))\Big\| \\
&\quad +\Big\|f(t_{s+i},y_{s+i},z_{si}^h(\cdot))-f(t_{s+i},y_{s+i},y_{si}^h(\cdot))\Big\|\Bigg)\Bigg)+h\|\varepsilon_{s+k}\| \\
&\leq h\sum_{i=0}^{k}\Bigg(|\beta_i|\Bigg(L_1\|\delta_{s+i}\|+L_2\max_{t\in[-\tau,t_{s+i}]}\Big\|z_{si}^h(t)-y_{si}^h(t)\Big\|\Bigg)\Bigg)+h\|\varepsilon_{s+k}\| \\
&\leq h\sum_{i=0}^{k}\Bigg(|\beta_i|\Bigg(L_1\|\delta_{s+i}\|+L_2\max_{t\in[-\tau,t_{s+i}]}\Big\|\prod\nolimits_{si}^h(t;\chi,z_1,z_2,\ldots,z_{s+k}) \\
&\quad -\prod\nolimits_{si}^h(t;\psi,y_1,y_2,\ldots,y_{s+k})\Big\|+L_2\max_{t\in[-\tau,t_{s+i}]}\|\gamma_{si}(t)\|\Bigg)\Bigg)+h\|\varepsilon_{s+k}\| \\
&\leq h\sum_{i=0}^{k}\Bigg(|\beta_i|\Bigg(L_1\|\delta_{s+i}\|+L_2c_\pi\max\Big\{\max_{1\leq\bar{i}\leq s+k}\|\delta_{\bar{i}}\|,\max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\|\Big\} \\
&\quad +L_2\max_{t\in[-\tau,t_{s+i}]}\|\gamma_{si}(t)\|\Bigg)\Bigg)+h\|\varepsilon_{s+k}\|,\quad s=\breve{k}-k,\breve{k}-k+1,\ldots,N-k.
\end{aligned} \tag{42}$$

Then, (42) yields

$$\begin{aligned}
\sum_{s=\breve{k}-k}^{n-k}\|\overline{b}_s\| &\leq h\sum_{i=0}^{k}\sum_{s=\breve{k}-k}^{n-k}\Big(|\beta_i|\Big(L_1\|\delta_{s+i}\|+L_2c_\pi\max\Big\{\max_{1\leq\bar{i}\leq s+k}\|\delta_{\bar{i}}\|,\max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\|\Big\}+L_2\max_{t\in[-\tau,t_{s+i}]}\|\gamma_{si}(t)\|\Big)\Big)+h\sum_{s=\breve{k}-k}^{n-k}\|\varepsilon_{s+k}\| \\
&\leq h\sum_{i=0}^{k}\sum_{j=\breve{k}-k}^{n}\Big(|\beta_i|L_1\|\delta_j\|\Big)+L_2c_\pi h\sum_{i=0}^{k}\sum_{s=\breve{k}-k}^{n-k}\Big(|\beta_i|\max\Big\{\max_{1\leq\bar{i}\leq s+k}\|\delta_{\bar{i}}\|,\max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\|\Big\}\Big) \\
&\quad +hL_2\sum_{i=0}^{k}\sum_{s=\breve{k}-k}^{n-k}\Big(|\beta_i|\max_{t\in[-\tau,t_{s+i}]}\|\gamma_{si}(t)\|\Big)+h\sum_{s=\breve{k}-k}^{n-k}\|\epsilon_{s+k}\|,\quad n=\breve{k},\breve{k}+1,\ldots,N.
\end{aligned} \tag{43}$$

Setting $\beta=\max_{0\leq i\leq k}|\beta_i|$ and noticing (43), we have

$$\begin{aligned}
&\sum_{s=\breve{k}-k}^{n-k}\|\overline{b}_s\|\leq(k+1)h\beta L_1\sum_{j=\breve{k}-k}^{n}\|\delta_j\|+(k+1)h\beta L_2c_\pi\sum_{s=\breve{k}-k}^{n-k}\max\Big\{\max_{1\leq\bar{i}\leq s+k}\|\delta_{\bar{i}}\|,\max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\|\Big\}+ \\
&(k+1)h\beta L_2\sum_{s=\breve{k}-k}^{n-k}\max_{0\leq i\leq k}\max_{t\in[-\tau,t_{s+i}]}\|\gamma_{si}(t)\|+t_n\max_{\breve{k}\leq j\leq n}\|\varepsilon_j\|,\quad n=\breve{k},\breve{k}+1,\ldots,N.
\end{aligned} \tag{44}$$

