Hindawi
Discrete Dynamics in Nature and Society
Volume 2021, Article ID 6996805, 18 pages
https://doi.org/10.1155/2021/6996805

Research Article

On the Limit Cycles for a Class of Perturbed Fifth-Order Autonomous Differential Equations

Nabil Sellami,[1] Romaissa Mellal,[1] Bahri Belkacem Cherif ,[2,3] and Sahar Ahmed Idris [4,5]

[1] *Department of Mathematics, University of Guelma, BO 401, Guelma 24000, Algeria*
[2] *Department of Mathematics, College of Sciences and Arts in Arrass, Qassim University, Buraydah, Saudi Arabia*
[3] *Preparatory Institute for Engineering Studies in Sfax, Sfax, Tunisia*
[4] *College of Industrial Engineering, King Khalid University, Abha, Saudi Arabia*
[5] *Department of Mathematics, College of Sciences, Juba University, Juba, Sudan*

Correspondence should be addressed to Sahar Ahmed Idris; sa6044690@gmail.com

Received 12 September 2021; Accepted 23 November 2021; Published 7 December 2021

Academic Editor: Sundarapandian Vaidyanathan

Copyright © 2021 Nabil Sellami et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study the limit cycles of the fifth-order differential equation $\overset{\cdot\cdot\cdot\cdot\cdot}{x} - e\ddddot{x} - \mathrm{d}\dddot{x} - c\ddot{x} - b\dot{x} - ax = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x})$ with $a = \lambda\mu\delta, b = -(\lambda\mu + \lambda\delta + \mu\delta), c = \lambda + \mu + \delta + \lambda\mu\delta, d = -(1 + \lambda\mu + \lambda\delta + \mu\delta), e = \lambda + \mu + \delta$, where ε is a small enough real parameter, λ, μ, and δ are real parameters, and $F \in C^2$ is a nonlinear function. Using the averaging theory of first order, we provide sufficient conditions for the existence of limit cycles of this equation.

1. Introduction and Statement of the Main Results

The study of the limit cycles is one of the main topics of the qualitative theory of differential equations and dynamical systems. A limit cycle of a differential equation is an isolated periodic orbit of this equation; it means that there is no periodic orbits in the vicinity of this limit cycle. There are several theories and methods for the study of the existence, uniqueness, or number and stability of limit cycles of differential equations which have been developed in trying to answer Hilbert's sixteenth problem posed in 1900 [1] about the maximum number of limit cycles that a planar polynomial differential system can have.

The averaging theory is one of the most important tools used actually to the study of limit cycles for second and higher order differential equations, you can see in [2–8]. More details on the averaging theory can be found in the books of Sanders and Verhulst [9] and of Verhulst [9].

In [7], the authors studied the limit cycles of the following third-order differential equation

$$\dddot{x} - \mu x + \dot{x} - \mu x = \varepsilon F(x, \dot{x}, x, t), \tag{1}$$

with $\mu \neq 0$; ε is a small real parameter; $F \in C^2$ is $2\pi-$ periodic in t.

In [5], the authors studied equation (1) with $F = F(x, \dot{x}, \ddot{x})$ which is autonomous. They studied the two cases $\mu \neq 0$ and $\mu = 0$.

In [6], the authors studied the following fourth-order differential equation:

$$\ddddot{x} - (\lambda + \mu)\dddot{x} + (1 + \lambda\mu)x - (\lambda + \mu)\dot{x} + \lambda\mu x = \varepsilon F(x, \dot{x}, x, \dddot{x}, t), \tag{2}$$

where λ and μ are real, ε is a small real parameter, and $F \in C^2$ is $2\pi-$ periodic in t.

In [4], the authors studied equation (2) with $F = F(x, \dot{x}, \ddot{x}, \dddot{x})$ which is autonomous.

In this paper, we shall use a result of the averaging theory to study the limit cycles of the following class of fifth-order autonomous ordinary differential equations:

$$\overset{\cdot\cdot\cdot\cdot\cdot}{x} - e\ddddot{x} - \mathrm{d}\dddot{x} - cx - b\dot{x} - ax = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \tag{3}$$

where

$$a = \lambda\mu\delta, b = -(\lambda\mu + \lambda\delta + \mu\delta), c = \lambda + \mu + \delta + \lambda\mu\delta,$$
$$d = -(1 + \lambda\mu + \lambda\delta + \mu\delta), e = \lambda + \mu + \delta, \quad (4)$$

where the dot means derivative with respect to an independent variable t, ε is a small enough parameter, and $F \in C^2$ is a nonlinear function. Here, the variable x and the parameters λ, μ, δ and ε are real.

In [8], the authors studied equation (3) with $F = F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}, t)$ which depends explicitly on the independent variable t. Here, we study the autonomous case using a different approach. Note that our results are distinct and new.

Now, we state our main results for the limit cycles of equation (3).

For the different values of the parameters λ, μ, and δ, we distinguish the five following cases.

Case 1. $\lambda\mu\delta \neq 0$ and $\lambda \neq \mu \neq \delta$.

Case 2. $\lambda = 0, \mu\delta \neq 0$, and $\mu \neq \delta$.

Case 3. $\lambda = 0$ and $\mu = \delta \neq 0$.

Case 4. $\lambda \neq 0$ and $\mu = \delta \neq 0$.

Case 5. $\lambda = \mu = \delta \neq 0$.

For each one of these cases, we will give a theorem which provides sufficient conditions for the existence of limit cycles of equation (3) and we provide also an application.

There are two other cases ($\lambda = \mu = 0$ and $\delta \neq 0$) and ($\lambda = \mu = \delta = 0$) that we cannot study because they are too much degenerated for Theorem 6.

1.1. Case 1: $\lambda\mu\delta \neq 0$ and $\lambda \neq \mu \neq \delta$. In order to state our results for this case, we define the function

$$\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta\, F(A_1, A_2, A_3, A_4, A_5)\,d\theta, \quad (5)$$

where

$$A_1 = \frac{((\lambda\mu\delta - \lambda - \mu - \delta)\cos\theta + (1 - \lambda\mu - \lambda\delta - \mu\delta)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)(1+\delta^2)},$$

$$A_2 = \frac{((1 - \lambda\mu - \lambda\delta - \mu\delta)\cos\theta + (\lambda + \mu + \delta - \lambda\mu\delta)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)(1+\delta^2)},$$

$$A_3 = \frac{((\lambda + \mu + \delta - \lambda\mu\delta)\cos\theta + (\lambda\mu + \lambda\delta + \mu\delta - 1)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)(1+\delta^2)},$$

$$A_4 = \frac{((\lambda\mu + \lambda\delta + \mu\delta - 1)\cos\theta + (\lambda\mu\delta - \lambda - \mu - \delta)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)(1+\delta^2)},$$

$$A_5 = \frac{((\lambda\mu\delta - \lambda - \mu - \delta)\cos\theta + (1 - \lambda\mu - \lambda\delta - \mu\delta)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)(1+\delta^2)}. \quad (6)$$

Our main result for this case is the following theorem.

Theorem 1. *Assume that $\lambda\mu\delta \neq 0$ and $\lambda \neq \mu \neq \delta$. For every positive simple zero r_0^* of the function $\mathcal{F}(r_0)$ given by (5) there is a limit cycle $x(t, \varepsilon)$ of equation (3) tending to the periodic solution*

$$x^*(t) = \frac{((\lambda\mu\delta - \lambda - \mu - \delta)\cos t + (1 - \lambda\mu - \lambda\delta - \mu\delta)\sin t)r_0^*}{(1+\lambda^2)(1+\mu^2)(1+\delta^2)}, \quad (7)$$

of

$$\overset{.....}{x} - (\lambda + \mu + \delta)\ddddot{x} + (1 + \lambda\mu + \lambda\delta + \mu\delta)\dddot{x} - (\lambda + \mu + \delta + \lambda\mu\delta)x + (\lambda\mu + \lambda\delta + \mu\delta)\dot{x} - \lambda\mu\delta x = 0, \quad (8)$$

when $\varepsilon \longrightarrow 0$.

Theorem 1 will be proved in Section 3.1.1.

An application of Theorem 1 is the following.

Corollary 1. *Assume that $\lambda\mu\delta \neq 0$, $\lambda \neq \mu \neq \delta$, $\lambda\mu + \lambda\delta + \mu\delta \neq 1$, and*

$$F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = x^5 - \dot{x}^4 - \ddot{x}^3 - \dddot{x}^2 - \ddddot{x} - 1. \quad (9)$$

Then, there is a limit cycle $x_1(t, \varepsilon)$ of equation (3) tending to the periodic solution

$$x_1^*(t) = \frac{2((\lambda\mu\delta - \lambda - \mu - \delta)\cos t + (1 - \lambda\mu - \lambda\delta - \mu\delta)\sin t)}{\sqrt{5(1+\lambda^2)(1+\mu^2)(1+\delta^2)}}, \quad (10)$$

of equation (8) when $\varepsilon \longrightarrow 0$.

Corollary 1 will be proved in Section 3.1.2.

1.2. Case 2: $\lambda = 0, \mu\delta \neq 0$, and $\mu \neq \delta$. We define the functions

$$\mathcal{F}_1(r_0, Z_0) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta\, F(B_1, B_2, B_3, B_4, B_5)\mathrm{d}\theta,$$
$$\mathcal{F}_2(r_0, Z_0) = \frac{1}{2\pi}\int_0^{2\pi} F(B_1, B_2, B_3, B_4, B_5)\mathrm{d}\theta, \tag{11}$$

and

$$B_1 = \frac{((-\mu-\delta)\cos\theta + (1-\mu\delta)\sin\theta)r_0}{(1+\mu^2)(1+\delta^2)} + \frac{Z_0}{\mu\delta},$$
$$B_2 = \frac{((1-\mu\delta)\cos\theta + (\mu+\delta)\sin\theta)r_0}{(1+\mu^2)(1+\delta^2)},$$
$$B_3 = \frac{((\mu+\delta)\cos\theta + (\mu\delta-1)\sin\theta)r_0}{(1+\mu^2)(1+\delta^2)}, \tag{12}$$
$$B_4 = \frac{((\mu\delta-1)\cos\theta - (\mu+\delta)\sin\theta)r_0}{(1+\mu^2)(1+\delta^2)},$$
$$B_5 = \frac{(-(\mu+\delta)\cos\theta + (1-\mu\delta)\sin\theta)r_0}{(1+\mu^2)(1+\delta^2)}.$$

Our main result for this case is the following theorem.

