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Research Article

Analysis of a Stochastic Competitive Model with Distributed Time Delays and Jumps in a Polluted Environment

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In this paper, a stochastic competitive model with distributed time delays and Lévy jumps is formulated. With or without a polluted environment, the model is denoted by (M) or (M_0), respectively. The existence of positive solution, persistence in mean, and extinction of species for (M) and (M_0) are both studied. The sufficient criteria of stability in distribution for model (M) is obtained. Finally, some numerical simulations are given to illustrate our theoretical results.

1. Introduction

The dynamics of the biological system has attracted many researchers and has no interruption in the past few decades. This includes the study of the persistence and extinction, stability in distribution of biological systems, optimal harvesting effects of renewable resources (for example, fish and plants), and so on. These studies have implications for the management of biological resources. The dynamics behaviors from the initial deterministic model to stochastic model have been extensively studied and a lot of nice results have been reported [1–4]. It has been verified that the growth rates of species are inevitably subject to white noise. And whether to consider the white noise is the difference between the stochastic model and the deterministic model. Following the method adopted in [4], we will model a stochastic system with white noise. For the biological system, usually there are three kinds of population relationship, i.e., predator-prey, mutualistic, and competitive scenarios, where the competitive scenario between populations is relatively popular [5, 6]. The general competitive model between two populations with white noise is as follows:

$$\begin{cases} \mathrm{d}y_1(t) = y_1(t)\left[r_{10} - a_{11}y_1(t) - a_{12}y_2(t)\right]\mathrm{d}t + \sigma_1 y_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}y_2(t) = y_2(t)\left[r_{20} - a_{21}y_1(t) - a_{22}y_2(t)\right]\mathrm{d}t + \sigma_2 y_2(t)\mathrm{d}B_2(t), \end{cases} \tag{1}$$

where $y_i(t)$ is the size of the i-th population at time t; r_{i0} represents the growth rate of i-th population; $a_{ii} > 0$ denotes the intraspecific competitive coefficients of y_i; a_{12} and a_{21} are positive and represent the competitive rates between y_1 and y_2, respectively; $B_i(t)$ stands for the standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$ contains all P null sets); σ_i^2 is the intensity of the white noise, $i = 1, 2$.

However, the world economy is developing more and more rapidly and the economic development will inevitably destroy the ecological environment. With the increasing toxins and pollution into the ecological environment, the quality of human living environment is becoming worse and worse. Therefore, the study on the impact of toxin importation and environmental pollution on biological populations has become one of the most popular topics in the world [7–12], which is of vital significance to the development of sustainable economy and the protection of human's only living environment. Based on model (1) and considering the environmental pollution factors, then we derive the following model:

$$\begin{cases} \mathrm{d}y_1(t) = y_1(t)\left[r_{10} - r_{11}C_1(t) - a_{11}y_1(t) - a_{12}y_2(t)\right]\mathrm{d}t + \sigma_1 y_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}y_2(t) = y_2(t)\left[r_{20} - r_{21}C_2(t) - a_{21}y_1(t) - a_{22}y_2(t)\right]\mathrm{d}t + \sigma_2 y_2(t)\mathrm{d}B_2(t), \\ \mathrm{d}C_1(t) = \left[k_1 C_E(t) - (g_1 + m_1)C_1(t)\right]\mathrm{d}t, \\ \mathrm{d}C_2(t) = \left[k_2 C_E(t) - (g_2 + m_2)C_2(t)\right]\mathrm{d}t, \\ \mathrm{d}C_E(t) = \left[-hC_E(t) + u(t)\right]\mathrm{d}t, \end{cases} \tag{2}$$

where $C_1(t), C_2(t)$, and $C_E(t)$ are the concentrations of the toxicant in the organism of species y_1 and y_2 and environment at time t, respectively; r_{11} and r_{21} denote the dose-response of species y_1 and y_2 to the organismal toxicant, respectively; k_i and g_i represent the absorbing and excretion rates of the toxicant from the environment respectively, $-m_i$ is depuration rate of the toxicant, $i = 1, 2$. $-hC_E(t)$ denotes the loss rate of the toxicant because of volatilization; $u(t)$ represents the exogenous rate of toxicant inputting into the environment and is always assumed to be bounded.

On the other hand, the behavior between predator and prey is often not always continuous. For example, in some cases, young predators cannot engage in predation; that is, young prey cannot be preyed on. These phenomena are called time delays. Similar phenomena include hibernation, pregnancy, and migration. In fact, time delays exist not only in biological systems, but in other domains as well. For example, R. Manivannan has studied a control system with probabilistic time-delay signals [13]. Therefore, time delays are very important to reveal the real world and should be taken into account in our system. Some scholars pointed out that discrete delays and continuous delays do not include each other, but the S-type distributed delays can be done [14, 15]. Therefore, taking S-type delays into account in above model is interesting. In addition, in nature there are some environmental perturbations such as earthquakes, epidemics, and hurricanes, which differ from white noise because of its sudden and destructive nature, so Lévy jumps are introduced to simulate them in mathematical modeling [16–21]. For example, Liu and Wang [18] studied the persistence and extinction of the two-species model with Lévy jumps. Liu and Bai [21] investigated the stability in distribution of a stochastic model with Lévy noises by Lyapunov functional approach.

Motivated by these, taking the *S*-type distributed time delays and Lévy noises into the above model, we get the following stochastic predator-prey model (M):

$$(M)\begin{cases} \mathrm{d}y_1(t) = y_1(t^-)\left[r_{10} - r_{11}C_1(t) - a_{11}y_1(t) - a_{12}\displaystyle\int_{-\tau_2}^{0} y_2(t^- + \theta)\mathrm{d}\mu_2(\theta)\right]\mathrm{d}t + \sigma_1 y_1(t)\mathrm{d}B_1(t) \\ + y_1(t^-)\displaystyle\int_{\mathbb{Z}} \gamma_1(u)\widetilde{\Gamma}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}y_2(t) = y_2(t^-)\left[r_{20} - r_{21}C_2(t) - a_{21}\displaystyle\int_{-\tau_1}^{0} y_1(t^- + \theta)\mathrm{d}\mu_1(\theta) - a_{22}y_2(t)\right]\mathrm{d}t + \sigma_2 y_2(t)\mathrm{d}B_2(t) \\ + y_2(t^-)\displaystyle\int_{\mathbb{Z}} \gamma_2(u)\widetilde{\Gamma}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}C_1(t) = \left[k_1 C_E(t) - (g_1 + m_1)C_1(t)\right]\mathrm{d}t, \\ \mathrm{d}C_2(t) = \left[k_2 C_E(t) - (g_2 + m_2)C_2(t)\right]\mathrm{d}t, \\ \mathrm{d}C_E(t) = \left[-hC_E(t) + u(t)\right]\mathrm{d}t, \end{cases} \tag{3}$$

with initial data

$$\begin{aligned} y_i(\theta) &= \xi_i(\theta), \quad \theta \in [-\tau, 0], \\ \tau &= \max\{\tau_i\}, \quad i = 1, 2, \end{aligned} \tag{4}$$

where $y_i(t^-)$ denotes the left limit of $y_i(t)$; $\widetilde{\Gamma}(\mathrm{d}t, \mathrm{d}u) = \Gamma(\mathrm{d}t, \mathrm{d}u) - \lambda(\mathrm{d}u)\mathrm{d}t$ represents a compensated Poisson process, where Γ is a Poisson counting measure, λ is the characteristic measure of Γ on a measurable subset $\mathbb{Z}$ in $(0, +\infty)$ with $\lambda(\mathbb{Z}) < +\infty$, $\lambda(\mathrm{d}u)$ is the measure of $\mathbb{Z}$; $\gamma_i(u)$ is a bounded function with $\gamma_i(u) > -1$, $u \in \mathbb{Z}$; the term $\int_{-\tau_i}^{0} y_i(t^- + \theta)\mathrm{d}\mu_i(\theta)$ denotes the Lebesgue–Stieltjes integral, where $\mu_i(\theta)$ denotes a nonnegative variation function defined on $[-\tau, 0]$ with $\tau = \max\{\tau_i\}$ satisfying $\int_{-\tau_i}^{0} d\mu_i(\theta) = 1$, $i = 1, 2$. The biological meanings of other parameters are the same as before. If $r_{11} = r_{21} = 0$, the corresponding model is denoted by (M_0), which means that the population is not contaminated.

We aim to study the dynamical behaviors of (M) and (M_0) such as the extinction and persistence in mean for all species and explore the impacts on the dynamics of time delays and Lévy noise.

The article is structured as follows. For preliminaries, we give some notations and important lemmas in Section 2. In Section 3, we establish the sufficient criteria for the persistence in mean and nonpersistence of M and M_0 and investigate the stable in distribution of (M). In Section 4, some numerical simulations are presented to verify our main results. Finally, conclusion and discussion are given to end this article in Section 5.

