Research Article

Heteroclinic Cycles Imply Chaos and Are Structurally Stable

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This paper is concerned with the chaos of discrete dynamical systems. A new concept of heteroclinic cycles connecting expanding periodic points is raised, and by a novel method, we prove an invariant subsystem is topologically conjugate to the one-side symbolic system. Thus, heteroclinic cycles imply chaos in the sense of Devaney. In addition, if a continuous differential map $h$ has heteroclinic cycles in $\mathbb{R}^n$, then $g$ has heteroclinic cycles with $\|h - g\|_{C^1}$ being sufficiently small. The results demonstrate $C^1$ structural stability of heteroclinic cycles. In the end, two examples are given to illustrate our theoretical results and applications.

1. Introduction

Since Li and Yorke first introduced the term “chaos” in 1975 [1], chaotic dynamics have been observed in various fields [2–10]. When chaotic theory was in its initial stage, Marotto generalized the results of Li and Yorke in interval mapping to multidimensional discrete systems and proved that a snapback repeller implies chaos [11]. In [2], Blanco demonstrated that $f : \mathbb{R}^n \to \mathbb{R}^n$ has positive topological entropy if $f$ has snapback repellers. In 1989, Devaney gave an explicit chaotic definition which is called Devaney’s chaos in [12]. In order to understand the relationship between various kinds of chaos, Huang and Ye obtained that Devaney’s chaos implies Li–Yorke chaos in [13]. Based on the research of Marotto, Shi and Chen raised the concept of snapback repellers in Banach spaces and complete metric spaces [14, 15]. As is well known, some homoclinic and heteroclinic cycles imply chaos in dynamical systems [6, 16]. Lin and Chen introduced the new chaotic criteria of heteroclinic repellers in $\mathbb{R}^n$ [17]. Recently, Li et al. generalized the definition of heteroclinic repellers to infinite dimensional dynamical systems [18].

From the view of application, we naturally ask whether a chaotic dynamical system still has chaotic behaviors under small perturbations. In [19], Marotto showed that the systems with snapback repellers under delayed perturbations still have chaotic behaviors. In [20], Li et al. showed the $C^1$ structural stability of snapback repellers in $\mathbb{R}^n$. Further, Chen et al. studied the structure stability of snapback repellers in Banach space [21]. Chen and Li provided a sufficient condition of a high-dimensional difference equation having symbolic embedding for enough small $C^1$ perturbations [22]. In 2020, Chen and Wu et al. showed that the system with heteroclinic repellers has the structural stability in [4, 16].

In this paper, we consider the following discrete dynamical systems:

$$x(n + 1) = f(x(n)), \quad n = 1, 2, \ldots ,$$

where $f$ is a continuous map from $\mathbb{R}^n$ into itself. Based on the concept of heteroclinic repellers in discrete dynamical systems, we introduce a new concept of heteroclinic cycles connecting expanding periodic points in $\mathbb{R}^n$; by a novel method, we can construct shift invariant sets and prove that heteroclinic cycles imply chaos in the sense of Devaney. It is very important to choose an appropriate recurrent time set in the proof of the theorem. In particular, it is easier to find the conditions of heteroclinic cycles than that of snapback repellers for the system in [3]. The following theorems are the main results.

Theorem 1. Let $h$ be a continuous differential map from $\mathbb{R}^n$ into itself. If $h$ has heteroclinic cycles connecting expanding periodic points, then $h$ is chaotic in the sense of Devaney.
Theorem 2. Let \( g, h \) be continuous differential maps from \( \mathbb{R}^n \) into itself. If \( h \) has heteroclinic cycles connecting expanding periodic points and \( \|g - h\|_{C^1} \) is sufficiently small, where \( \|g - h\|_{C^1} = \max \{\sup_{x \in X} \|g(x) - h(x)\|, \sup_{x \in X} \|Dg(x) - Dh(x)\|\} \), \( g \) also has heteroclinic cycles connecting expanding periodic points.

This paper is organized as follows: in Section 1, the relevant results of the previous studies are introduced. In Section 2, some definitions and necessary lemmas are given. A new concept of heteroclinic cycles in the sequel is raised. In Section 3, the main conclusions are drawn. At the end, some examples are presented to illustrate the main results and applications in Section 4.

