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Research Article

The Number of Blocks of a Graph with Given Minimum Degree

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A block of a graph is a nonseparable maximal subgraph of the graph. We denote by $b(G)$ the number of block of a graph G. We show that, for a connected graph G of order n with minimum degree $k \geq 1$, $b(G) < ((2k-3)/(k^2-k-1))n$. The bound is asymptotically tight. In addition, for a connected cubic graph G of order $n \geq 14$, $b(G) \leq (n/2) - 2$. The bound is tight.

1. Introduction

We consider finite, undirected, simple graphs only. Let $G = (V(G), E(G))$ be a graph. The numbers of vertices and edges of G are called the *order* and the *size* of G and denoted by $v(G)$ and $e(G)$, respectively. A vertex v is called a *cut vertex* if $\mathrm{com}(G-v) > \mathrm{com}(G)$, where $\mathrm{com}(G)$ denotes the number of components of G. $c(G)$ denotes the number of cut vertices of G. Rao [1] proved that, for a connected graph G of order n and size m,

$$c(G) \leq \max\left\{q\colon m \leq \binom{n-q}{2} + q\right\}, \tag{1}$$

characterized all extremal graphs. Rao and Rao [2] solved the corresponding problem for a strong digraph. Later, Achuthan and Rao [3] determined the maximum number of cut edges in a connected d-regular graph of order p.

Let $f(n,d) = \max\{c(G)\colon G$ is a connected k-regular graph of order $n\}$. Rao [4] determined $f(n,d)$ for $d \leq 4$. Nirmala and Rao [5] showed that $f(n,d) = ((2n-d-5)/(d+1)) - 1$ or $((2n-d-5)/(d+1)) - 2$ for odd $d \geq 5$ and have obtained an upper bound for $f(n,d)$ for even $d \geq 6$.

Alberten and Berman [6] proved that, for a graph G of order n and minimum degree $k \geq 2$,

$$c(G) < \frac{2k-2}{k^2-2}n. \tag{2}$$

This bound is asymptotically tight.

Hopkins and Staton [7] showed that every connected graph of order n contains no more than $(r/(2r-2))n$ cut vertices of degree r. Some related results are referred to [8, 9].

A *separation* of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. The common vertex is called a *separating vertex* of the graph. Since the graph G under consideration is simple, $v \in V(G)$ is a separating vertex if and only if it is a cut vertex. A block of a graph is a nonseparable maximal subgraph of the graph. We denote by $b(G)$ the number of blocks of a graph G.

It is clear that any two blocks of a graph have at most one vertex in common. Recall that the block tree $B(G)$ of G is the bipartite graph with bipartition $(\mathbb{B}, \mathbb{S})$, where $\mathbb{B}$ is the set of blocks of G and S, the set of separating vertices of G, and a block B, and a separating vertex v is joined by an edge in $B(G)$ if and only if B contains v. It is easy to see that if G is connected, $B(G)$ is a tree. Each leaf of $B(G)$ corresponds to an end block of G.

Inspired from the bound for the cut vertices, in the present paper, we consider the upper bound for the number of blocks, a connected graph of order n with given minimum degree. Let us begin with two easy cases when $\delta(G) = 1$ and $\delta(G) = 2$.

Proposition 1. *For a connected graph G of order $n \geq 2$, $b(G) \leq n-1$, with equality if and only if G is a tree.*

Proof. Our proof is induction on n. If $n = 2$, then $G \cong K_2$; thus, the result holds. Next, we assume that $n \geq 3$. If G has no cut vertex, then $b(G) = 1 < n - 1$. Now suppose G has a cut vertex. Let B be an end block of G and v be the cut vertex, which belongs to B. Let $G' = G - (V(B), \{v\})$. Clearly, G' is connected. By the induction hypothesis, $b(G') \leq \nu(G') - 1$. Since $b(G) = b(G') + 1$, $\nu(G') \leq \nu(G) - 1$, we have $b(G) \leq n - 1$, with equality only if $b(G') = \nu(G') - 1 = n - 2$ and $B \cong K_2$. By the induction hypothesis, G' is a tree, implying that G is a tree.