2. Preliminaries

For the simplicity, we first make the following notations:

$$\begin{aligned}
b_i &= r_{i0} - 0.5\sigma_i^2 - \int_{\mathbb{Z}} [\gamma_i(u) - \ln(1+\gamma_i(u))\lambda(du)], \quad i = 1, 2,\\
R_i(t) &= \int_0^t \int_{\mathbb{Z}} \ln(1+\gamma_i(u))\widetilde{\Gamma}(ds, du), \quad i = 1, 2,\\
\overline{f(t)} &= t^{-1}\int_0^t f(s)ds,\\
f^* &= \lim_{t\longrightarrow\infty} \sup f(t),\\
f_* &= \lim_{t\longrightarrow\infty} \inf f(t),\\
\overline{f(t)}^* &= \lim_{t\longrightarrow\infty} \sup t^{-1}\int_0^t f(s)ds,\\
\overline{f(t)}_* &= \lim_{t\longrightarrow\infty} \inf t^{-1}\int_0^t f(s)ds,\\
\Delta &= a_{11}a_{22} - a_{12}a_{21},\\
\Delta_1 &= a_{22}b_1 - a_{12}b_2,\\
\Delta_2 &= a_{11}b_2 - a_{21}b_1,\\
\Delta_{31} &= a_{22}\left(b_1 - r_{11}\overline{C_1(t)}^*\right) - a_{12}\left(b_2 - r_{21}\overline{C_2(t)}_*\right),\\
\Delta_{32} &= a_{22}\left(b_1 - r_{11}\overline{C_1(t)}_*\right) - a_{12}\left(b_2 - r_{21}\overline{C_2(t)}^*\right),\\
\Delta_{41} &= -a_{21}\left(b_1 - r_{11}\overline{C_1(t)}_*\right) + a_{11}\left(b_2 - r_{21}\overline{C_2(t)}^*\right),\\
\Delta_{42} &= -a_{21}\left(b_1 - r_{11}\overline{C_1(t)}^*\right) + a_{11}\left(b_2 - r_{21}\overline{C_2(t)}_*\right).
\end{aligned} \tag{5}$$

Our later discussion are based on the following technological hypotheses.

Assumption 1. There exists a positive constant L such that

$$\int_{\mathbb{Z}} [\ln(1+\gamma_i(u))]^2 \lambda(du) < L, \quad i = 1, 2. \tag{6}$$

Assumption 2. Suppose that $\Delta > 0$, which means the internal competition is greater than the external competition (see [22]).

The following lemmas are necessary for our later proof.

Lemma 1. *Let Assumption 1 hold, then for any given initial data* $(\xi_1(t), \xi_2(t)) \in C([-\tau, 0], R_+^2)$, *there exists a unique solution remaining in* R_+^2 *with probability 1.*

Proof. It is obvious that the coefficients of model (M) are locally Lipschitz. By [5], model (M) has a unique local solution $y(t) = (y_1(t), y_2(t)) \in R_+^2$ a.s. for any initial data $(\xi_1(t), \xi_2(t)) \in C([-\tau, 0], R_+^2)$ and $t \in [0, \tau_e]$, where $\tau_e > 0$ is the explosion time. It needs only to verify that $\tau_e = \infty$ a.s. Let $m_0 > 0$ be sufficiently large such that $(\xi_1(0), \xi_2(0)) \in [1/m_0, m_0]$ for each integer $m \geq m_0$. Define the stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e]:\ y_1(t) \notin \left(\frac{1}{m}, m\right), y_2(t) \notin \left(\frac{1}{m}, m\right)\right\}. \tag{7}$$

Clearly, τ_m is strictly increasing with m. Let $\tau_\infty = \lim_{m\longrightarrow\infty}\tau_m$ a.s., then $\tau_\infty \leq \tau_m$ a.s. Then, we only need to prove $\tau_\infty = \infty$ a.s. If the statement is not true, then there exist $T > 0$ and $0 < \varepsilon < 1$ such that $P(\tau_\infty < T) > \varepsilon$ and an integer $m_1 > m_0$ such that

$$P(\tau_m \le T) > \varepsilon, \quad \text{for any } m > m_1. \tag{8}$$

Define

$$V(y) = \beta V_1(y_1) + V_2(y_2) + V_3(t), \tag{9}$$

where

$$\begin{aligned} y &= (y_1, y_2), \\ V_1(y_1) &= y_1 - 1 - \ln y_1, \\ V_2(y_2) &= y_2 - 1 - \ln y_2, \\ V_3(t) &= \frac{\beta}{2n^2} a_{12} \int_{-\tau_2}^{0} \int_{t+\theta}^{t} y_2^2(s^-) \mathrm{d}s \mathrm{d}\mu_2(\theta) + \frac{a_{21}}{2n^2} \int_{-\tau_1}^{0} \int_{t+\theta}^{t} y_1^2(s^-) \mathrm{d}s \mathrm{d}\mu_1(\theta). \end{aligned} \tag{10}$$

Choose a constant $\beta > 0$ and integer $n > 0$ such that

$$\begin{aligned} &\frac{a_{21}}{2n^2} - \beta a_{11} < 0, \\ &\frac{\beta a_{12}}{2n^2} - a_{21} < 0. \end{aligned} \tag{11}$$

For model (M), by Ito's formula, we get

$$\begin{aligned} \mathrm{d}V(y) = &\left[\beta LV_1(y_1) + LV_2(y_2) + \frac{\mathrm{d}}{\mathrm{d}t} V_3(t)\right] \mathrm{d}t \\ &+ \beta\sigma_1(y_1 - 1)\mathrm{d}B_1(t) + \beta \int_{\mathbb{Z}} (y_1\gamma_1(u) - \ln(1 + \gamma_1(u)))\widetilde{\Gamma}(\mathrm{d}s, \mathrm{d}u) \\ &+ \sigma_2(y_2 - 1)\mathrm{d}B_2(t) + \int_{\mathbb{Z}} (y_2\gamma_2(u) - \ln(1 + \gamma_2(u)))\widetilde{\Gamma}(\mathrm{d}s, \mathrm{d}u), \end{aligned} \tag{12}$$

where

$$\begin{aligned} LV_1(y_1) = &(y_1 - 1)\left[r_{10} - r_{11}C_1(t) - a_{11}y_1(t^-) - a_{12} \int_{-\tau_2}^{0} y_2(t^- + \theta) d\mu_2(\theta)\right] \\ &+ 0.5\sigma_1^2 + \int_{\mathbb{Z}} (\gamma_1(u) - \ln(1 + \gamma_1(u)))\lambda(\mathrm{d}u), \\ LV_2(y_2) = &(y_2 - 1)\left[r_{20} - r_{21}C_2(t) - a_{21} \int_{-\tau_1}^{0} y_1(t^- + \theta)\mathrm{d}\mu_1(\theta) - a_{22}y_2(t^-)\right] \\ &+ 0.5\sigma_2^2 + \int_{\mathbb{Z}} (\gamma_2(u) - \ln(1 + \gamma_2(u)))\lambda(\mathrm{d}u). \end{aligned} \tag{13}$$

By basic inequality $a^2 + b^2 \ge 2ab$, we have

$$a_{12} \int_{-\tau_2}^{0} y_2(t^- + \theta)\mathrm{d}\mu_2(\theta) \le \frac{1}{2} a_{12}\left(n^2 + \frac{1}{n^2} \int_{-\tau_2}^{0} y_2^2(t^- + \theta)\mathrm{d}\mu_2(\theta)\right), \tag{14}$$

$$a_{21} \int_{-\tau_1}^{0} y_1(t^- + \theta)\mathrm{d}\mu_1(\theta) \le \frac{1}{2} a_{21}\left(n^2 + \frac{1}{n^2} \int_{-\tau_1}^{0} y_1^2(t^- + \theta)\mathrm{d}\mu_1(\theta)\right). \tag{15}$$