2. Definitions and Lemmas

In this section, some definitions and lemmas will be introduced.

Definition 1. Let \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous differentiable map. \( f \) is expanding at a point \( x_0 \) if the derivative operator \( Df(x_0) \) is invertible and the norm of each of its eigenvalues is larger than 1. \( f \) is expanding in a nonempty subset \( U \) of \( \mathbb{R}^n \) if there exists positive constant \( \lambda > 1 \) such that, for any \( x, y \in U \), \( \|f(x) - f(y)\| \geq \lambda \|x - y\| \).

It is well known that if \( f \) is differentiable, then \( f \) is expanding at a point \( x_0 \) which is equivalent to the fact that there exist a norm \( \|\cdot\| \) in \( \mathbb{R}^n \) and a constant \( \lambda > 1 \) such that \( \forall x \in \mathbb{R}^n, \|Df(x)\| \geq \lambda \|x\| \) [3]. The following heteroclinic repellers in \( \mathbb{R}^n \) were presented by Lin and Chen in [17] where \( t(i) \equiv i(\text{mod} k) \).

Definition 2 (see [17]). Let \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous differentiable map and the integer \( k \geq 2 \) be fixed. The fixed points \( x_1, x_2, \ldots, x_k \) of \( f \) are called heteroclinic repellers if for any integer \( 1 \leq i \leq k \), there are

1. \( f \) is expanding at the point \( x_i \)
2. \( f \) has a forward orbit \( \Gamma_i = \{z_i(-n)\}_{n \geq 0} \) of \( x_{t(i)+1} \) such that \( \forall n \in \mathbb{N} \cap [1, +\infty), f(z_i(-n)) = z_i(-n + 1) \) and \( z_i(0) = x_{t(i)+1}, \lim_{n \rightarrow +\infty} z_i(-n) = x_i \)
3. \( f \) is continuously differentiable in a neighborhood of any point \( z \) on \( \Gamma_i \) and \( Df(z) \) is invertible.

Based on Definition 2, we introduce the following new concept.

Definition 3. Let \( f \) be a continuous differentiable map from \( \mathbb{R}^n \) into itself and \( k \geq 1 \) be fixed integer. Let \( f \) have \( k \) different periodic points \( x_1, x_2, \ldots, x_k \) with periods \( n_1, n_2, \ldots, n_k \), respectively. For any \( 1 \leq m \leq n_i \) and \( i \neq j \), \( f^m(x_i) \neq x_j \). For any \( 1 \leq i \leq k \), suppose that

1. \( f \) is expanding at every point \( x \) which is in the periodic orbits of \( x_i \).
2. \( f \) has a forward orbit \( \Gamma_i = \{z_i(-n)\}_{n \geq 0} \) of \( x_{t(i)+1} \), connecting periodic points \( x_i \) and \( x_{t(i)+1} \). That is, it satisfies that \( \forall n \in \mathbb{N} \cap [1, +\infty), f(z_i(-n)) = z_i(-n + 1), z_i(0) = x_{t(i)+1} \) and there exists a positive integer \( m_i \) such that \( \lim_{n \rightarrow +\infty} z_i(-m_i - m_i) = x_i \).

For any point \( z \) on \( \Gamma_i \), the linear operator \( Df(z): \mathbb{R}^n \rightarrow \mathbb{R}^n \) is invertible.

Then, the set \( \bigcup_{i=1}^k \Gamma_i \) is called a heteroclinic cycle connecting expanding periodic points \( x_1, x_2, \ldots, x_k \).

Remark 1. By Definition 3, for any point \( z \) in heteroclinic cycles connecting expanding periodic points \( x_1, x_2, \ldots, x_k \), \( Df(z) \) is invertible and the norm of each of its eigenvalues is larger than 1. Thus, \( x_1, x_2, \ldots, x_k \) are heteroclinic repellers when every periodic point is fixed point in Definition 3 and \( k \geq 2 \). Example 1 shows that heteroclinic cycles connecting expanding periodic points are different from heteroclinic repellers.

To prove the \( C^1 \) structural stability of heteroclinic cycles connecting expanding periodic points, we need the following implicit function theorem with parametric variables and continuous dependence theorem of inverse mapping.

