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Research Article

A Study on $A-I-\Gamma$-Hyperideals and $(m,n)-\Gamma$-Hyperfilters in Ordered Γ-Semihypergroups

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The concept of almost interior Γ-hyperideals ($A-\mathbf{I}-\Gamma$-hyperideals) in ordered Γ-semihypergroups is a generalization of the concept of interior Γ-hyperideals ($\mathbf{I}-\Gamma$-hyperideals). In this study, the connections between $\mathbf{I}-\Gamma$-hyperideals and $A-\mathbf{I}-\Gamma$-hyperideals in ordered Γ-semihypergroups were presented. Also, we define the notion of (m,n)-Γ-hyperfilters of ordered Γ-semihypergroups and provide useful results on it. Moreover, we use the weak pseudoorders to construct quotient ordered Γ-semihypergroups by using ordered regular equivalence relations. These concepts lead us to a new research direction in ordered Γ-semihypergroups.

1. Introduction

The concept of hyperstructure was first introduced in 1934 by Marty [1] at the 8th Congress of Scandinavian Mathematicians. Moreover, the hyperrings were introduced after the hyperfields, by Krasner [2], for solving some problems in valuation of fields. From the early attempts [1, 2] to recent studies, hyperstructure theory has been used in diverse branches of mathematics [3, 4], physics [5, 6], chemistry [7], biology [8, 9], etc. In [5], Davvaz et al. showed that the leptons and gauge bosons along with the interactions between their members construct a weak algebraic hyperstructure (H_v-structure). Some applications of hyperstructure theory in mathematics, cryptography, codes, and other fields can be found in [10]. In [3], Jun studied algebraic and geometric aspects of Krasner hyperrings in detail. In 2018, Omidi and Davvaz [11] introduced and studied the notion of ordered regular equivalence relations in ordered semihyperrings.

Semihypergroups are as important in algebraic hyperstructures as the semigroups in algebraic structures. Recently, Daengsaen et al. [12] studied minimal and maximal hyperideals of n-ary semihypergroups. Ordered semihypergroups were introduced by Heidari and Davvaz [13] as a generalization of the ordered semigroups. In 2015, Davvaz et al. [14] presented a connection between ordered semihypergroups and ordered semigroups by using pseudoorder. In 2016, Gu and Tang [15] studied an open problem for ordered semihypergroups and established a partial solution. Next, Tang et al. [16] studied the further properties of ordered regular equivalence relations in ordered semihypergroups. Moreover, the authors gave a complete answer of the open problem given by Davvaz et al. in [14].

As a generalization of a semihypergroup, Anvariyeh et al. [17], in 2010, introduced the notion of Γ-semihypergroups and discussed the Γ-hyperideals of Γ-semihypergroups. In [18], the concept of (fuzzy) Γ-hyperideals of involution Γ-semihypergroups was investigated. Recently, fuzzy set theory has been well developed in the framework of ordered Γ-semihypergroup theory [19]. In 2020, Rao et al. [20] introduced and studied the concept of relative bi-(int-)Γ-hyperideals in ordered Γ-semihypergroups. The beauty of relative bi-(int-)Γ-hyperideals is that instead of involving all the elements of the ordered Γ-semihypergroup S, we deal with the elements belonging to the nonempty subsets of S. The concept of A-ideals of semigroups was introduced by Grosek and Satko [21] and was further investigated in [22]. In [23], Suebsung et al. studied A-hyperideals in semihypergroups.

Hyperfilter theory in ordered hyperstructures has been investigated by many mathematicians. In 2015, Tang et al. [24] studied (fuzzy) hyperfilters in ordered semihypergroups. In 2018, Omidi et al. [25] applied the hyperfilter theory to ordered Γ-semihypergroups and obtained some results in this respect. In 2019, Bouaziz and Yaqoob [26] investigated some properties of rough hyperfilters in po-LA-semihypergroups. The notion of (m,n)-hyperideals was introduced in various ordered hyperstructures and had been studied by many authors, for instance, Mahboob et al. [27], Omidi and Davvaz [28], and many others.

Motivated by the work of Kaopusek et al. [29] and Mahboob and Khan [30], we applied the concepts of almost interior ideals and (m,n)-hyperfilters to ordered Γ-semihypergroups. In Section 2, we give the preliminary concepts concerning ordered Γ-semihypergroups and $\mathbf{I}-\Gamma$-hyperideals. In Section 3, the notion of almost interior Γ-hyperideals (briefly, $A-\mathbf{I}-\Gamma$-hyperideal) is given. In addition, some properties of $(m,n)-\Gamma$-hyperfilters in ordered Γ-semihypergroups have been proven. Finally, we conclude in Section 4 with some arising examples. Theory of $A-\mathbf{I}-\Gamma$-hyperideals and $(m,n)-\Gamma$-hyperfilters is useful to explore new results associated with ordered Γ-semihypergroups.

2. Preliminaries

Let S be a nonempty set and $P^*(S)$ be the family of all nonempty subsets of S. A mapping $\gamma\colon S\times S\longrightarrow P^*(S)$ is called a hyperoperation on S. A hypergroupoid is a set S together with a (binary) hyperoperation. If A and B are two nonempty subsets of S and $x\in S$, then we denote

$$\begin{aligned}A\gamma B &= \bigcup_{\substack{a\in A\\ b\in B}} a\gamma b,\\ x\gamma A &= \{x\}\gamma A,\\ B\gamma x &= B\gamma\{x\}.\end{aligned}\tag{1}$$

A hypergroupoid (S,γ) is called a semihypergroup if for every a,b,c in S,

$$a\gamma(b\gamma c)=(a\gamma b)\gamma c.\tag{2}$$

The notion of Γ-semihypergroups has been introduced by Anvariyeh et al. [17] in 2010. In this section, we present some notions related to (ordered) Γ-semihypergroup theory.

Definition 1 (see [17]). Let S and Γ be two nonempty sets. Then, S is said to be a Γ-semihypergroup if every $\gamma\in\Gamma$ is a hyperoperation on S, i.e., $x\gamma y\subseteq S$ for every $x,y\in S$, and for every $\alpha,\beta\in\Gamma$ and $x,y,z\in S$, we have

$$x\alpha(y\beta z)=(x\alpha y)\beta z.\tag{3}$$

Let A and B be two nonempty subsets of S. We define

$$A\gamma B=\cup\{a\gamma b\mid a\in A, b\in B\}.\tag{4}$$

Also,

$$A\Gamma B=\cup\{a\gamma b\mid a\in A, b\in B\text{ and }\gamma\in\Gamma\}=\bigcup_{\gamma\in\Gamma}A\gamma B.\tag{5}$$

If every $\gamma\in\Gamma$ is an operation, then S is a Γ-semigroup. A nonempty subset A of S is called a sub Γ-semihypergroup of S if $x\beta y\subseteq A$ for every $x,y\in A$ and $\beta\in\Gamma$.