Substituting (14) and (15) into $LV_1(y_1), LV_2(y_2)$, then

$$\begin{aligned} LV_1(y_1) &\le y_1(r_{10} - r_{11}C_1(t)) - a_{11}y_1^2 - r_{10} + r_{11}C_1(t) + a_{11}y_1 + 0.5\sigma_1^2 \\ &\quad + \int_{\mathbb{Z}} (\gamma_1(u) - \ln(1+\gamma_1(u)))\lambda(\mathrm{d}u) + a_{12}\int_{-\tau_2}^{0} y_2(t^- + \theta)\mathrm{d}\mu_2(\theta) \\ &\le y_1(r_{10} - r_{11}C_1(t)) - a_{11}y_1^2 - r_{10} + r_{11}C_1(t) + a_{11}y_1 + 0.5\sigma_1^2 \\ &\quad + \int_{\mathbb{Z}} (\gamma_1(u) - \ln(1+\gamma_1(u)))\lambda(\mathrm{d}u) + \frac{n^2}{2}a_{12} + \frac{a_{12}}{2n^2}\int_{-\tau_2}^{0} y_2^2(t^- + \theta)\mathrm{d}\mu_2(\theta), \\ LV_2(y_2) &\le y_2(r_{20} - r_{21}C_2(t)) - a_{22}y_2^2 - r_{20} + r_{21}C_2(t) + a_{22}y_2 + 0.5\sigma_2^2 \\ &\quad + \int_{\mathbb{Z}} (\gamma_2(u) - \ln(1+\gamma_2(u)))\lambda(\mathrm{d}u) + \frac{n^2}{2}a_{21} + \frac{a_{21}}{2n^2}\int_{-\tau_1}^{0} y_1^2(t^- + \theta)\mathrm{d}\mu_1(\theta). \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} &\beta LV_1(y_1) + LV_2(y_2) + \frac{\mathrm{d}}{\mathrm{d}t}V_3(t) \\ &\le \beta\left[\begin{array}{c} r_{10}y_1 - r_{11}C_1(t)y_1 - a_{11}y_1^2 - r_{10} + r_{11}C_1(t) + a_{11}y_1 + 0.5\sigma_1^2 + \dfrac{n^2}{2}a_{12} \\ + \displaystyle\int_{\mathbb{Z}} (\gamma_1(u) - \ln(1+\gamma_1(u)))\lambda(\mathrm{d}u) + \frac{a_{12}}{2n^2}y_2^2 \end{array}\right] + r_{20}y_2 - r_{21}C_2(t)y_2 - a_{22}y_2^2 - r_{20} \\ &\quad + r_{21}C_2(t) + a_{22}y_2 + 0.5\sigma_2^2 + \int_{\mathbb{Z}} (\gamma_2(u) - \ln(1+\gamma_2(u)))\lambda(\mathrm{d}u) + \frac{n^2}{2}a_{21} + \frac{a_{21}}{2n^2}y_1^2. \end{aligned} \tag{17}$$

Substituting (17) into $\mathrm{d}V(y)$ and together with (11), there exists a constant $K > 0$ such that

$$\begin{aligned} \mathrm{d}V(y) &= K\mathrm{d}t + \beta\sigma_1(y_1 - 1)\mathrm{d}B_1(t) + \beta\int_{\mathbb{Z}} (y_1\gamma_1(u) - \ln(1+\gamma_1(u)))\widetilde{\Gamma}(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \sigma_2(y_2 - 1)\mathrm{d}B_2(t) + \int_{\mathbb{Z}} (y_2\gamma_2(u) - \ln(1+\gamma_2(u)))\widetilde{\Gamma}(\mathrm{d}s, \mathrm{d}u). \end{aligned} \tag{18}$$

By this result and according to the argument in [23], we have

$$\infty \le KT + V_0(y_1(0), y_2(0)) \le \infty, \tag{19}$$

which leads to a contradiction, and hence, $\tau_\infty = \infty$ a.s. Therefore, $\tau_e = \infty$ a.s. The proof is completed. □

Lemma 2. *Let $(y_1(t), y_2(t))$ be a positive solution of (14) initial data $(\xi_1(\theta), \xi_2(\theta)) \in C([-\tau, 0], R_+^2)$. Then, for any $p > 0$, there exists a constant $K_i(p) > 0$ such that*

$$\lim_{t\longrightarrow+\infty} \sup E\left[y_i^p(t)\right] \le K_i(p), \quad i = 1, 2. \tag{20}$$

Proof. We only prove $\lim_{t\longrightarrow+\infty}\sup E[y_1^p(t)] \le K_1(p)$. The rest is similar and omitted. Defining $Q_1(t) = e^t y_1^p(t)$, by Ito's formula, we have

$$\begin{aligned} \mathrm{d}Q_1(t) &= LQ_1(t)\mathrm{d}t + pe^t y_1^p \sigma_1 \mathrm{d}B_1(t) \\ &\quad + \int_{\mathbb{Z}} e^t y_1^p (1+\gamma_1(u))^p \widetilde{\Gamma}(\mathrm{d}s, \mathrm{d}u), \end{aligned} \tag{21}$$

where

$$LQ_1(t) = e^t y_1^p \left\{ 1 + \frac{p(p-1)\sigma_1^2}{2} + \int_{\mathbb{Z}} \left((1+\gamma_1(u))^p - p\gamma_1(u) \right) \lambda(\mathrm{d}u) + p\left[r_{10} - r_{11}C_1(t) - a_{11}y_1(t^-) - a_{12}\int_{-\tau_2}^{0} y_2(t^- + \theta)\mathrm{d}\mu_2(\theta) \right] \right\}. \quad (22)$$

Let $K_1(p) = \max_{y_1 \geq 0}\{y_1^p[1 + (p(p-1)\sigma_1^2/2) + \int_{\mathbb{Z}}((1+\gamma_1(u))^p - p\gamma_1(u))\lambda(\mathrm{d}u) + pr_{10}] - pa_{11}y_1^{p+1}\}$, then

$$LQ_1(t) \leq K_1(p)e^t. \quad (23)$$

Integrating both sides of (21) from 0 to t and taking expectation leads to

$$E\left(e^t y_1^p\right) - \xi_1^p(0) \leq K_1(p)\left(e^t - 1\right). \quad (24)$$

Then, $\lim_{t\longrightarrow+\infty}\sup E[y_1^p(t)] \leq K_1(p)$. The proof is completed. □

Lemma 3 (see [18]). *Let $Y \in C(R_+ \times \Omega, R_+)$, $Z \in C(R_+ \times \Omega, R)$, and $\rho \in C(R_+, R)$ satisfying $\lim_{t\longrightarrow\infty}(Z(t)/t) = 0$ a.s.*

(1) If there exist two constants $T > 0$ and $\rho_0 > 0$ such that
$\ln Y(t) = \int_0^t \rho(s)ds - \rho_0 \int_0^t Y(s)ds + Z(t)$ a.s. for all $t \geq T$,
then (i) $\lim_{t\longrightarrow\infty} Y(t) = 0$ and $\lim_{t\longrightarrow\infty}\sup(\ln Y(t)/t) \leq 0$ a.s. if $\overline{\rho} < 0$;
$\lim_{t\longrightarrow\infty}\overline{Y(t)} = \rho/\rho_0$ and $\lim_{t\longrightarrow\infty}(\ln Y(t)/t) = 0$ a.s. if $\overline{\rho} \geq 0$.

(2) If there exist two constants $T > 0$ and $\rho_0 > 0$ such that
$\ln Y(t) \leq \int_0^t \rho(s)ds - \rho_0 \int_0^t Y(s)ds + Z(t)$ a.s. for all $t \geq T$
then (i) $\lim_{t\longrightarrow\infty}\overline{Y(t)}^* \leq \rho/\rho_0$ a.s. if $\overline{\rho} \geq 0$;
$\lim_{t\longrightarrow\infty} Y(t) = 0$, a.s. if $\overline{\rho} < 0$.

(3) If there exist two constants $T > 0$ and $\rho_0 > 0$ such that for all

$$\ln Y(t) \geq \int_0^t \rho(s)\mathrm{d}s - \rho_0 \int_0^t Y(s)\mathrm{d}s + Z(t), \quad \text{a.s. } t \geq T. \quad (25)$$

Then, $\lim_{t\longrightarrow\infty}\overline{Y(t)}_* \geq \rho/\rho_0$ a.s.

Lemma 4. *Let $(y_1(t), y_2(t))$ be any positive solution of model (M), then*

(1) $\lim_{t\longrightarrow\infty}\sup(\ln y_i(t)/t) \leq 0$ a.s., $i = 1, 2$;
(2) For any positive constant τ, $\lim_{t\longrightarrow\infty} t^{-1}\int_{t-\tau}^{t} y_i(s)ds = 0$ a.s., $i = 1, 2$.

The proof of Lemma 4 is standard and is omitted (see, e.g., [24]).

