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Research Article

Several Different Types of Convergence for ND Random Variables under Sublinear Expectations

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The goal of this paper is to build average convergence and almost sure convergence for ND (negatively dependent) sequences of random variables under sublinear expectation space. By using the basic definition of sublinear expectation space, Markov inequality, and C_r inequality, we extend average convergence and almost sure convergence theorems for ND sequences of random variables under sublinear expectation space, and we provide a way to learn this subject.

1. Introduction

Classical probability theorems are widely used in many fields, which only hold on some occasions of model certainty. However, there are uncertainties, such as measures of risk, nonlinear stochastic calculus, and statistics in the process of finance. At this time, nonadditive probabilities and nonadditive expectations are useful tools for studying uncertainties and nonlinear stochastic calculus in the process of finance. In order to solve similar problems, Professor Shige Peng [1–3] proposed g-expectation and G-expectation theory in 2008, so that the sublinear expectation space has attracted a lot of scholars' attention.

The limit theorem of nonadditive probability or nonlinear expectation is a challenging question of interest. Under the framework of Peng, many limit theorems are gradually established, such as Zhang [4–8] studied some inequalities under sublinear expectation spaces, some limit theorems for sublinear expectation spaces, and Marcinkiewiczs strong law of large numbers for nonlinear expectations; Bayraktar and Munk [9] acquired an α-stable limit theorem under sublinear expectation; Xu and Zhang [10] achieved three-series theorem for independent random variables under sublinear expectations with applications; Wu and Jiang [11] researched strong law of large numbers and Chover's law of the iterated logarithm under sublinear expectations. In the last two years, the research of the convergence under the sublinear expectation space is still very hot. Ze and Zhou [12] discussed convergence of random variables under sublinear expectations; Gao et al. [13] researched a strong law of large number for negatively dependent and nonidentical distributed random variables in the framework of sublinear expectation; Wu and Lu [14] acquired another form of Chover's law of the iterated logarithm under sublinear expectations; Chen and Zhang [15] studied an elementary proof of Peng's central limit theorem under sublinear expectations.

Complete convergence is a strong convergence, which was first proposed by Hsu and Robbins [16] in 1947. In the classical probability space, complete convergence, almost sure convergence, average convergence, and convergence of probability have been widely used, and many scholars have studied these [17–20]. In the probability space, these convergence problems have been studied more thoroughly. The average convergence and almost sure convergence is needed to be made perfect under sublinear expectation. Because of the expectation subaddition, it has brought us some difficulties in dealing with these problems. We mainly establish the average convergence and almost sure convergence for ND random variables under sublinear expectation and generalize them [21] to the sublinear expectation space.

This paper is divided into four parts. The first part mainly introduces the background of sublinear expectation spaces and some existing research results. The second part mainly introduces definitions and lemmas of sublinear expectation spaces that we need to use. The third part mainly describes theorems and remarks. The fourth part mainly explains the proof process of the theorems.

2. Preliminaries

We use the framework and notions of Peng [2]. Let $(\Omega, \mathcal{F})$ be a given measurable space, and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le c\left(1 + |\mathbf{x}|^m + |\mathbf{y}|^m\right)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n, \tag{1}$$

for some $c > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case, we denote $X \in \mathcal{H}$.

In this paper, we define C as positive constants, and the positive constants represented by different places are different.

Definition 1 (see [2]). A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}}\colon \mathcal{H} \longrightarrow \overline{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have the following:

(a) Monotonicity: if$X \ge Y$, then $\hat{\mathbb{E}}X \ge \hat{\mathbb{E}}Y$

(b) Constant preserving: $\hat{\mathbb{E}}c = c$

(c) Subadditivity: $\hat{\mathbb{E}}(X + Y) \le \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$, whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$

(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X, \lambda \ge 0$

Here, $\overline{\mathbb{R}} = [-\infty, \infty]$, and the triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Given a sublinear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\hat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}X := -\hat{\mathbb{E}}(-X), \quad \forall X \in \mathcal{H}, \hat{\mathbb{E}}f \le \mathbb{V}(A) \le \hat{\mathbb{E}}g, \widehat{\mathcal{E}}f \le \mathcal{V}(A) \le \widehat{\mathcal{E}}g, \quad \text{if } f \le I(A) \le g, f, g \in \mathcal{H}. \tag{2}$$

From the definition, it is easily shown that, for all $X, Y \in \mathcal{H}$,

$$\widehat{\mathcal{E}}X \le \hat{\mathbb{E}}X, \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}X + c, \hat{\mathbb{E}}(X - Y) \ge \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y, \hat{\mathbb{E}}|X - Y| \ge |\hat{\mathbb{E}}X - \hat{\mathbb{E}}Y|. \tag{3}$$

Definition 2 (see [1]). If $\mathcal{G} \subset \mathcal{F}$, for the function $V\colon \mathcal{G} \longrightarrow [0, 1]$, there are the following:

(1) $V(\varnothing) = 0, V(\Omega) = 1$

(2) $V(A) \le V(B)$, $\forall A \subseteq B, A, B \in \mathcal{G}$

Then, call V as the capacity. If for any $A, B \in \mathcal{G}$ and $A \cup B \in \mathcal{G}$, there is $V(A \cup B) \le V(A) + V(B)$, and then it is said that V is subadditive. In the sublinear expected space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, the definition of upper capacity and lower capacity $(\mathbb{V}, \mathcal{V})$ is

$$\begin{aligned} \mathbb{V}(A) &:= \inf\left\{\hat{\mathbb{E}}\xi; I(A) \le \xi, \xi \in \mathcal{H}\right\}, \\ \mathcal{V}(A) &:= 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}, \end{aligned} \tag{4}$$

where $\mathbb{V}(A^c)$ is the complement of A.

$\mathbb{V}$ has subadditivity, and

$$\begin{aligned} \mathcal{V}(A) &\le \mathbb{V}(A), \quad \forall A \in \mathcal{F}, \\ \mathbb{V}(A) &= \hat{\mathbb{E}}(I(A)), \\ \mathcal{V}(A) &= \widehat{\mathcal{E}}(I(A)), \quad I(A) \in \mathcal{H}. \end{aligned} \tag{5}$$

If $f \le I(A) \le g, f, g \in \mathcal{H}$, then

$$\begin{aligned} \hat{\mathbb{E}}f &\le \mathbb{V}(A) \le \hat{\mathbb{E}}g, \\ \widehat{\mathcal{E}}f &\le \mathcal{V}(A) \le \widehat{\mathcal{E}}g, \end{aligned} \tag{6}$$

from this, for $X \in \mathcal{H}$, we can get the Markov inequality

$$\mathbb{V}(|X| \ge x) \le \frac{\hat{\mathbb{E}}(|X|^p)}{x^p}, \quad \forall x > 0, p > 0. \tag{7}$$

In addition to the Markov inequality, the C_r inequality is also used in the following chapters. Let $X_1, X_2, \ldots, X_n \in \mathcal{H}$ be random variables, then

$$\hat{\mathbb{E}}\left|X_1 + X_2 + \cdots + X_n\right|^r \le c_r\left(\hat{\mathbb{E}}\left|X_1\right|^r + \hat{\mathbb{E}}\left|X_2\right|^r + \cdots + \hat{\mathbb{E}}\left|X_n\right|^r\right), \tag{8}$$

and among them,

$$c_r = \begin{cases} 1, & 0 < r \le 1, \\ n^{r-1}, & r > 1. \end{cases} \tag{9}$$

Definition 3. (see [4]).

(1) $\hat{\mathbb{E}}\colon \mathcal{H} \longrightarrow \overline{\mathbb{R}}$, If there is $\forall X, X_n \in \mathcal{H}, X \ge 0, n \ge 1$,

$$\hat{\mathbb{E}}(X) \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}(X_n), \quad X \leq \sum_{n=1}^{\infty} X_n. \tag{10}$$

It is said that $\hat{\mathbb{E}}$ can be added several times.

(2) V: $\mathcal{F} \longrightarrow \mathbb{R}$, for $\forall A_n \in \mathcal{F}$, there is

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n). \tag{11}$$

It is said that V can be added several times.

Under normal circumstances, $\mathbb{V}$ has no countable additivity, so it is necessary to define the external capacity $\mathbb{V}^*$.

Definition 4 (see [4]). For $A \in \mathcal{F}$,

$$\mathbb{V}^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mathbb{V}(A_n): A \subset \bigcup_{n=1}^{\infty} A_n\right\}, \quad \mathcal{V}^*(A) = 1 - \mathbb{V}^*(A^c). \tag{12}$$

From the definition, it is known that $\mathbb{V}^*(A)$ can be added several times, and $\mathbb{V}^* \leq \mathbb{V}$ has the following properties:

(a) If $\mathbb{V}$ can be added several times, then $\mathbb{V}^* = \mathbb{V}$

(b) If there is $g \in \mathcal{H}$ for $I(A) \leq g$, then $\mathbb{V}^*(A) \leq \hat{\mathbb{E}}(g)$; furthermore, if $\hat{\mathbb{E}}$ can be added several times, then it is necessary for $f \leq I(A) \leq g, f, g \in \mathcal{H}$ has $\hat{\mathbb{E}}(f) \leq \mathbb{V}^*(A) \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}(g)$.

(c) When $I(A) \leq g \in \mathcal{H}, \mathbb{V}^*(A) \leq \hat{\mathbb{E}}(g)$, there is $V(A) \leq \mathbb{V}^*(A)$, that is, $\mathbb{V}^*$ is the largest and has a capacity of countable additivity.