On the contrary, if G is a tree, clearly, $b(G) = n - 1$. □

Proposition 2. *For a connected graph G of order $n \geq 4$ with $\delta(G) \geq 2$, $b(G) \leq n - 3$, with equality if and only if G is the graph obtained from P_{n-4} identifying each end with a vertex of separate K_3, as given in Figure 1.*

Proof. If G has no cut vertex, the result holds trivially. Next, we assume that G has cut vertices, and thus, it has at least two end blocks and $n \geq 5$. Let $B_1, \ldots, B_t$ be all end blocks of G. Let c_i be the cut vertex of G, which belongs to B_i for each $i \in \{1, \ldots, t\}$. Clearly, $\nu(B_i) \geq 3$ for any i. If $c_i = c_j$ for any two distinct i, j, then $b(G) = t$ and $n = \sum_{i=1}^{t} (\nu(B_i) - 1) + 1 \geq 2t + 1$. Therefore, $b(G) \leq (n-1)/2 \leq n - 3$.

Otherwise, G has at least two cut vertices. It follows that the order n' of $G' = G - \cup_{i=1}^{t} V(B_i), \{c_i\}$ is at least two. Hence, $n \geq n' + 2t$ and $b(G) = b(G') + t$. By Proposition 1, $b(G') \leq n' - 1$. Summing up the above, we have

$$b(G) = b(G') + t \leq n' - 1 + t \leq n - t - 1 \leq n - 3. \quad (3)$$

From the above, $b(G) = n - 3$ if and only if $t = 2$ and $G' = P_{n-4}$, $B_1 \cong K_3 \cong B_2$, as we promised. □

It is clear that, for a graph G of order n, $b(G)$ decreases when $\delta(G)$ increases. For a connected graph G of order n and minimum degree at least k, we have the following result, which is asymptotically best possible.

Theorem 1. *For a connected graph G of order n with $\delta(G) \geq k$, $b(G) < ((2k-3)/(k^2-k-1))n$.*

We show that the bound in the above theorem is asymptotically best possible. Let $k \geq 3$. Consider a tree T of order p with each vertex having degree k or 1. By the handshaking lemma, the number of leaves of this tree is

$$\frac{(k-2)p+2}{k-1}. \quad (4)$$

Let G be the graph obtained from identifying each leaf of T with a vertex of a clique of order $k+1$ separately. Therefore,

$$\nu(G) = p + \frac{(k-2)p+2}{k-1}k,$$
$$b(G) = p - 1 + \frac{(k-2)p+2}{k-1}. \quad (5)$$

So, we have

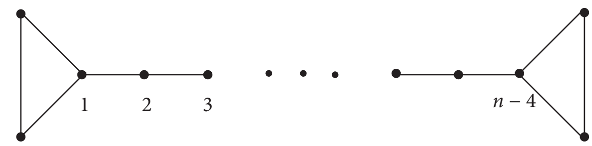

Figure 1: The extremal graph attaining the upper bound in Proposition 2.

$$\frac{b(G)}{\nu(G)} = \frac{(2k-3)p - k + 3}{(k^2-k-1)p + 2k} < \frac{2k-3}{k^2-k-1}. \quad (6)$$

However, as p gets larger, $b(G)/\nu(G)$ gets arbitrarily close to $(2k-3)/(k^2-k-1)$.

What happens for the k-regular graphs? The situation becomes complicated. We are just able to get an exact bound for a cubic graph G of order n: $b(G) < n/2$ (Theorem 2), whereas by Theorem 1, we have $b(G) < (3/5)n$ for a connected graph G of order n with $\delta(G) \geq 3$.

Theorem 2. *For a connected cubic graph G of order n,*

$$b(G) \leq \begin{cases} 1, & \text{if } n \leq 8, \\ 3, & \text{if } n = 12, \\ \dfrac{n}{2} - 2, & \text{otherwise.} \end{cases} \quad (7)$$

The bound is sharp.