3. Main Results

3.1. Persistence in Mean

Definition 1 (see [25]). The system is said to be persistence in mean if there are positive constants v_i and s_i $(i = 1, 2)$ such that

$$v_i \leq \overline{y_i(t)}_* \leq \overline{y_i(t)}^* \leq s_i, \quad i = 1, 2, \quad (26)$$

holds for any solution $(y_1(t), y_2(t))$ of model (M) with initial data

$$y(t) = \{(\xi_1(t), \xi_2(t)): -\tau \leq t \leq 0\} \in C\left([-\tau, 0]: R_+^2\right). \quad (27)$$

Theorem 1. *Assume $\Delta_1 > 0, \Delta_2 > 0$, then for any initial data $y(t) = \{(\xi_1(t), \xi_2(t)): -\tau \leq t \leq 0\} \in C([-\tau, 0]: R_+^2)$, the solution $(y_1(t), y_2(t))$ of model (M_0) has the following properties:*

$$\lim_{t\longrightarrow\infty}\overline{y_1(t)} = \frac{\Delta_1}{\Delta}, \qquad \lim_{t\longrightarrow\infty}\overline{y_2(t)} = \frac{\Delta_2}{\Delta}. \quad (28)$$

Proof. For $i = 1, 2$, we compute

$$\int_0^t \int_{-\tau_i}^0 y_i(s^- + \theta)\mathrm{d}\mu_i(\theta)\mathrm{d}s - \int_{-\tau_i}^0 \mathrm{d}\mu_i(\theta)\int_0^t y_i(s^-)\mathrm{d}s = \int_{-\tau_i}^0 \int_\theta^0 y_i(s^-)\mathrm{d}s\,\mathrm{d}\mu_i(\theta) - \int_{-\tau_i}^0 \int_{t+\theta}^t y_i(s^-)\mathrm{d}s\,\mathrm{d}\mu_i(\theta). \quad (29)$$

For model (M_0), noticing $\int_{-\tau_i}^{0} d\mu_i(\theta) = 1$ $(i = 1, 2)$ and using Ito's formula, we get

$$\begin{aligned}\ln y_1(t) &= b_1 t - a_{11}\int_0^t y_1(s^-)ds + \sigma_1 B_1(t) + \int_0^t\int_{\mathbb{Z}} \ln(1+\gamma_1(u))\widetilde{\Gamma}(ds, du) \\ &\quad + \ln y_1(0) - a_{12}\int_0^t\int_{-\tau_2}^{0} y_2(s^- + \theta))\mu_2(\theta)ds \\ &= b_1 t - a_{11}\int_0^t y_1(s^-)ds - a_{12}\int_0^t y_2(s^-)ds + \Phi_1(t),\end{aligned} \tag{30}$$

$$\ln y_2(t) = b_2 t - a_{22}\int_0^t y_2(s^-)ds - a_{21}\int_0^t y_1(s^-)ds + \Phi_2(t), \tag{31}$$

where

$$\begin{aligned}\Phi_1(t) &= a_{12}\int_{-\tau_2}^{0}\int_{t+\theta}^{t} y_2(s^-)dsd\mu_2(\theta) - a_{12}\int_{-\tau_2}^{0}\int_{\theta}^{0} y_2(s^-)dsd\mu_2(\theta) \\ &\quad + \ln y_1(0) + \sigma_1 B_1(t) + R_1(t),\end{aligned} \tag{32}$$

$$\begin{aligned}\Phi_2(t) &= a_{21}\int_{-\tau_1}^{0}\int_{t+\theta}^{t} y_1(s^-)dsd\mu_1(\theta) - a_{21}\int_{-\tau_1}^{0}\int_{\theta}^{0} y_1(s^-)dsd\mu_1(\theta) \\ &\quad + \ln y_2(0) + \sigma_2 B_2(t) + R_2(t).\end{aligned} \tag{33}$$

Since

$$\begin{aligned}\int_{-\tau_2}^{0}\int_{t+\theta}^{t} y_2(s^-)dsd\mu_2(\theta) &\le \int_{-\tau_2}^{0} d\mu_2(\theta)\int_{t-\tau_2}^{t} y_2(s^-)ds, \\ \int_{-\tau_2}^{0}\int_{\theta}^{0} y_2(s^-)dsd\mu_2(\theta) &\le \int_{-\tau_2}^{0} d\mu_2(\theta)\int_{-\tau_2}^{0} y_2(s^-)ds.\end{aligned} \tag{34}$$

By Lemma 4, we get

$$\begin{aligned}\lim_{t\longrightarrow\infty}\int_{-\tau_2}^{0}\int_{t+\theta}^{t} y_2(s^-)dsd\mu_2(\theta) &= 0, \\ \lim_{t\longrightarrow\infty}\int_{-\tau_2}^{0}\int_{\theta}^{0} y_2(s^-)dsd\mu_2(\theta) &= 0.\end{aligned} \tag{35}$$

By the same way, we can derive that

$$\begin{aligned}\lim_{t\longrightarrow\infty}\int_{-\tau_1}^{0}\int_{t+\theta}^{t} y_1(s^-)dsd\mu_1(\theta) &= 0, \\ \lim_{t\longrightarrow\infty}\int_{-\tau_1}^{0}\int_{\theta}^{0} y_1(s^-)dsd\mu_1(\theta) &= 0.\end{aligned} \tag{36}$$

Substituting (35) and (36) into (32) and (33), respectively, then

$$\begin{aligned}\lim_{t\longrightarrow\infty} t^{-1}\Phi_1(t) &= 0, \\ \lim_{t\longrightarrow\infty} t^{-1}\Phi_2(t) &= 0 \text{ a.s.}\end{aligned} \tag{37}$$

Computing

$$\begin{aligned}&a_{22}\ln y_1(t) - a_{12}\ln y_2(t) \\ &\quad = (a_{22}b_1 - a_{12}b_2)t - (a_{11}a_{22} - a_{12}a_{21})\int_0^t y_1(s^-)ds + \Phi_3(t) \\ &\quad = \Delta_1 t - \Delta\int_0^t y_1(s^-)ds + \Phi_3(t),\end{aligned} \tag{38}$$

where $\Phi_3(t) = a_{22}\Phi_1(t) - a_{12}\Phi_2(t)$. From (37), we can easily get $\lim_{t\longrightarrow\infty} t^{-1}\Phi_3(t) = 0$ a.s.

By Lemma 4 again, for any $\varepsilon > 0$, there is a $T_0 > 0$ such that

$$a_{22}\ln y_1(t) \le (\Delta_1 + a_{12}\varepsilon)t - \Delta\int_0^t y_1(s^-)ds + \Phi_3(t), \quad \text{for any } t > T_0. \tag{39}$$

According to Lemma 3, then

$$\lim_{t\longrightarrow\infty}\overline{y_1(t)}^{*} \le \frac{\Delta_1}{\Delta} \text{ a.s.} \tag{40}$$

Thus, for any $\varepsilon > 0$ and sufficiently large t, there is

$$a_{21}\overline{y_1(t)} \le a_{21}\overline{y_1(t)}^{*} + \varepsilon \le a_{21}\frac{\Delta_1}{\Delta} + \varepsilon. \tag{41}$$

Using (41) in (31), we get

$$\ln y_2(t) \geq \left(b_2 - a_{21}\frac{\Delta_1}{\Delta} - \varepsilon\right)t - a_{22}\int_0^t y_2(s^-)ds + \Phi_2(t)$$
$$= \left(\frac{a_{22}\Delta_2}{\Delta} - \varepsilon\right)t - a_{22}\int_0^t y_2(s^-)ds + \Phi_2(t), \tag{42}$$

Lemma 3 implies that

$$\lim_{t\longrightarrow\infty} \overline{y_2(t)}_* \geq \frac{\Delta_2}{\Delta} \text{ a.s.} \tag{43}$$

Similarly, we have

$$a_{11}\ln y_2(t) - a_{21}\ln y_1(t)$$
$$= (a_{11}b_2 - a_{21}b_1)t - (a_{11}a_{22} - a_{12}a_{21})\int_0^t y_2(s^-)ds + \Phi_4(t)$$
$$= \Delta_2 t - \Delta\int_0^t y_2(s^-)ds + \Phi_4(t), \tag{44}$$

where $\Phi_4(t) = a_{11}\Phi_2(t) - a_{21}\Phi_1(t)$. Obviously, $\lim_{t\longrightarrow\infty}\Phi_4(t) = 0$.

From Lemma 4, for any $\varepsilon > 0$, there is a $T_1 > 0$ such that

$$a_{11}\ln y_2(t) \leq (\Delta_2 + a_{21}\varepsilon)t - \Delta\int_0^t y_2(s^-)ds + \Phi_4(t), \quad \text{for } t > T_1. \tag{45}$$

It follows from Lemma 3 that

$$\lim_{t\longrightarrow\infty} \overline{y_2(t)}^* \leq \frac{\Delta_2}{\Delta} \text{ a.s.} \tag{46}$$

Combining (43) and (46) leads to

$$\lim_{t\longrightarrow\infty} \overline{y_2(t)} = \frac{\Delta_2}{\Delta} \text{ a.s.} \tag{47}$$

Substituting (47) into (30) and using Lemma 2, we get

$$\lim_{t\longrightarrow\infty} \overline{y_1(t)} = \frac{\Delta_1}{\Delta} \text{ a.s.} \tag{48}$$

The proof is completed. □

Next, let us consider model (M).