Definition 5. (see [5], negative dependence). In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}$ is said to have negative dependence (ND) to another random vector $\mathbf{X} = (X_1, \ldots, X_m), X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if for each pair of test functions $\varphi_1 \in C_{l,\mathrm{Lip}}(\mathbb{R}_m)$ and $\varphi_2 \in C_{l,\mathrm{Lip}}(\mathbb{R}_n)$, we have $\hat{\mathbb{E}}[\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})] \leq \hat{\mathbb{E}}[\varphi_1(\mathbf{X})]\hat{\mathbb{E}}[\varphi_2(\mathbf{Y})]$, whenever either φ_1 and φ_2 are coordinate-wise nondecreasing or φ_1 and φ_2 are coordinate-wise nonincreasing with $\varphi_1(\mathbf{X}) \geq 0, \hat{\mathbb{E}}[\varphi_2(\mathbf{Y})] \geq 0, \hat{\mathbb{E}}[|\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})|] < \infty, \hat{\mathbb{E}}[|\varphi_1(\mathbf{X})|] < \infty, \hat{\mathbb{E}}[|\varphi_2(\mathbf{Y})|] < \infty$.

ND random variables: a sequence of random variables $\{X_n; n \geq 1\}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}}).X_1, X_2, \ldots$ is said to be ND if X_{i+1} is ND to $(X_1, \ldots, X_i)$ for each $i \geq 1$.

Let $\{X_n; n \geq 1\}$ be a sequence of ND random variables and $f_1(x), f_2(x), \ldots \in C_{l,\mathrm{Lip}}(\mathbb{R})$ are nondecreasing (resp., nonincreasing) functions, then $f_n(X_n; n \geq 1)$ is also sequence of ND random variables.

Definition 6. Let $\{X_n; n \geq 1\}$ be a sequence of random variables in $\mathcal{H}$, we have several different types of convergence:

(1) (Ze [12]) A sequence of random variables $\{X_n; n \geq 1\}$ is said to completely converge to $X(p > 0)$ if any $\varepsilon > 0, \sum_{n=1}^{\infty} \mathbb{V}(|X_n - X| > \varepsilon) < \infty$, which is denoted by $X_n \longrightarrow^c .c\ X$.

(2) (Wu [11]) A sequence of random variables $\{X_n; n \geq 1\}$ is said to converge to X almost surely V (a.s. V), denoted by $X_n \longrightarrow X$ a.s. V as $n \longrightarrow \infty$ if $V(X_n \nrightarrow X) = 0$.

V can be replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively. By $\mathcal{V}(A) \leq \mathbb{V}(A)$ and $\mathcal{V}(A) + \mathbb{V}(A^c) = 1$, for any $A \in \mathcal{F}$, it is obvious that $X_n \longrightarrow X$ a.s. $\mathbb{V}$ implies $X_n \longrightarrow X$ a.s. $\mathcal{V}$, but $X_n \longrightarrow X$ a.s. $\mathcal{V}$ does not imply $X_n \longrightarrow X$ a.s. $\mathbb{V}$. Furthermore,

$$\begin{aligned} &X_n \longrightarrow X .\mathrm{s}.\mathbb{V} \Leftrightarrow \mathcal{V}(X_n \longrightarrow X) = 1 \Leftrightarrow \mathbb{V}\left(\left|X_n - X\right| \geq \varepsilon, \mathrm{i.o.}\right) = 0, \quad \text{for } \forall \varepsilon > 0, \\ &X_n \longrightarrow X\ \mathrm{a.s.}\mathcal{V} \Leftrightarrow \mathcal{V}(X_n \nrightarrow X) = 0 \Leftrightarrow \mathcal{V}(X_n \longrightarrow X) = 1. \end{aligned} \tag{13}$$

In the probability space, $P(A) + P(A^c) = 1$ can get $X_n \longrightarrow X$ a.s.$\Leftrightarrow P(X_n \longrightarrow X) = 1 \Leftrightarrow P(X_n \nrightarrow X) = 0$. But $\mathbb{V}(A) + \mathbb{V}(A^c) = 1$ does not necessarily hold under sublinear expectation, that is, to say, $\mathbb{V}(X_n \longrightarrow X) = 1 \not\Leftrightarrow \mathbb{V}(X_n \nrightarrow X) = 0$. We can actually get $\mathbb{V}(X_n \nrightarrow X) = 0 \Rightarrow \mathbb{V}(X_n \longrightarrow X) = 1$; because of $\mathbb{V}(X_n \longrightarrow X) = 1 \not\Rightarrow \mathbb{V}(X_n \nrightarrow X) = 0$, we cannot define $X_n \longrightarrow X$ a.s. with $\mathbb{V}(X_n \longrightarrow X) = 1$.

(3) (Ze [12]) A sequence of random variables $\{X_n; n \geq 1\}$ is said to L^p converge to $X(p > 0)$ if $\lim_{n\longrightarrow\infty} \hat{\mathbb{E}}[|X_n - X|^p] = 0$, which is denoted by $X_n \longrightarrow^{L^p} X$.

