Research Article

Application of Fixed-Point Index Theory for a Nonlinear Fractional Boundary Value Problem with an Advanced Argument

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In this paper, we investigate the existence of positive solutions of a class of fractional three-point boundary value problem with an advanced argument by using fixed-point index theory. Our results improve and extend some known results in the literature. Two examples are given to demonstrate the effectiveness of our results.

1. Introduction

Three-point boundary value problems arise in a variety of different applied mathematics and physics. In [1], Gupta studied the solvability of three-point BVPs for nonlinear second-order ordinary differential equations. Since then, many authors have investigated the existence and multiplicity of solutions for three-point boundary value problems for nonlinear integer-order ordinary differential equations and nonlinear fractional-order differential equations, see [2–19] and the references therein.

In [2], Ma considered the following three-point boundary value problem:

$$\begin{cases} u''(t) + a(t)f(u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \alpha u(\eta), \end{cases} \quad (1)$$

where $\alpha > 0$, $\eta \in (0, 1)$, $\alpha\eta < 1$, $a \in C([0, 1], t[0, \infty))$, and $f \in C([0, \infty), t[0, \infty))$ is suplinear or sublinear. The existence of at least one positive solution for BVP (1) is obtained by virtue of the Krasnosel'skii fixed-point theorem.

In [3], due to the well-known Guo–Krasnoselskii fixed-point theorem [20], Wang et al. proved the existence of a positive solution to the following three-point BVPs of

nonlinear fractional-order differential equation with an advanced argument:

$$\begin{cases} {}^C D^\alpha u(t) + a(t)f(u(\theta(t))) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \beta u(\eta), \end{cases} \quad (2)$$

where $2 < \alpha \leq 3$, $\eta \in (0, 1)$, $0 < \beta < (1/\eta)$, ${}^C D^\alpha$ is the Caputo fractional derivative, and $f: [0, \infty) \rightarrow [0, \infty)$ is a continuous function.

Recently, in [4], Pourhadi et al. considered the following three-point boundary value problem:

$$\begin{cases} {}^C D^\alpha x(t) = f(t, x(t), x'(t)), & a < t < b, 1 < \alpha < 2, \\ x(a) = 0, x(b) = \mu x(\eta), & a < \eta < b, \mu > \lambda, \end{cases} \quad (3)$$

where $\lambda = (b - a)/(\eta - a)$, ${}^C D^\alpha$ stands for the Caputo fractional derivative, and $f: [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. The existence of positive solutions for BVPs (3) is given by using the inequalities related with Green's function and two well-known fixed-point theorems.

Motivated by the mentioned works, in this paper, we consider the following fractional-order three-point boundary value problem:

$$\begin{cases} {}^C D^\alpha u(t) + a(t)f(u(\theta(t))) = 0, & 0 < t < 1, \\ u(0) - \beta u'(0) = u''(0) = 0, & u(1) = \gamma u(\eta), \end{cases} \quad (4)$$

where $2 < \alpha \leq 3$, $0 < \eta < 1$, $\beta \geq 0$, $0 < \gamma < ((1 + \beta)/(\beta + \eta))$, ${}^C D^\alpha$ is the Caputo fractional derivative, $a \in C([0, 1], t[0, \infty))$, and $f \in C^1([0, \infty), t[0, \infty))$.

Based on the fixed-point index theory, the existence of at least one positive solution to BVP (4) has been obtained. The main results of the paper by Wang et al. [3] have been extended and improved.

Remark 1. If $\beta = 0$, then the fractional differential equations under boundary value conditions (4) reduces to the BVPs (2).

The paper is organized in this order. In Section 2, we present basic definitions on fractional calculus theory and some lemmas which play an important role in the proof of our main results. In Section 3, under an adequate condition imposed on f , we prove the existence of positive solutions of BVPs (4). In Section 4, two examples are given to demonstrate our results.

2. Preliminaries

For convenience, some fundamental definitions on fractional calculus are given. For more details, we refer to [21].