To see the sharpness of the bound, we denote by K_4^* the graph obtained from K_4 by replacing an edge with a path of length two, as drawn in Figure 2.

The graphs G_n achieve the upper bound in Theorem 2, which are classified into three types in terms of $n \equiv 4 \pmod 6$, $n \equiv 0 \pmod 6$, and $n \equiv 2 \pmod 6$, respectively.

For an integer $n \equiv 4 \pmod 6$, $k = (n-4)/3$ is an even integer. Let T_k be a tree in which every vertex has degree 1 or 3. It is clear that T_k has exactly $(k+1)/2$ vertices of degree 1 (leaves) and $(k-1)/2$ vertices of degree 3. Let G_n be a graph obtained from identifying each leaf of T_k with the vertex of degree two of a separate K_4^*, as shown in Figure 3.

For an integer $n \equiv 0 \pmod 6$, let G_n be a cubic graph obtained from a graph G_{n-2} by replacing a vertex of degree three (not belongs to any K_4^*) with a triangle, as shown in Figure 3.

For an integer $n \equiv 2 \pmod 6$, let G_n be a cubic graph obtained from a graph G_{n-4} by inserting a $K_4 - e$ into an edge of G_{n-4} (not belongs to any K_4^*), as shown in Figure 3.

It can be checked that $\nu(G_n) = n$ and $b(G_n) = (n/2) - 2$ for any graph G_n constructed as above.

2. The Proof of Theorem 1

Suppose the result is not true and let G be a counterexample of minimum order n, i.e., $\delta(G) \geq k$ and $b(G) \geq ((2k-3)/(k^2-k-1))n$, but for any connected graph G' of order $n' < n$ with $\delta(G') \geq k$, $b(G') < ((2k-3)/(k^2-k-1))\, n'$.

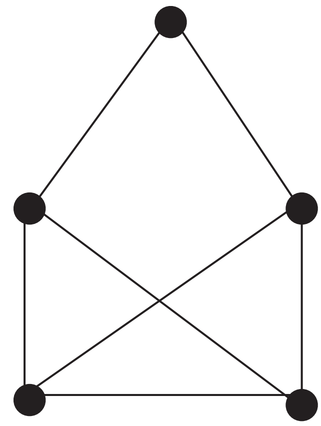

Figure 2: K_4^*.

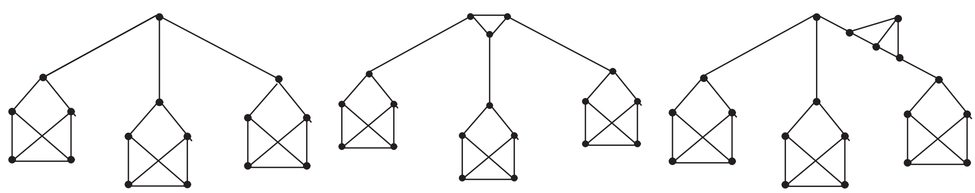

Figure 3: The extremal graph G of order n attaining the upper bound in Theorem 2, where (a) $n \equiv 4$, (b) $n \equiv 0$, and (c) $n \equiv 2 \pmod 6$, respectively.

If $k \in \{1, 2\}$, $((2k-3)/(k^2-k-1))n = n$. By Propositions 1 and 2, $b(G) \leq n-1 < n$. Hence, $k \geq 3$. Since $n \geq k+1$, we have $((2k-3)/(k^2-k-1))n \geq 1$, and thus, G has at least two blocks.

Claim 1. Every end block of G is a clique of order $k+1$.

Proof of Claim 1. If it is not, let B be an end block of G. Let G' be the graph obtained from G by replacing B with B' of order $k+1$. Clearly, $b(G) = b(G')$ and $\delta(G') \geq k$. By the choice of G,

$$b(G') < \frac{2k-3}{k^2-k-1} n'. \quad (8)$$

Combining the above facts, we conclude that $b(G) < ((2k-3)/(k^2-k-1))n$, contradicting the choice of G. □

Claim 2. No cut vertex of G belongs to at least two end blocks of G.