Theorem 2. *If $\Delta_{31} > 0, \Delta_{41} > 0$ hold, then for any initial $y(t) = \{(\xi_1(t), \xi_2(t)):\ -\tau \leq t \leq 0\} \in C([-\tau, 0]; R_+^2)$, the solution $(y_1(t), y_2(t))$ of model (M) has the properties that*

$$\frac{\Delta_{31}}{\Delta} \leq \overline{y_1(t)}_* \leq \overline{y_1(t)}^* \leq \frac{\Delta_{32}}{\Delta},$$
$$\frac{\Delta_{41}}{\Delta} \leq \overline{y_2(t)}_* \leq \overline{y_2(t)}^* \leq \frac{\Delta_{42}}{\Delta}. \tag{49}$$

That is to say, model (M) will be persistence in mean.

Proof. Using Ito's formula to compute $\ln y_i(t)$ $(i = 1, 2)$, we have

$$t^{-1}\ln y_1(t) = b_1 - r_{11}\overline{C_1(t)} - a_{11}\overline{y_1(t)} - a_{12}\overline{y_2(t)} + t^{-1}\Phi_1(t), \tag{50}$$

$$t^{-1}\ln y_2(t) = b_2 - r_{21}\overline{C_2(t)} - a_{21}\overline{y_1(t)} - a_{22}\overline{y_2(t)} + t^{-1}\Phi_2(t). \tag{51}$$

Then,

$$a_{11}t^{-1}\ln y_2(t) - a_{21}t^{-1}\ln y_1(t)$$
$$= a_{11}\left(b_2 - r_{21}\overline{C_2(t)}\right) - a_{21}\left(b_1 - r_{11}\overline{C_1(t)}\right) - (a_{11}a_{22} - a_{12}a_{21})\overline{y_2(t)} + t^{-1}\Phi_4(t)$$
$$\geq a_{11}\left(b_2 - r_{21}\overline{C_2(t)}^*\right) - a_{21}\left(b_1 - r_{11}\overline{C_1(t)}_*\right) - (a_{11}a_{22} - a_{12}a_{21})\overline{y_2(t)} + t^{-1}\Phi_4(t)$$
$$= \Delta_{41} - \Delta\overline{y_2(t)} + t^{-1}\Phi_4(t). \tag{52}$$

We can get from Lemma 3 that

$$\overline{y_2(t)}_* \geq \frac{\Delta_{41}}{\Delta} \text{ a.s.} \tag{53}$$

For any $\varepsilon > 0$, there is sufficiently $T_2 > 0$ such that

$$a_{12}\overline{y_2(t)} \geq a_{12}\overline{y_2(t)}_* - \varepsilon \geq a_{12}\frac{\Delta_{41}}{\Delta} - \varepsilon, \quad t \geq T_2. \tag{54}$$

Substituting (54) into (50), we get

$$t^{-1}\ln y_1(t) \leq b_1 - r_{11}\overline{C_1(t)}_* - a_{12}\frac{\Delta_{41}}{\Delta} + \varepsilon - a_{11}\overline{y_1(t)} + t^{-1}\Phi_1(t) \tag{55}$$
$$= \frac{a_{11}\Delta_{32}}{\Delta} + \varepsilon - a_{11}\overline{y_1(t)} + t^{-1}\Phi_1(t).$$

Therefore,

$$\overline{y_1(t)}^* \le \frac{\Delta_{32}}{\Delta} \text{ a.s.} \tag{56}$$

Similar computation leads to

$$\begin{aligned} a_{22}t^{-1}\ln y_1(t) - a_{12}t^{-1}\ln y_2(t) &= a_{22}\left(b_1 - r_{11}\overline{C_1(t)}\right) - a_{12}\left(b_2 - r_{21}\overline{C_2(t)}\right) - \left(a_{11}a_{22} - a_{12}a_{21}\right)\overline{y_1(t)} + t^{-1}\Phi_3(t) \\ &\ge \Delta_{31} - \Delta\overline{y_1(t)} + t^{-1}\Phi_3(t). \end{aligned} \tag{57}$$

In view of Lemmas 3 and 4, we have

$$\overline{y_1(t)}_* \ge \frac{\Delta_{31}}{\Delta} \text{ a.s.} \tag{58}$$

For any $\varepsilon > 0$, there is a $T_3 > 0$ such that

$$a_{21}\overline{y_1(t)} \ge a_{21}\overline{y_1(t)}_* - \varepsilon \ge a_{21}\frac{\Delta_{31}}{\Delta} - \varepsilon, \quad t \ge T_3. \tag{59}$$

Substituting (59) into (51), we get

$$\begin{aligned} t^{-1}\ln y_2(t) &\le b_2 - r_{21}\overline{C_2(t)}_* - a_{21}\frac{\Delta_{31}}{\Delta} + \varepsilon - a_{22}\overline{y_2(t)} + t^{-1}\Phi_2(t) \\ &= \frac{a_{22}\Delta_{42}}{\Delta} + \varepsilon - a_{22}\overline{y_2(t)} + t^{-1}\Phi_2(t), \end{aligned} \tag{60}$$

and then we have

$$\overline{y_2(t)}^* \le \frac{\Delta_{42}}{\Delta}. \tag{61}$$

Combing (53), (56), (58), and (61) leads to the result. The proof is completed. □

Remark 1. If the limit of $\overline{C_1(t)}$ and $\overline{C_2(t)}$ exist, that is, $\overline{C_1(t)}^* = \overline{C_1(t)}_*$ and $\overline{C_2(t)}^* = \overline{C_2(t)}_*$, then Theorem 2 will be simplified as the following case.

If $\Delta_{31} > 0, \Delta_{41} > 0$ hold, then model (M) will have the properties that

$$\begin{aligned} \lim_{t\to\infty}\overline{y_1(t)} &= \frac{\Delta_{31}}{\Delta}, \\ \lim_{t\to\infty}\overline{y_2(t)} &= \frac{\Delta_{41}}{\Delta}. \end{aligned} \tag{62}$$

3.2. Nonpersistence

Theorem 3. *If* $0 < (a_{11}/a_{21})(b_2 - r_{21}\overline{C_2(t)}_*) < b_1 - r_{11}\overline{C_1(t)}^* \le b_1 - r_{11}\overline{C_1(t)}_*$ *holds, then for any initial data* $y(t) = \{(\xi_1(t), \xi_2(t)) : -\tau \le t \le 0\} \in C([-\tau, 0] : R_+^2)$, *the solution* $(y_1(t), y_2(t))$ *of model (M) has the properties that*

$$\begin{aligned} &\frac{b_1 - r_{11}\overline{C_1(t)}^*}{a_{11}} \le \overline{y_1(t)}_* \le \overline{y_1(t)}^* \le \frac{b_1 - r_{11}\overline{C_1(t)}_*}{a_{11}}, \\ &\lim_{t\to\infty} y_2(t) = 0 \text{ a.s.} \end{aligned} \tag{63}$$

That is to say, model (M) is nonpersistent.

Proof. From (50), we get

$$t^{-1}\ln y_1(t) \le b_1 - r_{11}\overline{C_1(t)}_* - a_{11}\overline{y_1(t)} + t^{-1}\Phi_1(t). \tag{64}$$

According to Lemma 3, we get

$$\overline{y_1(t)}^* \le \frac{b_1 - r_{11}\overline{C_1(t)}_*}{a_{11}} \text{ a.s.} \tag{65}$$

We can easily compute

$$\begin{aligned} a_{11}t^{-1}\ln y_2(t) - a_{21}t^{-1}\ln y_1(t) &= a_{11}\left(b_2 - r_{21}\overline{C_2(t)}\right) - a_{21}\left(b_1 - r_{11}\overline{C_1(t)}\right) \\ &\quad - \Delta\overline{y_2(t)} + t^{-1}\Phi_4(t). \end{aligned} \tag{66}$$

By Lemma 4, for enough large t and any $\varepsilon > 0$, there is

$$\frac{\ln y_i(t)}{t} < \varepsilon, \quad i = 1, 2. \tag{67}$$

Then,

$$\begin{aligned} a_{11}t^{-1}\ln y_2(t) &\le a_{11}\left(b_2 - r_{21}\overline{C_2(t)}_*\right) - a_{21}\left(b_1 - r_{11}\overline{C_1(t)}^*\right) \\ &\quad - \Delta\overline{y_2(t)} + t^{-1}\Phi_4(t). \end{aligned} \tag{68}$$

By the assumption

$$\frac{a_{11}}{a_{21}}\left(b_2 - r_{21}\overline{C_2(t)}_*\right) < b_1 - r_{11}\overline{C_1(t)}^*, \tag{69}$$

and Lemma 3, we obtain from (68) that

$$\lim_{t\to\infty} y_2(t) = 0 \text{ a.s.} \tag{70}$$

Substituting (70) into (50), then

$$\begin{aligned} t^{-1}\ln y_1(t) &= b_1 - r_{11}\overline{C_1(t)} - a_{11}\overline{y_1(t)} + t^{-1}\Phi_1(t) \\ &\ge b_1 - r_{11}\overline{C_1(t)}^* - a_{11}\overline{y_1(t)} + t^{-1}\Phi_1(t). \end{aligned} \tag{71}$$

From Lemma 3, we get

$$\overline{y_1(t)}_* \geq \frac{b_1 - r_{11}\overline{C_1(t)}^*}{a_{11}}, \quad (72)$$

and thus,

$$\frac{b_1 - r_{11}\overline{C_1(t)}^*}{a_{11}} \leq \overline{y_1(t)}_* \leq \overline{y_1(t)}^* \leq \frac{b_1 - r_{11}\overline{C_1(t)}_*}{a_{11}}. \quad (73)$$

The proof is finished. □

Theorem 4. *If* $(a_{11}/a_{21})(b_2 - r_{21}\overline{C_2(t)}_*) < b_1 - r_{11}\overline{C_1(t)}^* \leq b_1 - r_{11}\overline{C_1(t)}_* < 0$ *holds, then for any initial data* $y(t) = \{(\xi_1(t), \xi_2(t)): -\tau \leq t \leq 0\} \in C([-\tau, 0]: R_+^2)$, *the solution* $(y_1(t), y_2(t))$ *of model (M) has the properties that*

$$\lim_{t\to\infty} y_1(t) = 0,$$
$$\lim_{t\to\infty} y_2(t) = 0 \text{ a.s.} \quad (74)$$

That is to say, model (M) is nonpersistent.