(4) (Ze [12]) A sequence of random variables $\{X_n; n \geq 1\}$ is said to converge to X in capacity, if any $\varepsilon > 0, \lim_{n\longrightarrow\infty} V(|X_n - X| > \varepsilon) = 0$, which is denoted by $X_n \longrightarrow^V X$.

By Borel–Cantelli's Lemma, we can get $X_n \longrightarrow^c .c.\ X \Rightarrow X_n \longrightarrow^a .s.\mathbb{V}\ X$.

By Markov inequality, we can obtain $X_n \longrightarrow^{L^p} X \Rightarrow X_n \longrightarrow^V X$.

Lemma 1 (see [4], Borel–Cantelli's lemma). *Let $\{A_n; n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably subadditive capacity. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then*

$$V(A_n\ \mathrm{i.o.}) = 0, \quad \textit{where}\ \{A_n\ \mathrm{i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i. \tag{14}$$

Lemma 2 (see [5], Corollary 2.2; [6], Theorem 1). *Let $\{X_1, \ldots, X_n\}$ be a sequence of random variables in $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}X_k \leq 0$. Suppose that X_{k+1} is negatively dependent to $(X_1, \ldots, X_k)$ for each $k = 1, \ldots, n-1, S_n = X_1 + \cdots + X_n$, $S_0 = 0$. Then,*

For $1 \leq p \leq 2$

$$\hat{\mathbb{E}}\left[(S_n^+)^p \right] \leq 2^{2-p} \sum_{k=1}^{n} \hat{\mathbb{E}}|X_k|^p, \tag{15}$$

for all $x > 0$

$$\mathbb{V}(S_n \geq x) \leq C \frac{\sum_{k=1}^{n} \hat{\mathbb{E}}X_k^2}{x^2}. \tag{16}$$

We know that the indicative function may be discontinuous, $\hat{\mathbb{E}}[X]$ is defined on $C_{l,\text{Lip}}$ under sublinear expectations space, so the constructor $g(x) \in C_{l,\text{Lip}}$ is needed to correct the discontinuity of the indicative function. Now, we define $g(x) \in C_{l,\text{Lip}}$, for $0 < \mu < 1$; let $g(x)$ be a nonincreasing function such that $0 \leq g(x) \leq 1$ for all x and $g(x) = 1$ if $x \leq \mu$, $g(x) = 0$ if $x > 1$. Then,

$$I(|x| \leq \mu) \leq g(|x|) \leq I(|x| \leq 1). \tag{17}$$

3. Main Results

Theorem 1. *Suppose that $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ are positive integer sequences, $\{X_{nk}; a_n \leq k \leq b_n, n \geq 1\}$ is an array of row-wise ND random variables and $\hat{\mathbb{E}}X_{nk} = \hat{\mathscr{E}}X_{nk}$. Let $\{h_n; n \geq 1\}$ and $\{k_n; n \geq 1\}$ be two increasing sequences of positive constants with $k_n \longrightarrow \infty, h_n \longrightarrow \infty$ and $(h_n/k_n) \longrightarrow 0$ as $n \longrightarrow \infty$. For some $1 \leq r < 2$, satisfying*

$$\sup_{n \geq 1} k_n^{-1} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}|X_{nk}|^r < \infty, \tag{18}$$

$$\lim_{n \longrightarrow \infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left(|X_{nk}| - h_n^{(1/r)}\right)\left(1 - g\left(\frac{|X_{nk}|}{h_n^{(1/r)}}\right)\right) = 0, \tag{19}$$

we can get

$$k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(X_{nk} - \hat{\mathbb{E}}X_{nk}\right) \xrightarrow{L^r} 0, \tag{20}$$

that is, to say,

$$\hat{\mathbb{E}}\left| k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(X_{nk} - \hat{\mathbb{E}}X_{nk}\right) \right|^r \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{21}$$

Remark 1. We extend the main conclusions of Shen's [21] article to sublinear expectation spaces because the inequality of Lemma 2 is about ND random variables, and the conclusion of Shen [21] is not extended to END random variables. Condition (19) is different from $\lim_{n\longrightarrow\infty} k_n^{-1} \sum_{i=u_n}^{v_n} E(|X_{ni}| - h_n^{(1/r)})^r I(|X_{ni}|^r > h(n)) = 0$ of Shen's [21] because the indicator function does not necessarily exist in the sublinear expected space, so it needs to be replaced by the $g(\cdot)$ function.