Definition 2 (see [25]). Let $\leq$ be an ordered relation on a nonempty set S. By an ordered Γ-semihypergroup we mean an algebraic hyperstructure $(S,\Gamma,\leq)$ in which the following are satisfied:

(1) (S,Γ) is a Γ-semihypergroup.
(2) $(S,\leq)$ is a (partially) ordered set.
(3) For any $a,x,y\in S$ and $\beta\in\Gamma$, $x\leq y$ implies $a\beta x\preceq a\beta y$ and $x\beta a\preceq y\beta a$. If U and V are nonempty subsets of S, then

$$U\preceq V\Longleftrightarrow\forall u\in U,\quad\exists v\in V; u\leq v.\tag{6}$$

Let σ be an equivalence relation on an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. If U and V are nonempty subsets of S, then

(1) $U\overrightarrow{\sigma}V\Longleftrightarrow\forall u\in U, \exists v\in V; u\sigma v$.
(2) $U\overleftarrow{\sigma}V\Longleftrightarrow\forall v'\in V, \exists u'\in U; u'\sigma v'$.
(3) $U\tilde{\sigma}V\Longleftrightarrow\forall u\in U, \exists v\in V; u\sigma v$ and $v\sigma u$ & $\forall\ v'\in V$, $\exists u'\in U; v'\sigma u'$ and $u'\sigma v'$.
(4) $U\overline{\sigma}V\Longleftrightarrow U\overrightarrow{\sigma}V$ and $U\overleftarrow{\sigma}V$.
(5) $U\overline{\overline{\sigma}}V\Longleftrightarrow\forall u\in U, \forall v\in V, u\sigma v$.

A relation σ on an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called a pseudoorder [25] on S if (1) $\leq\subseteq\sigma$; (2) $a\sigma b$ and $b\sigma c$ imply $a\sigma c$ for all $a,b,c\in S$; (3) $a\sigma b$ implies $a\gamma c\overline{\overline{\sigma}}b\gamma c$ and $c\gamma a\overline{\overline{\sigma}}c\gamma b$ for all $a,b,c\in S$ and $\gamma\in\Gamma$.

Theorem 1 (see [25]). *Let $(S,\Gamma,\leq)$ be an ordered Γ-semihypergroup and σ a pseudoorder on S. Then, there exists a strongly regular equivalence relation $\sigma^*=\{(a,b)\in S\times S|\ a\sigma b\text{ and }b\sigma a\}$ on S such that $((S/\sigma^*),\Gamma_{\sigma^*},\preceq_{\sigma^*})$ is an ordered Γ_{σ^*}-semigroup, where $\preceq_{\sigma^*}:=\{(\sigma^*(x),\ \sigma^*(y))\in (S/\sigma^*)\times(S/\sigma^*)|\ \exists a\in\sigma^*(x),\exists b\in\sigma^*(y)\text{ such that }(a,b)\in\sigma\}$.*

Let H be a nonempty subset of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. $(H]$ is defined as follows:

$$(H]:=\{x\in S|x\leq h\text{ for some }h\in H\}.\tag{7}$$

For convenience, given $h\in H$, we write $(\{h\}]=(h]$. Let A and B be nonempty subsets of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. Then,

(1) $A\subseteq(A]$.
(2) If $A\subseteq B$, then $(A]\subseteq(B]$.
(3) $(A]\Gamma(B]\subseteq(A\Gamma B]$ and $((A]\Gamma(B]]=(A\Gamma B]$.

(4) $((A]] = (A]$.

Let S be an ordered Γ-semihypergroup. By a sub Γ-semihypergroup of S we mean a nonempty subset A of S such that $(A\Gamma A] \subseteq (A]$. We denote

$$A^n = \underbrace{A\Gamma A\Gamma \cdots \Gamma A}_{n-\text{copies}}. \tag{8}$$

The interior Γ-hyperideal (in short $\mathbf{I}-\Gamma$-hyperideal) is defined as follows.

Definition 3 (see [19]). Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A sub Γ-semihypergroup A of S is called an interior Γ-hyperideal (in short $\mathbf{I}-\Gamma$-hyperideal) of S if

(1) $S\Gamma A\Gamma S \subseteq A$.

(2) $(A] \subseteq A$.

An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called regular [31] if for every $a \in S$ there exist $x \in S$, $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. This is equivalent to saying that $a \in (a\Gamma S\Gamma a]$, for every $a \in S$ or $A \subseteq (A\Gamma S\Gamma A]$, for every $A \subseteq S$.

3. $A-\mathbf{I}-\Gamma$-Hyperideals and $(m,n)-\Gamma$-Hyperfilters

Throughout the rest of this paper: S will be an ordered Γ-semihypergroup. We begin this section with the definition of an almost interior Γ-hyperideal (in short $A-\mathbf{I}-\Gamma$-hyperideal) on an ordered Γ-semihypergroup $(S, \Gamma, \leq)$.

Definition 4. Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A nonempty subset K of S is called an $A-\mathbf{I}-\Gamma$-hyperideal of S if

(1) $(x\Gamma K\Gamma y] \cap K \neq \varnothing$ for any $x, y \in S$.

(2) $(K] \subseteq K$.

Example 1. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma, \beta\}$. Define hyperoperations γ and β on S by the following tables:

γ	a	b	c	d
a	a	$\{b,d\}$	c	d
b	$\{b,d\}$	b	$\{b,d\}$	d
c	c	$\{b,d\}$	a	d
d	d	d	d	d

,

β	a	b	c	d
a	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	d
b	$\{b,d\}$	b	$\{b,d\}$	d
c	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	d
d	d	d	d	d

. (9)

Then, (S, Γ) is a Γ-semihypergroup [32]. We have that $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup where the (partial) order relation $\leq$ is defined by

$$\leq \; := \{(a,a), (b,b), (c,c), (d,a), (d,c), (d,d)\}. \tag{10}$$

We give the covering relation $\prec$ as

$$\prec \; = \{(d,a), (d,c)\}. \tag{11}$$

The Hasse diagram of S is shown in Figure 1.

Here, $I = \{a, c, d\}$ is an $A-\mathbf{I}-\Gamma$-hyperideal of S.