Proof. Under the assumption that $b_1 - r_{11}\overline{C_1(t)}_* < 0$, we can get from Lemma 3 and (64) that

$$\lim_{t\to\infty} y_1(t) = 0 \text{ a.s.} \quad (75)$$

Substituting (75) into (51) reads

$$\begin{aligned} t^{-1}\ln y_2(t) &= b_2 - r_{21}\overline{C_2(t)} - a_{22}\overline{y_2(t)} + t^{-1}\Phi_2(t) \\ &\leq b_2 - r_{21}\overline{C_2(t)}_* - a_{22}\overline{y_2(t)} + t^{-1}\Phi_2(t). \end{aligned} \quad (76)$$

The assumption $(a_{11}/a_{21})(b_2 - r_{21}\overline{C_2(t)}_*) < b_1 - r_{11}\overline{C_1(t)}^* < 0$ implies that $b_2 - r_{21}\overline{C_2(t)}_* < 0$. Then, from (76) and Lemma 3, it is easy to verify that

$$\lim_{t\to\infty} y_2(t) = 0 \text{ a.s.} \quad (77)$$

This completes the proof. □

3.3. Stable in Distribution

Theorem 5. *If all the assumptions hold, then model (M) is stable in distribution.*

Proof. The proof of this result is divided into three steps.

Step 1. We first prove model (M) is globally attractive.

Let $y(t) = (y_1(t;\varphi), y_2(t;\varphi)), y^\wedge(t) = (y_1(t;\varphi^\wedge), y_2(t;\varphi^\wedge))$ be any two solutions of (M) with initial data $\varphi, \varphi^\wedge \in C([-\tau, 0], R_+^2)$, respectively. Denote $\tilde{y}_i(t) = y_i(t;\varphi) - y_i(t;\varphi^\wedge)$, $i = 1, 2$. We only need to verify that

$$\lim_{t\to+\infty} E|\tilde{y}_i(t)| = \lim_{t\to+\infty} E|y_i(t;\varphi) - y_i(t;\varphi^\wedge)| = 0, \quad i = 1, 2. \quad (78)$$

Define $V_i(t;\varphi,\varphi^\wedge) = |\ln y_i(t;\varphi) - \ln y_i(t;\varphi^\wedge)|$, $i = 1, 2$. By Ito's formula, we get

$$\begin{aligned} LV_1(t;\varphi,\varphi^\wedge) &= \operatorname{sign}(\tilde{y}_1(t))[-a_{11}(y_1(t;\varphi) - y_1(t;\varphi^\wedge)) - a_{12}\int_{-\tau_2}^{0}(y_2(t+\theta;\varphi) - y_2(t+\theta;\varphi^\wedge))\mathrm{d}\mu_2(\theta) \\ &\leq -a_{11}|\tilde{y}_1(t)| + a_{12}\int_{-\tau_2}^{0}|\tilde{y}_2(t+\theta)|\mathrm{d}\mu_2(\theta), \\ LV_2(t;\varphi,\varphi^\wedge) &= \operatorname{sign}(\tilde{y}_2(t))[-a_{22}(y_2(t;\varphi) - y_2(t;\varphi^\wedge)) - a_{21}\int_{-\tau_1}^{0}(y_1(t+\theta;\varphi) - y_1(t+\theta;\varphi^\wedge))\mathrm{d}\mu_1(\theta) \\ &\leq -a_{22}|\tilde{y}_2(t)| + a_{21}\int_{-\tau_1}^{0}|\tilde{y}_1(t+\theta)|\mathrm{d}\mu_1(\theta). \end{aligned} \quad (79)$$

Let

$$V(t;\varphi,\varphi^\wedge) = \sum_{i=1}^{2} V_i(t;\varphi,\varphi^\wedge) + V_3(t;\varphi,\varphi^\wedge), \quad (80)$$

where $V_3(t;\varphi,\varphi^\wedge) = a_{12}\int_{-\tau_2}^{0}\int_{t+\theta}^{t}|\tilde{y}_2(s)|\mathrm{d}\mu_2(\theta)\mathrm{d}s + a_{21}\int_{-\tau_1}^{0}\int_{t+\theta}^{t}|\tilde{y}_1(s)|\mathrm{d}\mu_1(\theta)\mathrm{d}s$.

Using Ito's formula to (80), we can easily compute that

$$\begin{aligned} LV(t;\varphi,\varphi^\wedge) &= LV_1(t;\varphi,\varphi^\wedge) + LV_2(t;\varphi,\varphi^\wedge) + \frac{\mathrm{d}V_3(t;\varphi,\varphi^\wedge)}{\mathrm{d}t} \\ &\leq -\left(a_{11} - a_{21}\int_{-\tau_1}^{0}\mathrm{d}\mu_1(\theta)\right)|\tilde{y}_1(t)| - \left(a_{22} - a_{12}\int_{-\tau_2}^{0}\mathrm{d}\mu_2(\theta)\right)|\tilde{y}_2(t)|. \end{aligned} \quad (81)$$

According to (81), we get

$$E\left(V\left(t;\varphi,\varphi^{\wedge}\right)\right)\leq V\left(0;\varphi,\varphi^{\wedge}\right)-\left(a_{11}-a_{21}\int_{-\tau_1}^{0}\mathrm{d}\mu_1(\theta)\right)\int_0^t E\left|\tilde{y}_1(s)\right|\mathrm{d}s-\left(a_{22}-a_{12}\int_{-\tau_2}^{0}\mathrm{d}\mu_2(\theta)\right)\int_0^t E\left|\tilde{y}_2(s)\right|\mathrm{d}s, \tag{82}$$

which means

$$E\left(V\left(t;\varphi,\varphi^{\wedge}\right)\right)+\left(a_{11}-a_{21}\int_{-\tau_1}^{0}d\mu_1(\theta)\right)\int_0^t E\left|\tilde{y}_1(s)\right|\mathrm{d}s+\left(a_{22}-a_{12}\int_{-\tau_2}^{0}\mathrm{d}\mu_2(\theta)\right)\int_0^t E\left|\tilde{y}_2(s)\right|\mathrm{d}s \leq V\left(0;\varphi,\varphi^{\wedge}\right)<+\infty. \tag{83}$$

Consequently,

$$E\left|\tilde{y}_i(t)\right|\in L^1[0,+\infty),\quad i=1,2. \tag{84}$$

Furthermore, considering the continuity of $E(y_i(t))$, $i=1,2$, and combining (M), we have

$$E\left(y_1(t)\right)=y_1(0)+\int_0^t E\left[r_{10}y_1(s)-r_{11}C_1(t)y_1(s)-a_{11}y_1^2(s)-a_{12}y_1(s)\int_{-\tau_2}^{0}y_2(s+\theta)\mathrm{d}\mu_2(\theta)\right]\mathrm{d}s. \tag{86}$$

$$E\left(B_i(t)\right)=0, E\left(R_i(t)\right)=0,\quad i=1,2. \tag{85}$$

Therefore,

It is not difficult to see that $E(y_1(t))$ is differential. From Lemma 2, we get

$$\frac{\mathrm{d}E\left(y_1(t)\right)}{\mathrm{d}t}=E\left[r_{10}y_1(t)-r_{11}C_1(t)y_1(t)-a_{11}y_1^2(t)-a_{12}y_1(t)\int_{-\tau_2}^{0}y_2(t+\theta)\mathrm{d}\mu_2(\theta)\right]\leq E\left(y_1(t)\right)r_{10}\leq r_{10}K_1, \tag{87}$$

