Hindawi
Discrete Dynamics in Nature and Society
Volume 2021, Article ID 7113065, 9 pages
https://doi.org/10.1155/2021/7113065

Research Article

A Positive Answer for 3IM+1CM Problem with a General Difference Polynomial

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Received 28 September 2021; Accepted 5 November 2021; Published 27 November 2021

Academic Editor: Li Li

In this paper, the 3IM+1CM theorem with a general difference polynomial $L(z, f)$ will be established by using new methods and technologies. Note that the obtained result is valid when the sum of the coefficient of $L(z, f)$ is equal to zero or not. Thus, the theorem with the condition that the sum of the coefficient of $L(z, f)$ is equal to zero is also a good extension for recent results. However, it is new for the case that the sum of the coefficient of $L(z, f)$ is not equal to zero. In fact, the main difficulty of proof is also from this case, which causes the traditional theorem invalid. On the other hand, it is more interesting that the nonconstant finite-order meromorphic function f can be exactly expressed for the case $f \equiv -L(z, f)$. Furthermore, the sharpness of our conditions and the existence of the main result are illustrated by examples. In particular, the main result is also valid for the discrete analytic functions.

1. Introduction and Results

It is well known that any polynomial is uniquely determined by its zero points (the set on which the polynomial takes zeros) except for a nonconstant factor, but it is not true for transcendental entire or meromorphic functions. For example, functions e^z and e^{-z} have the same $\pm 1, 0$ and ∞ points. Thus, it is interesting and complex to uniquely determine a meromorphic function. It is well known that the classical uniqueness results of the value distribution theory of meromorphic functions are the 5IM theorem (or five-point theorem) and the 4CM theorem (or four-point theorem) which had been obtained in the study by Nevanlinna [1], where IM means ignoring multiplicities and CM is counting multiplicities.

Let f and g be two nonconstant meromorphic functions on the complex plane $\mathbb{C}$ and a be a finite or infinite complex number in the whole complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, we say that f and g share a CM (counting multiplicities). If not considering the multiplicities, we say that f and g share a IM (ignoring multiplicities). Clearly, they also share a IM if f and g share a CM. In this case, the famous five-point theorem and four-points theorem mean that if two nonconstant meromorphic functions f and g share five distinct values IM, then $f = g$, and if two nonconstant meromorphic functions f and g share four distinct values CM, then f is a factional linear transformation of g, respectively.

The 4CM theorem in the study by Nevanlinna [1] can also be understood to be the 4CM+0 IM theorem which motivates us to consider the other cases as the 3CM+1IM theorem, the 2CM+2IM theorem, the 3IM+1CM theorem, or the 0CM+4IM theorem. Obviously, it is very good if the 0CM+4IM theorem is a fact. Unfortunately, Gundersen [2] has constructed counterexamples which show that the conclusion of the four-point theorem is invalid if CM is replaced by IM; that is, the 0CM+4IM theorem is false. However, the 2CM+2IM theorem is valid; please see the study by Gundersen [3]. It is known that the 2CM+2IM theorem implies that the 3CM+1IM theorem holds (see Gundersen [2, 3]). Thus, in [4], Gundersen proposed the following well-known question.

Problem 1. (3IM+1CM open problem [4]). If two nonconstant meromorphic functions share three values IM and share a fourth value CM, then do the functions necessarily share all four values CM?

It is very difficult to completely solve Problem 1 (see Gundersen [4]). In the past years, some researchers established 3IM+1CM theorems with some conditions, for example, the restriction of zero points for $f-g$ (see Mues [5–7], Reinders [8, 9], Wang [10], Ueda [11, 12], Yi and Zhou [13], Czubiak and Gundersen [14], Qiu [15], Wang [16, 17], Huang [18, 19], Huang and Du [20], Ishizaki [21], Li [22], Yao [23], and Yang and Yi [24]), the periodicity hypothesis (see Lin and Ishizaki [25] and Banerjee and Ahamed [26]), or the differential limitation (see Lahiri and Sinha [27], Gundersen [28, 29], Mues and Steinmetz [30, 31], Mues and Reinders [32], and Frank and Hua [33]).