Theorem 2. *Assume that $\lambda = 0, \mu\delta \neq 0$, and $\mu \neq \delta$. For every zero (r_0^*, Z_0^*) of the system $\mathcal{F}_1(r_0, Z_0) = \mathcal{F}_2(r_0, Z_0) = 0$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by (11) such that*

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(r_0, Z_0)}\right)\Big|_{(r_0,Z_0)=(r_0^*,Z_0^*)} \neq 0, \tag{13}$$

there is a limit cycle $x(t,\varepsilon)$ of equation (3) tending to the periodic solution

$$x^*(t) = \frac{((-\mu-\delta)\cos t + (1-\mu\delta)\sin t)r_0^*}{(1+\mu^2)(1+\delta^2)} + \frac{Z_0^*}{\mu\delta}, \tag{14}$$

of

$$x^{(5)} - (\mu+\delta)x^{(4)} + (1+\mu\delta)\dddot{x} - (\mu+\delta)\ddot{x} + \mu\delta\dot{x} = 0, \tag{15}$$

when $\varepsilon \longrightarrow 0$.

Theorem 2 will be proved in Section 3.2.1.

An application of Theorem 2 is the following.

Corollary 2. *Assume that $\lambda = 0, \mu\delta \neq 0$, $\mu \neq \delta$, $\mu\delta \neq 1$, and $F(x, \dot{x}, \ddot{x}, \dddot{x}, x) = x^2 - 2x + \dot{x}^2$, then there is a limit cycle $x_2(t,\varepsilon)$ of equation (3) tending to the periodic solution*

$$x_2^*(t) = \frac{((-\mu-\delta)\cos t + (1-\mu\delta)\sin t)}{\sqrt{(1+\mu^2)(1+\delta^2)}} + 1, \tag{16}$$

of equation (15) when $\varepsilon \longrightarrow 0$.

Corollary 2 will be proved in Section 3.

1.3. Case 3: $\lambda = 0$ and $\mu = \delta \neq 0$. We define the functions

$$\mathcal{F}_1(r_0, Z_0) = \frac{1}{2\pi}\int_0^{2\pi} -\sin\theta\, F(C_1, C_2, C_3, C_4, C_5)\mathrm{d}\theta,$$
$$\mathcal{F}_2(r_0, Z_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi} F(C_1, C_2, C_3, C_4, C_5)\mathrm{d}\theta, \tag{17}$$

where

$$C_1 = \frac{(2\mu\cos\theta + (\mu^2-1)\sin\theta)r_0}{(1+\mu^2)^2} + \frac{Z_0}{\mu^2},$$
$$C_2 = \frac{((\mu^2-1)\cos\theta - 2\mu\sin\theta)r_0}{(1+\mu^2)^2},$$
$$C_3 = \frac{(-2\mu\cos\theta - (\mu^2-1)\sin\theta)r_0}{(1+\mu^2)^2}, \tag{18}$$
$$C_4 = \frac{(-(\mu^2-1)\cos\theta + 2\mu\sin\theta)r_0}{(1+\mu^2)^2},$$
$$C_5 = \frac{(2\mu\cos\theta + (\mu^2-1)\sin\theta)r_0}{(1+\mu^2)^2}.$$

Our main result for this case is the following theorem.

Theorem 3. *Assume that $\lambda = 0$ and $\mu = \delta \neq 0$. For every zero (r_0^*, Z_0^*) of the system $\mathcal{F}_1(r_0, Z_0) = \mathcal{F}_2(r_0, Z_0) = 0$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by (17) such that*

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(r_0, Z_0)}\right)\Big|_{(r_0,Z_0)=(r_0^*,Z_0^*)} \neq 0, \tag{19}$$

there is a limit cycle $x(t,\varepsilon)$ of equation (3) tending to the periodic solution

$$x^*(t) = \frac{(2\mu\cos t + (\mu^2-1)\sin t)r_0^*}{(1+\mu^2)^2} + \frac{Z_0^*}{\mu^2}, \tag{20}$$

of

$$x^{(5)} - 2\mu x^{(4)} + (1+\mu^2)\dddot{x} - 2\mu\ddot{x} + \mu^2\dot{x} = 0, \tag{21}$$

when $\varepsilon \longrightarrow 0$.

Theorem 3 will be proved in Section 3.3.1.

An application of Theorem 3 is the following.

Corollary 3. *Assume that $\lambda = 0$, $\delta = \mu \neq 0, \mu \neq -1 \pm \sqrt{5}/2$, and $F(x, \dot{x}, \ddot{x}, \dddot{x}, x) = x^2 - x\dot{x} - 1$, then there is a limit cycle $x_3(t,\varepsilon)$ of equation (3) tending to the periodic solution*

$$x_3^*(t) = \frac{(2\mu\cos t + (\mu^2-1)\sin t)\sqrt{2}}{1+\mu^2}, \tag{22}$$

of equation (21) when $\varepsilon \longrightarrow 0$.

Corollary 3 will be proved in Section 3.3.2.

1.4. Case 4: $\lambda \neq 0$ and $\mu = \delta \neq 0$. We define the function

$$\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} -\sin\theta F(D_1, D_2, D_3, D_4, D_5)d\theta, \tag{23}$$

where

$$\begin{aligned}
D_1 &= \frac{\left(\left(\lambda + 2\mu - \lambda\mu^2\right)\cos\theta + \left(\mu^2 + 2\lambda\mu - 1\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)\left(1+\mu^2\right)^2},\\
D_2 &= \frac{\left(\left(\mu^2 + 2\lambda\mu - 1\right)\cos\theta - \left(\lambda + 2\mu - \lambda\mu^2\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)\left(1+\mu^2\right)^2},\\
D_3 &= \frac{\left(-\left(\lambda + 2\mu - \lambda\mu^2\right)\cos\theta - \left(\mu^2 + 2\lambda\mu - 1\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)\left(1+\mu^2\right)^2},\\
D_4 &= \frac{\left(-\left(\mu^2 + 2\lambda\mu - 1\right)\cos\theta + \left(\lambda + 2\mu - \lambda\mu^2\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)\left(1+\mu^2\right)^2},\\
D_5 &= \frac{\left(\left(\lambda + 2\mu - \lambda\mu^2\right)\cos\theta + \left(\mu^2 + 2\lambda\mu - 1\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)\left(1+\mu^2\right)^2}.
\end{aligned} \tag{24}$$

Our main result for this case is the following theorem.

Theorem 4. *Assume that $\lambda \neq 0$ and $\delta = \mu \neq 0$. For every positive simple zero r_0^* of the function $\mathcal{F}(r_0)$ given by (23), there is a limit cycle $x(t,\varepsilon)$ of equation (3) tending to the periodic solution*

$$x^*(t) = \frac{\left(\left(\lambda + 2\mu - \lambda\mu^2\right)\cos t + \left(\mu^2 + 2\lambda\mu - 1\right)\sin t\right)r_0^*}{\left(1+\lambda^2\right)\left(1+\mu^2\right)^2}, \tag{25}$$

of

$$\begin{aligned}
&x^{(5)} - (\lambda + 2\mu)x^{(4)} + \left(1 + 2\lambda\mu + \mu^2\right)\dddot{x} - \left(\lambda + 2\mu + \lambda\mu^2\right)x\\
&\quad + \left(2\lambda\mu + \mu^2\right)\dot{x} - \lambda\mu^2 x = 0,
\end{aligned} \tag{26}$$

when $\varepsilon \longrightarrow 0$.

Theorem 4 will be proved in Section 3.4.1.

An application of Theorem 4 is the following.

Corollary 4. *Assume that $\lambda \neq 0$, $\delta = \mu \neq 0$, $\mu^2 + 2\lambda\mu - 1 \neq 0$, and $F(x,\dot{x},\ddot{x},\dddot{x},x) = x^3 - x - 1$, then there is a limit cycle $x_4(t,\varepsilon)$ of equation (3) tending to the periodic solution*

$$x_4^*(t) = \frac{2\sqrt{3}\left(\left(\lambda + 2\mu - \lambda\mu^2\right)\cos t + \left(\mu^2 + 2\lambda\mu - 1\right)\sin t\right)}{3\sqrt{1+\lambda^2}\left(1+\mu^2\right)}, \tag{27}$$

of equation (26) when $\varepsilon \longrightarrow 0$.

Corollary 4 will be proved in Section 3.4.2.

1.5. Case 5: $\lambda = \mu = \delta \neq 0$. In order to state our result for this case, we define the function

$$\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta F(E_1, E_2, E_3, E_4, E_5)d\theta, \tag{28}$$

where

$$\begin{aligned}
E_1 &= \frac{\left(\left(\lambda^3 - 3\lambda\right)\cos\theta + \left(1 - 3\lambda^2\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)^3},\\
E_2 &= \frac{\left(\left(1 - 3\lambda^2\right)\cos\theta - \left(\lambda^3 - 3\lambda\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)^3},\\
E_3 &= \frac{\left(-\left(\lambda^3 - 3\lambda\right)\cos\theta - \left(1 - 3\lambda^2\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)^3},\\
E_4 &= \frac{\left(-\left(1 - 3\lambda^2\right)\cos\theta + \left(\lambda^3 - 3\lambda\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)^3},\\
E_5 &= \frac{\left(\left(\lambda^3 - 3\lambda\right)\cos\theta + \left(1 - 3\lambda^2\right)\sin\theta\right)r_0}{\left(1+\lambda^2\right)^3}.
\end{aligned} \tag{29}$$

Our main result for this case is the following theorem.