Definition 1. Let $\alpha \geq 0$ and f be a real function defined on $[a, b]$. The Riemann–Liouville fractional integral of order α is defined by $I^\alpha f \equiv f$ and

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, t \in [0, 1]. \quad (5)$$

Definition 2. The Caputo derivative of fractional order $\alpha \geq 0$ is defined by ${}^C D^\alpha f \equiv f$ and

$$({}^C D^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad \alpha > 0, t \in [0, 1], \quad (6)$$

where n is the smallest integer greater or equal to α . In the following, we state some properties of fractional operators that will be used in proving our main results.

Lemma 1 (see [21]). Let $\alpha, \beta \in \mathbb{R}^+$ and $u \in L_1[0, 1]$; then, $I^\alpha I^\beta u(x) = I^{\alpha+\beta} u(x)$ holds almost everywhere on $[0, 1]$.

Lemma 2 (see [21]). Let $\alpha \in \mathbb{R}^+$ and $u \in C[0, 1]$; then, ${}^C D^\alpha I^\alpha u(x) = u(x)$.

Lemma 3 (see [21]). If $\alpha > 0$, n is the smallest integer greater than or equal to α , and $u \in AC^n[0, 1]$, then

$$I^{\alpha C} D^\alpha u(x) = u(x) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \quad (7)$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$.

Lemma 4. Let $2 < \alpha \leq 3$ and $g \in C^1[0, 1]$. A function u is a solution of the following boundary value problem:

$${}^C D^\alpha u(t) + g(t) = 0, \quad 0 < t < 1, \quad (8)$$

$$\begin{aligned} u(0) - \beta u'(0) &= u''(0) = 0, \\ u(1) &= \gamma u(\eta), \end{aligned} \quad (9)$$

if and only if u satisfies the integral equation

$$u(t) = \int_0^1 G(t, s) g(s) ds, \quad (10)$$

where

$$G(t, s) = \begin{cases} \frac{(t+\beta)(1-s)^{\alpha-1} - (t+\beta)\gamma(\eta-s)^{\alpha-1} - (t-s)^{\alpha-1}(1+\beta-(\beta+\gamma)\eta)}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 \leq s \leq t \leq 1, s \leq \eta, \\ \frac{(t+\beta)(1-s)^{\alpha-1} - (t-s)^{\alpha-1}(1+\beta-(\beta+\gamma)\eta)}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 < \eta \leq s \leq t \leq 1, \\ \frac{(t+\beta)(1-s)^{\alpha-1} - (t+\beta)\gamma(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 \leq t \leq s \leq \eta < 1, \\ \frac{(t+\beta)(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 \leq t \leq s \leq 1, \eta \leq s, \end{cases} \quad (11)$$

is the Green function of BVP (8)-(9).

Proof. In view of Lemma 3, we may reduce (8) to an equivalent integral equation:

$$u(t) = c_1 + c_2 t + c_3 t^2 - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds. \quad (12)$$

According to Lemmas 1 and 2, we obtain

$$u'(t) = c_2 + 2c_3 t - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s) ds, \quad (13)$$

$$u''(t) = 2c_3 - \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} g(s) ds.$$

From $u''(0) = 0$, we know that $c_3 = 0$. By $u(0) - \beta u'(0) = 0$, it follows that $c_1 = \beta c_2$. Using the boundary condition $u(1) = \gamma u(\eta)$, we have

$$c_2 = -\frac{1}{1+\beta-(\beta+\eta)\gamma} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds + \frac{\gamma}{1+\beta-(\beta+\eta)\gamma} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds. \quad (14)$$

Therefore, the solution of BVP (8)-(9) is

$$\begin{aligned} u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds + \frac{t+\beta}{1+\beta-(\beta+\eta)\gamma} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds \\ &\quad - \frac{\gamma(t+\beta)}{1+\beta-(\beta+\eta)\gamma} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds \\ &= \int_0^1 G(t,s) g(s) ds, \end{aligned} \quad (15)$$

where the Green function $G(t,s)$ is defined by (11). $\square$

Remark 2. For Caputo-type fractional problems, Cichon and Salem in [22] studied the problem of the lack of equivalent between differential and integral forms and pointed out that many papers contain an error in the proof of the equivalent of the fractional-type differential problems and the corresponding integral forms. If $g \in C^1[0,1]$, then g is absolutely continuous on $[0,1]$, and we know from [22] that (8) and (12) are equivalent.