Lemma 1. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, every $\mathbf{I}-\Gamma$-hyperideal of S is an $A-\mathbf{I}-\Gamma$-hyperideal of S.*

Proof. Let K be an $\mathbf{I}-\Gamma$-hyperideal of S. Then, $S\Gamma K\Gamma S \subseteq K$. If $a, b \in S$, then

$$a\Gamma K\Gamma b \subseteq S\Gamma K\Gamma S \subseteq K. \tag{12}$$

It implies that $(a\Gamma K\Gamma b] \subseteq (K] \subseteq K$. So, $(a\Gamma K\Gamma b] \cap K \neq \varnothing$ for any $a, b \in S$. Therefore, K is an $A-\mathbf{I}-\Gamma$-hyperideal of S. □

In Section 4, we show that the converse of Lemma 1 is not true in general, i.e., an $A-\mathbf{I}-\Gamma$-hyperideal may not be an $\mathbf{I}-\Gamma$-hyperideal of S.

Lemma 2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. If K_1 and K_2 are $A-\mathbf{I}-\Gamma$-hyperideals of S, then $K_1 \cup K_2$ is an A-I-Gamma-hyperideal of S.*

Proof. Clearly, $K_1 \subseteq K_1 \cup K_2$. If $a, b \in S$, then

$$a\Gamma K_1\Gamma b \subseteq a\Gamma(K_1 \cup K_2)\Gamma b. \tag{13}$$

So, $(a\Gamma K_1\Gamma b] \subseteq (a\Gamma(K_1 \cup K_2)\Gamma b)$. It implies that

$$(a\Gamma K_1\Gamma b] \cap K_1 \subseteq (a\Gamma(K_1 \cup K_2)\Gamma b] \cap (K_1 \cup K_2). \tag{14}$$

Since K_1 is an $A-\mathbf{I}-\Gamma$-hyperideal of S, it follows that $(a\Gamma K_1\Gamma b] \cap K_1 \neq \varnothing$. This means that $(a\Gamma(K_1 \cup K_2)\Gamma b] \cap (K_1 \cup K_2) \neq \varnothing$. Therefore, $K_1 \cup K_2$ is an $A-\mathbf{I}-\Gamma$-hyperideal of S.

Theorem 2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup and $|S| > 1$. Then, the following assertions are equivalent:*

(1) S has no proper $A-\mathbf{I}-\Gamma$-hyperideal.

(2) For any $x \in S$, there exists $a_x \in S$ such that $(a_x\Gamma(S\backslash\{x\})\Gamma a_x] = \{x\}$.

Proof

(1) $\Rightarrow$ (2): Assume that (1) holds. Let $x \in S$. Then, $S\backslash\{x\}$ is not a $A-\mathbf{I}-\Gamma$-hyperideal. So, there exists $a_x \in S$ such that $(a_x\Gamma(S\backslash\{x\})\Gamma a_x] \cap (S\backslash\{x\}) = \varnothing$. It implies that $(a_x\Gamma(S\backslash\{x\})\Gamma a_x] = \{x\}$.

(2) $\Rightarrow$ (1): Suppose that for any $x \in S$, there exists $a_x \in S$ such that $(a_x\Gamma(S\backslash\{x\})\Gamma a_x] = \{x\}$. This gives

$$(a_x\Gamma(S\backslash\{x\})\Gamma a_x] \cap (S\backslash\{x\}) = \{x\} \cap (S\backslash\{x\}) = \varnothing. \tag{15}$$

Therefore, $S\backslash\{x\}$ is not an $A-\mathbf{I}-\Gamma$-hyperideal for every $x \in S$. Now, let K be a proper $A-\mathbf{I}-\Gamma$-hyperideal of S.

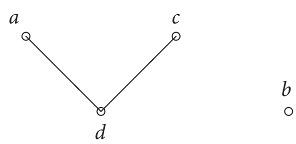

Figure 1: Figure of S for Example 1.

Then, $K \subseteq S\backslash\{x\}$ for some $x \in S$. Thus, $(x\Gamma K\Gamma x] \subseteq (x\Gamma(S\backslash\{x\})\Gamma x)$, and hence $(x\Gamma K\Gamma x) \cap K \subseteq (x\Gamma(S\backslash\{x\})\ \Gamma x] \cap (S\backslash\{x\})$. Since K is an $A - \mathbf{I} - \Gamma$-hyperideal of S, it follows that $\emptyset \neq (x\Gamma K\Gamma x] \cap K$. So, $\emptyset \neq (x\Gamma(S\backslash\{x\})\ \Gamma x] \cap (S\backslash\{x\})$. This means that $S\backslash\{x\}$ is an $A - \mathbf{I} - \Gamma$-hyperideal of S, which is a contradiction. This completes the proof.

In the following, as a continuation of the previous study [25], the authors discuss $(m,n) - \Gamma$-hyperfilter in ordered Γ-semihypergroups. The $(m,n) - \Gamma$-hyperfilter of ordered Γ-semihypergroups is defined as follows.

Definition 5. A sub Γ-semihypergroup F of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called a left $m - \Gamma$-hyperfilter (resp. right $n - \Gamma$-hyperfilter) of S if

(1) For all $a, b \in S$ and $\gamma \in \Gamma$, $a\gamma b \cap F \neq \emptyset \Rightarrow a^m \subseteq F$ (resp. $b^n \subseteq F$).

(2) For all $a \in F$ and $c \in S$, $a \leq c \Rightarrow c \in F$.

Here, m, n are positive integers. Note that if F is both a left $m - \Gamma$-hyperfilter and a right $n - \Gamma$-hyperfilter of S, then F is called an $(m,n) - \Gamma$-hyperfilter of S.

Let K be a sub Γ-semihypergroup of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. We define

$$[K) \coloneqq \{x \in S | x \geq k \text{ for some } k \in K\}. \tag{16}$$

For $K = \{k\}$, we write $[k)$ instead of $[\{k\})$. Note that condition (2) in Definition 5 is equivalent to $[F) \subseteq F$.

The concept of an $(m,n) - \Gamma$-hyperfilter is a generalization of the concept of a Γ-hyperfilter of S. An $(m,n) - \Gamma$-hyperfilter of S is a $(k,l) - \Gamma$-hyperfilter of S for all $k \geq m$ and $l \geq n$. For $m = 1 = n$, F is a Γ-hyperfilter of S. The following example shows that any $(m,n) - \Gamma$-hyperfilter of an ordered Γ-semihypergroup S need not always be a Γ-hyperfilter of S. Also, see Example 4 in Section 4.