In recent years, the difference variant of the Nevanlinna theory has been established in [34–37]. Using these theories, some mathematicians began to consider the uniqueness of meromorphic functions sharing values with their shifts or difference operators and produced many fine works; for example, see Banerjee and Bhattacharyya [38], Ahamed [39, 40], Ma et al. [41], Jiang et al. [42], Charak et al. [43], Lin et al. [44, 45], and Li et al. [46, 47].

In this paper, we will establish the 3IM+1CM theorem with the difference condition. Actually, Li et al. [46] obtained the following uniqueness result when f and $\Delta_\eta^n f$ fulfill the condition 3IM+1CM, where

$$\begin{aligned}\Delta_\eta f(z) &= f(z+\eta)-f(z),\\ \Delta_\eta^n f(z) &= \Delta_\eta\left(\Delta_\eta^{n-1} f(z)\right),\end{aligned} \tag{1}$$

for $n=2,3,\ldots$.

Theorem 1 (see [46]). *Let f be a nonconstant meromorphic function with finite order, and let η be a nonzero complex number. Suppose that f and $\Delta_\eta^n f$ share a_1, a_2, a_3 IM and share ∞ CM, where a_1, a_2, and a_3 are three distinct finite values. Then, $f(z)=\Delta_\eta^n f$ for all $z\in\mathbb{C}$.*

Noting that

$$\Delta_\eta^n f(z) = \sum_{i=0}^{n}\binom{n}{j}(-1)^{n-j} f(z+j\eta), \tag{2}$$

naturally, a more general case

$$L(z,f) \triangleq \sum_{j=0}^{n}\alpha_j f\left(z+d_j\right), \tag{3}$$

should be considered, where $n\geq 1$ is a positive integer, $d_j\,(j=0,\ldots,n)$ are distinct finite values, and $\alpha_j\,(j=0,\ldots,n)$ are nonzero constants. In the case, Problem 1 turns to be the following problem.

Problem 2. When a nonconstant meromorphic function f and its general linear difference polynomial $L(z,f)$ satisfy the condition 3IM+1CM, does the uniqueness result still hold?

In the following, we will give our main result, some corollaries, and remarks.

Theorem 2. *Let f be a nonconstant meromorphic function with finite order; let $L(z,f)$ be defined as (3). If f and $L(z,f)$ share a_1, a_2, a_3 IM and share ∞ CM, where a_1, a_2, and a_3 are three distinct finite values, then for all $z\in\mathbb{C}$, one of the following results holds:*

(i) $f=L(z,f)$;

(ii) $f=-L(z,f)$; furthermore,

$$f(z) = c\frac{1+e^{az+b}}{1-e^{az+b}}, \tag{4}$$

and $\sum_{j=0}^{n}\alpha_j=-1$, where $c\in\{a_1,a_2,a_3\}$, and $a(\neq 0), b$ are constants.

To explain a nonconstant meromorphic function f with finite order, we need the following definition.

Definition 1 (see [24, 48]). For a nonconstant meromorphic function f, the order of f, denoted by $\rho(f)$, is defined as

$$\rho(f) = \limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}, \tag{5}$$

where

$$\begin{aligned}T(r,f) &= m(r,f)+N(r,f),\\ m(r,f) &= \int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|\mathrm{d}\theta,\\ N(r,f) &= \int_0^r\frac{n(t,f)-n(0,f)}{t}\mathrm{d}t+n(0,f)\log r,\end{aligned} \tag{6}$$

and $n(t,f)$ denotes the number of poles of f (counting multiplicities) in $|z|<t$.

A nonconstant meromorphic function f with finite order means that $\rho(f)<\infty$.

Remark 1. The following example shows that Theorem 2 (ii) can happen. Let

$$\begin{aligned}f(z) &= a_1\frac{1+e^{az+b}}{1-e^{az+b}},\\ L(z,f(z)) &= \sum_{j=0}^{n}\alpha_j a_1\frac{1+e^{a(z+j\eta)+b}}{1-e^{a(z-j\eta)+b}}\\ &= a_1\sum_{j=0}^{n}\alpha_j\frac{1+e^{az+b}}{1-e^{az+b}} = -a_1\frac{1+e^{az+b}}{1-e^{az+b}},\end{aligned} \tag{7}$$

where $a_1 \neq 0$, $e^{a\eta} = 1$, and $\alpha_j (j = 0, 1 \ldots, n)$ are nonzero constants satisfying $\sum_{j=0}^{n} \alpha_j = -1$, and then f and $L(z, f)$ share $a_1, -a_1, 0, \infty$ CM, and $f = -L(z, f)$.