Denote

$$Q_0=\|\delta_0\|,Q_n=\max\Big\{\|\delta_n\|,\max_{0\leq i\leq n-1}\|\delta_i\|\Big\}\leq\|\delta_n\|+Q_{n-1},\quad n\geq 1. \tag{45}$$

Combining (44) with (45), we conclude

$$\sum_{s=\breve{k}-k}^{n-k}\left\|\overline{b}_s\right\| \leq (k+1)h\beta L_1 \sum_{j=\breve{k}-k}^{n} Q_j + (k+1)h\beta L_2 c_\pi \sum_{j=\breve{k}}^{n} Q_j + (k+1)h\beta L_2 c_\pi \sum_{s=\breve{k}-k}^{n-k} \max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\| + (k+1)h\beta L_2 \sum_{s=\breve{k}-k}^{n-k} \max_{0\leq i\leq k} \max_{t\in[-\tau,t_{s+i}]}\left\|\gamma_{si}(t)\right\| + t_n \max_{\breve{k}\leq j\leq n}\left\|\varepsilon_j\right\|, \quad n=\breve{k},\breve{k}+1,\ldots,N. \tag{46}$$

From inequalities (41) and (46), noticing Remark 2, we obtain

$$\left\|\delta_n\right\| \leq M\left\|\left|W^{-1}\right|\right\| \max_{\breve{k}-k\leq j\leq \breve{k}-1}\left\|\delta_j\right\| + (k+1)h\beta M\left(L_1+L_2c_\pi\right)\sum_{j=\breve{k}-k}^{n} Q_j + (k+1)\beta L_2 c_\pi M t_n \max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\| + (k+1)\beta L_2 M t_n \max_{\breve{k}-k\leq s\leq n-k}\max_{0\leq i\leq k}\max_{t\in[-\tau,t_{s+i}]}\left\|\gamma_{si}(t)\right\| + M t_n \max_{\breve{k}\leq j\leq n}\left\|\varepsilon_j\right\|, \quad n=\breve{k}-k,\breve{k}-k+1,\ldots,N. \tag{47}$$

By (45), we have

$$\sum_{j=\breve{k}-k}^{n} Q_j = \sum_{j=\breve{k}-k}^{n-1} Q_j + Q_n \leq 2\sum_{j=\breve{k}-k}^{n-1} Q_j + \left\|\delta_n\right\|, \quad n=\breve{k},\breve{k}+1,\ldots,N. \tag{48}$$

It can be concluded from (47) and (48) that

$$\left\|\delta_n\right\| \leq M\left\|\left|W^{-1}\right|\right\| \max_{\breve{k}-k\leq j\leq \breve{k}-1}\left\|\delta_j\right\| + (k+1)\beta L_2 c_\pi M t_n \max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\| + (k+1)\beta L_2 M t_n \max_{\breve{k}-k\leq s\leq n-k}\max_{0\leq i\leq k}\max_{t\in[-\tau,t_{s+i}]}\left\|\gamma_{si}(t)\right\| + M t_n \max_{\breve{k}\leq j\leq n}\left\|\varepsilon_j\right\| + (k+1)h\beta M\left(L_1+L_2c_\pi\right)\left\|\delta_n\right\| + 2(k+1)h\beta M\left(L_1+L_2c_\pi\right)\sum_{j=\breve{k}-k}^{n-1} Q_j, \quad n=\breve{k},\breve{k}+1,\ldots,N. \tag{49}$$