Example 2. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$. We define

γ	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a,b\}$	a
d	a	a	$\{a,b\}$	$\{a,b\}$

(17)

Then, S is a Γ-semihypergroup. We have that $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by

$$\leq \coloneqq \{(a,a), (a,b), (a,c), (b,b), (c,c), (d,d)\}. \tag{18}$$

We give the covering relation $\prec$ as

$$\prec = \{(a,b), (a,c)\}. \tag{19}$$

The figure of S is shown in Figure 2.

Let $\mathscr{F} = \{a, b, c\}$. Clearly, $\mathscr{F}$ is an $(m,n) - \Gamma$-hyperfilter of S for $m, n \geq 2$. But $\mathscr{F}$ is not a Γ-hyperfilter of S. Indeed: since $d\gamma x = \{a\}$ or $\{a,b\} \subseteq \mathscr{F}$, $x\gamma d = \{a\}$ or $\{a,b\} \subseteq \mathscr{F}$ for any $x \in S$, $d^m \subseteq \mathscr{F}$ for $m \geq 2$ and $d \notin \mathscr{F}$, we have that $\mathscr{F}$ is an $(m,n) - \Gamma$-hyperfilter of S for $m, n \geq 2$ but not a Γ-hyperfilter of S.

Lemma 3. *Let $\{F_\lambda | \lambda \in \Lambda\}$ be a family of $(m,n) - \Gamma$-hyperfilters of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If $\cap_{\lambda\in\Lambda} F_\lambda \neq \emptyset$, then $\cap_{\lambda\in\Lambda} F_\lambda$ is an $(m,n) - \Gamma$-hyperfilter of S.*

Proof. Let $a, b \in \cap_{\lambda\in\Lambda} F_\lambda$. Then, $a, b \in F_\lambda$ for each $\lambda \in \Lambda$. Since F_λ is a sub Γ-semihypergroup of S for each $\lambda \in \Lambda$, it follows that $a\gamma b \subseteq \cap_{\lambda\in\Lambda} F_k$ for all $\gamma \in \Gamma$. Hence, $\cap_{\lambda\in\Lambda} F_\lambda$ is a sub Γ-semihypergroup of S. Now, let $a, b \in S$, $\gamma \in \Gamma$ and $a\gamma b \cap (\cap_{\lambda\in\Lambda} F_\lambda) \neq \emptyset$. Then, there exists $x \in \cap_{\lambda\in\Lambda} F_\lambda$ for some $x \in a\gamma b$. Since $x \in \cap_{\lambda\in\Lambda} F_\lambda$, it follows that $x \in F_\lambda$ for each $\lambda \in \Lambda$. Since F_λ is an $(m,n) - \Gamma$-hyperfilter of S for each $\lambda \in \Lambda$, we get $x^m, y^n \subseteq F_\lambda$ for each $\lambda \in \Lambda$. It implies that $x^m, y^n \subseteq \cap_{\lambda\in\Lambda} F_\lambda$. Now, let $a \in \cap_{\lambda\in\Lambda} F_\lambda$ and $a \leq c \in S$. Then, $a \in F_\lambda$ for each $\lambda \in \Lambda$. Since F_λ is an $(m,n) - \Gamma$-hyperfilter of S for all $\lambda \in \Lambda$, it follows that $c \in F_\lambda$ for all $\lambda \in \Lambda$. So, $c \in \cap_{\lambda\in\Lambda} F_\lambda$. Therefore, $\cap_{\lambda\in\Lambda} F_\lambda$ is an $(m,n) - \Gamma$-hyperfilter of S.

One of the distinguished properties of the $(m,n) - \Gamma$-hyperfilters is that their union is not an $(m,n) - \Gamma$-hyperfilter in general, for example, see [25, 30].

Let $(S_i, \Gamma_i, \leq_i)$ be an ordered Γ_i-semihypergroup for all $i \in \Omega$. Define $\odot\colon (\prod_{i\in I} S_i) \times (\prod_{i\in I} \Gamma_i) \times (\prod_{i\in I} S_i) \longrightarrow \mathscr{P}^*(\prod_{i\in I} S_i)$ by

$$(a_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (b_i)_{i\in\Omega} = \{(c_i)_{i\in\Omega} | c_i \in a_i\gamma_i b_i\}, \tag{20}$$

for all $(a_i)_{i\in\Omega}, (b_i)_{i\in\Omega} \in \prod_{i\in\Omega} S_i$, and $(\gamma_i)_{i\in\Omega} \in \prod_{i\in\Omega} \Gamma_i$. Now, we put $(a_i)_{i\in\Omega} \leq (b_i)_{i\in\Omega}$ if and only if, for all $i \in \Omega$, $a_i \leq_i b_i$.

Then, $(\prod_{i\in\Omega} S_i = \{(a_i)_{i\in\Omega} | a_i \in S_i\}, \prod_{i\in\Omega} \Gamma_i, \leq)$ is an ordered $\prod_{i\in\Omega} \Gamma_i$-semihypergroup [33]. In the following, we study the behavior of $(m,n) - \Gamma$-hyperfilters on the product of ordered Γ-semihypergroups. □

Theorem 3. *Let F_i be an $(m,n) - \Gamma$-hyperfilter on the ordered Γ_i-semihypergroup $(S_i, \Gamma_i, \leq_i)$ for all $i \in \Omega$. Then, $F = \prod_{i\in\Omega} F_i$ is an $(m,n) - \Gamma$-hyperfilter on $\prod_{i\in\Omega} S_i$.*

Proof. We divide the proof into three steps.

Step 1. We first show that F is a sub Γ-semihypergroup of $\prod_{i\in\Omega} S_i$.

Let $(a_i)_{i\in\Omega}, (b_i)_{i\in\Omega} \in F = \prod_{i\in\Omega} F_i$. Then, $a_i, b_i \in F_i$ for each $i \in \Omega$. As F_i's is a sub Γ-semihypergroup of S_i, we have $a_i\gamma_i b_i \subseteq F_i$. So, we have

$$(a_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (b_i)_{i\in\Omega} = (a_i\gamma_i b_i)_{i\in\Omega} \in \prod_{i\in\Omega} F_i = F. \tag{21}$$

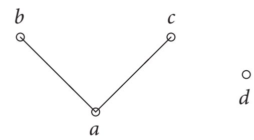

Figure 2: Figure of S for Example 2.

Thus, F is a sub Γ-semihypergroup of $\prod_{i\in\Omega}S_i$.