Lemma 5. The Green function $G(t,s)$, defined as in Lemma 4, has the following properties:

- (i) $G(t,s)$ is a continuous function.
- (ii)

$$G(t,s) \geq H(t,s) = \begin{cases} \frac{(t+\beta)s\gamma\eta^{2-\alpha}(\alpha-1)(1-\eta)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 \leq s \leq t \leq 1, s \leq \eta, \\ \frac{(t+\beta)\gamma\eta(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 < \eta \leq s \leq t \leq 1, \\ \frac{(t+\beta)(1-\gamma\eta)(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 \leq t \leq s \leq \eta < 1, \\ \frac{(t+\beta)(1-s)^{\alpha-1}}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)}, & 0 \leq t \leq s \leq 1, \eta \leq s. \end{cases} \quad (16)$$

Proof. It is obvious that (i) holds by the expression of $G(t,s)$. Now, we prove that (ii) holds. When $t = 0$, $G(0,s) = 0$. When $t \neq 0$, we show three different cases.

Case 1. For $0 \leq s \leq t \leq 1$, $s < \eta$, let

$$g_1(t,s) = (t+\beta)(1-s)^{\alpha-1} - (t+\beta)\gamma(\eta-s)^{\alpha-1} - (t-s)^{\alpha-1}(1+\beta-(\beta+\eta)\gamma). \quad (17)$$

Then, by Remark 3,

$$\begin{aligned} g_1(t,s) &= (t+\beta)(1-s)^{\alpha-1} - (t+\beta)\gamma\eta^{\alpha-1} \left(1 - \frac{s}{\eta}\right)^{\alpha-1} \\ &\quad - t^{\alpha-1} \left(1 - \frac{s}{t}\right)^{\alpha-1} (1+\beta-(\beta+\eta)\gamma) \end{aligned}$$

$$\begin{aligned}
&\geq (t + \beta)(1 - s)^{\alpha-1} - (t + \beta)\gamma\eta^{\alpha-1}\left(1 - \frac{s}{\eta}\right)^{\alpha-1} \\
&\quad - t^{\alpha-1}(1 - s)^{\alpha-1}(1 + \beta - (\beta + \eta)\gamma) \\
&\geq (t + \beta)(1 - s)^{\alpha-1} - (t + \beta)\gamma\eta\left(1 - \frac{s}{\eta}\right)^{\alpha-1} \\
&\quad - t(1 - s)^{\alpha-1}(1 + \beta - (\beta + \eta)\gamma) \\
&= (1 - s)^{\alpha-1}[(t + \beta) - t(1 + \beta - (\beta + \eta)\gamma)] \\
&\quad - (t + \beta)\gamma\eta\left(1 - \frac{s}{\eta}\right)^{\alpha-1} \\
&= (1 - s)^{\alpha-1}[t\eta\gamma + \beta(1 - t + t\gamma)] - (t + \beta)\gamma\eta\left(1 - \frac{s}{\eta}\right)^{\alpha-1} \\
&\geq (1 - s)^{\alpha-1}(t\eta\gamma + \beta\gamma\eta) - (t + \beta)\gamma\eta\left(1 - \frac{s}{\eta}\right)^{\alpha-1} \\
&= (t + \beta)\gamma\eta\left[(1 - s)^{\alpha-1} - \left(1 - \frac{s}{\eta}\right)^{\alpha-1}\right] \\
&= (t + \beta)\gamma\eta(\alpha - 1) \int_{1-(s/\eta)}^{1-s} x^{\alpha-2} dx \\
&\geq (t + \beta)\gamma\eta(\alpha - 1)\left(1 - \frac{s}{\eta}\right)^{\alpha-2}\left[(1 - s) - \left(1 - \frac{s}{\eta}\right)\right] \\
&= (t + \beta)s\gamma\eta^{2-\alpha}(\alpha - 1)(1 - \eta)(\eta - s)^{\alpha-2}.
\end{aligned} \tag{18}$$