Let

$$\widetilde{\alpha}_n = M\left\|\left|W^{-1}\right|\right\| \max_{\breve{k}-k\leq j\leq \breve{k}-1}\left\|\delta_j\right\| + (k+1)\beta L_2 c_\pi M t_n \max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\| + (k+1)\beta L_2 M t_n \max_{\breve{k}-k\leq s\leq n-k}\max_{0\leq i\leq k}\max_{t\in[-\tau,t_{s+i}]}\left\|\gamma_{si}(t)\right\| + M t_n \max_{\breve{k}\leq j\leq n}\left\|\varepsilon_j\right\| + \max_{t\in[-\tau,0]}\|\chi(t)-\psi(t)\| + \max_{1\leq j\leq \breve{k}-1}\left\|\delta_j\right\|, \quad n=0,1,\ldots,N. \tag{50}$$

From (49) and (50), we know that, for $n = 0, 1, \ldots, N$,

$$\|\delta_n\| \le \tilde{\alpha}_n + (k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\|\delta_n\| + 2(k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\sum_{j=\breve{k}-k}^{n-1} Q_j, \tag{51}$$

$$\left(1 - (k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\right)\|\delta_n\| \le \tilde{\alpha}_n + 2(k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\sum_{j=\breve{k}-k}^{n-1} Q_j. \tag{52}$$

For any given $v_1 \in (0, 1)$, set

$$h_0 = \min\left\{\frac{1 - v_1}{(k+1)\beta M\left(L_1 + L_2 c_\pi\right)}, h_\varphi\right\}, \tag{53}$$

which depends on method (6), L_1 and L_2, where h_φ is defined in Section 2. Noticing $\breve{k} - k \ge 0$, when $0 < h \le h_0$, from (52), we obtain

$$\begin{aligned}\|\delta_n\| &\le v_1^{-1}\tilde{\alpha}_n + 2v_1^{-1}(k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\sum_{j=\breve{k}-k}^{n-1} Q_j \\ &\le v_1^{-1}\tilde{\alpha}_n + 2v_1^{-1}(k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\sum_{j=0}^{n-1} Q_j, \quad n = 0, 1, \ldots, N.\end{aligned} \tag{54}$$

Because $\tilde{\alpha}_n$ does not decrease as n increases, when $n \ge 1$ and $0 \le i \le n - 1$, from (54), we know

$$\|\delta_i\| \le v_1^{-1}\tilde{\alpha}_n + 2v_1^{-1}(k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\sum_{j=0}^{n-1} Q_j, \quad 0 < h \le h_0. \tag{55}$$

It is concluded from (54) and (55) that

$$Q_n \le v_1^{-1}\tilde{\alpha}_n + 2v_1^{-1}(k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\sum_{j=0}^{n-1} Q_j, \quad 0 < h \le h_0, n = 0, 1, \ldots, N, \tag{56}$$

which implies

$$Q_i \le v_1^{-1}\tilde{\alpha}_n + 2v_1^{-1}(k+1)h\beta M\left(L_1 + L_2 c_\pi\right)\sum_{j=0}^{i-1} Q_j, \quad 0 < h \le h_0, i = 0, 1, \ldots, n, n = 0, 1, \ldots, N. \tag{57}$$

From (57) and Proposition 4, we have

$$Q_n \le v_1^{-1}\tilde{\alpha}_n e^{2v_1^{-1}(k+1)hn\beta M\left(L_1 + L_2 c_\pi\right)}, \quad 0 < h \le h_0, n = 0, 1, \ldots, N, \tag{58}$$

so

$$\delta_n \le Q_n \le \nu_1^{-1} \widetilde{\alpha}_n e^{2\nu_1^{-1}(k+1)t_n \beta M (L_1 + L_2 c_\pi)}, \quad 0 < h \le h_0, n = 0, 1, \ldots, N, \tag{59}$$