Step 2. Now, let $(a_i)_{i\in\Omega}, (b_i)_{i\in\Omega} \in \prod_{i\in\Omega}S_i$ and $((a_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (b_i)_{i\in\Omega}) \cap F \neq \varnothing$. Then,

$$\begin{aligned}
&((a_i)_{i\in\Omega} \odot (\gamma_i)_{i\in\Omega} \odot (b_i)_{i\in\Omega}) \cap F \neq \varnothing \\
&\quad \Rightarrow (a_i\gamma_i b_i)_{i\in\Omega} \cap F \neq \varnothing \\
&\quad \Rightarrow a_i\gamma_i b_i \cap F_i \neq \varnothing, \quad \forall i \in \Omega \\
&\quad \Rightarrow a_i^m \subseteq F_i, b_i^n \subseteq F_i, \quad \forall i \in \Omega \\
&\quad \Rightarrow (a_i^m)_{i\in\Omega} \subseteq F, (b_i^n)_{i\in\Omega} \subseteq F \\
&\quad \Rightarrow ((a_i)_{i\in\Omega})^m \subseteq F, ((b_i)_{i\in\Omega})^n \subseteq F.
\end{aligned} \tag{22}$$

Step 3. Let $(a_i)_{i\in\Omega} \in F$ and $(c_i)_{i\in\Omega} \in \prod_{i\in\Omega}S_i$ such that $((a_i)_{i\in\Omega}, (c_i)_{i\in\Omega}) \in \leq$. Then, for all $i \in \Omega$, we have $(a_i, c_i) \in \leq_i$. Since F_i is an $(m,n)-\Gamma$-hyperfilter of S_i for each $i \in \Omega$, we get $c_i \in F_i$ for each $i \in \Omega$. It implies that $(c_i)_{i\in\Omega} \in \prod_{i\in\Omega}F_i = F$. Therefore, F is an $(m,n)-\Gamma$-hyperfilter of $\prod_{i\in\Omega}S_i$.

A mapping $\varphi\colon S \longrightarrow T$ of an ordered Γ-semihypergroup $(S, \Gamma, \leq_S)$ into an ordered Γ-semihypergroup $(T, \Gamma', \leq_T)$ is said to be a normal Γ-homomorphism if (1) $\varphi(a\gamma b) = \varphi(a)\gamma'\varphi(b)$ for all $a, b \in S, \gamma \in \Gamma$ and $\gamma' \in \Gamma'$; (2) φ is isotone, i.e., for any $a, b \in S$, $(a,b) \in \leq_S$ implies $(\varphi(a), \varphi(b)) \in \leq_T$.

Theorem 4. *Let $\varphi\colon S \longrightarrow T$ be a normal Γ-homomorphism of ordered Γ-semihypergroups $(S, \Gamma, \leq_S)$ and $(T, \Gamma', \leq_T)$. If F is an $(m,n)-\Gamma$-hyperfilter of T, then*

$$\varphi^{-1}(F) = \{x \in S \mid \varphi(x) \in F\}, \tag{23}$$

is an $(m,n)-\Gamma$-hyperfilter of S.

Proof. We divide the proof into three steps.

Step 1. Let $x, y \in \varphi^{-1}(F)$, $\gamma \in \Gamma$, and $\gamma' \in \Gamma'$. Then, $\varphi(x), \varphi(y) \in F$. Since F is a sub Γ-semihypergroup of T and φ is normal Γ-homomorphism, we have

$$\begin{aligned}
\varphi(x\gamma y) &= \varphi(x)\gamma'\varphi(y) \\
&\subseteq F\Gamma F \\
&\subseteq F.
\end{aligned} \tag{24}$$

It implies that $x\gamma y \subseteq \varphi^{-1}(F)$. Thus, $\varphi^{-1}(F)$ is sub Γ-semihypergroup of S.

Step 2. Let $x, y \in S, \gamma \in \Gamma$, and $x\gamma y \cap \varphi^{-1}(F) \neq \varnothing$. Then,

$$\begin{aligned}
&x\gamma y \cap \varphi^{-1}(F) \neq \varnothing \\
&\quad \Rightarrow \varphi(x\gamma y) \cap F \neq \varnothing \\
&\quad \Rightarrow (\varphi(x)\gamma'\varphi(y)) \cap F \neq \varnothing \\
&\quad \Rightarrow (\varphi(x))^m \subseteq F, (\varphi(y))^n \subseteq F \\
&\quad \Rightarrow \underbrace{\varphi(x)\gamma'\varphi(x)\gamma' \cdots \gamma'\varphi(x)}_{m-\text{copies}} \subseteq F, \\
&\qquad \underbrace{\varphi(y)\gamma'\varphi(y)\gamma' \cdots \gamma'\varphi(y)}_{n-\text{copies}} \subseteq F \\
&\quad \Rightarrow \varphi\left(\underbrace{x\gamma' x\gamma' \cdots \gamma' x}_{m-\text{copies}}\right) \subseteq F, \varphi\left(\underbrace{y\gamma' y\gamma' \cdots \gamma' y}_{n-\text{copies}}\right) \subseteq F \\
&\quad \Rightarrow \varphi(x^m) \subseteq F, \varphi(y^n) \subseteq F \\
&\quad \Rightarrow x^m \subseteq \varphi^{-1}(F), y^n \subseteq \varphi^{-1}(F).
\end{aligned} \tag{25}$$

Step 3. Now, suppose that $a \in \varphi^{-1}(F)$ and $c \in S$ such that $a \leq_S c$. Then, $\varphi(a) \in F$. Since $a \leq_S c$ and φ is normal Γ-homomorphism, we have $\varphi(a) \leq_T \varphi(c)$. Since F is an $(m,n)-\Gamma$-hyperfilter of T, we get $\varphi(c) \in F$. So, $c \in \varphi^{-1}(F)$. Therefore, $\varphi^{-1}(F)$ is an $(m,n)-\Gamma$-hyperfilter of S.

Let K be a sub Γ-semihypergroup (resp. nonempty subset) of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, K is called an $(m,n)-\Gamma$-hyperideal (resp. generalized $(m,n)-\Gamma$-hyperideal) [34] of S if (1) $K^m\Gamma S\Gamma K^n \subseteq K$ and (2) $(K] \subseteq K$. Moreover, an $(m,n)-\Gamma$-hyperideal K of an ordered Γ-semihypergroup S is said to be completely prime if for each $x, y \in S$ and $\gamma \in \Gamma$ such that $x\gamma y \cap K \neq \varnothing$, then $x \in K$ or $y \in K$. An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called (m,n)-regular if for every $a \in S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a^m\alpha x\beta a^n$. □

Theorem 5. *Let F be a nonempty subset of a commutative (m,n)-regular ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, the following assertions are equivalent:*

(1) F is an $(m,n)-\Gamma$-hyperfilter of S.