Remark 2. The conditions of Theorem 2 are sharp. From the example in the following, we know that the assumption 3IM+1CM cannot be relaxed to 4IM in Theorem 2. For example, see Steinmetz [49]. Let $\wp$ denote the Weierstrass $\wp$-function with a pair of primitive periods $2\omega - \omega'$ and $\omega + \omega'$, and set

$$f(z) = \frac{(\wp(z) - \wp(\omega/2))(\wp(z) - \wp(3\omega/2))^2}{\wp(z) - \wp(\omega)}, \qquad (8)$$
$$L(z, f) = f(z + \omega).$$

Then, the functions f and $L(z, f)$ share the values $0, \infty$ and some values A and B IM, where $A/B = (1/2)(1 \pm \sqrt{3}\, i)$. We note that it is showed that three meromorphic functions $f(z), L(z, f)$, and $f(z + \omega')$ share the four values $0, \infty, A$, and B IM in [49]. However, $f \neq L(z, f)$ or $f \neq -L(z, f)$.

Corollary 1. *Let f be a nonconstant meromorphic function with finite order; let $L(z, f)$ be defined as (3). Assume that f and $L(z, f)$ share a_1, a_2, a_3 IM and share ∞ CM, where a_1, a_2, a_3 are three distinct finite values. If $\sum_{j=0}^{n} \alpha_j \neq -1$, then $f = L(z, f)$ for all $z \in \mathbb{C}$.*

Corollary 2. *Let f be a nonconstant meromorphic function with finite order; let $L(z, f)$ be defined as (3). Assume that f and $L(z, f)$ share a_1, a_2, a_3 IM and share ∞ CM, where a_1, a_2, a_3 are three distinct finite values. If $\sum_{j=0}^{n} \alpha_j = 0$, then $f = L(z, f)$ for all $z \in \mathbb{C}$.*

Remark 3. Note that

$$\Delta_\eta^n f(z) = \sum_{i=0}^{n} \binom{n}{j} (-1)^{n-j} f(z + j\eta), \qquad (9)$$

and the sum of its coefficients $\sum_{i=0}^{n} \binom{n}{j} (-1)^{n-j}$ is zero. So, even if $\sum_{j=0}^{n} \alpha_j = 0$ in $L(z, f)$, $\Delta_\eta^n f(z)$ is still a special case of $L(z, f)$. Thus, Theorem 2 is a large extension of Theorem 1.2 in Li et al. [46]; naturally, the corresponding results in the study by Heittokangas et al. [50], Li et al. [46, 47], and so on are also the special cases of Theorem 2. However, our result is new when $\sum_{j=0}^{n} \alpha_j \neq 0$.

Remark 4. Choosing some special α_j and d_j for $j = 0, 1, \ldots, n$, we can obtain some difference equations of the form $f(z) = \sum_{j=0}^{n} \alpha_j f(z + d_j)$. Thus, the main result is also valid for the discrete analytic functions (see [51–55]). For example, let

$$\frac{f(z + 1 + i) - f(z)}{i + 1} = \frac{f(z + i) - f(z + 1)}{i - 1}, \qquad (10)$$

$$\text{or } f(z) = i f(z + i) - i f(z - 1) + f(z + 1 + i), \qquad (11)$$

which is the special case of (3). Thus, Theorem 2 is also valid for (10) or (11). When $z = m + in$ and $m, n \in \mathbb{Z}$, equation (10) is the definition of discrete analytic and harmonic functions (see Hundhausen [56]). At this time, we can obtain the uniqueness of (10) or (11). Another definition of discrete analytic and harmonic functions:

$$f(z + 1) - f(z) = \frac{f(z + i) - f(z)}{i}, \qquad (12)$$

$$\text{or } f(z) = \frac{1 - i}{2} f(z + 1) + \frac{1 + i}{2} f(z + i), \qquad (13)$$

can be seen in Harman [57]. Clearly, our theorem is also valid for (12) or (13).