Theorem 5. *Assume that $\mu = \delta = \lambda \neq 0$. For every positive simple zero r_0^* of the function $\mathcal{F}(r_0)$ given by (28), there is a limit cycle $x(t,\varepsilon)$ of equation (3) tending to the periodic solution*

$$x^*(t) = \frac{\left(\left(\lambda^3 - 3\lambda\right)\cos t + \left(1 - 3\lambda^2\right)\sin t\right)r_0^*}{\left(1+\lambda^2\right)^3}, \tag{30}$$

of

$$x^{(5)} - 3\lambda x^{(4)} + \left(1 + 3\lambda^2\right)\dddot{x} - \left(3\lambda + \lambda^3\right)\ddot{x} + 3\lambda^2\dot{x} - \lambda^3 x = 0, \tag{31}$$

when $\varepsilon \longrightarrow 0$.

Theorem 5 will be proved in Section 3.5.1.

An application of Theorem 5 is the following.

Corollary 5. *Assume that $\mu = \delta = \lambda \neq 0$, $\lambda \neq \pm\sqrt{3}$, $\lambda \neq \pm\sqrt{3}/3$, and $F(x,\dot{x},\ddot{x},\dddot{x},x) = -\ddot{x}^2 x^{(4)} + x^3 + x^2 - \dot{x}\ddot{x} + \dddot{x} - 1$, then there is a limit cycle $x_5(t,\varepsilon)$ of equation (3) tending to the periodic solution*

$$x_5^*(t) = \frac{2\sqrt{3}\sqrt{(3\lambda^2-1)\lambda(\lambda^4-2\lambda^2-3)}\left((\lambda^3-3\lambda)\cos t + (1-3\lambda^2)\sin t\right)}{3(3\lambda^2-1)(1+\lambda^2)^2}, \quad (32)$$

of equation (31) when $\varepsilon \longrightarrow 0$.

Corollary 5 will be proved in Section 3.5.2.

2. The Main Tool (First-Order Averaging Theory)

In this section, we present the basic result from the averaging theory that we need for proving the main results of this article.

We consider the problem of the bifurcation of T–periodic solutions from the differential system

$$\dot{x}(t) = F_0(\mathbf{x}, t) + \varepsilon F_1(\mathbf{x}, t) + \varepsilon^2 F_2(\mathbf{x}, t, \varepsilon), \quad (33)$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^n$ and $F_2\colon \Omega \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathbb{R}^n$ are $\mathscr{C}^2$ functions, T–periodic is in the variable t, and Ω is an open subset of $\mathbb{R}^n$. We suppose that the unperturbed system

$$\dot{x}(t) = F_0(\mathbf{x}, t), \quad (34)$$

has a k-dimensional submanifold $\mathscr{Z}$ of periodic solutions.

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (34) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. The linearisation of system (34) along the periodic solution $\mathbf{x}(t, \mathbf{z})$ is written as

$$\dot{y} = D_x F_0(\mathbf{x}(t, \mathbf{z}), t) y. \quad (35)$$

We denote by $M_{\mathbf{z}}(t)$ some fundamental matrices of the linear differential system (35) and by $\xi\colon \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e., $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

Theorem 6. *Let $\mathbb{V} \subset \mathbb{R}^k$ be open and bounded and $\beta_0\colon Cl(\mathbb{V}) \longrightarrow \mathbb{R}^{n-k}$ be a $\mathscr{C}^2$ function. We assume that*

(i) $\mathscr{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)), \alpha \in Cl(\mathbb{V})\} \subset \Omega$, and that for each $z_\alpha \in \mathscr{Z}$, the solution $x(t, \mathbf{z}_\alpha)$ of (34) is T–periodic.

(ii) For each $\mathbf{z}_\alpha \in \mathscr{Z}$, there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (35) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a matrix $\Delta_\alpha((n-k) \times (n-k))$ with $\det\Delta_\alpha \neq 0$. We consider the function $\mathscr{F}\colon Cl(\mathbb{V}) \longrightarrow \mathbb{R}^k$

$$\mathscr{F}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(\mathbf{x}(t, \mathbf{z}_\alpha), t)\,\mathrm{d}t\right). \quad (36)$$

If there exists $a \in V$ with $\mathscr{F}(a) = 0$ and $\det((d\mathscr{F}/d\alpha)(a)) \neq 0$, then there is a T–periodic solution $\varphi(t, \varepsilon)$ of the system (33) such that $\varphi(0, \varepsilon) \longrightarrow z_a$ as $\varepsilon \longrightarrow 0$.

Theorem 1 goes back to [10] and [11]; for a shorter proof, see [12].

Note that the periodic orbits provided by Theorem 6 are limit cycles.

3. Proofs of the Results

3.1. Proofs of the Results in Case 1: $\lambda\mu\delta \neq 0$ and $\lambda \neq \mu \neq \delta$

3.1.1. Proof of Theorem 1. We consider equation (3) and put $y = \dot{x}$, $z = \ddot{x}$, $u = \dddot{x}$, and $v = x$, then equation (3) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = ax + by + cz + \mathrm{d}u + ev + \varepsilon F(x, y, z, u, v). \end{cases} \quad (37)$$

System (37) with $\varepsilon = 0$ has a unique singular point at the origin and the linear part of this system has the eigenvalues $\pm i$, λ, μ, and δ. Using the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} \lambda\mu\delta & -\lambda\mu - \lambda\delta - \mu\delta & \lambda + \mu + \delta & -1 & 0 \\ 0 & -\lambda\mu\delta & \lambda\mu + \lambda\delta + \mu\delta & -\lambda - \mu - \delta & 1 \\ \mu\delta & -\mu - \delta & 1 + \mu\delta & -\mu - \delta & 1 \\ \lambda\delta & -\lambda - \delta & 1 + \lambda\delta & -\lambda - \delta & 1 \\ \lambda\mu & -\lambda - \mu & 1 + \lambda\mu & -\lambda - \mu & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \quad (38)$$

we transform system (37) into the following system:

$$\begin{cases} \dot{X} = -Y, \\ \dot{Y} = X + \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{Z} = \lambda Z + \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{U} = \mu U + \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{V} = \delta V + \varepsilon\widetilde{F}(X,Y,Z,U,V), \end{cases} \tag{39}$$

where

$$\begin{aligned}
&\widetilde{F} = \widetilde{F}(X,Y,Z,U,V) = F(a_1,a_2,a_3,a_4,a_5),\\
&a_1 = \frac{(\lambda\mu\delta-\lambda-\mu-\delta)X}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{(1-\lambda\mu-\lambda\delta-\mu\delta)Y}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{Z}{(\lambda^2+1)(\lambda-\mu)(\lambda-\delta)} - \frac{U}{(\mu^2+1)(\lambda-\mu)(\mu-\delta)} + \frac{V}{(\delta^2+1)(\lambda-\delta)(\mu-\delta)},\\
&a_2 = \frac{(1-\lambda\mu-\lambda\delta-\mu\delta)X}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} - \frac{(\lambda\mu\delta-\lambda-\mu-\delta)Y}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{\lambda Z}{(\lambda^2+1)(\lambda-\mu)(\lambda-\delta)} - \frac{\mu U}{(\mu^2+1)(\lambda-\mu)(\mu-\delta)} + \frac{\delta V}{(\delta^2+1)(\lambda-\delta)(\mu-\delta)},\\
&a_3 = -\frac{(\lambda\mu\delta-\lambda-\mu-\delta)X}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} - \frac{(1-\lambda\mu-\lambda\delta-\mu\delta)Y}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{\lambda^2 Z}{(\lambda^2+1)(\lambda-\mu)(\lambda-\delta)} - \frac{\mu^2 U}{(\mu^2+1)(\lambda-\mu)(\mu-\delta)} + \frac{\delta^2 V}{(\delta^2+1)(\lambda-\delta)(\mu-\delta)},\\
&a_4 = -\frac{(1-\lambda\mu-\lambda\delta-\mu\delta)X}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{(\lambda\mu\delta-\lambda-\mu-\delta)Y}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{\lambda^3 Z}{(\lambda^2+1)(\lambda-\mu)(\lambda-\delta)} - \frac{\mu^3 U}{(\mu^2+1)(\lambda-\mu)(\mu-\delta)} + \frac{\delta^3 V}{(\delta^2+1)(\lambda-\delta)(\mu-\delta)},\\
&a_5 = \frac{(\lambda\mu\delta-\lambda-\mu-\delta)X}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{(1-\lambda\mu-\lambda\delta-\mu\delta)Y}{(\lambda^2+1)(\mu^2+1)(\delta^2+1)} + \frac{\lambda^4 Z}{(\lambda^2+1)(\lambda-\mu)(\lambda-\delta)} - \frac{\mu^4 U}{(\mu^2+1)(\lambda-\mu)(\mu-\delta)} + \frac{\delta^4 V}{(\delta^2+1)(\lambda-\delta)(\mu-\delta)}.
\end{aligned} \tag{40}$$

Note that the linear part of system (39) is in the real normal Jordan form of the linear part of system (37). We pass now from the Cartesian coordinates (X,Y,Z,U,V) to the cylindrical ones (r,θ,Z,U,V) with $X = r\cos\theta$, $Y = r\sin\theta$, and we obtain

$$\begin{cases} \dot{r} = \varepsilon\sin\theta\, G(r,\theta,Z,U,V), \\ \dot{\theta} = 1 + \dfrac{\varepsilon}{r}\cos\theta\, G(r,\theta,Z,U,V), \\ \dot{Z} = \lambda Z + \varepsilon G(r,\theta,Z,U,V), \\ \dot{U} = \mu U + \varepsilon G(r,\theta,Z,U,V), \\ \dot{V} = \delta V + \varepsilon G(r,\theta,Z,U,V), \end{cases} \tag{41}$$

where $G(r,\theta,Z,U,V) = \widetilde{F}(r\cos\theta, r\sin\theta, Z,U,V)$.