(2) $SF = \varnothing$ or $S\backslash F$ is a completely prime generalized $(m,n)-\Gamma$-hyperideal of S.

Proof

(1) $\Rightarrow$ (2): Assume that (1) holds. Let $S\backslash F \neq \varnothing$. First of all, we show that $S\backslash F$ is a generalized $(m,n)-\Gamma$-hyperideal of S, i.e., $(S\backslash F)^m\Gamma S\Gamma(S\backslash F)^n \subseteq S\backslash F$. Let $(S\backslash F)^m\Gamma S\Gamma(S\backslash F)^n \subseteq F$. Since S is (m,n)-regular and $(F] = F$, we get

$$S\backslash F \subseteq (((S\backslash F)^m\Gamma S\Gamma(S\backslash F)^n) \subseteq (F] = F, \tag{26}$$

which is a contradiction. Suppose that $a \in S\backslash F$ and $x \leq a$, where $x \in S$. If $x \in F$, then, since F is an $(m,n)-\Gamma$-hyperfilter of S, we get $a \in F$, a contradiction. It implies that $x \in S\backslash F$, and hence $(S\backslash F] \subseteq S\backslash F$. Thus, $S\backslash F$ is a generalized $(m,n)-\Gamma$-hyperideal of S. Now, we assert that $S\backslash F$ is a completely prime generalized $(m,n)-\Gamma$-hyperideal of S. Let $x, y \in S$, $\gamma \in \Gamma$, and $x\gamma y \cap (S\backslash F) \neq \emptyset$. Then, there exists $a \in x\gamma y$ such that $a \in S\backslash F$. If $x \in F$ and $y \in F$, then, since F is a sub Γ-semihypergroup of S, we get $x\gamma y \subseteq F$, a contradiction. Hence, $x \in S\backslash F$ or $y \in S\backslash F$. Therefore, $S\backslash F$ is a completely prime generalized $(m,n)-\Gamma$-hyperideal of S.

(2) $\Rightarrow$(1): If $S\backslash F = \emptyset$, then $F = S$, and hence F is an $(m,n)-\Gamma$-hyperfilter of S. Let $S\backslash F$ be a completely prime generalized $(m,n)-\Gamma$-hyperideal of S. Now, let $x, y \in F$ and $\gamma \in \Gamma$. If $x\gamma y \nsubseteq F$, then $x\gamma y \cap (S\backslash F) \neq \emptyset$. Since $S\backslash F$ is completely prime, we get $x \in S\backslash F$ or $y \in S\backslash F$, which is a contradiction. Thus, $x\gamma y \subseteq F$ and so F is a sub Γ-semihypergroup of S. Now, let $a, b \in S$, $\gamma \in \Gamma$, and $a\gamma b \cap F \neq \emptyset$. Let $a^m \subseteq S\backslash F$ or $b^n \subseteq S\backslash F$. Without any loss of generality, we assume that $a^m \subseteq S\backslash F$. Since S is (m,n)-regular, we get $a \leq a^m \alpha x \beta a^n$. Then,

$$\begin{aligned} a\gamma b &\leq (a^m \alpha x \beta a^n)\gamma b \\ &\leq ((a^m)^m \lambda y \mu (a^m)^n)\alpha x \beta a^n)\gamma b. \end{aligned} \tag{27}$$

Since S is commutative and $S\backslash F$ is an $(m,n)-\Gamma$-hyperideal of S, we have

$$a\gamma b \subseteq (S\backslash F)^m \Gamma S \Gamma (S\backslash F)^n \subseteq S\backslash F. \tag{28}$$

It implies that $(S\backslash F) \cap F \neq \emptyset$, which is a contradiction. Thus, $a^m \subseteq F$ and $b^n \subseteq F$. Next, let $a \in F$ and $a \leq c \in S$. If $c \in S\backslash F$, then, since $S\backslash F$ is an $(m,n)-\Gamma$-hyperideal of S, we get $a \in S\backslash F$, a contradiction. So, $c \in F$, and thus F is an $(m,n)-\Gamma$-hyperfilter of S.

4. Construction of Ordered Γ-Semihypergroups

In this section, we consider some ordered Γ-semihypergroups, where we define an ordered regular equivalence relation σ^* such that the quotient is an ordered Γ-semihypergroup. More exactly, starting with an ordered Γ-semihypergroup and using σ^*, we can construct an ordered Γ-semihypergroup structure on the quotient set.

Example 3. Let

$$S = \{w, x, y, z\}. \tag{29}$$

Define the hyperoperation $\Gamma = \{\gamma\}$ and (partial) order relation $\leq$ on S as follows:

$$\begin{array}{c|cccc} \gamma & w & x & y & z \\ \hline w & w & y & y & y \\ x & y & \{y,z\} & y & \{y,z\} \\ y & y & y & y & y \\ z & y & \{y,z\} & y & z \end{array}, \tag{30}$$

$$\begin{aligned} \leq \;:= \{&(w,w), (w,x), (w,y), (w,z), (x,x), \\ &(x,z), (y,y), (y,z), (z,z)\}. \end{aligned}$$

Then, $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup. We give the covering relation $\prec$ as

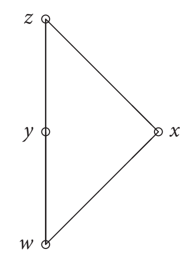

Figure 3: Figure of S for Example 3.

$$\prec\; = \{(w,x), (w,y), (x,z), (y,z)\}. \tag{31}$$

The figure of S is shown in Figure 3.

Here, it is a routine matter to verify that $K = \{w, y\}$ is an $A - \mathbf{I} - \Gamma$-hyperideal of S.

Example 4. Suppose that

$$S = \{a, b, c, d, e\}, \tag{32}$$

and $\Gamma = \{\beta\}$. Define

$$\begin{array}{c|ccccc} \beta & a & b & c & d & e \\ \hline a & \{b,d\} & \{b,d\} & d & d & d \\ b & \{b,d\} & \{b,d\} & d & d & d \\ c & d & d & c & d & c \\ d & d & d & d & d & d \\ e & d & d & c & d & c \end{array}. \tag{33}$$

We consider the ordered Γ-semihypergroup $(S, \Gamma, \leq)$, where the (partial) order relation $\leq$ is defined by the following table:

$$\leq\; := \{(a,a), (a,b), (b,b), (c,c), (d,b), (d,c), (d,d), (e,e)\}. \tag{34}$$

We give the covering relation $\prec$ as

$$\prec\; = \{(a,b), (d,b), (d,c)\}. \tag{35}$$

The figure of S is shown in Figure 4.