Lemma 1 (see [21]). Let \( (X, \|\cdot\|) \) and \( (Y, \|\cdot\|) \) be Banach spaces, \( (\Lambda, d) \) be a metric space, and \( U \) be an open set of \( \Lambda \times X \). Suppose that \( F: U \rightarrow Y \) is a continuous differentiable map and there exists a point \((\lambda_0, x_0) \in U \) satisfying the following conditions:

1. \( DF_\lambda(x_0) \), the Fréchet partial derivative of \( F(\lambda, x) \) with respect to \( x \), is continuous with respect to \( x \) in some neighborhood of \((\lambda_0, x_0) \).
2. \( DF_\lambda(\lambda_0, x_0): X \rightarrow Y \) is an invertible linear operator.
3. \( F(\lambda_0, x_0) = 0 \).

Then, there exist open ball \( B_\varepsilon(x_0) = \{x\|x - x_0\| < r_1\} \) and \( B_\delta(\lambda_0) = \{\lambda |d(\lambda, \lambda_0) < \delta\} \), where \( r_1 > 0, \delta > 0 \), such that for any \( \lambda \in B_\delta(\lambda_0) \), \( F(\lambda, x) = 0 \), the unique continuous solution \( x = h(\lambda) \) of \( F(\lambda, x) \) exists and \( x_0 = h(\lambda_0) \) is satisfied.

Lemma 2 (see [21]). Let \( (Y, \|\cdot\|) \) and \( (Z, \|\cdot\|) \) be two Banach spaces and \( V_0 = B_r(y_0) \) be a closed ball neighborhood of \( y_0 \). Assume that \( h \) is a \( C^\rho \) map from \( V_0 \) into \( Z \) such that \( h(y_0) = z_0(p \geq 1) \) and \( Dh(y_0) \) is an invertible linear operator from \( Y \) into \( Z \). Then, there are constants \( r_0 > 0, \delta_0 > 0, \lambda_0 > 0 \) and a map \( f \) from \( V(h, \lambda_0) \times B_\varepsilon(z_0) \) into \( B_\varepsilon(y_0) \) satisfying the following conditions:

1. For any \( \phi \in V(h, \lambda_0), \phi \) is one to one on \( B_\varepsilon(y_0) \), and \( D\phi(y) \) is invertible for every \( y \in B_\varepsilon(y_0) \).
2. For any \( \phi \in V(h, \lambda_0) \) and \( z \in B_\varepsilon(z_0) \), there is a unique \( f(\phi, z) \in B_\varepsilon(y_0) \) satisfying \( \phi(f(\phi, z)) = z \).
3. The map \( f \) is a continuous map from \( V(h, \lambda_0) \times B_\varepsilon(z_0) \) into \( B_\varepsilon(y_0) \).
4. For every \( \phi \in V(h, \lambda_0), f(\phi, \cdot): B_\varepsilon(z_0) \rightarrow B_\varepsilon(y_0) \) is a \( p \)-order continuous differentiable map, and \( D_2 f(\phi, z) = (Df(f(\phi, z)))^{-1} \).
where
\[ \mathcal{V}(h, \lambda_0) = \{ \varphi \| \varphi - h \|_{C^1} < \lambda_0, \; \varphi \in C^p(V_0, Z) \}, \]
denotes the \( \lambda_0 \)-\( C^1 \)-open ball neighborhood of \( h \) and \( C^p(V_0, Z) \) is a \( p \)-order continuous differentiable map set.

**Lemma 3.** Let \( h \) be a continuous differentiable map from \( \mathbb{R}^n \) into itself. Let \( h \) have \( k \geq 2 \) periodic points \( x_1, x_2, \ldots, x_k \) with periods \( n_1, n_2, \ldots, n_k \), respectively. If \( h \) is expanding at every point \( x \) which is in the periodic orbit of \( x_i \), then for all \( 1 \leq i \leq k \), there exist positive constants \( r_i \) and \( \lambda > 0 \) such that \( \varphi^{m_i} \) is expanding on \( \mathbb{R}^n \) and \( \mathbb{R}^n \) is chaotic in the sense of Devaney.