$$\begin{aligned}
\max_{0\le i\le N} \|\delta_i\| &\le \nu_1^{-1} \widetilde{\alpha}_N e^{2\nu_1^{-1}(k+1)T\beta M (L_1+L_2c_\pi)} \le \nu_1^{-1} e^{2\nu_1^{-1}(k+1)T\beta M(L_1+L_2c_\pi)} \Big(\left(M\left\|W^{-1}\right\| + 1 \right) \max_{1\le j\le \breve{k}-1} \left\|\delta_j\right\| \\
&+ \left((k+1)\beta L_2 c_\pi MT + M\left\|W^{-1}\right\| + 1 \right) \max_{t\in[-\tau,0]} \|\chi(t) - \psi(t)\| \\
&+ (k+1)\beta L_2 MT \max_{\breve{k}-k\le s\le N-k} \max_{0\le i\le k} \max_{t\in[-\tau, t_{s+i}]} \left\|\gamma_{si}(t)\right\| + MT \max_{\breve{k}\le j\le N} \left\|\varepsilon_j\right\| \Big) \\
&\le C \left(\max_{1\le j\le \breve{k}-1} \left\|\delta_j\right\| + \max_{t\in[-\tau,0]} \|\chi(t) - \psi(t)\| + \max_{\breve{k}-k\le s\le N-k} \max_{0\le i\le k} \max_{t\in[-\tau,t_{s+i}]} \left\|\gamma_{si}(t)\right\| + \max_{\breve{k}\le j\le N} \left\|\varepsilon_j\right\| \right), \quad 0 < h \le h_0,
\end{aligned} \tag{60}$$

where

$$C = \nu_1^{-1} e^{2\nu_1^{-1}(k+1)T\beta M(L_1+L_2c_\pi)} \max\left\{ M\left\|W^{-1}\right\| + 1, (k+1)\beta L_2 c_\pi MT + M\left\|W^{-1}\right\| + 1, (k+1)\beta L_2 MT, MT \right\}, \tag{61}$$

which is independent of h. This completes the proof. □

4. Consistency and Convergence Analyses

In this section, for equation (1), we assume the mapping f possesses sufficiently high-order continuous partial derivatives, and its true solution $y(t)$ possesses sufficiently high-order continuous derivatives on the interval $[0, T]$ and the constants M_i used later which are defined as

$$M_i = \sup_{0\le t\le T} \left\| \frac{d^i y(t)}{dt^i} \right\| \tag{62}$$

are all of moderate size except for some special cases such as in the transient phase of a stiff problem.

Definition 3. It is said the piecewise Lagrangian interpolation operator Π_{ni}^h in (6) is consistent of order $p\,(p \ge 1)$ if, for any given function u: $[-\tau, T] \longrightarrow \mathbf{R}^m$ which is sufficiently differentiable on the subinterval $[0, T]$, there exists

$$\max_{-\tau\le t\le t_{n+i}} \left\| u(t) - \prod_{ni}^{h} \left(t; \phi, u(t_1), u(t_2), \ldots, u(t_{n+k})\right) \right\| \le \widehat{g} h^p, \tag{63}$$

where the function ϕ is a restriction of the function $u(t)$ on the subinterval $[-\tau, 0]$ and the constant $\widehat{g}$ only depends on some $\widetilde{M}_j$ defined as $\widetilde{M}_j = \sup_{t\in[0,T]} (\|d^j u(t)/dt^j\|)$.

Remark 4. By Definition 3 and the Lagrangian interpolation theorem, we know the piecewise interpolation operators Π_{ni}^h in (6) which based on $p\,(p \ge 0)$-degree Lagrangian interpolation polynomial are all consistent of order $p + 1$.

Definition 4. It is said method (6) is consistent of order p if its mother method (5) and the piecewise interpolation operators $\Pi_{ni}^h\,(n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k, i = 0, 1, \ldots, k)$ are all consistent of order p.

Remark 5. As for the consistency of the mother method (5), please see references [35, 38, 40].

Definition 5. Method (6) is said to be convergent of order p if, for the sequence $\left\{ y_{\breve{k}}, y_{\breve{k}+1}, \ldots, y_N \right\}$ determined by method (6) applied to any given equations (1) in $\mathbb{D}(L_1, L_2)$ with the starting data $\psi \in \mathbf{C}_m[-\tau, 0]$, $y_0 := \psi(0), y_1, y_2, \ldots, y_{\breve{k}-1} \in \mathbf{R}^m$ (the method (6) starts with $n = \breve{k} - k$), there exists a sufficiently small positive number $\widetilde{h}$, such that

$$\left\|y(t_n) - y_n\right\| \le C_0(t_n) \max\left\{ \max_{-\tau\le t\le 0} \|\varphi(t) - \psi(t)\|, \max_{1\le j\le \breve{k}-1} \left\| y(t_j) - y_j \right\| \right\} + C_1(t_n) h^p, \quad n = \breve{k}, \breve{k}+1, \ldots, N, 0 < h \le \widetilde{h}, \tag{64}$$

where the continuous functions $C_0(t)$ and $C_1(t)$ depend on t, method (6), L_1, L_2, and some M_i .