Theorem 2. *Assume that $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ are positive integer sequences, $\{X_{nk}; a_n \leq k \leq b_n, n \geq 1\}$ is an array of row-wise ND random variables, and $\mathbb{V}$ is countably subadditive. Let $\{h_n; n \geq 1\}$ and $\{k_n; n \geq 1\}$ be two increasing sequences of positive constants with $k_n \longrightarrow \infty, h_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\sum_{n=1}^{\infty} (h_n/k_n)^{(2-r/r)} < \infty$. For some $1 \leq r < 2$, satisfying (18) and*

$$\sum_{n=1}^{\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left(|X_{nk}| - h_n^{(1/r)}\right)\left(1 - g\left(\frac{|X_{nk}|}{h_n^{(1/r)}}\right)\right) < \infty, \tag{22}$$

we can have

$$\limsup_{n \longrightarrow \infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(X_{nk} - \hat{\mathbb{E}}X_{nk}\right) \leq 0 \text{ a.s. } \mathbb{V}, \tag{23}$$

$$\liminf_{n \longrightarrow \infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(X_{nk} - \hat{\mathscr{E}}X_{nk}\right) \geq 0 \text{ a.s. } \mathbb{V}. \tag{24}$$

In particular, if $\hat{\mathbb{E}}X_{nk} = \hat{\mathscr{E}}X_{nk}$, then

$$\lim_{n \longrightarrow \infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(X_{nk} - \hat{\mathbb{E}}X_{nk}\right) = 0 \text{ a.s. } \mathbb{V}. \tag{25}$$

Remark 2. The almost sure convergence under the sublinear expectation space is defined by the convergence of the capacity; capacity is divided into upper capacity and lower capacity; the almost sure convergence of the upper capacity can be pushed almost sure converges of the lower capacity; otherwise, it does not hold. We prove that almost sure convergence under sublinear expectation space is the proof of the upper capacity of almost sure convergence. In order to adapt to the sublinear expectation space and better prove Theorem 2, combined with the g function, we change $\lim_{n\longrightarrow\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > \varepsilon) = 0$ of Shen's [21] to formula (22). In the sublinear expectation space, almost sure convergence is different from the probability space. Generally speaking, the limit does not exist. Only under condition $\hat{\mathbb{E}}X_{nk} = \hat{\mathscr{E}}X_{nk}$ can there be a limit. Therefore, our conclusion is divided into three parts, namely, formulas (23)–(25).

4. Proof

Proof of Theorem 1. For convenience, $x \ll y$ denotes that there exists a constant $c > 0$ such that $x \leq cy$ for n sufficiently large. For an array of row-wise ND random variables $\{X_{nk}; a_n \leq k \leq b_n, n \geq 1\}$, to ensure the truncated random variables are also ND, we demand that truncated functions belong to $C_{l,\text{Lip}}$. For all $a_n \leq k \leq b_n, n \geq 1$, we define that

$$\begin{aligned}
Y_{nk} &= -h_n^{(1/r)} I\{X_{nk} < -h_n^{(1/r)}\} + X_{nk} I\{|X_{nk}| \le h_n^{(1/r)}\} + h_n^{(1/r)} I\{X_{nk} > h_n^{(1/r)}\},\\
Z_{nk} &= X_{nk} - Y_{nk} = (X_{nk} + h_n^{(1/r)}) I\{X_{nk} < -h_n^{(1/r)}\} + (X_{nk} - h_n^{(1/r)}) I\{X_{nk} > h_n^{(1/r)}\}.
\end{aligned} \tag{26}$$

By (17), we can easily draw that

$$|Z_{nk}| = |X_{nk} - Y_{nk}| = (|X_{nk}| - h_n^{(1/r)}) I\{|X_{nk}| > h_n^{(1/r)}\} \le (|X_{nk}| - h_n^{(1/r)})\left(1 - g\left(\frac{|X_{nk}|}{h_n^{(1/r)}}\right)\right). \tag{27}$$

We know

$$\left|k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (X_{nk} - \hat{\mathbb{E}}X_{nk})\right| = \left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (X_{nk} - \hat{\mathbb{E}}X_{nk})\right)^+ + \left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (X_{nk} - \hat{\mathbb{E}}X_{nk})\right)^-. \tag{28}$$

Paying attention to $(x + y)^+ \le x^+ + |y|$ for $x, y \in \mathbb{R}$, it is easy to see that