We note that the main tool of proof for Theorem 1.2 in Li et al. [46] is the traditional Theorem 1.62 in [24]; however, it is invalid for Theorem 2 because $\sum_{j=0}^{n} \alpha_j$ can be nonzero. We specially thank Corollary 1.105 in [58] which will be our key tool and listed in Section 2 as Lemma 8. On the other hand, our methods and technologies are also different with Li et al. [46]; for example, the proof is divided into the six cases in Li et al. [46], but it is only two cases in the proof of Theorem 2. For the other detail cases, please see the proof in Section 3. However, it is worth mentioning that the meromorphic function f can be exactly expressed by

$$f(z) = c \frac{1 + e^{az+b}}{1 - e^{az+b}}, \qquad (14)$$

in some special cases. Certainly, some preliminaries of the difference value distribution theory [34–37] are also used in the proof of Theorem 2.

In the final, the present paper will be organized as follows. In Section 2, we will give some preliminaries which can be seen in the listed references. And the main result will be proved in the final section.

2. Some Lemmas

In this paper, the value distribution theory established by R. Nevanlinna is the main tool for the studies. For convenience of the reader who might not be familiar with Nevanlinna theory, we list here some results from Nevanlinna theory (see, e.g., [24, 48]).

The following known results are important in the value distribution theory (see, e.g., [24, 48]).

(i) The arithmetic properties of $T(r, f)$ and $m(r, f)$ are as follows:

$$T(r, fg) \leq T(r, f) + T(r, g),$$
$$T(r, f + g) \leq T(r, f) + T(r, g) + O(1). \qquad (15)$$

The same inequalities hold for $m(r, f)$.

(ii) The Nevanlinna first fundamental theorem is as follows: $T(r, f) = T(r, (1/f)) + O(1)$.

(iii) The logarithmic derivative lemma is as follows: $m(r, (f'/f)) = O(\log r)$, if the order $\rho(f)$ is finite.

In the following, we present some lemmas, which will be needed in the sequel.

Lemma 1 (see Theorem 3 in [59]). *Let f and g be two nonconstant rational functions. If f and g share a_1, a_2, a_3, a_4 IM, where a_1, a_2, a_3, a_4 are four distinct values in the extended complex plane, then $f = g$.*

Lemma 2 (see Lemma 1 in [4]). *Let f and g be two distinct nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4 be four distinct values in the extended complex. If f and g share a_1, a_2, a_3, a_4 IM, then for a set with finite linear measure E,*

(i) $T(r, f) = T(r, g) + O(\log(rT(r, f)))$, *as* $r \notin E$ *and* $r \longrightarrow \infty$

(ii) $2T(r, f) = \sum_{j=1}^{4} N(r, 1/(f - a_j)) + O(\log(rT(r, f)))$, *as as* $r \notin E$ *and* $r \longrightarrow \infty$, *where* $N(r, 1/(f - \infty)) = N(r, f)$

Remark 5. Under the assumptions of Lemma 2, if f and g in Lemma 2 are of finite order, by the context of paper [26] and the proof of Lemma 1 in [26], we can find that the conclusion of Lemma 2 can be changed to

(i) $T(r, f) = T(r, g) + O(\log r)$, as $r \longrightarrow \infty$

(ii) $2T(r, f) = \sum_{j=1}^{4} N(r, 1/(f - a_j))O(\log r)$, as $r \longrightarrow \infty$

Lemma 3 (see Lemma 2 and Corollary 1 in [3]). *Let f and g be distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3, and a_4 IM, where $a_4 = \infty$. Then, for a set with finite linear measure E, the following statements hold:*

(i) $N_0(r, (1/f')) = O(\log(rT(r, f)))$ *and* $N_0(r, (1/g')) = O(\log(rT(r, f)))$ *as* $r \notin E$, $r \longrightarrow \infty$, *and E is a set with finite linear measure, where* $N_0(r, (1/f'))$ *and* $N_0(r, (1/g'))$*count, respectively, only those points in* $N(r, (1/f'))$ *and* $N(r, (1/f'))$ *which do not occur when* $f(z) = g(z) = a_j$ *for some* $j = 1, 2, 3, 4$.

(ii) For $j = 1, 2, 3, 4$, *we next let* $N_2(r, a_j)$ *refer only to those* a_j*-points that are multiple for both f and g and count each such point the number of times of the smaller of the two multiplicities. Then,* $\sum_{j=1}^{4} N_2(r, a_j) = O(\log(rT(r, f)))$ *as* $r \notin E$ *and* $r \longrightarrow \infty$.