After dividing by $\dot{\theta}$ and simplifying, we find

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \varepsilon\sin\theta\, G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{\mathrm{d}Z}{\mathrm{d}\theta} = \lambda Z + \varepsilon\left(1 - \dfrac{\lambda Z}{r}\cos\theta\right)G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{\mathrm{d}U}{\mathrm{d}\theta} = \mu U + \varepsilon\left(1 - \dfrac{\mu U}{r}\cos\theta\right)G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{\mathrm{d}V}{\mathrm{d}\theta} = \delta V + \varepsilon\left(1 - \dfrac{\delta V}{r}\cos\theta\right)G(r,\theta,Z,U,V) + o(\varepsilon^2). \end{cases} \tag{42}$$

System (42) is now of the same form as system (33) with

$$\mathbf{x} = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta,$$

$$F_0(\mathbf{x}, \theta) = \begin{pmatrix} 0 \\ \lambda Z \\ \mu U \\ \delta V \end{pmatrix},$$

$$F_1(\mathbf{x}, \theta) = \begin{pmatrix} \sin\theta G(r, \theta, Z, U, V) \\ \left(1 - \frac{\lambda Z}{r}\cos\theta\right) G(r, \theta, Z, U, V) \\ \left(1 - \frac{\mu U}{r}\cos\theta\right) G(r, \theta, Z, U, V) \\ \left(1 - \frac{\delta V}{r}\cos\theta\right) G(r, \theta, Z, U, V) \end{pmatrix}. \tag{43}$$

We shall apply Theorem 6 to system (42). System (42) with $\varepsilon = 0$ has the 2π–periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall r_0 > 0. \tag{44}$$

By the notations of Theorem 6, we have that $k = 1$ and $n = 4$. Let $r_1 > 0$ and $r_2 > 0$; we take $\mathbb{V} =]r_1, r_2[\subset \mathbb{R}$, $\alpha = r_0 \in [r_1, r_2]$, and

$$\begin{aligned} \beta_0 &: [r_1, r_2] \longrightarrow \mathbb{R}^3, \\ &: r_0 \mapsto \beta_0(r_0) = (0, 0, 0). \end{aligned} \tag{45}$$

We also take

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, 0, 0, 0), r_0 \in [r_1, r_2]\}. \tag{46}$$

The fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear system (42) with $\varepsilon = 0$ with respect to the periodic solution $\mathbf{z}_\alpha = (r_0, 0, 0, 0)$ satisfying that $M_{\mathbf{z}_\alpha}(0)$ is the identity matrix is

$$M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\lambda\theta} & 0 & 0 \\ 0 & 0 & e^{\mu\theta} & 0 \\ 0 & 0 & 0 & e^{\delta\theta} \end{pmatrix}. \tag{47}$$

We have

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - e^{-2\pi\lambda} & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi\mu} & 0 \\ 0 & 0 & 0 & 1 - e^{-2\pi\delta} \end{pmatrix}, \tag{48}$$

which satisfy the assumption (ii) of Theorem 6. Taking

$$\begin{aligned} \xi &: \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \\ &: (r, Z, U, V) \mapsto \xi(r, Z, U, V) = r, \end{aligned} \tag{49}$$

we must compute the function $\mathcal{F}(\alpha)$ given by (36), and we obtain

$$\begin{aligned} \mathcal{F}(\alpha) = \mathcal{F}(r_0) &= \frac{1}{2\pi}\int_0^{2\pi} \sin\theta G(r_0, \theta, 0, 0, 0) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sin\theta F(A_1, A_2, A_3, A_4, A_5) d\theta, \end{aligned} \tag{50}$$

and A_1, A_2, A_3, A_4, and A_5 are given by (6). Then, by Theorem 6, for every simple zero r_0^* of the function $\mathcal{F}(r_0)$, there exists a limit cycle $(r, Z, U, V)(\theta, \varepsilon)$ of system (42) such that

$$(r, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0), \text{ when } \varepsilon \longrightarrow 0. \tag{51}$$

Going back through the change of coordinates, we obtain a limit cycle $(r, \theta, Z, U, V)(t, \varepsilon)$ of system (41) such that

$$(r, \theta, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0, 0), \text{ when } \varepsilon \longrightarrow 0. \tag{52}$$

We have a limit cycle $(X, Y, Z, U, V)(t, \varepsilon)$ of system (39) such that

$$(X, Y, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0, 0) \text{ when } \varepsilon \longrightarrow 0. \tag{53}$$

Finally, we obtain a limit cycle $x(t, \varepsilon)$ of equation (3) tending to the periodic solution (7) of equation (8) when $\varepsilon \longrightarrow 0$.

Theorem 1 is proved.

3.1.2. Proof of Corollary 1. If $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = x^5 - \dot{x}^4 - \ddot{x}^3 - \dddot{x}^2 - \ddddot{x} - x - 1$, then we have

$$\begin{aligned} \mathcal{F}(r_0) = &\frac{r_0(-1 + \lambda\mu + \lambda\delta + \mu\delta)}{16(1 + \lambda^2)^3(1 + \mu^2)^3(1 + \delta^2)^3} \\ &\cdot \left(-5r_0^2 + 4(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)\right) \\ &\cdot \left(r_0^2 + 2(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)\right), \end{aligned} \tag{54}$$

which have the real positive simple zero $r_0^* = 2\sqrt{5}/5\sqrt{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)}$ with

$$\frac{df}{dr_0}(r_0^*) = -\frac{34}{9}\frac{-1 + \lambda\mu + \lambda\delta + \mu\delta}{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)} \neq 0. \tag{55}$$

The proof of Corollary 1 follows directly by applying Theorem 1 and (10) is obtained by substituting r_0^* in (7).

3.2. Proofs of the Results in Case 2: $\lambda = 0, \mu\delta \neq 0$, and $\mu \neq \delta$

3.2.1. Proof of Theorem 2. If $\lambda = 0$, equation (3) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = -\mu\delta y + (\mu+\delta)z - (1+\mu\delta)u + (\mu+\delta)v + \varepsilon F(x,y,z,u,v). \end{cases} \tag{56}$$

System (56) has a unique singular point at the origin and the eigenvalues of the linear part of this system are $\pm i$, 0, μ, and δ. By the linear transformation

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} 0 & -\mu\delta & \mu+\delta & -1 & 0 \\ 0 & 0 & \mu\delta & -\mu-\delta & 1 \\ \mu\delta & -\mu-\delta & 1+\mu\delta & -\mu-\delta & 1 \\ 0 & -\delta & 1 & -\delta & 1 \\ 0 & -\mu & 1 & -\mu & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \tag{57}$$

we transform system (56) into the following system:

$$\begin{cases} \dot{X} = -Y, \\ \dot{Y} = X + \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{Z} = \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{U} = \mu U + \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{V} = \delta V + \varepsilon\widetilde{F}(X,Y,Z,U,V), \end{cases} \tag{58}$$

where

$$\begin{aligned} \widetilde{F} &= \widetilde{F}(X,Y,Z,U,V) = F(b_1,b_2,b_3,b_4,b_5), \\ b_1 &= \frac{(-\mu-\delta)X}{(\mu^2+1)(\delta^2+1)} + \frac{(1-\mu\delta)Y}{(\mu^2+1)(\delta^2+1)} + \frac{Z}{\mu\delta} + \frac{U}{\mu(\mu^2+1)(\mu-\delta)} - \frac{V}{\delta(\delta^2+1)(\mu-\delta)}, \\ b_2 &= \frac{(1-\mu\delta)X}{(\mu^2+1)(\delta^2+1)} - \frac{(-\mu-\delta)Y}{(\mu^2+1)(\delta^2+1)} + \frac{U}{(\mu^2+1)(\mu-\delta)} - \frac{V}{(\delta^2+1)(\mu-\delta)}, \\ b_3 &= -\frac{(-\mu-\delta)X}{(\mu^2+1)(\delta^2+1)} - \frac{(1-\mu\delta)Y}{(\mu^2+1)(\delta^2+1)} + \frac{\mu U}{(\mu^2+1)(\mu-\delta)} - \frac{\delta V}{(\delta^2+1)(\mu-\delta)}, \\ b_4 &= -\frac{(1-\mu\delta)X}{(\mu^2+1)(\delta^2+1)} + \frac{(-\mu-\delta)Y}{(\mu^2+1)(\delta^2+1)} + \frac{\mu^2 U}{(\mu^2+1)(\mu-\delta)} - \frac{\delta^2 V}{(\delta^2+1)(\mu-\delta)}, \\ b_5 &= \frac{(-\mu-\delta)X}{(\mu^2+1)(\delta^2+1)} + \frac{(1-\mu\delta)Y}{(\mu^2+1)(\delta^2+1)} + \frac{\mu^3 U}{(\mu^2+1)(\mu-\delta)} - \frac{\delta^3 V}{(\delta^2+1)(\mu-\delta)}. \end{aligned} \tag{59}$$

Note that the linear part of system (58) is in the real Jordan normal form of the linear part of system (56). We pass now from the Cartesian coordinates (X, Y, Z, U, V) to the cylindrical ones (r, θ, Z, U, V) with $X = r\cos\theta$, $Y = r\sin\theta$, and we obtain

$$\begin{cases} \dot{r} = \varepsilon \sin\theta\, G(r, \theta, Z, U, V), \\ \dot{\theta} = 1 + \dfrac{\varepsilon}{r}\cos\theta\, G(r, \theta, Z, U, V), \\ \dot{Z} = \varepsilon G(r, \theta, Z, U, V), \\ \dot{U} = \mu U + \varepsilon G(r, \theta, Z, U, V), \\ \dot{V} = \delta V + \varepsilon G(r, \theta, Z, U, V), \end{cases} \tag{60}$$

where $G(r, \theta, Z, U, V) = \widetilde{F}(r\cos\theta, r\sin\theta, Z, U, V)$.