Proof of Claim 2. Let B and B' be two end blocks of G containing the same cut vertex v of G. Let $G' = G - (V(B'), \{v\})$. By Claim 1, $v(B) = v(B') = k+1$, and thus, $v(G') = n-k$ and $\delta(G') \geq k$. By the minimality of G,

$$b(G') < \frac{2k-3}{k^2-k-1}(n-k). \quad (9)$$

Combining (9) with the fact that $b(G) = b(G') + 1$, we have a contradiction:

$$b(G) < \frac{2k-3}{k^2-k-1}(n-k) + 1 < \frac{2k-3}{k^2-k-1} n, \quad (10)$$

□

Claim 3. Let c be a cut vertex lying on an end block B_c. If $B \neq B_c$ is a block containing c, then $B \cong K_2$.

Proof of Claim 3. It suffices to show that $v(B) = 2$.

First suppose that $v(B) \geq k+1$. Let G' be the graph obtained from G, $(V(B_c), \{c\})$ and joining c to every vertex in $V(B), \{c\}$. Clearly, G' is a connected graph with $\delta(G') \geq k$. Moreover, by Claim 1, $v(G') = n-k$. Again, by the minimality of G,

$$b(G') < \frac{2k-3}{k^2-k-1}(n-k). \quad (11)$$

Combining (11) with the fact that $b(G) = b(G') + 1$, we have a contradiction.

$$b(G) < \frac{2k-3}{k^2-k-1}(n-k) + 1 < \frac{2k-3}{k^2-k-1} n, \quad (12)$$

Now assume that $v(B) \in \{3, \ldots, k\}$. Let $V(B) = \{v_1, \ldots, v_r\}$, where $v_1 = c$. Since $\delta(G) \geq k$ and $r \leq k$, each v_i is a cut vertex of G. Let G' be the graph obtained from G by identifying all vertices in $\{v_2, \ldots, v_r\}$. Clearly, $\delta(G') \geq k$, $n' < n$, and $b(G') = b(G)$. By the choice of G, $b(G') < ((2k-3)/(k^2-k-1))n'$. Thus,

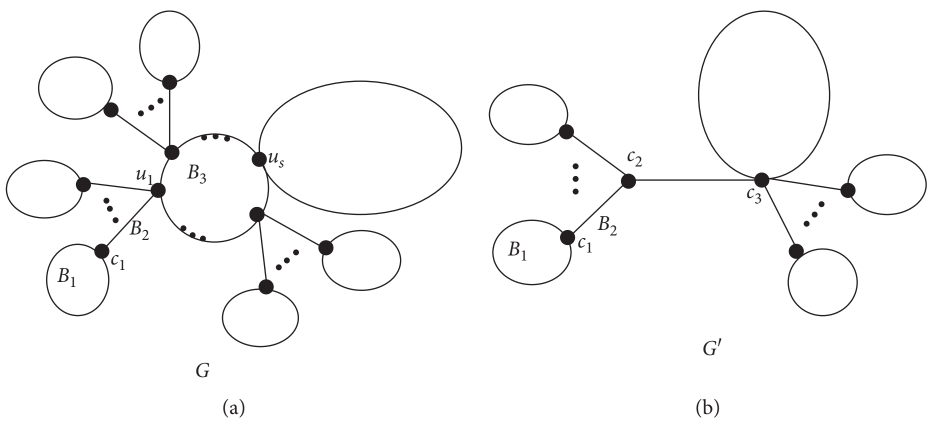

FIGURE 4: (a) G and (b) G' in case 3.2.

$$b(G) < \frac{2k-3}{k^2-k-1}n, \tag{13}$$

contradicting the choice of G. This proves the claim.