Suppose that $F = \{a, b, c, d\}$. Clearly, F is an $(m,n)-$ Γ-hyperfilter of S for $m, n \geq 2$. Indeed: since $\{e\}\beta x = \{c\}$ or $\{d\} \in F$, $x\beta\{e\} = \{c\}$ or $\{d\} \in F$ for any $x \in S$ and $\{e\}^m = \{c\} \subseteq F$ for $m \geq 2$, we have that F is an $(m,n)-\Gamma$-hyperfilter of S for $m, n \geq 2$.

But F is not a Γ-hyperfilter of S. Indeed: since $\{e\}\beta x = \{c\}$ or $\{d\} \in F$, $x\beta\{e\} = \{c\}$ or $\{d\} \in F$ for any $x \in S$ and $\{e\} \notin \mathscr{F}$, we have that F is not a Γ-hyperfilter of S.

Definition 6. Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A relation σ on S is called a weak pseudoorder on S if

(1) $\leq\, \subseteq \sigma$.

(2) $a\sigma b$ and $b\sigma c$ imply $a\sigma c$ for all $a, b, c \in S$.

(3) $a\sigma b$ implies $a\gamma c \overrightarrow{\sigma} b\gamma c$ and $c\gamma a \overrightarrow{\sigma} c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

(4) $a\sigma b$ and $b\sigma a$ imply $a\gamma c \widetilde{\sigma} b\gamma c$ and $c\gamma a \widetilde{\sigma} c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

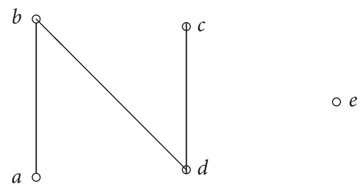

Figure 4: Figure of S for Example 4.

Clearly, every pseudoorder relation on an ordered Γ-semihypergroup S is a weak pseudoorder. The converse is not true, in general, that is, a weak pseudoorder may not be a pseudoorder of S (see Example 6).

Theorem 6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup and σ a weak pseudoorder on S. Then, there exists a regular equivalence relation*

$$\sigma^* = \{(a, b) \in S \times S | a\sigma b \text{ and } b\sigma a\}, \quad (36)$$

on S such that $((S/\sigma^*), \Gamma_{\sigma^*}, \prec_{\sigma^*})$ is an ordered Γ_{σ^*}-semihypergroup, where

$$\prec_{\sigma^*} := \{(\sigma^*(x), \sigma^*(y)) \in \frac{S}{\sigma^*} \times \frac{S}{\sigma^*} | \quad \exists a \in \sigma^*(x), \exists b \in \sigma^*(y), \quad (37)$$

such that $(a, b) \in \sigma\}$.

Proof. The proof is similar to the proof of Theorem 4 in [16]. □

Example 5. Let

$$S = \{a, b, c, d, e\}, \quad (38)$$

and $\Gamma = \{\gamma, \beta\}$. Define the hyperoperations γ, β and (partial) order relation $\leq$ on S as follows:

γ_{θ^*}	w_1	w_2	w_3
w_1	$\{w_1, w_2\}$	w_2	w_3
w_2	w_2	w_2	w_3
w_3	w_3	w_3	w_3

,

β_{θ^*}	w_1	w_2	w_3
w_1	w_2	w_2	w_3
w_2	w_2	w_2	w_3
w_3	w_3	w_3	w_3

,

$$\leq := \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c), (d, d), (e, e)\}. \quad (39)$$

Then, $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup. We give the covering relation $\prec$ as

$$\prec = \{(a, b), (b, c)\}. \quad (40)$$

The figure of S is shown in Figure 5.

Here, $F = \{a, b, c\}$ is a Γ-hyperfilter of S. Let us consider the following relation in S:

$$\theta = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, b), (c, c), (d, d), (d, e), (e, d), (e, e)\}. \quad (41)$$

Now, we consider a regular equivalence relation θ^* on S as follows:

$$\theta^* = \{(x, y) \in S \times S | (x, y) \in \theta \text{ and } (y, x) \in \theta\}. \quad (42)$$

By definition of θ^*, we get

$$\theta^* = \{(a, a), (b, b), (b, c), (c, b), (c, c), (d, d), (d, e), (e, d), (e, e)\}. \quad (43)$$

If we take

$$w_1 = \{a\},$$
$$w_2 = \{b, c\}, \quad (44)$$
$$w_3 = \{d, e\},$$

then, by Theorem 6, $(S/\theta^*) = \{w_1, w_2, w_3\}$ is an ordered Γ_{θ^*}-semihypergroup, where $\Gamma_{\theta^*} = \{\gamma_{\theta^*}, \beta_{\theta^*}\}$ and $\prec_{\theta^*}$ are defined by

γ	a	b	c	d	e
a	$\{a, b\}$	$\{b, c\}$	c	$\{d, e\}$	e
b	$\{b, c\}$	c	c	$\{d, e\}$	e
c	c	c	c	$\{d, e\}$	e
d	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	d	e
e	e	e	e	e	e

,

β	a	b	c	d	e
a	$\{b, c\}$	c	c	$\{d, e\}$	e
b	c	c	c	$\{d, e\}$	e
c	c	c	c	$\{d, e\}$	e
d	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	d	e
e	e	e	e	e	e

,

$$\prec_{\theta^*} := \{(\theta^*(x), \theta^*(y)) \in \frac{S}{\theta^*} \times \frac{S}{\theta^*} | \quad \exists a \in \theta^*(x), \exists b \in \theta^*(y), \quad (45)$$

such that $(a, b) \in \theta\} = \{(w_1, w_1), (w_1, w_2), (w_2, w_2), (w_3, w_3)\}$.

We give the covering relation $\prec_{\theta^*}$ as

$$\prec_{\theta^*} = \{(w_1, w_2)\}. \quad (46)$$

The figure of S/θ^* is shown in Figure 6.

Example 6. Let

$$S = \{a, b, c, d, e, f\}, \quad (47)$$

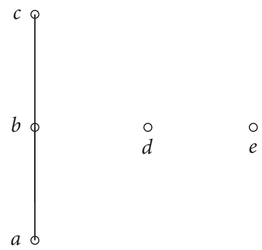

Figure 5: Figure of S for Example 5.