Theorem 2. *If method (6) is consistent of order p and satisfies the root condition, it is convergent of order p.*

Proof. Assume method (6) is consistent of order p and satisfies the root condition. In this proving course, we always assume $0 < h \le h_0$, where h_0 is defined by (53). Using method (6) to solve any given equations (1) $\in \mathbb{D}(L_1, L_2)$, we easily know the true solution $y(t)$ obviously satisfies the perturbation equations:

$$\begin{cases} y_{ni}(t) = \prod_{ni}^{h}(t;\varphi, y(t_1), y(t_2), \ldots, y(t_{n+k})) + \gamma_{ni}(t), & -\tau \le t \le t_{n+i}, i = 0, 1, \ldots, k, \\ \sum_{i=0}^{k} \alpha_i y(t_{n+i}) = h\left(\left(\sum_{i=0}^{k} \beta_i f(t_{n+i}, y(t_{n+i}), y_{ni}(\cdot))\right) + \frac{1}{h} T_{n+k}\right), & n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k, \end{cases} \tag{65}$$

where

$$\gamma_{ni}(t) = y(t) - \prod_{ni}^{h}(t;\varphi, y(t_1), y(t_2), \ldots, y(t_{n+k})), \quad -\tau \le t \le t_{n+i}, i = 0, 1, \ldots, k, \tag{66}$$

$$T_{n+k} = \sum_{i=0}^{k}(\alpha_i y(t_{n+i}) - h\beta_i f(t_{n+i}, y(t_{n+i}), y_{ni}(\cdot))). \tag{67}$$

From (1) and (67), we conclude

$$T_{n+k} = \sum_{i=0}^{k}(\alpha_i y(t_{n+i}) - h\beta_i y'(t_{n+i})), \quad n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k. \tag{68}$$

Since the mother method (5) is consistent of order p, equation (68) implies [35, 38, 40]

$$\left\|T_{n+k}\right\| \le \widetilde{C}h^{p+1}, \quad n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k, \tag{69}$$

where the constant $\widetilde{C}$ only depends on the mother method (5) and M_{p+1} defined above. Because all the interpolation operators $\Pi_{ni}^{h}(n = \breve{k} - k, \breve{k} - k + 1, \ldots, N - k, i = 0, 1, \ldots, k)$ are consistent of order p, by Definition 3, we obtain

$$\max_{\breve{k}-k \le n \le N-k} \max_{0 \le i \le k} \max_{-\tau \le t \le t_{n+i}} \left\|\gamma_{ni}(t)\right\| \le \widehat{g}h^{p}, \tag{70}$$

where the constant $\widehat{g}$ only depends on some M_i defined above. By Theorem 1 and assumption that method (6) satisfies the root condition, we know method (6) is zero-stable, which tells us that inequality (59) holds. From (6), (27), (50), (59), (65), (69), and (70), we further conclude that