Take a longest path P of $B(G)$. Let B_1 be an end block of G, which corresponds to a terminal vertex of $P = B_1c_1B_2c_2B_3c_3\cdots$, where c_1 be the unique cut vertex of G which belongs to B_1. By Claim 3, $B_2 \cong K_2$. Next, we consider three possible cases in terms of the order $v(B_3)$ of B_3. □

2.1. Case 1: $v(B_3) \geq k+1$. Let G' be the graph obtained from $G, V(B_1)$ and joining c_2 to each vertex of $V(B_3)$. It is clear that $\delta(G') \geq k$, and by Claim 1, $v(G') = n-k-1$. By the minimality of G, $b(G') < ((2k-3)/(k^2-k-1))(n-k-1)$. Since $b(G) = b(G') + 2$, we have

$$b(G) < \frac{2k-3}{k^2-k-1}(n-k-1) + 2 < \frac{2k-3}{k^2-k-1}n. \tag{14}$$

2.2. Case 2: $v(B_3) = 2$. By the choice of P, each block $B^* \neq B_3$ of G containing c_2 is isomorphic to K_2. In addition, the end block containing the other end of B^* is a leaf of $B(G)$. Since $d(c_2) \geq k$, there are $k-1$ such end block $B_1^*, \ldots, B_{k-1}^*$, each of which are jointed c_2 with an edge. Let G' be the graph obtained from $G - \cup_{i=1}^{k-1} V(B_i^*)$ by identifying a vertex of a new clique of order $k+1$ with c_2. It is clear that $\delta(G') \geq k$, $n' = v(G') = n - (k-1)(k+1) + k$, and $b(G') = b(G) - 2(k-1) + 1$. So,

$$\begin{aligned} b(G) &= b(G') + 2(k-1) - 1 \\ &< \frac{2k-3}{k^2-k-1}n' + 2(k-1) - 1 \\ &= \frac{2k-3}{k^2-k-1}(n-(k-1)(k+1)+k) + 2(k-1) - 1 \\ &= \frac{2k-3}{k^2-k-1}n \end{aligned} \tag{15}$$

is a contradiction.

2.3. Case 3: $3 \leq v(B_3) \leq k$. Since $\delta(G) \geq k$, each vertex of B_3 is a cut vertex of G. We distinguish two subcases in terms of $v(B_3)$.

2.3.1. Case 3.1: $v(B_3) = k$. Since $\delta(G) \geq k$, every vertex $v \in V(B_3)$ has a neighbor not in $V(B_3)$, which belongs to distinct blocks of G. Let G' be the graph obtained from G by contracting B_3 to a vertex v'. It can be seen that $\delta(G') \geq k$, $b(G) = b(G') + 1$, and $n' = n-k+1$. So,

$$b(G) = b(G') + 1 < \frac{2k-3}{k^2-k-1}(n-k+1) + 1 < \frac{2k-3}{k^2-k-1}n. \tag{16}$$

2.3.2. Case 3.2: $3 \leq v(B_3) \leq k-1$. Let $s = v(B_3)$ and $V(B_3) = \{u_1, \ldots, u_s\}$, where $u_1 = c_2$ and $u_s = c_3$. Since $\delta(G) \geq k$, for any $i \in \{1, \ldots, s-1\}$, there are at least $k-s+1$ blocks containing u_i, each of which is isomorphic to K_2, as illustrated in Figure 4.

Let G' be the graph obtained from joining each component of $G - \{u_2, \ldots, u_{s-1}\}$ to c_2 or c_3 such that $d_{G'}(c_j) \geq k$ for each $j \in \{2,3\}$. In addition, add an edge c_2c_3 if $c_2c_3 \notin E(G)$. Note that G' is a connected graph of order $n' < n$ with $\delta(G') \geq k$ and $b(G') = b(G)$. By the minimality of G, $b(G') < ((2k-3)/(k^2-k-1))n'$. Therefore, $b(G) < ((2k-3)/(k^2-k-1))n$, contradicting the choice of G.

The proof of Theorem 1 is completed.