**Proof.** For any \( 1 \leq i \leq k \), because \( h \) is expanding at every point \( x \) which is in the periodic orbit of \( x_i \), by the continuity of \( h \), there exists some closed neighborhood \( \mathcal{B}_r(x_i) \) of \( x_i \) such that \( h \) is expanding in \( h^m(\mathcal{B}_r(x_i)) \) for \( 0 \leq m \leq n_i - 1 \). Therefore, there exists a positive constant \( \lambda > 0 \) such that, for all \( 0 \leq m \leq n_i - 1 \) and for any \( x, y \in h^m(\mathcal{B}_r(x_i)) \), there is
\[ \| h(x) - h(y) \| \geq \lambda \| x - y \|. \]

Let \( \lambda_1 \) be a positive integer in the interval \( (1, \lambda) \). Then, for every \( \varphi \in \mathcal{V}(h, \lambda_1) \) and \( x, y \in \mathcal{B}_r(x_i) \), there is
\[ \| \varphi(x) - \varphi(y) \| \geq \| h(x) - h(y) \|. \]

The proof of the lemma is done. \( \square \)

### 3. Heteroclinic Cycles Connecting Expanding Periodic Points Imply Chaos and Are Structurally Stable

In this section, we are going to consider that heteroclinic cycles imply Devaney’s chaos and are structurally stable.

**Theorem 3.** Let \( h \) be a continuous differentiable map from \( \mathbb{R}^n \) into itself. If \( h \) has heteroclinic cycles connecting expanding periodic points, \( h \) is chaotic in the sense of Devaney.

**Proof.** Let \( x_1, x_2, \ldots, x_k \) be \( k \geq 1 \) different periodic points with periods \( n_1, n_2, \ldots, n_k \), respectively, and \( h \) be expanding at \( x_1, x_2, \ldots, x_k \), respectively. Let \( \bigcup_{i=1}^{k} \Gamma_i = \{ x_i(-m) \}_{m \in \mathbb{Z}} \) be heteroclinic cycles connecting periodic points \( x_1, x_2, \ldots, x_k \).

For any integer \( i \in [1, k] \) and any integer \( j \geq 0 \), by Definition 3, \( D_h(\mathbb{R}^n) \) is invertible and the norm of each of its eigenvalues is larger than 1. Therefore, there exist constants \( r_i > 0, \lambda > 1 \) such that, for any integer \( m \in [0, n_i - 1] \) and \( x, y \in \mathcal{B}_{r_i}(h^m(x_i)) \), \( \| h(x) - h(y) \| \geq \lambda \| x - y \| \). Therefore, there exists a positive integer \( m \) such that \( x_{i}(-m) \in \mathcal{B}_{r_i}(x_i) \) and \( \lim_{m \to -\infty} x_{i}(-m) = x_i \). Without loss of generality, for any integer \( i, j \in [1, k], m \in [1, m_1], l \in [1, m_2] \), let \( \mathcal{B}_l(h^m(x_i)) \cap \mathcal{B}_j(h^l(x_j)) = \emptyset \) if \( h^m(x_i) \neq h^l(x_j) \). Since \( D_h^m(x_i) \) is invertible, there exist positive constants...
$0 < \sigma < \min\{1/3\|x_i - z_i - (m_i)\|, (r_i/2)\}, \mu_i > 0$ such that $h^{m_i}$ is a homeomorphism on $\mathcal{B}_{\sigma_i}(z_i - (m_i))$ and $\forall x, y \in \mathcal{B}_{\sigma_i}(z_i - (m_i))$,
\begin{equation}
|h^{m_i}(x) - h^{m_i}(y)| \geq \mu_i^{m_i}|x - y|.
\end{equation}

By the continuity of $h$, there exists a closed ball $\mathcal{B}_{\sigma_i}(x_i) \subset \mathcal{B}_{\sigma_i}(x_i)$ such that
\begin{equation}
\left(h|_{\mathcal{B}_{\sigma_i}^{-1}(z_i - (m_i))}^{-m_i} \left(\mathcal{B}_{\sigma_i}(x_i)\right) \subset \mathcal{B}_{\sigma_i}(z_i - (m_i)),
\mathcal{B}_{\sigma_i}(x_i) \cap \mathcal{B}_{\sigma_i}(z_i - (m_i)) = \emptyset,\right.
\end{equation}
where $(h|_{\mathcal{B}_{\sigma_i}^{-1}(z_i - (m_i))}^{-m_i}$ is the inverse of $h^{m_i}$ restricted to $\mathcal{B}_{\sigma_i}(z_i - (m_i))$. Similarly, for any integer $i \in [2,k]$, there exists $\mathcal{B}_{\sigma_i}(x_i) \subset \mathcal{B}_{\sigma_i}(x_i)$ such that
\begin{equation}
\left(h|_{\mathcal{B}_{\sigma_i}^{-1}(z_i - (m_i))}^{-m_i} \left(\mathcal{B}_{\sigma_i}(x_i)\right) \subset \mathcal{B}_{\sigma_i}(z_i - (m_i)).
\end{equation}