After dividing by $\dot{\theta}$ and simplifying, we find

$$\begin{cases} \dfrac{dr}{d\theta} = \varepsilon \sin\theta\, G(r, \theta, Z, U, V) + o(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = \varepsilon G(r, \theta, Z, U, V) + o(\varepsilon^2), \\ \dfrac{dU}{d\theta} = \mu U + \varepsilon\left(1 - \dfrac{\mu U}{r}\cos\theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2), \\ \dfrac{dV}{d\theta} = \delta V + \varepsilon\left(1 - \dfrac{\delta V}{r}\cos\theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2). \end{cases} \tag{61}$$

System (61) is now of the same form as system (33) with

$$\mathbf{x} = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta,$$
$$F_0(\mathbf{x}, \theta) = \begin{pmatrix} 0 \\ 0 \\ \mu U \\ \delta V \end{pmatrix},$$
$$F_1(\mathbf{x}, \theta) = \begin{pmatrix} \sin\theta\, G(r, \theta, Z, U, V) \\ G(r, \theta, Z, U, V) \\ \left(1 - \dfrac{\mu U}{r}\cos\theta\right) G(r, \theta, Z, U, V) \\ \left(1 - \dfrac{\delta V}{r}\cos\theta\right) G(r, \theta, Z, U, V) \end{pmatrix}. \tag{62}$$

We shall apply Theorem 6 to system (61). System (61) with $\varepsilon = 0$ has the 2π–periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ Z_0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall r_0 > 0,\ \forall Z_0 \in \mathbb{R}. \tag{63}$$

By the notations of Theorem 6, we have that $k = 2$ and $n = 4$. Let $R > 0$; we take

$$\mathbb{V} = \left\{(r_0, Z_0) \colon 0 < r_0^2 + Z_0^2 < R\right\} \subset \mathbb{R}^2, \tag{64}$$

$\alpha = (r_0, Z_0) \in \mathbb{V}$, and

$$\begin{aligned} \beta_0 \colon & Cl(\mathbb{V}) \longrightarrow \mathbb{R}^2, \\ & \colon (r_0, Z_0) \longmapsto \beta_0(r_0, Z_0) = (0, 0). \end{aligned} \tag{65}$$

We also take

$$\mathscr{Z} = \{\mathbf{z}_\alpha = (r_0, Z_0, 0, 0),\ (r_0, Z_0) \in \mathbb{V}\}. \quad (66)$$

The fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear system (61) with $\varepsilon = 0$ with respect to the periodic solution $\mathbf{z}_\alpha = (r_0, Z_0, 0, 0)$ satisfying that $M_{\mathbf{z}_\alpha}(0)$ is the identity matrix is

$$M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\mu\theta} & 0 \\ 0 & 0 & 0 & e^{\delta\theta} \end{pmatrix}. \quad (67)$$

We have

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi\mu} & 0 \\ 0 & 0 & 0 & 1 - e^{-2\pi\delta} \end{pmatrix}, \quad (68)$$

which satisfy the assumption (ii) of Theorem 6. Taking

$$\begin{aligned} \xi &: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \\ &: (r, Z, U, V) \mapsto \xi(r, Z, U, V) = (r, Z), \end{aligned} \quad (69)$$

we must compute the function $\mathscr{F}(\alpha)$ given by (36), and we obtain

$$\begin{aligned} \mathscr{F}(\alpha) = \mathscr{F}(r_0, Z_0) &= \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} \sin\theta\, G(r_0, \theta, Z_0, 0, 0) \\ G(r_0, \theta, Z_0, 0, 0) \end{pmatrix} \\ \mathrm{d}\theta &= \begin{pmatrix} \mathscr{F}_1(r_0, Z_0) \\ \mathscr{F}_2(r_0, Z_0) \end{pmatrix}, \end{aligned} \quad (70)$$

where

$$\begin{aligned} \mathscr{F}_1(r_0, Z_0) &= \frac{1}{2\pi}\int_0^{2\pi} \sin\theta\, F(B_1, B_2, B_3, B_4, B_5)\mathrm{d}\theta, \\ \mathscr{F}_2(r_0, Z_0) &= \frac{1}{2\pi}\int_0^{2\pi} F(B_1, B_2, B_3, B_4, B_5)\mathrm{d}\theta, \end{aligned} \quad (71)$$

and B_1, B_2, B_3, B_4, and B_5 are given by (12). Then, by Theorem 6, for every simple zero (r_0^*, Z_0^*) of the function $\mathscr{F}(r_0, Z_0)$ there exists a limit cycle $(r, Z, U, V)(\theta, \varepsilon)$ of system (61) such that

$$(r, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, Z_0^*, 0, 0), \text{when } \varepsilon \longrightarrow 0. \quad (72)$$

Going back through the change of coordinates, we obtain a limit cycle $(r, \theta, Z, U, V)(t, \varepsilon)$ of system (60) such that

$$(r, \theta, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, Z_0^*, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \quad (73)$$

We have a limit cycle $(X, Y, Z, U, V)(t, \varepsilon)$ of system (58) such that

$$(X, Y, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, Z_0^*, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \quad (74)$$

Finally, we obtain a limit cycle $x(t, \varepsilon)$ of equation (3) tending to the periodic solution (14) of equation (15) when $\varepsilon \longrightarrow 0$.

Theorem 2 is proved.

3.2.2. Proof of Corollary 2. If $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = x^2 - 2x + \dot{x}^2$, then we have

$$\begin{aligned} \mathscr{F}_1(r_0, Z_0) &= \frac{r_0(\mu\delta - 1)(\mu\delta - Z_0)}{\mu\delta(1 + \mu^2)(1 + \delta^2)}, \\ \mathscr{F}_2(r_0, Z_0) &= \frac{r_0^2\mu^2\delta^2 - Z_0(1 + \mu^2)(1 + \delta^2)(2\mu\delta - Z_0)}{\mu^2\delta^2(1 + \mu^2)(1 + \delta^2)}. \end{aligned} \quad (75)$$

The system $\mathscr{F}_1(r_0, Z_0) = \mathscr{F}_2(r_0, Z_0) = 0$ has the zero $(r_0^*, Z_0^*) = (\sqrt{(1 + \mu^2)(1 + \delta^2)}, \mu\delta)$ such that

$$\det\left(\frac{\partial(\mathscr{F}_1, \mathscr{F}_2)}{\partial(r_0, Z_0)}\right)\Big|_{(r_0, Z_0) = (r_0^*, Z_0^*)} = \frac{2(\mu\delta - 1)}{\mu\delta(1 + \mu^2)(1 + \delta^2)} \neq 0. \quad (76)$$

The proof of Corollary 2 follows directly by applying Theorem 2 and (16) is obtained by substituting (r_0^*, Z_0^*) in (14).

3.3. Proofs in Case 3: $\lambda = 0$

3.3.1. Proof of Theorem 3. If $\lambda = 0$, $\mu = \delta \neq 0$, then equation (3) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = -\mu^2 y + 2\mu z - (1 + \mu^2)u + 2\mu v + \varepsilon F(x, y, z, u, v). \end{cases} \quad (77)$$

System (77) has a unique singular point at the origin and the eigenvalues of the linear part of this system are $\pm i$, 0, and μ. By the linear transformation

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} 0 & \mu^2 & -2\mu & 1 & 0 \\ 0 & 0 & -\mu^2 & 2\mu & -1 \\ \mu^2 & -2\mu & 1 + \mu^2 & -2\mu & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -\mu & 1 & -\mu & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \quad (78)$$

we transform system (77) into the following system:

$$\begin{cases} \dot{X} = -Y, \\ \dot{Y} = X - \varepsilon \widetilde{F}(X,Y,Z,U,V), \\ \dot{Z} = \varepsilon \widetilde{F}(X,Y,Z,U,V), \\ \dot{U} = \mu U + V, \\ \dot{V} = \mu V + \varepsilon \widetilde{F}(X,Y,Z,U,V), \end{cases} \tag{79}$$

where

$$\begin{aligned} \widetilde{F} &= \widetilde{F}(X,Y,Z,U,V) = F(c_1,c_2,c_3,c_4,c_5), \\ c_1 &= \frac{2\mu X}{(\mu^2+1)^2} + \frac{(\mu^2-1)Y}{(\mu^2+1)^2} + \frac{Z}{\mu^2} + \frac{U}{\mu(\mu^2+1)} - \frac{(3\mu^2+1)V}{\mu^2(\mu^2+1)^2}, \\ c_2 &= \frac{(\mu^2-1)X}{(\mu^2+1)^2} - \frac{2\mu Y}{(\mu^2+1)^2} + \frac{U}{\mu^2+1} - \frac{2\mu V}{(\mu^2+1)^2}, \\ c_3 &= \frac{-2\mu X}{(\mu^2+1)^2} - \frac{(\mu^2-1)Y}{(\mu^2+1)^2} + \frac{\mu U}{\mu^2+1} - \frac{(\mu^2-1)V}{(\mu^2+1)^2}, \\ c_4 &= \frac{-(\mu^2-1)X}{(\mu^2+1)^2} + \frac{2\mu Y}{(\mu^2+1)^2} + \frac{\mu^2 U}{\mu^2+1} + \frac{2\mu V}{(\mu^2+1)^2}, \\ c_5 &= \frac{2\mu X}{(\mu^2+1)^2} + \frac{(\mu^2-1)Y}{(\mu^2+1)^2} + \frac{\mu^3 U}{\mu^2+1} + \frac{\mu^2(\mu^2+3)V}{(\mu^2+1)^2}. \end{aligned} \tag{80}$$

Note that the linear part of system (81) is in the real Jordan normal form of the linear part of system (77). We pass now from the Cartesian coordinates (X,Y,Z,U,V) to the cylindrical ones (r,θ,Z,U,V) with $X = r\cos\theta$, $Y = r\sin\theta$, and we obtain

$$\begin{cases} \dot{r} = -\varepsilon \sin\theta\, G(r,\theta,Z,U,V), \\ \dot{\theta} = 1 + \dfrac{\varepsilon}{r}\cos\theta\, G(r,\theta,Z,U,V), \\ \dot{Z} = \varepsilon G(r,\theta,Z,U,V), \\ \dot{U} = \mu U + V, \\ \dot{V} = \mu V + \varepsilon G(r,\theta,Z,U,V), \end{cases} \tag{81}$$

where $G(r,\theta,Z,U,V) = \widetilde{F}(r\cos\theta, r\sin\theta, Z,U,V)$.