3. Proof of Theorem 2

Suppose the result is not true and let G be a counterexample of minimum order n. The following fact is clear:

(1) G must contain cut vertex.
   Since no cubic graph of order ≤ 8 has a cut vertex, $n \geq 10$.

(2) Moreover, $n \geq 14$. If $10 \leq n \leq 12$, it is not hard to check that $b(G) \leq 3$.

Claim 4. Every end block of G is a K_4^*.

Proof of Claim 4. If it is not, let B be an end block of G. Let G' be the graph obtained from G by replacing B with K_4^*. Clearly, G' is a connected cubic graph of order $n' < n$ and $b(G') = b(G)$. By the minimality of G,

$$b(G') \leq \frac{n'}{2} - 2. \tag{17}$$

Combining the above facts, we conclude that $b(G) \leq (n/2) - 2$, contradicting the choice of G.

Take a longest path P of $B(G)$. Let B_1 be an end block of G, which corresponds to a terminal vertex of $P = B_1 c_1 B_2 c_2 B_3 c_3 \cdots$, where c_1 be the unique cut vertex of G which belongs to B_1. Since G is a cubic graph, $B_2 \cong K_2$. Next we consider three possible cases. □

3.1. Case 1: $\nu(B_3) = k \geq 4$. If $\nu \in V(B_3)$ is a cut vertex of G, then ν belongs to another block which is isomorphic to K_2. In addition, the end block containing the other end of the K_2 is a leaf of $B(G)$. We may assume B_3 has r cut vertices except c_3, which belong to $B_1', B_2', \ldots, B_r'$, respectively, where $B_1' = B_1$.

Let G' be the graph obtained from $G - \cup_{i=1}^{r} V(B_i') - (V(B_3), \{c_3\})$ by identifying c_3 with the vertex of degree two of a new K_4^*. It is clear that G' is a cubic graph of order $n' = n - 5r - k + 5$ and $b(G') = b(G) - 2r$. By the induction hypothesis, $b(G') \leq (1/2)(n - 5r - k + 5) - 2$. Thus,

$$\begin{aligned} b(G) &= b(G') + 2r \\ &\leq \frac{1}{2}(n - 5r - k + 5) - 2 + 2r \\ &= \frac{n}{2} - \frac{r}{2} - \frac{k}{2} + \frac{1}{2}, \quad (k \geq 4), \\ &\leq \frac{n}{2} - 2 + \frac{1}{2} - \frac{r}{2} \\ &\leq \frac{n}{2} - 2. \end{aligned} \tag{18}$$

3.2. Case 2: $\nu(B_3) = 3$. It follows that $B_3 \cong K_3$. Every vertex of B_3 is cut vertex. Let G' be the graph obtained by the same operation as in the proof of Case 1. We have $n' = n - 8$ and $b(G') = b(G) - 4$. Therefore,

$$b(G) = b(G') + 4 \leq \frac{1}{2}(n - 8) - 2 + 4 = \frac{1}{2}n - 2. \tag{19}$$

3.3. Case 3: $\nu(B_3) = 2$. By the choice of P and $\nu(B_3) = 2$, one can find another longest path $P' = B_1' c_1' B_2' c_2 B_3 c_3 \cdots$ of $B(G)$. Let G' be the graph obtained from identifying c_2 of $G - V(B_1) - V(B_1')$ with the vertex of degree two of a new K_4^*. Note that $b(G') = b(G) - 3$ and $n' = n - 6$. By the induction hypothesis, $b(G') \leq (n'/2) - 2$. Therefore,

$$b(G) = b(G') + 3 \leq \frac{1}{2}(n - 6) - 2 + 3 = \frac{1}{2}n - 2. \tag{20}$$

The proof is completed.

4. Conclusions and Future Work

By arguing the properties of a minimum counterexample to the assertion of the main theorems and by using several kinds of graph transformation, we arrive at a contradiction, and thereby, we show our results. However, the upper bound for $b(G)$ remains open if G is a k-regular graph with $k \geq 4$. One of the referees pointed out the possibility of the obtained results to some real-life applications and other fields (see [10–12] for instance).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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