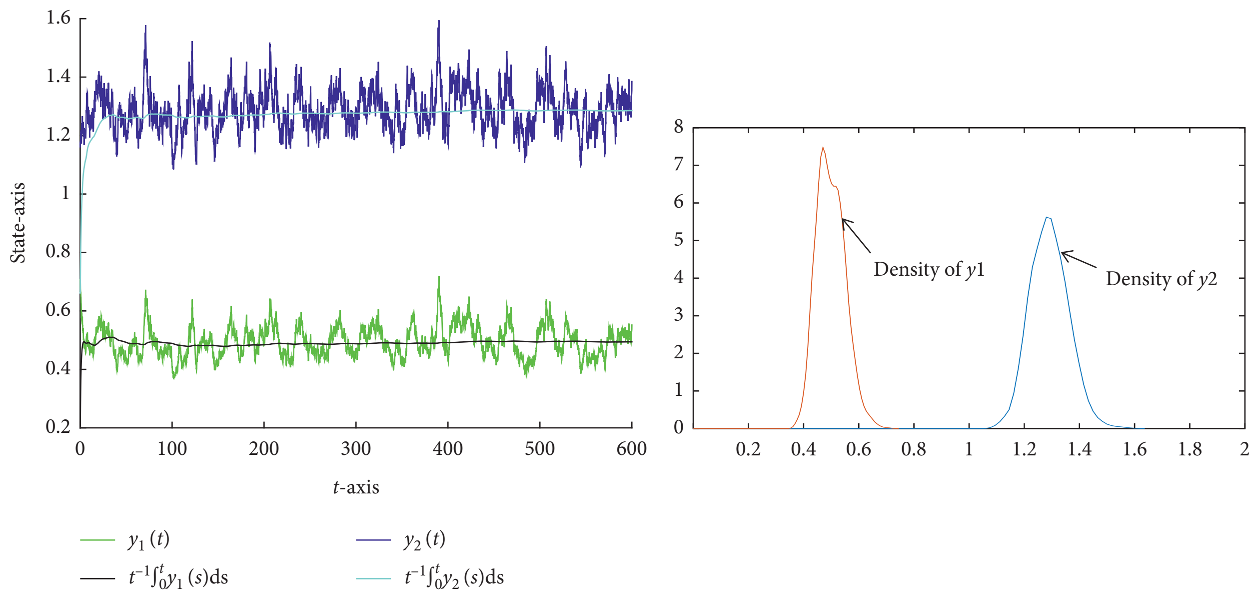
where K_1 is a positive constant. Therefore, $E(y_1(t))$ is uniformly continuous. Similarly, we can also obtain that $E(y_2(t))$ is uniformly continuous. By virtue of (84) and Barbalat's conclusion in [26], we get

$$\lim_{t\longrightarrow+\infty}E\left|\tilde{y}_i(t)\right|=0,\quad i=1,2. \tag{88}$$

Step 2. For any $\varphi\in C([-\tau,0],R_+^2)$, it is denoted by $p(t,\varphi,\mathrm{d}z)$ the transition probability of the process $z(t)$, $P(t,\varphi,R_+^2)$ the probability of $(y_1(t;\varphi),y_2(t;\varphi))^T\in R_+^2$, and $\mathscr{P}([-\tau,0],R_+^2)$ the space of all probability measures on $C([-\tau,0],R_+^2)$. For any $P_1,P_2\in\mathscr{P}([-\tau,0],R_+^2)$, define

$$\mathrm{d}_{\mathrm{BL}}\left(P_1,P_2\right)=\sup_{g\in\mathrm{BL}}\left|\int_{R_+^2}g(z)P_1(\mathrm{d}z)-\int_{R_+^2}g(z)P_2(\mathrm{d}z)\right|, \tag{89}$$

where $\mathrm{BL}=\{g\colon C([-\tau,0],R_+^2)\longrightarrow R\colon |g(z_1)-g(z_2)|\leq \|z_1-z_2\|, |g(\cdot)|\leq 1\}$. Thanks to Lemma 2 and Chebyshev's inequality, for any $\varphi\in C([-\tau,0],R_+^2)$, the family $p(t,\varphi,\mathrm{d}z)$



(a) (b)

Figure 1: The persistent in mean of model (M_0) with $\gamma_1(u) = 1.0187, \gamma_2(u) = 0.1332$. (a) The time series of y_1, y_2, respectively; (b) the probability density function of $\gamma_1(u) = 1.0429, \gamma_2(u) = 1.6876$, respectively.

is tight. That is, for any $\varepsilon \in (0,1)$, there exists a compact subset $D \subseteq R_+^2$ such that $P(t,\varphi,D) \geq 1-\varepsilon$ for any $t \geq 0$..

We compute

$$\begin{aligned}
&\mathrm{d}_{\mathrm{BL}}(p(t+s,\varphi,\cdot),p(t,\varphi,\cdot)) \\
&\quad = \sup_{g\in \mathrm{BL}}\left|\int_{R_+^2} g(z(t+s;\varphi))p(t+s,\varphi,\mathrm{d}z) - \int_{R_+^2} g(z(t;\varphi))p(t,\varphi,\mathrm{d}z)\right| \\
&\quad = \sup_{g\in \mathrm{BL}}|E[g(z(t+s;\varphi))] - E[g(z(t;\varphi))]| \\
&\quad = \sup_{g\in \mathrm{BL}}\left|E[E[g(z(t+s;\varphi))|\mathscr{F}_s]] - E[g(z(t;\varphi))]\right| \\
&\quad = \sup_{g\in \mathrm{BL}}\left|\int_{R_+^2} E[g(z(t;\psi))]p(s,\varphi,\mathrm{d}\psi) - E[g(z(t;\varphi))]\right| \\
&\quad = \sup_{g\in \mathrm{BL}}\left|\int_{R_+^2} E[g(z(t;\psi)) - g(z(t;\varphi))]p(s,\varphi,\mathrm{d}\psi)\right| \\
&\quad \leq \sup_{g\in \mathrm{BL}}\int_{R_+^2} E[|g(z(t;\psi)) - g(z(t;\varphi))|]p(s,\varphi,\mathrm{d}\psi) \\
&\quad \leq \sup_{g\in \mathrm{BL}}\int_{U_B} E[|g(z(t;\psi)) - g(z(t;\varphi))|]p(s,\varphi,\mathrm{d}\psi) \\
&\qquad + \sup_{g\in \mathrm{BL}}\int_{R_+^2\setminus U_B} E[|g(z(t;\psi)) - g(z(t;\varphi))|]p(s,\varphi,\mathrm{d}\psi),
\end{aligned} \tag{90}$$

where $U_B = \left\{(x,y)^T \in R_+^2 \colon \sqrt{x^2+y^2} \leq B\right\}$.

$$\begin{aligned}
&\sup_{g\in \mathrm{BL}}\int_{R_+^2\setminus U_B} E[|g(z(t;\psi)) - g(z(t;\varphi))|] \\
&\quad \cdot p(s,\varphi,\mathrm{d}\psi) \leq 2P\left(s,\varphi,R_+^2\setminus U_B\right) \leq 2\varepsilon.
\end{aligned} \tag{91}$$

Therefore, for sufficiently large t and any $\varepsilon > 0$, we can derive that

$$\mathrm{d}_{\mathrm{BL}}(p(t+s,\varphi,\cdot),p(t,\varphi,\cdot)) \leq 3\varepsilon. \tag{92}$$

That is to say, $\{p(t,\varphi,\cdot)\colon t\geq 0\}$ is Cauchy in $\mathscr{P}([-\tau,0],R_+^2)$ with initial data $\varphi \in C([-\tau,0],R_+^2)$.

Step 3. We prove $\lim_{t\longrightarrow+\infty}\mathrm{d}_{\mathrm{BL}}(p(t,\varphi,\cdot),\nu(\cdot)) = 0$.