After dividing by $\dot{\theta}$ and simplifying, we find

$$\begin{cases} \dfrac{dr}{d\theta} = -\varepsilon \sin\theta\, G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = \varepsilon G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dU}{d\theta} = \mu U + V + \varepsilon \dfrac{\mu U + V}{r}\cos\theta\, G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dV}{d\theta} = \mu V + \varepsilon\left(1 + \dfrac{\mu V}{r}\cos\theta\right) G(r,\theta,Z,U,V) + o(\varepsilon^2). \end{cases} \tag{82}$$

System (82) is now of the same form as system (33) with

$$\mathbf{x} = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta,$$

$$F_0(\mathbf{x},\theta) = \begin{pmatrix} 0 \\ 0 \\ \mu U + V \\ \mu V \end{pmatrix}, \quad F_1(\mathbf{x},\theta) = \begin{pmatrix} -\sin\theta\, G(r,\theta,Z,U,V) \\ G(r,\theta,Z,U,V) \\ \dfrac{\mu U + V}{r}\cos\theta\, G(r,\theta,Z,U,V) \\ \left(1 + \dfrac{\mu V}{r}\cos\theta\right) G(r,\theta,Z,U,V) \end{pmatrix}. \tag{83}$$

We shall apply Theorem 6 to system (82). System (82) with $\varepsilon = 0$ has the 2π–periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ Z_0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall r_0 > 0, \forall Z_0 \in \mathbb{R}. \tag{84}$$

By the notations of Theorem 6, we have that $k = 2$ and $n = 4$. Let $R > 0$; we take

$$\mathbb{V} = \{(r_0, Z_0):\ 0 < r_0^2 + Z_0^2 < R\} \subset \mathbb{R}^2. \tag{85}$$

$\alpha = (r_0, Z_0) \in \mathbb{V}$ and

$$\begin{aligned} &\beta_0\colon Cl(\mathbb{V}) \longrightarrow \mathbb{R}^2,\\ &:\ (r_0, Z_0) \mapsto \beta_0(r_0, Z_0) = (0,0). \end{aligned} \tag{86}$$

We also take

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, Z_0, 0, 0),\ (r_0, Z_0) \in \mathbb{V}\}. \tag{87}$$

The fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear system (82) with $\varepsilon = 0$ with respect to the periodic solution $\mathbf{z}_\alpha = (r_0, Z_0, 0, 0)$ satisfying that $M_{\mathbf{z}_\alpha}(0)$ is the identity matrix is

$$M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & e^{\mu\theta} & \theta e^{\mu\theta}\\ 0 & 0 & 0 & e^{\mu\theta} \end{pmatrix}. \tag{88}$$

We have

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 - e^{-2\pi\mu} & 2\pi e^{-2\pi\mu}\\ 0 & 0 & 0 & 1 - e^{-2\pi\mu} \end{pmatrix}, \tag{89}$$

which satisfy the assumption (ii) of Theorem 6. Taking

$$\begin{aligned} &\xi\colon \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\\ &:\ (r, Z, U, V) \mapsto \xi(r, Z, U, V) = (r, Z), \end{aligned} \tag{90}$$

we must compute the function $\mathcal{F}(\alpha)$ given by (36), and we obtain

$$\begin{aligned} \mathcal{F}(\alpha) = \mathcal{F}(r_0, Z_0) &= \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} -\sin\theta\, G(r_0, \theta, Z_0, 0, 0)\\ G(r_0, \theta, Z_0, 0, 0) \end{pmatrix} d\theta\\ &= \begin{pmatrix} \mathcal{F}_1(r_0, Z_0)\\ \mathcal{F}_2(r_0, Z_0) \end{pmatrix}, \end{aligned} \tag{91}$$

where

$$\begin{aligned} \mathcal{F}_1(r_0, Z_0) &= \frac{1}{2\pi}\int_0^{2\pi} -\sin\theta\, F(C_1, C_2, C_3, C_4, C_5)\, d\theta,\\ \mathcal{F}_2(r_0, Z_0) &= \frac{1}{2\pi}\int_0^{2\pi} F(C_1, C_2, C_3, C_4, C_5)\, d\theta, \end{aligned} \tag{92}$$

and C_1, C_2, C_3, C_4, and C_5 are given by (18). Then, by Theorem 6, for every simple zero (r_0^*, Z_0^*) of the function $\mathcal{F}(r_0, Z_0)$, there exists a limit cycle $(r, Z, U, V)(\theta, \varepsilon)$ of system (82) such that

$$(r, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, Z_0^*, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{93}$$

Going back through the change of coordinates, we obtain a limit cycle $(r, \theta, Z, U, V)(t, \varepsilon)$ of system (81) such that

$$(r, \theta, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, Z_0^*, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{94}$$

We have a limit cycle $(X, Y, Z, U, V)(t, \varepsilon)$ of system (79) such that

$$(X, Y, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, Z_0^*, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{95}$$

Finally, we obtain a limit cycle $x(t, \varepsilon)$ of equation (3) tending to the periodic solution (20) of equation (21) when $\varepsilon \longrightarrow 0$.

Theorem 3 is proved.

3.3.2. Proof of Corollary 3. If $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = x^2 - x\dot{x} - 1$, then we have

$$\begin{aligned} \mathcal{F}_1(r_0, Z_0) &= \frac{(1 - \mu - \mu^2) r_0 Z_0}{\mu^2 (1 + \mu^2)^2},\\ \mathcal{F}_2(r_0, Z_0) &= \frac{r_0^2}{2(1 + \mu^2)^2} + \frac{Z_0^2}{\mu^4} + 1. \end{aligned} \tag{96}$$

The system $\mathcal{F}_1(r_0, Z_0) = \mathcal{F}_2(r_0, Z_0) = 0$ has the zero $(r_0^*, Z_0^*) = (\sqrt{2}\,(1 + \mu^2), 0)$ such that

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(r_0, Z_0)}\right)\Big|_{(r_0, Z_0) = (r_0^*, Z_0^*)} = \frac{2(\mu^2 + \mu - 1)}{\mu^2 (1 + \mu^2)^2} \neq 0. \tag{97}$$

The proof of Corollary 3 follows directly by applying Theorem 3 and (22) is obtained by substituting (r_0^*, Z_0^*) in (20).

3.4. Proofs in Case 4: $\lambda \neq 0$ and $\mu = \delta \neq 0$

3.4.1. Proof of Theorem 4. If $\lambda \neq 0$ and $\mu = \delta \neq 0$, then equation (3) can be written as

$$\begin{cases} \dot{x} = y,\\ \dot{y} = z,\\ \dot{z} = u,\\ \dot{u} = v,\\ \dot{v} = \lambda\mu^2 x - (\mu^2 + 2\lambda\mu) y + (\lambda + 2\mu + \lambda\mu^2) z\\ \quad - (\mu^2 + 2\lambda\mu + 1) u + (\lambda + 2\mu) v + \varepsilon F(x, y, z, u, v). \end{cases} \tag{98}$$

System (98) with $\varepsilon = 0$ has a unique singular point at the origin and the linear part of this system has the eigenvalues $\pm i$, λ, and μ. Using the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} -\lambda\mu^2 & \mu^2+2\lambda\mu & -\lambda-2\mu & 1 & 0 \\ 0 & \lambda\mu^2 & -\mu^2-2\lambda\mu & \lambda+2\mu & -1 \\ \mu^2 & -2\mu & \mu^2+1 & -2\mu & 1 \\ -\lambda & 1 & -\lambda & 1 & 0 \\ \lambda\mu & -\lambda-\mu & 1+\lambda\mu & -\lambda-\mu & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \tag{99}$$

we transform the system (98) into the following system:

$$\begin{cases} \dot{X} = -Y, \\ \dot{Y} = X - \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{Z} = \lambda Z + \varepsilon\widetilde{F}(X,Y,Z,U,V), \\ \dot{U} = \mu U + V, \\ \dot{V} = \mu V + \varepsilon\widetilde{F}(X,Y,Z,U,V), \end{cases} \tag{100}$$

where

$$\begin{aligned}
\widetilde{F} &= \widetilde{F}(X,Y,Z,U,V) = F(d_1,d_2,d_3,d_4,d_5), \\
d_1 &= \frac{(-\lambda\mu^2+2\mu+\lambda)X}{(\lambda^2+1)(\mu^2+1)^2} + \frac{(\mu^2+2\lambda\mu-1)Y}{(\lambda^2+1)(\mu^2+1)^2} + \frac{Z}{(\lambda^2+1)(\lambda-\mu)^2} - \frac{U}{(\mu^2+1)(\lambda-\mu)} - \frac{(3\mu^2-2\lambda\mu+1)V}{(\mu^2+1)^2(\lambda-\mu)^2}, \\
d_2 &= \frac{(\mu^2+2\lambda\mu-1)X}{(\lambda^2+1)(\mu^2+1)^2} - \frac{(-\lambda\mu^2+2\mu+\lambda)Y}{(\lambda^2+1)(\mu^2+1)^2} + \frac{\lambda Z}{(\lambda^2+1)(\lambda-\mu)^2} - \frac{\mu U}{(\mu^2+1)(\lambda-\mu)} - \frac{(2\mu^3-\lambda\mu^2+\lambda)V}{(\mu^2+1)^2(\lambda-\mu)^2}, \\
d_3 &= -\frac{(-\lambda\mu^2+2\mu+\lambda)X}{(\lambda^2+1)(\mu^2+1)^2} - \frac{(\mu^2+2\lambda\mu-1)Y}{(\lambda^2+1)(\mu^2+1)^2} + \frac{\lambda^2 Z}{(\lambda^2+1)(\lambda-\mu)^2} - \frac{\mu^2 U}{(\mu^2+1)(\lambda-\mu)} - \frac{\mu(\mu^3-\mu+2\lambda)V}{(\mu^2+1)^2(\lambda-\mu)^2}, \\
d_4 &= -\frac{(\mu^2+2\lambda\mu-1)X}{(\lambda^2+1)(\mu^2+1)^2} + \frac{(-\lambda\mu^2+2\mu+\lambda)Y}{(\lambda^2+1)(\mu^2+1)^2} + \frac{\lambda^3 Z}{(\lambda^2+1)(\lambda-\mu)^2} - \frac{\mu^3 U}{(\mu^2+1)(\lambda-\mu)} - \frac{\mu^2(\lambda\mu^2-2\mu+3\lambda)V}{(\mu^2+1)^2(\lambda-\mu)^2}, \\
d_5 &= \frac{(-\lambda\mu^2+2\mu+\lambda)X}{(\lambda^2+1)(\mu^2+1)^2} + \frac{(\mu^2+2\lambda\mu-1)Y}{(\lambda^2+1)(\mu^2+1)^2} + \frac{\lambda^4 Z}{(\lambda^2+1)(\lambda-\mu)^2} - \frac{\mu^4 U}{(\mu^2+1)(\lambda-\mu)} - \frac{\mu^3(-\mu^3+2\lambda\mu^2-3\mu+4\lambda)V}{(\mu^2+1)^2(\lambda-\mu)^2}.
\end{aligned} \tag{101}$$

Note that the linear part of system (100) is in the real normal Jordan form of the linear part of system (98). We pass now from the Cartesian coordinates (X,Y,Z,U,V) to the cylindrical ones (r,θ,Z,U,V) with $X = r\cos\theta$, $Y = r\sin\theta$, and we obtain

$$\begin{cases} \dot{r} = -\varepsilon\sin\theta\, G(r,\theta,Z,U,V), \\ \dot{\theta} = 1 + \dfrac{\varepsilon}{r}\cos\theta\, G(r,\theta,Z,U,V), \\ \dot{Z} = \lambda Z + \varepsilon G(r,\theta,Z,U,V), \\ \dot{U} = \mu U + V, \\ \dot{V} = \mu V + \varepsilon G(r,\theta,Z,U,V), \end{cases} \tag{102}$$

where $G(r,\theta,Z,U,V) = \widetilde{F}(r\cos\theta, r\sin\theta, Z,U,V)$.