Case 2. For $0 < \eta \leq s \leq t \leq 1$, let

$$g_2(t, s) = (t + \beta)(1 - s)^{\alpha-1} - (t - s)^{\alpha-1}(1 + \beta - (\beta + \eta)\gamma). \tag{19}$$

Then, by Remark 3,

$$\begin{aligned}
g_2(t, s) &= (t + \beta)(1 - s)^{\alpha-1} - t^{\alpha-1}\left(1 - \frac{s}{t}\right)^{\alpha-1}(1 + \beta - (\beta + \eta)\gamma) \\
&\geq (t + \beta)(1 - s)^{\alpha-1} - t(1 - s)^{\alpha-1}(1 + \beta - (\beta + \eta)\gamma) \\
&= (1 - s)^{\alpha-1}[(t + \beta) - t(1 + \beta - (\beta + \eta)\gamma)] \\
&\geq (t + \beta)\gamma\eta(1 - s)^{\alpha-1}.
\end{aligned} \tag{20}$$

Case 3. For $0 \leq t \leq s \leq \eta < 1$, let

$$g_3(t, s) = (t + \beta)(1 - s)^{\alpha-1} - (t + \beta)\gamma(\eta - s)^{\alpha-1}. \tag{21}$$

Then,

$$\begin{aligned}
g_3(t, s) &= (t + \beta)\left[(1 - s)^{\alpha-1} - \gamma\eta^{\alpha-1}\left(1 - \frac{s}{\eta}\right)^{\alpha-1}\right] \\
&\geq (t + \beta)\left[(1 - s)^{\alpha-1} - \gamma\eta^{\alpha-1}(1 - s)^{\alpha-1}\right] \\
&\geq (t + \beta)(1 - \gamma\eta)(1 - s)^{\alpha-1}.
\end{aligned} \tag{22}$$

The proof is complete.

Remark 3. We have inequality (18) by using $1 - t + t\gamma \geq \gamma\eta$. In fact, if $\gamma \geq 1$, since $0 < \gamma < ((1 + \beta)/(\beta + \eta))$ that $1 - \gamma\eta > \beta(\gamma - 1) \geq 0 \geq t(1 - \gamma)$, which implies that $1 - t + t\gamma \geq \gamma\eta$. If $0 < \gamma < 1$, then $1 - \gamma\eta > 1 - \gamma \geq t(1 - \gamma)$, which also yields that $1 - t + t\gamma \geq \gamma\eta$.

Remark 4. From Lemma 5, we know that $G(t, s) \geq H(t, s) \geq 0$, for all $t, s \in [0, 1]$.

Lemma 6. Let $2 < \alpha \leq 3$ and $0 < \gamma < ((1 + \beta)/(\beta + \eta))$. Assume $g \in C^1([0, 1], t[0, \infty))$; then, the unique solution u of (8)-(9) satisfies $u(t) \geq 0$ for any $t \in [0, 1]$. Moreover, the unique solution u of (8)-(9) satisfies

$$\inf_{t \in [\eta, 1]} u(t) \geq \delta \|u\|, \tag{23}$$

where $\delta = \min\{\gamma\eta, (\gamma(1 - \eta)/(1 - \gamma\eta)), \eta\}$.

Proof. By Lemmas 4 and 5 and Remark 4, we know that $u(t) \geq 0$, $\forall t \in [0, 1]$. From Lemma 4, one has

$$u''(t) = - \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} g(s) ds \leq 0. \tag{24}$$

By applying the concavity of u , similar to the proof of Lemma 4 in [2], we can obtain that (23) holds.