$$\begin{aligned} \left\|y(t_n) - y_n\right\| &\le v_1^{-1} e^{2v_1^{-1}(k+1)t_n\beta M(L_1+L_2c_\pi)} \Big(M\left\|W^{-1}\right\| \max_{\breve{k}-k \le j \le \breve{k}-1} \left\|y(t_j) - y_j\right\| \\ &\quad + (k+1)\beta L_2 c_\pi M t_n \max_{t\in[-\tau,0]} \|\varphi(t) - \psi(t)\| + (k+1)\beta L_2 M t_n \max_{\breve{k}-k \le s \le n-k} \max_{0 \le i \le k} \max_{t \in [-\tau, t_{s+i}]} \left\|\gamma_{si}(t)\right\| \\ &\quad + \frac{Mt_n}{h} \max_{\breve{k} \le j \le n} \left\|T_j\right\| + \max_{t\in[-\tau,0]} \|\varphi(t) - \psi(t)\| + \max_{1 \le j \le \breve{k}-1} \left\|y(t_j) - y_j\right\| \Big) \\ &\le v_1^{-1} e^{2v_1^{-1}(k+1)t_n\beta M(L_1+L_2c_\pi)} \Big(\max_{t\in[-\tau,0]} \|\varphi(t) - \psi(t)\| + \max_{1 \le j \le \breve{k}-1} \left\|y(t_j) - y_j\right\| + M\left\|W^{-1}\right\| \max_{\breve{k}-k \le j \le \breve{k}-1} \left\|y(t_j) - y_j\right\| \\ &\quad + (k+1)\beta L_2 c_\pi M t_n \max_{t\in[-\tau,0]} \|\varphi(t) - \psi(t)\| + (k+1)\beta L_2 M t_n \widehat{g} h^p + M t_n \widetilde{C} h^p \Big) \\ &\le \widehat{C}_0(t_n) \max\left\{ \max_{t\in[-\tau,0]} \|\varphi(t) - \psi(t)\|, \max_{1 \le j \le \breve{k}-1} \left\|y(t_j) - y_j\right\| \right\} + \widehat{C}_1(t_n) h^p, \quad n = \breve{k}, \breve{k}+1, \ldots, N, \end{aligned} \tag{71}$$

where

$$\widehat{C}_0(t) = v_1^{-1}\left(M\left\|\left|W^{-1}\right|\right\| + (k+1)\beta L_2 c_\pi M t + 2\right)e^{2v_1^{-1}(k+1)t\beta M\left(L_1+L_2c_\pi\right)},$$
$$\widehat{C}_1(t) = v_1^{-1}\left((k+1)\beta L_2 M\widehat{g} + M\widetilde{C}\right)te^{2v_1^{-1}(k+1)t\beta M\left(L_1+L_2c_\pi\right)}, \tag{72}$$

and $v_1, M, \||W^{-1}|\|$, and β defined in the course of proving Theorem 1 in Section 3. It can be seen that the continuous functions $\widehat{C}_0(t)$ and $\widehat{C}_1(t)$ depend on t, method (6), L_1, L_2, and some M_i. This completes the proof. □

Remark 6. In this paper, the strict condition (2) can be generally weakened as the inequalities in (2) hold only in some neighborhood of the true solution $y(t)$ of the equations [35, 36, 38, 41].

5. Numerical Examples

In this section, the linear multistep methods formed as (6) are used to solve VFDEs formed as (1) $\in \mathbb{D}(L_1, L_2)$, and for convenience, we give the starting data $\psi(t), y_0, y_1, y_2, \ldots, y_{\breve{k}-1}$ according to true solution, i.e., let

$$\psi(t) = \varphi(t), -\tau \le t \le 0,$$
$$y_0 = \varphi(0), y_i = y(t_i), \quad i = 1, 2, \ldots, \breve{k}-1. \tag{73}$$

For a method formed as (6) whose stepsize is h, we denote the maximum global error:

$$E(h) = \max_{0\le n\le N}\max_{1\le i\le m}\left|y_i(t_n) - y_{ni}\right|, \tag{74}$$

where $y_i(t_n)$ and y_{ni} are the ith components of the true solution $y(t_n)$ and the approximation solution y_n, respectively, and furthermore, according to [36, 41], the observing order $\widetilde{p}$ of the numerical method is defined as

$$\widetilde{p} = \log_2\frac{E(h)}{E(h/2)}. \tag{75}$$

In order to adequately verify the correctness of the theory presented in this paper, we use various different zero-stable mother methods including the Adams–Bashforth methods [38, 42], the Adams–Moulton method [38, 43], the Backward Differentiation Formula (BDF) method, and the Milne-Simpson method [44, 45] with corresponding piecewise Lagrangian interpolation operator to build the methods formed as (6), where each mother method and its corresponding interpolation operator are both consistent of the same order $p\,(2 \le p \le 4)$. According to Definition 4, these methods are consistent of order p, and by Theorem 2, we know they are convergent of order p. For convenience, these methods are named the same as their corresponding mother methods. For simplicity, let AB p denote the Adams–Bashforth method which is consistent of order p, let AM p denote the Adams–Moulton method which is consistent of order p, let B p denote the BDF method which is consistent of order p, and let MS4 denote the Milne-Simpson method which is consistent of order 4.