After dividing by $\dot{\theta}$ and simplifying, we find

$$\begin{cases} \dfrac{dr}{d\theta} = -\varepsilon\sin\theta\, G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = \lambda Z + \varepsilon\left(1 + \dfrac{\lambda Z}{r}\cos\theta\right)G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dU}{d\theta} = \mu U + V + \varepsilon\left(\dfrac{\mu U + V}{r}\cos\theta\right)G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dV}{d\theta} = \mu V + \varepsilon\left(1 + \dfrac{\mu V}{r}\cos\theta\right)G(r,\theta,Z,U,V) + o(\varepsilon^2). \end{cases} \tag{103}$$

System (103) is now of the same form as system (33) with

$$\mathbf{x} = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta,$$

$$F_0(\mathbf{x}, \theta) = \begin{pmatrix} 0 \\ \lambda Z \\ \mu U + V \\ \mu V \end{pmatrix}, \tag{104}$$

$$F_1(\mathbf{x}, \theta) = \begin{pmatrix} -\sin\theta G(r,\theta,Z,U,V) \\ \left(1 + \frac{\lambda Z}{r}\cos\theta\right) G(r,\theta,Z,U,V) \\ \left(\frac{\mu U + V}{r}\cos\theta\right) G(r,\theta,Z,U,V) \\ \left(1 + \frac{\mu V}{r}\cos\theta\right) G(r,\theta,Z,U,V) \end{pmatrix}.$$

We shall apply Theorem 6 to system (103). System (103) with $\varepsilon = 0$ has the 2π−periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall r_0 > 0. \tag{105}$$

By the notations of Theorem 6, we have that $k = 1$ and $n = 4$. Let $r_1 > 0$ and $r_2 > 0$; we take $\mathbb{V} =]r_1, r_2[\subset \mathbb{R}$, $\alpha = r_0 \in [r_1, r_2]$, and

$$\begin{aligned} \beta_0 &: [r_1, r_2] \longrightarrow \mathbb{R}^3, \\ &: r_0 \mapsto \beta_0(r_0) = (0,0,0). \end{aligned} \tag{106}$$

We also take

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, 0, 0, 0), \quad r_0 \in [r_1, r_2]\}. \tag{107}$$

The fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear system (103) with $\varepsilon = 0$ with respect to the periodic solution $\mathbf{z}_\alpha = (r_0, 0, 0, 0)$ satisfying that $M_{\mathbf{z}_\alpha}(0)$ is the identity matrix is

$$M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\lambda\theta} & 0 & 0 \\ 0 & 0 & e^{\mu\theta} & \theta e^{\mu\theta} \\ 0 & 0 & 0 & e^{\mu\theta} \end{pmatrix}. \tag{108}$$

We have

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - e^{-2\pi\lambda} & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi\mu} & 2\pi e^{-2\pi\mu} \\ 0 & 0 & 0 & 1 - e^{-2\pi\mu} \end{pmatrix}, \tag{109}$$

which satisfy the assumption (ii) of Theorem 6. Taking

$$\begin{aligned} \xi &: \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \\ &: (r, Z, U, V) \mapsto \xi(r, Z, U, V) = r, \end{aligned} \tag{110}$$

we must compute the function $\mathcal{F}(\alpha)$ given by (34), and we obtain

$$\begin{aligned} \mathcal{F}(\alpha) &= \frac{1}{2\pi} \int_0^{2\pi} -\sin\theta G(r_0, \theta, 0, 0, 0)\, d\theta, \\ \mathcal{F}(r_0) &= \frac{1}{2\pi} \int_0^{2\pi} -\sin\theta F(D_1, D_2, D_3, D_4, D_5) d\theta, \end{aligned} \tag{111}$$

and D_1, D_2, D_3, D_4, and D_5 are given by (24). Then, by Theorem 6, for every simple zero r_0^* of the function $\mathcal{F}(r_0)$, there exists a limit cycle $(r, Z, U, V)(\theta, \varepsilon)$ of system (103) such that

$$(r, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{112}$$

Going back through the change of coordinates, we obtain a limit cycle $(r, \theta, Z, U, V)(t, \varepsilon)$ of system (102) such that

$$(r, \theta, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{113}$$

We have a limit cycle $(X, Y, Z, U, V)(t, \varepsilon)$ of system such that

$$(X, Y, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{114}$$

Finally, we obtain a limit cycle $x(t, \varepsilon)$ of equation (3) tending to the periodic solution (25) of equation (26) when $\varepsilon \longrightarrow 0$.

Theorem 4 is proved.

3.4.2. Proof of Corollary 4. If $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = x^3 - x - 1$, then we have

$$\mathcal{F}(r_0) = \frac{r_0\left(-1 + 2\lambda\mu + \mu^2\right)\left(4\left(1+\lambda^2\right)\left(1+\mu^2\right)^2 - 3r_0^2\right)}{8\left(1+\lambda^2\right)^2\left(1+\mu^2\right)^4}, \tag{115}$$

which have the real positive simple zero $r_0^* = 2\sqrt{3}/3\sqrt{(1+\lambda^2)}(1+\mu^2)$ with

$$\frac{df}{dr_0}(r_0^*) = \frac{1 - 2\lambda\mu - \mu^2}{\left(1+\lambda^2\right)\left(1+\mu^2\right)^2} \neq 0. \tag{116}$$

The proof of Corollary 4 follows directly by applying Theorem 4 and (27) is obtained by substituting r_0^* in (25).

3.5. Proofs in Case 5: $\lambda = \mu = \delta \neq 0$

3.5.1. Proof of Theorem 5. If $\lambda = \mu = \delta \neq 0$, then equation (3) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = \lambda^3 x - 3\lambda^2 y + \left(3\lambda + \lambda^3\right) z - \left(3\lambda^2 + 1\right) u + 3\lambda v + \varepsilon F(x, y, z, u, v). \end{cases} \tag{117}$$

System (117) with $\varepsilon = 0$ has a unique singular point at the origin and the linear part of this system has the eigenvalues $\pm i$ and λ. Using the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} \lambda^3 & -3\lambda^2 & 3\lambda & -1 & 0 \\ 0 & -\lambda^3 & 3\lambda^2 & -3\lambda & 1 \\ 1 & 0 & 1 & 0 & 0 \\ -\lambda & 1 & -\lambda & 1 & 0 \\ \lambda^2 & -2\lambda & \lambda^2 + 1 & -2\lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \tag{118}$$

we transform system (117) into the following system:

$$\begin{cases} \dot{X} = -Y, \\ \dot{Y} = X + \varepsilon \widetilde{F}(X, Y, Z, U, V), \\ \dot{Z} = \lambda Z + U, \\ \dot{U} = \lambda U + V, \\ \dot{V} = \lambda V + \varepsilon \widetilde{F}(X, Y, Z, U, V), \end{cases} \tag{119}$$

where

$$\begin{aligned}
\widetilde{F} &= \widetilde{F}(X, Y, Z, U, V) = F(e_1, e_2, e_3, e_4, e_5), \\
e_1 &= \frac{\lambda\left(\lambda^2 - 3\right) X + \left(1 - 3\lambda^2\right) Y + \left(\lambda^2 + 1\right)^2 Z - 2\lambda\left(\lambda^2 + 1\right) U + \left(3\lambda^2 - 1\right) V}{\left(\lambda^2 + 1\right)^3}, \\
e_2 &= \frac{\left(1 - 3\lambda^2\right) X - \lambda\left(\lambda^2 - 3\right) Y + \lambda\left(\lambda^2 + 1\right)^2 Z + \left(1 - \lambda^4\right) U + \lambda\left(\lambda^2 - 3\right) V}{\left(\lambda^2 + 1\right)^3}, \\
e_3 &= \frac{-\lambda\left(\lambda^2 - 3\right) X - \left(1 - 3\lambda^2\right) Y + \lambda^2\left(\lambda^2 + 1\right)^2 Z + 2\lambda\left(\lambda^2 + 1\right) U + \left(1 - 3\lambda^2\right) V}{\left(\lambda^2 + 1\right)^3}, \\
e_4 &= \frac{-\left(1 - 3\lambda^2\right) X + \lambda\left(\lambda^2 - 3\right) Y + \lambda^3\left(\lambda^2 + 1\right)^2 Z + \lambda^2\left(\lambda^2 + 3\right)\left(\lambda^2 + 1\right) U - \lambda\left(\lambda^2 - 3\right) V}{\left(\lambda^2 + 1\right)^3}, \\
e_5 &= \frac{\lambda\left(\lambda^2 - 3\right) X + \left(1 - 3\lambda^2\right) Y + \lambda^4\left(\lambda^2 + 1\right)^2 Z + 2\lambda^3\left(\lambda^2 + 2\right)\left(\lambda^2 + 1\right) U + \lambda^2\left(3\lambda^2 + \lambda^4 + 6\right) V}{\left(\lambda^2 + 1\right)^3}.
\end{aligned} \tag{120}$$

Note that the linear part of system (119) is in the real normal Jordan form of the linear part of system (51). We pass now from the Cartesian coordinates (X, Y, Z, U, V) to the cylindrical ones (r, θ, Z, U, V) with $X = r\cos\theta$, $Y = r\sin\theta$, and we obtain

$$\begin{cases} \dot{r} = \varepsilon \sin\theta\, G(r,\theta,Z,U,V), \\ \dot{\theta} = 1 + \dfrac{\varepsilon}{r}\cos\theta\, G(r,\theta,Z,U,V), \\ \dot{Z} = \lambda Z + U, \\ \dot{U} = \lambda U + V, \\ \dot{V} = \lambda V + \varepsilon G(r,\theta,Z,U,V), \end{cases} \tag{121}$$

where $G(r,\theta,Z,U,V) = \widetilde{F}(r\cos\theta, r\sin\theta, Z, U, V)$.