Next we prove there exist two nonempty bounded closed subsets $V_0$ and $V_1$ with $V_0 \cap V_1 = \emptyset$ such that, for a positive integer $n_0$,
$h^{n_0}(V_j) > V_0 \cup V_1$, $j = 0, 1$ and $h^{n_0}$ restricted to $V_0$ and $V_1$, respectively, is expanding. Because $h^{n_0}$ is expanding on $\mathcal{B}_{\sigma_i}(x_i)$ and $x_i$ is a fixed point of $h^{n_0}$ for every $1 \leq i \leq k$, there are
\begin{enumerate}
\item There exists a positive integer $\tilde{n}_1$ such that
$h^{\tilde{n}_1}(\mathcal{B}_{\sigma_i}(x_i)) \subset \mathcal{B}_{\sigma_i}(x_i).
\item For every $2 \leq i \leq k$, there exists a positive integer $n_i$ such that
$h^{\tilde{n}_1}(\mathcal{B}_{\sigma_i}(x_i)) \subset \mathcal{B}_{\sigma_i}(x_i).
\end{enumerate}

Let $\lambda = \min\{|\lambda_i|, 1 \leq i \leq k|$, then, there is $\lambda > 1$. There exists a positive integer $n_0 > \tilde{n}_1 + \sum_{i=1}^{k}(\tilde{n}_i n_i + m_i)$ such that $n_0 - \sum_{i=1}^{k}(n_i n_i + m_i) \equiv 0 \pmod{\mu}$ and $\lambda^{n_0} - \sum_{i=1}^{k} m_i > 1$. Let
\begin{equation}
V_0 = \left(\mathcal{B}_{\sigma_i}(h^{n_0-\tilde{n}_1}(x_i)) \right) \cap \left(\mathcal{B}_{\sigma_i}(x_i)\right),
\end{equation}
\begin{equation}
V_1 = \left(\mathcal{B}_{\sigma_i}(h^{n_0}(x_i)) \right) \cap \left(\mathcal{B}_{\sigma_i}(x_i)\right),
\end{equation}
Then, $V_0$ and $V_1$ are two nonempty bounded closed subsets of $\mathbb{R}^n$ and
\begin{equation}
V_0 \subset \mathcal{B}_{\sigma_i}(h^{n_0-\tilde{n}_1}(x_i)),
V_1 \subset \mathcal{B}_{\sigma_i}(x_i).
\end{equation}
Therefore, $V_0 \cap V_1 = \emptyset$. From definitions of $V_0$ and $V_1$, we have
\begin{equation}
\begin{align*}
h^{n_0}(V_0) &= \mathcal{B}_{\sigma_i}(x_i) \supset V_0 \cup V_1, \\
h^{n_0}(V_1) &= \mathcal{B}_{\sigma_i}(x_i) \supset V_0 \cup V_1.
\end{align*}
\end{equation}
In addition, we have that
\begin{equation}
\begin{align*}
\|h_0^{n_0}(x) - h_0^{n_0}(y)\| > \lambda^{n_0}\|x - y\|, \forall x, y \in V_0, \\
\|h_0^{n_0}(x) - h_0^{n_0}(y)\| > \lambda^{n_0} - \sum_{i=1}^{k} m_i \prod_{i=1}^{k} \mu_i^{m_i} \|x - y\|, \forall x, y \in V_1.
\end{align*}
\end{equation}
We claim that
\begin{equation}
\begin{align*}
\forall (i_0, i_1, \ldots, i_m) \in \Sigma^*_2, \\
\#\left(\bigcap_{s=0}^{\infty} h^{-n_0s}(V_{i_s})\right) \leq 1,
\end{align*}
\end{equation}
where $\#(\cdot)$ denotes the cardinality of a set.
Proceed the proof by contradiction. Suppose $x_1 \neq x_2$ and $x_1, x_2 \in \cap_{n=0}^{\infty} h^{-n}(V_i)$. Let $(i_0, i_1, \ldots, i_n) \in \Sigma^*_i$; then, there exists a subsequence $\{i_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ such that for any $n \in \mathbb{N}, i_n = i_{n+1}$. Without loss of generality, for any $n \in \mathbb{N}, i_n = 0$. By (17), we have
\[
\|h^{n_{i_n}}(x_1) - h^{n_{i_n}}(x_2)\| \geq \lambda^n \cdot \left( \sum_{i=1}^{n_{i_n}} \prod_{j=1}^{k} \mu_{i_j} \right)^{\lambda^{-n}} \cdot \|x_1 - x_2\| > \lambda^n \|x_1 - x_2\| \rightarrow +\infty,
\]
which is contrary to the fact that $h^{n_{i_n}}(x_1)$ and $h^{n_{i_n}}(x_2)$ belong to the bounded set $V_0$. Therefore, there exist an invariant set $\Lambda_0 \subset V_0$ and a positive integer $n_0$ such that $(\Lambda_0, h^n)$ is topologically conjugate to the one-side symbolic system $(\Sigma^*_1, \sigma)$. That is, $h^n$ is chaotic in the sense of Devaney and so is $h$.