According to (92), for $\varphi_0 \in C([-\tau,0],R_+^2)$, $\{p(t,\varphi_0,\cdot)\colon t\geq 0\}$ is Cauchy in $\mathscr{P}([-\tau,0],R_+^2)$, then there exists a unique $\nu(\cdot)$ such that

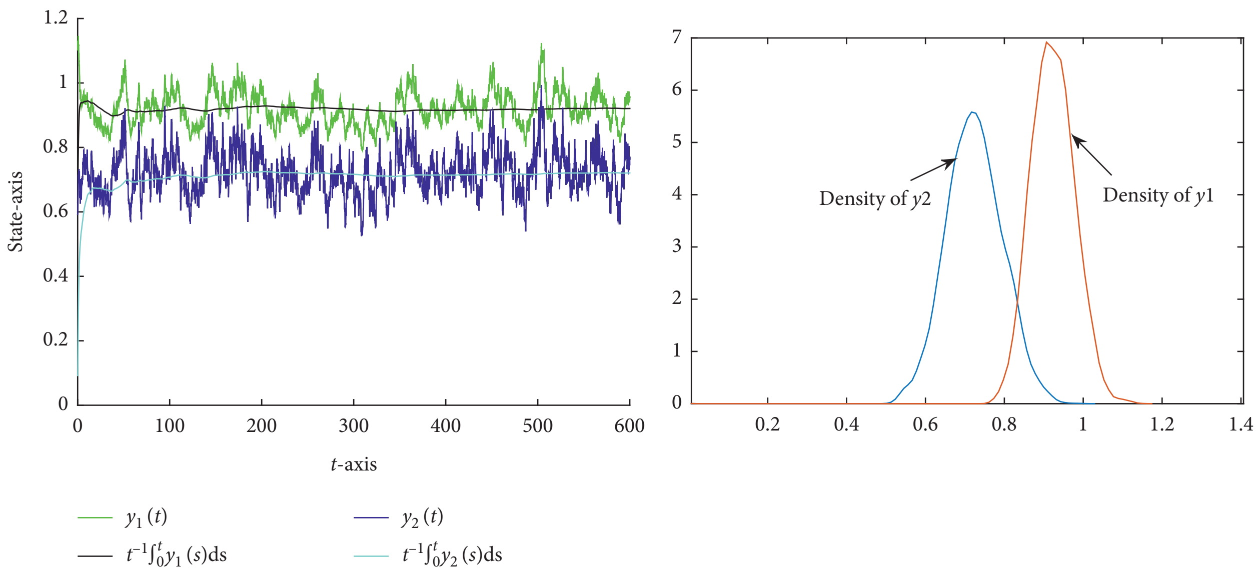


Figure 2: The persistent in mean of model (M) with $\gamma_1(u) = 0.1303, \gamma_2(u) = 0.1837$. (a) Time series of y_1 and y_2; (b) the probability density function of y_1 and y_2.

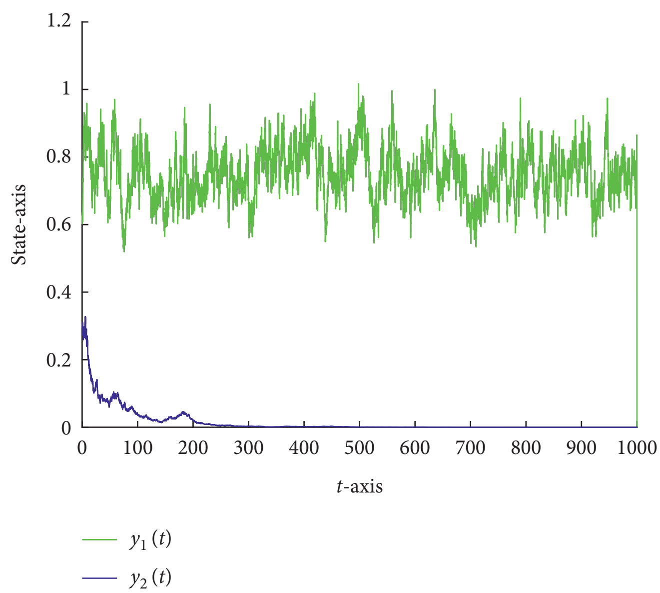


Figure 3: The nonpersistent in mean for model (M) with $\gamma_1(u) = 0.1303$, $\gamma_2(u) = 0.1837$. That is, specie y_2 will die out at some point.

$$\lim_{t\longrightarrow+\infty} \mathrm{d}_{\mathrm{BL}}\left(p\left(t,\varphi_0,\cdot\right),\nu(\cdot)\right) = 0. \tag{93}$$

By virtue of (78), we derive

$$\begin{aligned}
\lim_{t\longrightarrow+\infty} \mathrm{d}_{\mathrm{BL}}\left(p\left(t,\varphi,\cdot\right),p\left(t,\varphi_0,\cdot\right)\right) &= \sup_{g\in BL}\left|E\left[g\left(z\left(t;\varphi\right)\right)\right] - E\left[g\left(z\left(t;\varphi_0\right)\right)\right]\right| \\
&\leq \sup_{g\in BL} E\left[g\left(z\left(t;\varphi\right)\right) - g\left(z\left(t;\varphi_0\right)\right)\right] \\
&\leq \lim_{t\longrightarrow+\infty} E\left[\left\|z\left(t;\varphi\right) - z\left(t;\varphi_0\right)\right\|\right] \\
&= 0.
\end{aligned} \tag{94}$$

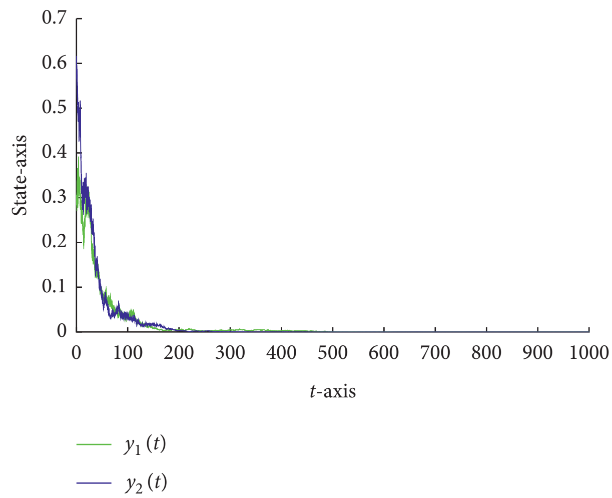


FIGURE 4: The extinction of model (M) with $\gamma_1(u) = 1.0429, \gamma_2(u) = 1.6876$. Both species y_1 and y_2 will die out at some point.

By the triangle inequality and together with (93) and (94), we have

$$\lim_{t\longrightarrow+\infty} d_{BL}(p(t,\varphi,\cdot),\nu(\cdot)) \le \lim_{t\longrightarrow+\infty} d_{BL}(p(t,\varphi,\cdot),p(t,\varphi_0,\cdot)) + \lim_{t\longrightarrow+\infty} d_{BL}(p(t,\varphi_0,\cdot),\nu(\cdot)) = 0. \tag{95}$$

This completes the proof. □

4. Numerical Simulations

In this section, some numerical simulations are given to verify our theoretical results. Take $a_{11} = 0.52, a_{12} = 0.02$, $a_{21} = 0.03, a_{22} = 0.8, r_{10} = 0.71, r_{20} = 1.35, \sigma_1^2 = 0.22, \sigma_2^2 = 0.59$, $\tau_1 = 0.1, \tau_2 = 0.1$. It is easy to check that $\Delta = 0.4154$, which means Assumption 2 holds.

(1) Set $\gamma_1(u) = 1.0187, \gamma_2(u) = 0.1332$, and then it is easy to count that $\Delta_1 = 0.2061, \Delta_2 = 0.5358$. Theorem 1 implies $\lim_{t\longrightarrow\infty}\overline{y_1(t)} = 0.4961$, $\lim_{t\longrightarrow\infty}\overline{y_2(t)} = 1.2900$, that is, the species are both persistent in mean. Simulation also validates the result, see Figure 1.

(2) Set $\gamma_1(u) = 0.1303, \gamma_2(u) = 0.1837, r_{11} = 0.48, r_{21} = 0.86, C_1(t) = 0.1 + 0.1 \sin t, C_2(t) = 0.4 + 0.1 \sin t$. By computation, we have $\Delta_{31} = 0.3813, \Delta_{41} = 0.2994$. By Theorem 2 and Remark 1, model (M) is persistent in mean, shown in Figure 2.

(3) Set $\gamma_1(u) = 0.5528, \gamma_2(u) = 1.0391, r_{11} = 0.48, r_{21} = 0.86, C_1(t) = 0.1 + 0.1 \sin t, C_2(t) = 0.4 + 0.1 \sin t$, and then these parameters satisfy the conditions of Theorem 3. By Theorem 3, we know that y_1 is permanent, but y_2 will die out, see Figure 3.

(4) Set $\gamma_1(u) = 1.0429, \gamma_2(u) = 1.6876, r_{11} = 0.1, r_{21} = 0.56, C_1(t) = 0.4 + 0.1 \sin t, C_2(t) = 0.4 + 0.1 \sin t$, $\sigma_1^2 = 0.87, \sigma_2^2 = 1.35$. Obviously, Theorem 4 shows that both y_1 and y_2 will die out, see Figure 4.

5. Conclusions

The study of biological dynamics has been a popular topic in the field of biomathematics in recent years. With the development of economy, the environmental pollution is becoming more and more serious, which has become an important factor affecting the population relationship. Time delays are also important factors affecting the relationship. In this paper, we formulate a delayed predator-prey model with Lévy noise. Theorem 1 and 2 give the sufficient criteria of persistent in mean for cases (M) and (M_0), respectively. Theorem 3 and 4 obtain the sufficient conditions of nonpersistence. Theorem 5 investigates the stable in distribution. Finally, numerical simulations are given to validate our conclusion.

In view of the complexity of the environments, other factors such as the telephone noise and impulsive input may bring important influence to the dynamics, which needs further research in the future.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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