After dividing by $\dot{\theta}$ and simplifying, we find

$$\begin{cases} \dfrac{dr}{d\theta} = \varepsilon \sin\theta\, G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = \lambda Z + U - \varepsilon\left(\dfrac{\lambda Z + U}{r}\cos\theta\right) G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dU}{d\theta} = \lambda U + V - \varepsilon\left(\dfrac{\lambda U + V}{r}\cos\theta\right) G(r,\theta,Z,U,V) + o(\varepsilon^2), \\ \dfrac{dV}{d\theta} = \lambda V + \varepsilon\left(1 - \dfrac{\lambda V}{r}\cos\theta\right) G(r,\theta,Z,U,V) + o(\varepsilon^2). \end{cases} \tag{122}$$

System (122) is now of the same form as system (33) with

$$\mathbf{x} = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta,$$
$$F_0(\mathbf{x},\theta) = \begin{pmatrix} 0 \\ \lambda Z + U \\ \lambda U + V \\ \lambda V \end{pmatrix},$$
$$F_1(\mathbf{x},\theta) = \begin{pmatrix} -\sin\theta\, G(r,\theta,Z,U,V) \\ \left(-\dfrac{\lambda Z + U}{r}\cos\theta\right) G(r,\theta,Z,U,V) \\ \left(-\dfrac{\lambda U + V}{r}\cos\theta\right) G(r,\theta,Z,U,V) \\ \left(1 - \dfrac{\lambda V}{r}\cos\theta\right) G(r,\theta,Z,U,V) \end{pmatrix}. \tag{123}$$

We shall apply Theorem 6 to system (122). System (122) with $\varepsilon = 0$ has the 2π−periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall r_0 > 0. \tag{124}$$

By the notations of Theorem 6, we have that $k = 1$ and $n = 4$. Let $r_1 > 0$ and $r_2 > 0$; we take $\mathbb{V} =]r_1, r_2[\subset \mathbb{R}$, $\alpha = r_0 \in [r_1, r_2]$, and

$$\begin{aligned} \beta_0 &: [r_1, r_2] \longrightarrow \mathbb{R}^3, \\ &: r_0 \mapsto \beta_0(r_0) = (0,0,0). \end{aligned} \tag{125}$$

We also take

$$\mathscr{Z} = \{\mathbf{z}_\alpha = (r_0, 0, 0, 0), \quad r_0 \in [r_1, r_2]\}. \tag{126}$$

The fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear system (122) with $\varepsilon = 0$ with respect to the periodic solution $\mathbf{z}_\alpha = (r_0, 0, 0, 0)$ satisfying that $M_{\mathbf{z}_\alpha}(0)$ is the identity matrix is

$$M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\lambda\theta} & \theta e^{\lambda\theta} & \dfrac{\theta^2}{2} e^{\lambda\theta} \\ 0 & 0 & e^{\lambda\theta} & \theta e^{\lambda\theta} \\ 0 & 0 & 0 & e^{\lambda\theta} \end{pmatrix}. \tag{127}$$

We have

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - e^{-2\pi\lambda} & 2\pi e^{-2\pi\lambda} & -2\pi^2 e^{-2\pi\lambda} \\ 0 & 0 & 1 - e^{-2\pi\lambda} & 2\pi e^{-2\pi\lambda} \\ 0 & 0 & 0 & 1 - e^{-2\pi\lambda} \end{pmatrix}, \tag{128}$$

which satisfy the assumption (ii) of Theorem 6. Taking

$$\begin{aligned} \xi &: \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}, \\ &: (r, Z, U, V) \mapsto \xi(r, Z, U, V) = r, \end{aligned} \tag{129}$$

we must compute the function $\mathscr{F}(\alpha)$ given by (36), and we obtain

$$\begin{aligned} \mathscr{F}(r_0) &= \frac{1}{2\pi} \int_0^{2\pi} \sin\theta\, G(r_0, \theta, 0, 0, 0)\, d\theta, \\ \mathscr{F}(\alpha) &= \frac{1}{2\pi} \int_0^{2\pi} \sin\theta\, F(E_1, E_2, E_3, E_4, E_5)\, d\theta, \end{aligned} \tag{130}$$

and E_1, E_2, E_3, E_4, and E_5 are given by (29). Then, by Theorem 6, for every simple zero r_0^* of the function $\mathscr{F}(r_0)$, there exists a limit cycle $(r, Z, U, V)(\theta, \varepsilon)$ of system (122) such that

$$(r, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{131}$$

Going back through the change of coordinates, we obtain a limit cycle $(r, \theta, Z, U, V)(t, \varepsilon)$ of system (121) such that

$$(r, \theta, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{132}$$

We have a limit cycle $(X, Y, Z, U, V)(t, \varepsilon)$ of system (119) such that

$$(X, Y, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, 0, 0, 0), \text{when } \varepsilon \longrightarrow 0. \tag{133}$$

Finally, we obtain a limit cycle $x(t, \varepsilon)$ of equation (3) tending to the periodic solution (30) of equation (31) when $\varepsilon \longrightarrow 0$.

Theorem 5 is proved.

3.5.2. Proof of Corollary 5. If $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = -\ddot{x}^2 \ddddot{x}^2 + x^3 + x^2 - \dot{x}\ddot{x} + \dddot{x} - 1$, then we have

$$\mathscr{F}(r_0) = \frac{r_0\left(3r_0^2(3\lambda^2 - 1) - 4\lambda(\lambda^2 - 3)(1 + \lambda^2)^3\right)}{8(1 + \lambda^2)^6}, \tag{134}$$

which have the real positive simple zero $r_0^* = 2\sqrt{3}/3\left(\sqrt{\lambda(3\lambda^2 - 1)((\lambda^2 - 3))(1 + \lambda^2)}\,(1 + \lambda^2)/3\lambda^2 - 1\right)$ with

$$\frac{df}{dr_0}(r_0^*) = \frac{\lambda(\lambda^2 - 3)}{(1 + \lambda^2)^3} \neq 0. \tag{135}$$

The proof of Corollary 5 follows directly by applying Theorem 5 and (32) is obtained by substituting r_0^* in (30).

4. Conclusion

There are several theories and methods for the study of the existence, uniqueness, or number and stability of limit cycles of differential equations which have been developed in trying to answer Hilbert's sixteenth problem posed in 1900 (see reference [1]) about the maximum number of limit cycles that a planar polynomial differential system can obtain. In this work, we study the limit cycles of the fifth-order differential equation by using the averaging theory of first order [6, 7], and we provide sufficient conditions for the existence of limit cycles of equation (12); in the next work, we will try to apply the same method on higher order differential equations.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Group Project under Grant no. (R.G.P-2/53/42).

References

[1] D. Hilbert, "Nachr. Ges. Wiss. Goett. Math.-Phys. Kl," *Bulletin of the American Mathematical Society*, vol. 8, pp. 437–479, 1900, English transl.

[2] A. Menaceur and S. Boulaaras, "A number of limit cycle of sextic polynomial differential systems via the averaging theory," *Boletim da Sociedade Paranaense de Matemática*, vol. 39, no. 4, pp. 181–197, 2021.

[3] J. Llibre and P. C. Ernesto, "Limit cycles of a class of perturbed differential systems via the first-order averaging method," *Complexity*, vol. 2021, Article ID 5581423, 6 pages, 2021.

[4] A. Menaceur, S. Boulaaras, A. Makhlouf, K. Rajagobal, and M. Abdalla, "Limit cycles for fourth-order autonomous differential equations," *Electronic Journal of Differential Equations*, vol. 2012, no. 22, pp. 1–17, 2012.

[5] J. Llibre and L. Roberto, "On the periodic orbits of the third-order differential equation $x''' - \mu x'' + x' - \mu x = \varepsilon F(x, x', x'')$," *Applied Mathematics Letters*, vol. 26, no. 4, pp. 425–430, 2013.

[6] A. Menaceur, S. Boulaaras, and S. Alkhalaf, "Jain limit cycles of a class of polynomial differential systems bifurcating from the periodic orbits of a linear center," *Symmetry*, vol. 12, no. 8, p. 1346, 2020.

[7] J. Llibre, J. Yu, and X. Zhang, "Limit cycles for a class of third order differential equations," *Rocky Mountain Journal of Mathematics*, vol. 40, pp. 581–594, 2010.

[8] N. Sellami and A. Makhlouf, "Limit cycles for a class of fifth-order differential equations," *Annals of Differential Equations*, vol. 28, no. 2, pp. 202–219, 2012.

[9] J. A. Sanders and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems*, Vol. 59, Springer, New York, NY, USA, 1985.

[10] I. G. Malkin, *Some Problems of the Theory of Nonlinear Oscillations*, Gostexizdat, Moscow, Russia, 1956.

[11] M. Roseau, *Vibrations Non-linéaires et Théorie de la Stabilité*, vol. 8, Springer-Verlag, Berlin, Germany, 1966, French.

[12] A. Buica, J. P. Francoise, and J. Llibre, "Periodic solutions of nonlinear periodic differential systems with a small parameter," *Communications on Pure and Applied Analysis*, vol. 6, pp. 103–111, 2007.