Lemma 7 (see [23]). Let P be a reproducing cone in a real Banach space E and $L: E \rightarrow E$ be a compact linear operator with $L(P) \subset P$ and spectral radius $r(L)$. If $r(L) > 0$, then there exists $\varphi \in P \setminus \{0\}$ such that $L\varphi = r(L)\varphi$.

Theorem 1 (see [23]). Assume that E is a Banach space, $P \subset E$ is a cone, and $\Omega(P)$ is a bounded open subset in P . Moreover, assume that $T: \Omega(P) \rightarrow P$ is a completely continuous operator. Then, the following conclusions hold:

- (i) If there exists $u_0 \in P \setminus \{0\}$ such that $u \neq Tu + \lambda u_0$, for all $u \in \partial\Omega(P)$ and $\lambda \geq 0$, then the fixed-point index $i(T, \Omega(P), P) = 0$
- (ii) If $0 \in \Omega(P)$ and $Tu \neq \lambda u$, for all $u \in \partial\Omega(P)$ and $\lambda \geq 1$, then the fixed-point index $i(T, \Omega(P), P) = 1$

3. Main Results

Let $E = C[0, 1]$ be the Banach space endowed with the sup norm and define the cone:

$$P = \left\{ x \in E: x \geq 0, \inf_{t \in [\eta, 1]} x(t) \geq \delta \|x\| \right\}, \tag{25}$$

where δ is defined in Lemma 6. Define the linear operator $L: P \rightarrow P$ as follows:

$$Lu(t) = \int_0^1 G(t, s)a(s)u(\theta(s))ds. \tag{26}$$

Lemma 8. Assume that

(H1) $a \in C^1([0, 1], t[0, \infty))$ and a does not vanish identically on any subinterval

(H2) The advanced argument $\theta \in C((0, 1), (0, 1))$ and $t \leq \theta(t) \leq 1$, $\forall t \in (0, 1)$

hold. Then, the spectral radius of the operator L is positive, that is, $r(L) > 0$.

Proof. Taking $u(t) = t^{\alpha-1}$, then $\|u\| = 1$. By (H1), (H2), and Lemma 5, we have

$$\begin{aligned} Lu(t) &= \int_0^1 G(t, s)a(s)u(\theta(s))ds \geq \int_0^1 H(t, s)a(s)\theta(s)^{\alpha-1}ds \\ &\geq \int_0^1 H(t, s)a(s)s^{\alpha-1}ds \\ &\geq \int_0^\eta \frac{(t+\beta)s\gamma\eta^{2-\alpha}(\alpha-1)(1-\eta)(\eta-s)^{\alpha-2}}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)} a(s)s^{\alpha-1}ds \\ &\geq t^{\alpha-1} \frac{\gamma\eta^{2-\alpha}(\alpha-1)(1-\eta)}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)} \int_0^\eta a(s)s^\alpha(\eta-s)^{\alpha-2}ds \\ &:= \omega t^{\alpha-1} > 0. \end{aligned} \quad (27)$$

Since $L: P \rightarrow P$, in view of the monotonicity of L , we obtain

$$L^2u(t) = L(Lu(t)) > L(\omega t^{\alpha-1}) > \omega L(t^{\alpha-1}) > \omega^2 t^{\alpha-1}. \quad (28)$$

Repeating the process gives $L^n u(t) > \omega^n t^{\alpha-1}$. Hence, we get $\|L^n\| > \omega^n$. So,

$$\begin{aligned} \|L^n\|^{1/n} &> \omega, \\ r(L) &= \lim_{n \rightarrow \infty} \|L^n\|^{1/n} > \omega > 0. \end{aligned} \quad (29)$$

The proof is completed.