Remark 7. We list the mother methods used in this section as follows:

(i) The Adams–Bashforth method which is consistent of order 2 takes the form

$$-y_{n+1} + y_{n+2} = h\left(-\frac{1}{2}f(t_n, y_n) + \frac{3}{2}f(t_{n+1}, y_{n+1})\right). \tag{76}$$

(ii) The Adams–Bashforth method which is consistent of order 3 takes the form

$$-y_{n+2} + y_{n+3} = h\left(\frac{5}{12}f(t_n, y_n) - \frac{16}{12}f(t_{n+1}, y_{n+1}) + \frac{23}{12}f(t_{n+2}, y_{n+2})\right). \tag{77}$$

(iii) The Adams–Moulton method which is consistent of order 2 takes the form

$$-y_n + y_{n+1} = h\left(\frac{1}{2}f(t_n, y_n) + \frac{1}{2}f(t_{n+1}, y_{n+1})\right). \tag{78}$$

(iv) The BDF method which is consistent of order 2 takes the form

$$\frac{1}{2}y_n - 2y_{n+1} + \frac{3}{2}y_{n+2} = hf(t_{n+2}, y_{n+2}). \tag{79}$$

(v) The Milne-Simpson method takes the form

$$-y_n + y_{n+2} = h\left(\frac{1}{3}f(t_n, y_n) + \frac{4}{3}f(t_{n+1}, y_{n+1}) + \frac{1}{3}f(t_{n+2}, y_{n+2})\right). \tag{80}$$

Example 1. Consider the pantograph equations [36]:

$$\begin{cases} y_1'(t) = -2.8y_1(t) + 3.6y_2(t) - 0.4\sin(0.5t)y_1(0.5t) + 0.8\sin(0.5t)y_2(0.5t) + 1 + 10\sin t, & 0 \le t \le 1, \\ y_2'(t) = 3.6y_1(t) - 8.2y_2(t) + 0.8\sin(0.5t)y_1(0.5t) - 1.6\sin(0.5t)y_2(0.5t) - 2 - 20\sin t, & 0 \le t \le 1, \\ y_1(0) = 2,\ y_2(0) = 1, \end{cases} \tag{81}$$

whose unique true solution is $y_1(t) = \sin t + 2e^{-t}$, $y_2(t) = -2\sin t + e^{-t}$. Equation (81) is a special case $\in \mathbb{D}(L_1, L_2)$, where $L_1 = 10$ and $L_2 = 2\sin 0.5 \approx 0.9588$[36]. We use the methods AB2, AM2, and B2 with $h = (1/(80 \times 2^i))(i = 0, 1, \ldots, 5)$ to solve (81), respectively. The maximum global errors $E(h)$ and the observing orders $\widetilde{p}$ are listed in Table 1.

From Table 1, we can see for the pantograph equation (81) the observing orders of the three methods AB2, AM2, and B2 are all very close to the theoretical convergent order 2.

Example 2. Consider the nonlinear VIDEs [46]:

$$\begin{cases} y_1'(t) = 2y_2(t) - \dfrac{1}{3}t^4 + \cos(y_1(t)) - 1 + \displaystyle\int_0^t (2s\sin(y_1(s)) + sty_2(s))\mathrm{d}s, & 0 \le t \le 1, \\ y_2'(t) = 1 - t\sin(y_2(t)) - \dfrac{1}{2}t^2\sin(y_1(t)) + \displaystyle\int_0^t \left(st^2\cos(y_1(s)) + t\cos(y_2(s))\right)\mathrm{d}s, & 0 \le t \le 1, \\ y_1(0) = 0,\ y_2(0) = 0. \end{cases} \tag{82}$$