From Theorem 3, we obtain the following corollary.

**Corollary 1.** Let $h$ be a continuous differentiable map from $\mathbb{R}^n$ into itself. If $h$ has a heteroclinic repellers $x_1, x_2, \ldots, x_k$ with $k \geq 2$, then for every neighborhood $U_i$ of $x_i$, there exist an invariant subset $\Lambda_i$ of $U_i$ and a positive integer $n_i$ such that $(\Lambda_i, h^n)$ is topologically conjugate to the one-side symbolic system $(\Sigma^*_1, \sigma)$. Therefore, $h^n$ is chaotic in the sense of Devaney and so is $h$.

**Theorem 4.** Let $g, h$ be continuous differentiable maps from $\mathbb{R}^n$ into itself. If $h$ has heteroclinic cycles connecting expanding periodic points and $\|g - h\|_{C^1}$ is sufficiently small, then $g$ also has heteroclinic cycles connecting expanding periodic points.

**Proof.** Let $x_1, x_2, \ldots, x_k$ be $k \geq 1$ different periodic points with periods $n_1, n_2, \ldots, n_k$, respectively. Let $\cup_{j \geq 0} \Gamma_j = \{z(-n)\}_{n \geq 0}$ be heteroclinic cycles connecting periodic points $x_1, x_2, \ldots, x_k$. By Definition 3, there exist
\[
D_{h}(h, z_0) = \frac{\partial h(g, z)}{\partial z}_{g=h, z_0=(z_{-m_0}, x_1, z_1, \ldots, z_{-m_k})} = \begin{bmatrix}
Dh^{n_0}(z_0) - I & 0 & 0 & \cdots & 0 \\
0 & Dh^{n_1}(x_1) - I & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & Dh^{n_k}(x_k) - I
\end{bmatrix}
\]
is also invertible. Here we write
\[
z = (v_k, u_1, v_1, \ldots, u_k) \in V_{k_0} \times B_{r_1}(x_1) \times V_{10} \times B_{r_2}(x_2) \times V_{20} \times \cdots \times V_{(k-1)0} \times B_{r_k}(x_k).\] (23)
(1) $B_k(x_i, (m_j)) \subset V_{g_k} B_k(x_i, (m_j)) \cap B_{r_k}(x_i) = \emptyset$, $\cup_{m=1} B_m^g (B_{r_m}(x_i)) \cap B_{r_m}(x_i) = \emptyset$.