Define the operator $T: P \rightarrow E$ by

$$(Tu)(t) = \int_0^1 G(t, s)a(s)f(u(\theta(s)))ds, \quad 0 \leq t \leq 1. \quad (30)$$

For convenience, set

$$\begin{aligned} f_0 &= \lim_{u \rightarrow 0^+} \frac{f(u)}{u}, \\ f_\infty &= \lim_{u \rightarrow +\infty} \frac{f(u)}{u}, \end{aligned} \quad (31)$$

$$P_c = \{u \in P: \|u\| < c\}.$$

Lemma 9. Let $f \in C^1([0, \infty), t[0, \infty))$. Assume that (H1) and (H2) hold and $0 \leq f_0 < (1/r(L))$. Then, there exists $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0]$, $i(T, P_\rho, P) = 1$.

Proof. From Lemma 6, we know that $TP \subset P$. It is easy to check that the operator $T: P \rightarrow P$ is completely continuous. It follows from $f_0 < (1/r(L))$ that there exist $\varepsilon > 0$ and $\rho_0 > 0$ such that, for $0 \leq u \leq \rho_0$,

$$f(u) \leq \left(\frac{1}{r(L)} - \varepsilon \right) u. \quad (32)$$

For $0 < \rho < \rho_0$, we will show that $\lambda u \neq Tu$, for all $u \in \partial P_\rho$ and $\lambda \geq 1$. Otherwise, there exist $u_0 \in \partial P_\rho$ and $\lambda_0 \geq 1$ such that

$$Tu_0 = \lambda_0 u_0. \quad (33)$$

By (32), one has

$$\begin{aligned} \lambda_0 u_0(t) &= Tu_0(t) = \int_0^1 G(t, s)a(s)f(u_0(\theta(s)))ds \\ &\leq \left(\frac{1}{r(L)} - \varepsilon \right) Lu_0(t), \end{aligned} \quad (34)$$

that is,

$$\begin{aligned} u_0(t) &\leq \frac{(1/r(L)) - \varepsilon}{\lambda_0} Lu_0(t) \leq \left(\frac{(1/r(L)) - \varepsilon}{\lambda_0} \right)^2 L^2 u_0(t) \\ &\leq \dots \leq \left(\frac{(1/r(L)) - \varepsilon}{\lambda_0} \right)^n L^n u_0(t) \leq \left(\frac{(1/r(L)) - \varepsilon}{\lambda_0} \right)^n \|L^n\| \|u_0\|. \end{aligned} \quad (35)$$

Thus,

$$\|u_0\| \leq \left(\frac{(1/r(L)) - \varepsilon}{\lambda_0} \right)^n \|L^n\| \|u_0\|, \quad (36)$$

which implies that

$$\frac{\lambda_0}{(1/r(L)) - \varepsilon} \leq \|L^n\|^{1/n} \rightarrow r(L), \quad n \rightarrow \infty. \quad (37)$$

It means that

$$1 \leq \lambda_0 \leq 1 - \varepsilon r(L). \quad (38)$$

This is a contradiction. By Theorem 1 (ii), we obtain that $i(T, P_\rho, P) = 1$, for each $\rho \in (0, \rho_0]$. The proof is completed.

Lemma 10. Let $f \in C^1([0, \infty), t[0, \infty))$. Assume that (H1) and (H2) hold and $(1/r(L)) < f_\infty \leq \infty$. Then, there exists $\tau_0 > 0$ such that, for $\tau > \tau_0$, $i(T, P_\tau, P) = 0$.

Proof. Since $f_\infty > (1/r(L))$, there exist $\varepsilon > 0$ and $\tau_0 > 0$ such that, for $u > \tau_0$,

$$f(u) \geq \left(\frac{1}{r(L)} + \varepsilon \right) u. \quad (39)$$

For $\tau > \tau_0$, assume that $u \neq Tu$ and $u \in \partial P_\tau$. By Lemma 7 and Theorem 1 (i), we need only to prove that

$$u \neq Tu + \lambda \varphi, \quad \lambda > 0, \quad (40)$$

where $\varphi \in P \setminus \{0\}$ with $L\varphi = r(L)\varphi$. Otherwise, there exist $u_0 \in \partial P_\tau$ and $\lambda_0 > 0$ such that