$$\begin{aligned}
&\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (X_{nk} - \hat{\mathbb{E}}X_{nk})\right)^+\right)^r\\
&= \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (Y_{nk} + Z_{nk} - \hat{\mathbb{E}}Y_{nk} + \hat{\mathbb{E}}Y_{nk} - \hat{\mathbb{E}}Z_{nk} + \hat{\mathbb{E}}Z_{nk} - \hat{\mathbb{E}}X_{nk})\right)^+\right)^r\\
&= \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (Y_{nk} - \hat{\mathbb{E}}Y_{nk} + Z_{nk} - \hat{\mathbb{E}}Z_{nk} + \hat{\mathbb{E}}Y_{nk} + \hat{\mathbb{E}}Z_{nk} - \hat{\mathbb{E}}X_{nk})\right)^+\right)^r\\
&\ll \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (Y_{nk} - \hat{\mathbb{E}}Y_{nk} + Z_{nk} - \hat{\mathbb{E}}Z_{nk})\right)^+\right)^r + \hat{\mathbb{E}}\left|k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (\hat{\mathbb{E}}Z_{nk} + \hat{\mathbb{E}}Y_{nk} - \hat{\mathbb{E}}X_{nk})\right|^r\\
&\ll \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (Y_{nk} - \hat{\mathbb{E}}Y_{nk})\right)^+\right)^r + \hat{\mathbb{E}}\left|k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (Z_{nk} - \hat{\mathbb{E}}Z_{nk})\right|^r + \hat{\mathbb{E}}\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left|\hat{\mathbb{E}}Z_{nk} + \hat{\mathbb{E}}Y_{nk} - \hat{\mathbb{E}}X_{nk}\right|\right)^r\\
&=: S_n + W_n + T_n.
\end{aligned} \tag{29}$$

If we want to get (21) in L^r, we first show that $S_n \longrightarrow 0$ as $n \longrightarrow \infty$, according to $1 \le r < 2$, it suffices to show $\hat{\mathbb{E}}((k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (Y_{nk} - \hat{\mathbb{E}}Y_{nk}))^+)^2 \longrightarrow 0$ as $n \longrightarrow \infty$. Noting that $(h_n/k_n) \longrightarrow 0$ as $n \longrightarrow \infty$, $\hat{\mathbb{E}}(Y_{nk} - \hat{\mathbb{E}}Y_{nk})^2 \le 4\hat{\mathbb{E}}Y_{nk}^2$, $|Y_{nk}| \le |X_{nk}|$ and $|Y_{nk}| \le h_n^{(1/r)}$, by (15) and (18),

$$\begin{aligned}\hat{\mathbb{E}}\left(\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Y_{nk}-\hat{\mathbb{E}}Y_{nk}\right)\right)^+\right)^2\right) &\leq 2^{2-2}\cdot k_n^{-(2/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Y_{nk}-\hat{\mathbb{E}}Y_{nk}\right|^2\\ &\ll k_n^{-(2/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}Y_{nk}^2\\ &\leq k_n^{-(2/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Y_{nk}\right|^2\left(\frac{h_n^{(1/r)}}{\left|Y_{nk}\right|}\right)^{2-r}\\ &\leq\left(\frac{h_n}{k_n}\right)^{(2-r/r)}k_n^{-1}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Y_{nk}\right|^r\\ &\leq\left(\frac{h_n}{k_n}\right)^{(2-r/r)}\cdot\sup_{n\geq1}k_n^{-1}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|X_{nk}\right|^r\\ &\longrightarrow 0\,(n\longrightarrow\infty).\end{aligned}\tag{30}$$

Therefore, $S_n \longrightarrow 0$ as $n \longrightarrow \infty$. Next, we will prove $W_n \longrightarrow 0$ as $n \longrightarrow \infty$; similar to (28), using C_r inequality, we have

$$\begin{aligned}W_n &= \hat{\mathbb{E}}\left|k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Z_{nk}-\hat{\mathbb{E}}Z_{nk}\right)\right|^r\\ &\ll \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Z_{nk}-\hat{\mathbb{E}}Z_{nk}\right)\right)^+\right)^r+\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Z_{nk}-\hat{\mathbb{E}}Z_{nk}\right)\right)^-\right)^r\\ &=:W_n'+W_n''.\end{aligned}\tag{31}$$

For W_n', combining (15), (27), and (19), we can obtain

$$\begin{aligned}W_n' &\leq 2^{2-r}\cdot k_n^{-1}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Z_{nk}-\hat{\mathbb{E}}Z_{nk}\right|^r\\ &\ll k_n^{-1}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Z_{nk}\right|^r\leq\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Z_{nk}\right|\right)^r\\ &\leq\left\{k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left(\left|X_{nk}\right|-h_n^{(1/r)}\right)\left(1-g\left(\frac{\left|X_{nk}\right|}{h_n^{(1/r)}}\right)\right)\right\}^r\\ &\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\end{aligned}\tag{32}$$

We know $\{-Z_{nk}; a_n \leq k \leq b_n, n \geq 1\}$ is an array of rowwise ND random variables, using $-Z_{nk}$ instead of Z_{nk} in W_n'. There is

$$\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(-Z_{nk}-\hat{\mathbb{E}}\left(-Z_{nk}\right)\right)\right)^+\right)^r\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\tag{33}$$