(2) For every $g \in \mathcal{F}(h, \lambda_0)$, $\varphi(g) \approx (\varphi_1(g), \varphi_2(g), \ldots, \varphi_{2k}(g))$ is the unique solution to equations $g^{n_i}(v_{k_j}) = u_1, g^{n_i}(u_i) = u_i, g^{n_i}(v_{k_1}) = u_2, g^{n_i}(u_2) = u_3, \ldots, g^{n_i-1}(v_{k_{j+1}}) = u_{k_j},$ and $g^{n_i}(u_{k_j}) = u_i,$ that is, $g^{n_i}(\varphi_1(g)) = \varphi_2(g), g^{n_i}(\varphi_2(g)) = \varphi_3(g), \ldots, g^{n_i}(\varphi_{2k}(g)) = \varphi_{2k}(g)$.

Therefore, for any integer $j \in [1, k], \varphi_2(g) = \varphi_2(z) \ldots, \varphi_{2k}(g) = \varphi_{2k}(z)$.

4. Examples

In the following, two examples are given which illustrate our results and applications.

Example 1. Consider the one-dimensional map
\[
f(x) = \begin{cases} -4(x(1-x)), & x \in [0, 1], \\ -4(x(1+x)), & x \in [-1, 0]. 
\end{cases}
\]

(29) $f$ has only one fixed point 0, and thus $f$ does not have heteroclinic repellers. Directing calculation, we have

(1) $f$ has three 2-periodic points which are not in the same orbit $x_1 \approx 0.3454915, x_2 = 0.75, x_3 \approx 0.9045084.$

(2) $f^2$ is expanding in $(0.16325, 0.4684), (0.5315, 0.8367), (0.8682, 1).$

(3) $f^2(0.25) = 0.75, f^2(0.6545085) = 0.3454915.$

Therefore, $f$ has heteroclinic cycles connecting expanding periodic points $x_1 \approx 0.3454915$ and $x_2 = 0.75$ (see Figure 1).

Remark 2. Example 1 shows that $f$ does not have heteroclinic repellers but $f$ has heteroclinic cycles connecting expanding periodic points. So, heteroclinic cycles connecting expanding periodic points really contain heteroclinic repellers.

Example 2. Recall the wave equation with a van der Pol boundary condition [3], as follows:
\[
\begin{align*}
\nu_t - \nu_{xx} &= 0, & x &\in (0, 1), t > 0, \\
\nu_x (0, t) &= -\mu \nu_t (0, t), & \eta &\neq 1, \\
\nu_x (1, t) &= \alpha \nu_t (1, t) - \beta \nu_t (1, t), & 0 < \alpha < 1, \beta > 0, t > 0, \\
\nu (x, 0) &= \nu_0 (x), \nu_t (x, 0) = \nu_t (x), & 0 < x < 1.
\end{align*}
\]

(30) Let $G_{\eta}(x) = \frac{\eta + 1}{\eta - 1} x, \eta \neq 1.$

(31) $F_{a}(x) = p(x) + x,$

where $y = p(x)$ is the unique real solution of the cubic equation
\[
\beta y^3 + (1 - \alpha) y + 2x = 0.
\]

(32) Let
\[
u (x, t) = \left( (\nu_x (x, t) + \nu_x (x, t))/2, \nu (x, t) = \left( (\nu_x (x, t) - \nu_x (x, t))/2. \right. \right.
\]

Then, the solution $(\nu, \psi)$ of system (30) is completely characterized by the interval maps $G_{\eta} \circ F_a (\cdot)$ and $F_a \circ G_{\eta} (\cdot),$ between which there is topological conjugacy. If $G_{\eta} \circ F_a (\cdot)$ is chaotic on some invariant interval, we say that the gradient $\omega$ of system (30) is chaotic. Chen, et al. have analyzed the chaos caused by snapback repellers in system (30) (see [3]). For example, when $\alpha = 0.6, \beta = 1$ and $\eta = 0.82, (G_{\eta} \circ F_a )^4 (\cdot)$ has a snapback repeller. It can be checked that $G_{\eta} \circ F_a (\cdot)$ has
heteroclinic cycles connecting expanding periodic points. In particular, it is easier to find the conditions that $G_n\circ F_n(\cdot)$ has heterocyclic cycles.

5. Conclusions

In this paper, a new criterion of chaos is established in $\mathbb{R}^n$. By constructing shift invariant sets, we prove that heterocyclic cycles imply chaos in the sense of Devaney. Heterocyclic cycles connecting expanding periodic points is $C^1$ structural stability.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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