$$u_0 = Tu_0 + \lambda_0 \varphi, \quad (41)$$

which implies that $u_0 \geq Tu_0$ and $u_0 \geq \lambda_0 \varphi$. So, by (39), we obtain that

$$\begin{aligned}
Tu_0(t) &= \int_0^1 G(t,s)a(s)f(u_0(\theta(s)))ds \geq \left(\frac{1}{r(L)} + \varepsilon\right)Lu_0(t) \geq \left(\frac{1}{r(L)} + \varepsilon\right)\lambda_0 L\varphi(t) \\
&= \left(\frac{1}{r(L)} + \varepsilon\right)\lambda_0 r(L)\varphi(t) > \lambda_0 \varphi(t),
\end{aligned} \tag{42}$$

by $((1/r(L)) + \varepsilon)r(L) > 1$. Together with (41), one has $u_0 \geq 2\lambda_0\varphi$. From (42), we have that $Tu_0 \geq 2\lambda_0\varphi$. Thus, $u_0 \geq 3\lambda_0\varphi$. Repeating this process, we get that $u_0 \geq n\lambda_0\varphi$; this means that $\|u_0\| \geq n\lambda_0\|\varphi\| \rightarrow \infty$, as $n \rightarrow \infty$. This is a contradiction. It follows from Theorem 1 (i) that $i(T, P_\tau, P) = 0$, for each $\tau > \tau_0$. The proof is completed. By using Lemmas 9 and 10 and the properties of index, we have the following.

Theorem 2. Let $2 < \alpha \leq 3$, $0 < \eta < 1$, $\beta \geq 0$, and $\gamma > 0$ with $\gamma < ((1 + \beta)/(\eta + \beta))$. Suppose $f \in C^1([0, \infty), t[0, \infty))$. Assume that (H1) and (H2) hold. If $0 \leq f_0 < (1/r(L))$ and $(1/r(L)) < f_\infty \leq \infty$, then problem (4) has at least one positive solution on $[0, 1]$.

Remark 5. Let $\beta = 0$, $f_0 = 0$, and $f_\infty = \infty$; then, Theorem 2 reduces to Theorem 1.2 in [3].

By using Theorem 1, similar to the proof of Lemmas 8 and 9 as in [10], we can prove the following lemmas.

Lemma 11. Let $f \in C^1([0, \infty), t[0, \infty))$. Assume that (H1) and (H2) hold and $(1/r(L)) < f_0 \leq \infty$. Then, there exists $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0]$, $i(T, P_\rho, P) = 0$.

Lemma 12. Let $f \in C^1([0, \infty), t[0, \infty))$. Assume that (H1) and (H2) hold and $0 \leq f_\infty < (1/r(L))$. Then, there exists $\tau_0 > 0$ such that, for $\tau > \tau_0$, $i(T, P_\tau, P) = 1$.

By using Lemmas 11 and 12 and the properties of index, we have the following.

Theorem 3. Let $2 < \alpha \leq 3$, $0 < \eta < 1$, $\beta \geq 0$, and $\gamma > 0$ with $\gamma < ((1 + \beta)/(\eta + \beta))$. Suppose $f \in C^1([0, \infty), t[0, \infty))$. Assume that (H1) and (H2) hold. If $(1/r(L)) < f_0 \leq \infty$ and $0 \leq f_\infty < (1/r(L))$, then problem (4) has at least one positive solution on $[0, 1]$.

Remark 6. Let $\beta = 0$, $f_0 = \infty$, and $f_\infty = 0$; then, Theorem 3 reduces to Theorem 1.1 in [3].

4. Examples

In this section, two examples are given to illustrate the existence of positive solutions of BVP (4).