For W_n'', it is easy to see $(x+y)^- \leq x^- + |y|$ and $(-x)^- = x^+$ for $x, y \in \mathbb{R}$; according to C_r inequality, (19), (27), and (33), we have

$$\begin{aligned}
W_n'' &= \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Z_{nk}-\hat{\mathcal{E}}Z_{nk}+\hat{\mathcal{E}}Z_{nk}-\hat{\mathbb{E}}Z_{nk}\right)\right)^{-}\right)^{r}\\
&\ll \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Z_{nk}-\hat{\mathcal{E}}Z_{nk}\right)\right)^{-}\right)^{r}+\hat{\mathbb{E}}\left|k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(\hat{\mathcal{E}}Z_{nk}-\hat{\mathbb{E}}Z_{nk}\right)\right|^{r}\\
&\le \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Z_{nk}-\hat{\mathcal{E}}Z_{nk}\right)\right)^{-}\right)^{r}+\hat{\mathbb{E}}\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left|\hat{\mathcal{E}}Z_{nk}-\hat{\mathbb{E}}Z_{nk}\right|\right)^{r}\\
&\le \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(Z_{nk}+\hat{\mathbb{E}}(-Z_{nk})\right)\right)^{-}\right)^{r}+\hat{\mathbb{E}}\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Z_{nk}\right|\right)^{r}\\
&\le \hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(-Z_{nk}-\hat{\mathbb{E}}(-Z_{nk})\right)\right)^{+}\right)^{r}+\left\{k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left(\left|X_{nk}\right|-h_n^{(1/r)}\right)\left(1-g\left(\frac{\left|X_{nk}\right|}{h_n^{(1/r)}}\right)\right)\right\}^{r}\\
&\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.
\end{aligned}\tag{34}$$

For T_n, note that $|\hat{\mathbb{E}}X-\hat{\mathbb{E}}Y|\le\hat{\mathbb{E}}|X-Y|$, (19), and (27). We conclude that

$$T_n\le\hat{\mathbb{E}}\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(\hat{\mathbb{E}}\left|Z_{nk}\right|+\hat{\mathbb{E}}\left|Y_{nk}-X_{nk}\right|\right)\right)^{r}\ll\hat{\mathbb{E}}\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left|Z_{nk}\right|\right)^{r},\tag{35}$$

$$\le\left\{k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\hat{\mathbb{E}}\left(\left|X_{nk}\right|-h_n^{(1/r)}\right)\left(1-g\left(\frac{\left|X_{nk}\right|}{h_n^{(1/r)}}\right)\right)\right\}^{r}\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\tag{36}$$

From (30), (32), (34), and (35), we can easily get $S_n\longrightarrow 0$, $W_n\longrightarrow 0$ and $T_n\longrightarrow 0$ as $n\longrightarrow\infty$ in L^r, so

$$\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(X_{nk}-\hat{\mathbb{E}}X_{nk}\right)\right)^{+}\right)^{r}\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\tag{37}$$

Finally, we have to prove $\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}(X_{nk}-\hat{\mathbb{E}}X_{nk})\right)^{-}\right)^{r}\longrightarrow 0$ as $n\longrightarrow\infty$. It shall be noted that $\{-X_{nk};a_n\le k\le b_n,n\ge 1\}$ is an array of rowwise ND random variables, using $-X_{nk}$ instead of X_{nk} in (37), so we can get

$$\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(-X_{nk}-\hat{\mathbb{E}}(-X_{nk})\right)\right)^{+}\right)^{r}\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\tag{38}$$

According to $\hat{\mathbb{E}}X_{nk}=-\hat{\mathcal{E}}(-X_{nk})$ and condition of $\hat{\mathbb{E}}X_{nk}=\hat{\mathcal{E}}X_{nk}$,

$$\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(X_{nk}-\hat{\mathbb{E}}X_{nk}\right)\right)^{-}\right)^{r}=\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}\left(-X_{nk}-\hat{\mathbb{E}}(-X_{nk})\right)\right)^{+}\right)^{r}\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty.\tag{39}$$

So, we have $\hat{\mathbb{E}}\left(\left(k_n^{-(1/r)}\sum_{k=a_n}^{b_n}(X_{nk}-\hat{\mathbb{E}}X_{nk})\right)^{-}\right)^{r}\longrightarrow 0$, as $n\longrightarrow\infty$. Combining (28), we can get (21). In other words, Theorem 1 is proved. □

Proof of Theorem 2. In the process of Theorem 2, we still use the mark of Theorem 1. In order to prove the establishment of (23), we need to show

$$I_1 = \limsup_{n\longrightarrow\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(Y_{nk} - \hat{\mathbb{E}}Y_{nk}\right) \le 0 \text{ a.s. } \mathbb{V},$$

$$I_2 = \limsup_{n\longrightarrow\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} Z_{nk} \le 0 \text{ a.s. } \mathbb{V}, \quad (40)$$

$$I_3 = \lim_{n\longrightarrow\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(\hat{\mathbb{E}}Y_{nk} - \hat{\mathbb{E}}X_{nk}\right) = 0.$$