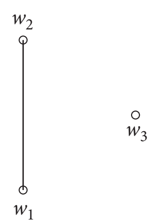

Figure 6: Figure of S/θ^* for Example 5.

and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows:

γ	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	$\{a,c\}$	a	a	$\{a,f\}$
d	a	b	$\{a,e\}$	a	a	$\{a,d\}$
e	a	b	$\{a,e\}$	a	a	$\{a,d\}$
f	a	b	$\{a,c\}$	a	a	$\{a,f\}$

,

β	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	a	a	a	a
d	a	b	a	$\{a,d\}$	$\{a,e\}$	a
e	a	b	a	a	a	a
f	a	b	a	$\{a,f\}$	$\{a,c\}$	a

. (48)

We have that $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup where the (partial) order relation $\leq$ is defined by

$$\leq \,:= \{(a,a), (b,a), (b,b), (b,c), (c,c), (d,d), (e,e), (f,f)\}. \tag{49}$$

We give the covering relation $\prec$ as

$$\prec \,= \{(b,a), (b,c)\}. \tag{50}$$

The figure of S is shown in Figure 7.

S has no proper $(m,n)-\Gamma$-hyperfilter. Now, we put

$$\rho := \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,c), (d,d), (e,e), (f,f)\}. \tag{51}$$

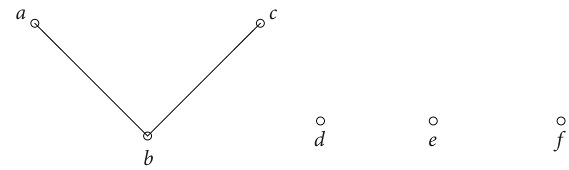

Figure 7: Figure of S for Example 6.

Clearly, ρ is not a pseudoorder, since $a\gamma f\overline{\overline{\rho}}c\gamma f$ does not hold. Indeed:

$$\begin{aligned} &(a,c) \in \rho, \\ &a\gamma f = a, \\ &c\gamma f = \{a,f\} \text{ but } (a,f) \notin \rho. \end{aligned} \tag{52}$$

We have

$$\begin{aligned} \rho^* &= \{(x,y) \in S \times S \mid \quad (x,y) \in \rho, (y,x) \in \rho\} \\ &= \{(a,a), (a,b), (b,a), (b,b), (c,c), (d,d), (e,e), (f,f)\}. \end{aligned} \tag{53}$$

Then, $S/\rho^* = \{u_1, u_2, u_3, u_4, u_5\}$, where $u_1 = \{a,b\}$, $u_2 = \{c\}$, $u_3 = \{d\}$, $u_4 = \{e\}$, and $u_5 = \{f\}$. By Theorem 6, $((S/\rho^*), \Gamma_{\rho^*}, \prec_{\rho^*})$ is an ordered Γ_{ρ^*}-semihypergroup, where γ_{ρ^*}, β_{ρ^*}, and $\prec_{\rho^*}$ are defined by

γ_{ρ^*}	u_1	u_2	u_3	u_4	u_5
u_1	u_1	u_1	u_1	u_1	u_1
u_2	u_1	$\{u_1,u_2\}$	u_1	u_1	$\{u_1,u_5\}$
u_3	u_1	$\{u_1,u_4\}$	u_1	u_1	$\{u_1,u_3\}$
u_4	u_1	$\{u_1,u_4\}$	u_1	u_1	$\{u_1,u_3\}$
u_5	u_1	$\{u_1,u_2\}$	u_1	u_1	$\{u_1,u_5\}$

,

β_{ρ^*}	u_1	u_2	u_3	u_4	u_5
u_1	u_1	u_1	u_1	u_1	u_1
u_2	u_1	u_1	u_1	u_1	u_1
u_3	u_1	u_1	$\{u_1,u_3\}$	$\{u_1,u_4\}$	u_1
u_4	u_1	u_1	u_1	u_1	u_1
u_5	u_1	u_1	$\{u_1,u_5\}$	$\{u_1,u_2\}$	u_1

,

$$\prec_{\rho^*} = \{(u_1,u_1), (u_1,u_2), (u_2,u_2), (u_3,u_3), (u_4,u_4), (u_5,u_5)\}. \tag{54}$$

We give the covering relation $\prec_{\rho^*}$ as

$$\prec_{\rho^*} = \{(u_1,u_2)\}. \tag{55}$$

The figure of (S/ρ^*) is shown in Figure 8.

In the following example, we show that the converse of Lemma 1 is not true in general, i.e., an $A-\mathbf{I}-\Gamma$-hyperideal may not be an $\mathbf{I}-\Gamma$-hyperideal of an ordered Γ-semihypergroup S.

Example 7. We come back to Example 6 and consider ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Put $K = \{a,b,c\}$. Clearly, K is a sub Γ-semihypergroup of S. We have

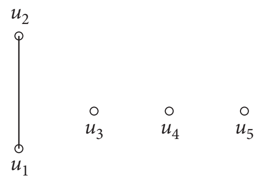

Figure 8: Figure of (S/ρ^*) for Example 6.

$(x\Gamma K\Gamma y] \cap K \neq \varnothing$ for all $x, y \in S$.

$S\Gamma K\Gamma S = S \nsubseteq K$.

$(K] = K$.

Therefore, K is an $A - \mathbf{I} - \Gamma$-hyperideal of S but is not $\mathbf{I} - \Gamma$-hyperideal of S.

5. Conclusions

One of the most important research areas in ordered Γ-semihypergroup theory is the investigation of hyperfilters [25]. Generalization of hyperfilters in (ordered) Γ-semihypergroups is necessary for further study of (ordered) Γ-semihypergroups. In this paper, we studied some properties of $A - \mathbf{I} - \Gamma$-hyperideals and $(m, n) - \Gamma$-hyperfilters of ordered Γ-semihypergroups. For future work, it will be interesting to study relative $A - \mathbf{I} - \Gamma$-hyperideals and intuitionistic fuzzy $A - \mathbf{I} - \Gamma$-hyperideals in ordered hyperstructures such as ordered Γ-semihypergroups, ordered Krasner hyperrings, and ordered semihyperrings. We expect further research efforts in this direction. These observations motivate the following future works: describe relations between fuzzy $(m, n) - \Gamma$-hyperfilters and $(\in, \in\vee q_k)$-fuzzy $(m, n) - \Gamma$-hyperfilters.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

Acknowledgments

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