The unique true solution of equation (82) is $y_1(t) = t^2$ and $y_2(t) = t$. Let

$$\begin{aligned} y(t) &= [y_1(t), y_2(t)]^T, \\ f(t, u, \psi(\cdot)) &= \begin{bmatrix} 2u_2 - \dfrac{1}{3}t^4 + \cos u_1 - 1 + \displaystyle\int_0^t (2s\sin(\psi_1(s)) + st\psi_2(s))\mathrm{d}s, \\ 1 - t\sin u_2 - \dfrac{1}{2}t^2\sin u_1 + \displaystyle\int_0^t \left(st^2\cos(\psi_1(s)) + t\cos(\psi_2(s))\right)\mathrm{d}s \end{bmatrix}, \end{aligned} \tag{83}$$

where $0 \le t \le 1$, $u = [u_1, u_2]^T \in \mathbf{R}^2$, and $\psi \in \mathbf{C}_2[0, 1]$. Then, equation (82) can be written in form (1)$\in \mathbb{D}(L_1, L_2)$, where L_1 and L_2 are both of moderate size. We use the method AB2 with $h = (1/(80 \times 2^i))(i = 0, 1, \ldots, 6)$ to solve (82), respectively. The maximum global errors $E(h)$ and the observing orders $\widetilde{p}$ are listed in Table 2.

From Table 2, we can see, for the nonlinear VIDEs (82), the observing orders of the method AB2 are all very close to the theoretical convergent order 2.

Example 3. Consider the nonlinear VDIDEs [36]:

$$\begin{cases} y'(t) = y^2(t) + 2y(t) - 2y\left(t - \dfrac{\pi}{2}\right) + 2y\left(\dfrac{t}{2}\right)y\left(\dfrac{t-\pi}{2}\right) + \displaystyle\int_{t-\pi}^{t-(\pi/2)} y(z)\mathrm{d}z + \cos^2 t - 1, & 0 \le t \le 6, \\ y(t) = \sin t, & -\pi \le t \le 0. \end{cases} \tag{84}$$

The unique true solution of equation (84) is $y(t) = \sin t$. Equation (84) $\in \mathbb{D}(L_1, L_2)$ in the sense of Remark 6, where L_1 and L_2 are both of moderate size [36]. We use MS4 with $h = (0.01/2^i)(i = 0, 1, 2, 3)$ to solve (84), respectively, and list the maximum global errors $E(h)$ and the observing orders $\widetilde{p}$ in Table 3.

Table 1: The maximum global errors $E(h)$ and the observing orders $\widetilde{p}$ produced by AB2, AM2, and B2 when applied to equation (81).

h	AB2		AM2		B2	
	$E(h)$	$\widetilde{p}$	$E(h)$	$\widetilde{p}$	$E(h)$	$\widetilde{p}$
1/80	5.179403×10^{-5}		9.921322×10^{-6}		4.066459×10^{-5}	
1/160	1.297365×10^{-5}	1.997201	2.497546×10^{-6}	1.990021	1.021635×10^{-5}	1.992893
1/320	3.246682×10^{-6}	1.998546	6.265474×10^{-7}	1.995016	2.560343×10^{-6}	1.996471
1/640	8.120817×10^{-7}	1.999269	1.569068×10^{-7}	1.997516	6.408666×10^{-7}	1.998241
1/1280	2.030719×10^{-7}	1.999634	3.926050×10^{-8}	1.998757	1.603144×10^{-7}	1.999120
1/2560	5.077445×10^{-8}	1.999816	9.819351×10^{-9}	1.999379	4.009081×10^{-8}	1.999561

Table 2: The maximum global errors $E(h)$ and the observing orders $\widetilde{p}$ produced by AB2 when applied to equation (82).

h	$E(h)$	$\widetilde{p}$
1/80	2.567537×10^{-6}	
1/160	6.421382×10^{-7}	1.999429
1/320	1.605505×10^{-7}	1.999857
1/640	4.013861×10^{-8}	1.999965
1/1280	1.003472×10^{-8}	1.999990
1/2560	2.508682×10^{-9}	1.999999
1/5120	6.271714×10^{-10}	1.999998

Table 3: The maximum global errors $E(h)$ and the observing orders $\widetilde{p}$ produced by MS4 when applied to equation (84).

h	$E(h)$	$\widetilde{p}$
0.01	6.757335×10^{-7}	
0.005	4.041475×10^{-8}	4.063501
0.0025	2.483531×10^{-9}	4.024417
0.00125	1.524486×10^{-10}	4.025998

From Table 3, we can see, for the nonlinear VDIDEs (84), the observing orders of the method MS4 are all very close to the theoretical convergent order 4.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare there are no conflicts of interest.

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