We consider I_1, using Markov inequality and (16). Noting that $|Y_{nk}| \le |X_{nk}|, |Y_{nk}| \le h_n^{(1/r)}$, $\sum_{n=1}^{\infty} (h_n/k_n)^{(2-r/r)} < \infty, \hat{\mathbb{E}}(Y_{nk} - \hat{\mathbb{E}}Y_{nk})^2 \le 4\hat{\mathbb{E}}Y_{nk}^2$ and (18), for some $1 \le r < 2$, any $\delta > 0$, we have

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{k=a_n}^{b_n} \left(Y_{nk} - \hat{\mathbb{E}}Y_{nk}\right) > \delta k_n^{(1/r)}\right) \\ &\ll \sum_{n=1}^{\infty} \frac{\sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left(Y_{nk} - \hat{\mathbb{E}}Y_{nk}\right)^2}{\delta^2 k_n^{(2/r)}} \\ &\ll \sum_{n=1}^{\infty} k_n^{-(2/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}Y_{nk}^2 \\ &\le \sum_{n=1}^{\infty} k_n^{-(2/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left|Y_{nk}\right|^2 \left(\frac{h_n^{(1/r)}}{\left|Y_{nk}\right|}\right)^{2-r} \\ &\le \sum_{n=1}^{\infty} \left(\frac{h_n}{k_n}\right)^{(2-r/r)} \cdot \sup_{n\ge1} k_n^{-1} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left|X_{nk}\right|^r < \infty. \end{aligned} \quad (41)$$

We know $\mathbb{V}$ is countably subadditive, combining Lemma 1 (Borel–Cantellis Lemma). Let $\delta \longrightarrow 0$, we have $\limsup_{n\longrightarrow\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} (Y_{nk} - \hat{\mathbb{E}}Y_{nk}) \le 0$, a.s. $\mathbb{V}$. Now, it suffices to verify that

$$I_2 = \sum_{n=1}^{\infty} \mathbb{V}\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(Z_{nk} - \hat{\mathbb{E}}Z_{nk}\right) > \delta - k_n^{-((1/r))} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}Z_{nk}\right) < \infty. \quad (42)$$

By (22) and (27), we conclude that

$$\begin{aligned} \lim_{n\longrightarrow\infty} \left|k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}Z_{nk}\right| &\le \lim_{n\longrightarrow\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left|Z_{nk}\right| \\ &\le \lim_{n\longrightarrow\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left(\left|X_{nk}\right| - h_n^{(1/r)}\right) \\ &\quad \cdot \left(1 - g\left(\frac{\left|X_{nk}\right|}{h_n^{(1/r)}}\right)\right) \\ &= 0. \end{aligned} \quad (43)$$

Hence, there exists $k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}Z_{nk} < (\delta/2)$, we use Markov inequality, (15) of Lemma 2 and C_r inequality because of (22) and (27); then, for any $\delta > 0$,

$$\begin{aligned} I_2 &\le \sum_{n=1}^{\infty} \mathbb{V}\left(k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \left(Z_{nk} - \hat{\mathbb{E}}Z_{nk}\right) > \frac{\delta}{2}\right) \\ &\le 2\delta^{-1} \sum_{n=1}^{\infty} k_n^{-(1/r)} \hat{\mathbb{E}}\left(\sum_{k=a_n}^{b_n} \left(Z_{nk} - \hat{\mathbb{E}}Z_{nk}\right)\right) \\ &\ll \sum_{n=1}^{\infty} k_n^{-(1/r)} \hat{\mathbb{E}}\left(\left(\sum_{k=a_n}^{b_n} \left(Z_{nk} - \hat{\mathbb{E}}Z_{nk}\right)\right)^+\right) \\ &\le 2^{2-1} \cdot \sum_{n=1}^{\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left|Z_{nk} - \hat{\mathbb{E}}Z_{nk}\right| \\ &\ll \sum_{n=1}^{\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left|Z_{nk}\right| \\ &\le \sum_{n=1}^{\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} \hat{\mathbb{E}}\left(\left|X_{nk}\right| - h_n^{(1/r)}\right)\left(1 - g\left(\frac{\left|X_{nk}\right|}{h_n^{(1/r)}}\right)\right) \\ &< \infty. \end{aligned} \quad (44)$$

We know $\mathbb{V}$ is countably subadditive and arbitrary of δ, which together with Lemma 1 (Borel–Cantellis Lemma) implies $\limsup_{n\longrightarrow\infty} k_n^{-(1/r)} \sum_{k=a_n}^{b_n} Z_{nk} \le 0$, a.s. $\mathbb{V}$.

Finally, we prove $\lim_{n\longrightarrow\infty} I_3 = 0$. Similarly, combining (43), we obtain $\lim_{n\longrightarrow\infty} I_3 = 0$. Then, we obtain (23). We use $-X_{nk}$ instead of X_{nk} in (23), so we can get (24). When $\hat{\mathbb{E}}X_{nk} = \hat{\mathscr{E}}X_{nk}$, there is (25), that is, Theorem 2 is proved. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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