Example 1. Consider the following boundary value problem:

$$\begin{cases} {}^C D^{5/2} u(t) + t^{5/2} f(u(\theta(t))) = 0, & 0 < t < 1, \\ u(0) - \frac{2}{3} u'(0) = u''(0) = 0, & u(1) = \frac{6}{5} u\left(\frac{1}{3}\right). \end{cases} \tag{43}$$

where $\alpha = 5/2$, $\beta = 2/3$, $\gamma = 6/5$, $\eta = 1/3$, $a(t) = t^{5/2}$, $\theta(t) = \sqrt{t}$, and

$$f(u) = \begin{cases} 3320\sqrt{u^3}, & 0 \leq u \leq 1, \\ 3320(2u - \sqrt{u}), & u > 1. \end{cases} \tag{44}$$

Clearly, $f \in C^1([0, \infty), t[0, \infty))$. A direct computation shows

$$\begin{aligned}
\omega &= \frac{\gamma\eta^{2-\alpha}(\alpha-1)(1-\eta)}{\Gamma(\alpha)(1+\beta-(\beta+\eta)\gamma)} \int_0^\eta a(s)s^\alpha(\eta-s)^{\alpha-2} ds \\
&= \frac{18\sqrt{3}}{7\Gamma(5/2)} \int_0^{1/3} s^5 \left(\frac{1}{3} - s\right)^{1/2} ds \\
&= \frac{18\sqrt{3}}{7\Gamma(5/2)} \frac{\Gamma(6)\Gamma(3/2)}{\Gamma(15/2)} \left(\frac{1}{3}\right)^{(13/2)} \approx \frac{1}{6631}.
\end{aligned} \tag{45}$$

Note that conditions (H1) and (H2) hold. Through a simple calculation, we can obtain $f_0 = 0$ and $f_\infty = 6640 > 6631 = (1/\omega) > (1/r(L))$ (see Lemma 8). Thus, by Theorem 2, we see that problem (43) has at least one positive solution.

Example 2. Consider the following boundary value problem:

$$\begin{cases} {}^C D^{8/3} u(t) + \sqrt{[3]} t^4 f(u(\theta(t))) = 0, & 0 < t < 1, \\ u(0) - \frac{1}{3} u'(0) = u''(0) = 0, & u(1) = \frac{3}{5} u\left(\frac{2}{3}\right). \end{cases} \tag{46}$$

Here, $\alpha = 8/3$, $\beta = 1/3$, $\gamma = 3/5$, $\eta = 2/3$, $a(t) = \sqrt{[3]}t^4$, $\theta(t) = \sqrt{[3]}t$, and

$$f(u) = \begin{cases} 93\left(\frac{2}{\pi^2}u^2 + \frac{11}{2}\sin u\right), & 0 \leq u \leq \frac{\pi}{2}, \\ 186\left(\sqrt{\frac{2}{\pi}}\sqrt{u} + 2\sin^2 u\right), & u > \frac{\pi}{2}. \end{cases} \tag{47}$$

Obviously, $f \in C^1([0, \infty), t[0, \infty))$. A direct computation shows

$$\begin{aligned}
\omega &= \frac{\gamma \eta^{2-\alpha} (\alpha-1) (1-\eta)}{\Gamma(\alpha) (1+\beta - (\beta+\eta)\gamma)} \int_0^\eta a(s) s^\alpha (\eta-s)^{\alpha-2} ds \\
&= \frac{5(3/2)^{2/3}}{11 \cdot \Gamma(8/3)} \int_0^{2/3} s^4 \left(\frac{2}{3}-s\right)^{2/3} ds \\
&= \frac{5(3/2)^{2/3}}{11 \cdot \Gamma(8/3)} \frac{\Gamma(5)\Gamma(5/3)}{\Gamma(20/3)} \left(\frac{2}{3}\right)^{17/3} \approx 0.0022.
\end{aligned} \quad (48)$$

Note that conditions (H1) and (H2) hold. Obviously, we can get $f_0 = 511.5 > 454.5455 = (1/\omega) > (1/r(L))$ and $f_\infty = 0$. Thus, by Theorem 3, we see that problem (46) has at least one positive solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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