Remark 6. Under the assumptions of Lemma 3, if f and g in Lemma 3 are of finite order, by the context of paper [38] and the proof of Lemma 2 and Corollary 1 in [38], we can find that the conclusion of Lemma 3 can be changed into

(i) $N_0(r, (1/f')) = O(\log r)$ and $N_0(r, (1/g')) = O(\log r)$, as $r \longrightarrow \infty$

(ii) $\sum_{j=1}^{4} N_2(r, a_j) = O(\log r)$, as $r \longrightarrow \infty$

Lemma 4 (see Corollary 2.5 in [34]). *Let f be a nonconstant meromorphic function of finite order and η be a fixed nonzero complex number. Then, for each $\varepsilon > 0$,*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\left(r^{\rho-1+\varepsilon}\right), \quad \text{as } r \longrightarrow \infty. \tag{16}$$

Lemma 5 (see Theorem 1 in [4]). *Let f and g be two nonconstant meromorphic functions that share a_1, a_2, a_3 IM and a_4 CM, where a_1, a_2, a_3, and a_4 are four distinct values in the extended complex plane. Suppose that there exists some real constant $\mu > 4/5$ and some set $I \subset (0, +\infty)$ that has infinite linear measure such that*

$$\frac{N(r, (1/(f - a_4)))}{T(r, f)} \geq \mu, \tag{17}$$

for all $r \in I$, where $N(r, (1/(f - a_4))) = N(r, f)$, if $a_4 = \infty$. Then, f and g share all four values CM.

Lemma 6 (see Theorem 4.3 in [24]). *Let f and g be two distinct nonconstant meromorphic functions and $a_j\,(j = 1, 2, 3, 4)$ be four distinct values in the extend complex plane. If f and g share $a_j\,(j = 1, 2, 3, 4)$ CM, then $f(z) = T(g(z))$, where T is a Mobius transformation such that two of the four values are fixed points and another two (are Picard exceptional values of f and g) exchange each other under T.*

Lemma 7 (see Theorem 1.62 in [24]). *Let $f_1, f_2, \ldots, f_n$ be nonconstant meromorphic functions, and let $f_{n+1} \not\equiv 0$ be a meromorphic function such that $\sum_{j=1}^{n+1} f_j = 1$. Suppose that there exists a subset $I \subseteq R^+$ whose linear measure is ∞ such that*

$$\sum_{i=1}^{n+1} N\left(r, \frac{1}{f_i}\right) + n \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \overline{N}(r, f_i) < (\lambda + o(1))T(r, f_j), \quad j = 1, 2, \ldots, n, \tag{18}$$

As $r \in I$ and $r \longrightarrow \infty$, where $0 \leq \lambda < 1$, then $f_{n+1} \equiv 1$.

Lemma 8 (see Corollary 1.105(iii) in [58]). *Assume that entire functions $f_0, f_1, \ldots, f_n$ vanish nowhere on $\mathbb{C}^m$ such that*

$$f_0 + f_1 + \cdots + f_n = 0. \tag{19}$$

Partition the index set $I = \{0, 1, \ldots, n\}$ into subsets I_α, $I = \cup_{\alpha=0}^{k} I_\alpha$, putting two indices i and j in the same subset I_α if and only if (f_i/f_j) is a constant. Then, we have

$$\sum_{i \in I_\alpha} f_i = 0, \quad \alpha = 0, 1, \ldots, k. \tag{20}$$

3. Proof of Theorem 2

If $L(z, f)$ is a constant, by the assumption that f and $L(z, f)$ share a_1, a_2, a_3 IM and share ∞ CM, we know that f has more than two Picard values. And it is impossible, so $L(z, f)$

is nonconstant. Assume that f is a nonconstant rational function; according to the definition of $L(z,f)$, we can find that $L(z,f)$ is also a nonconstant rational function, and then using Lemma 1, we deduce that $L(z,f)=f$. Next, we suppose that f is a nonconstant transcendental meromorphic function. Since $\rho(f)<\infty$ and f and $L(z,f)$ share a_1,a_2,a_3 IM and share ∞ CM, applying Remark 5, we obtain

$$T(r,f)=T(r,L(z,f))+S(r,f). \tag{21}$$

Then, we can see that $L(z,f)$ is also a transcendental meromorphic function. Next, we assume $L(z,f)\not\equiv f$.

Under the assumption that f and $L(z,f)$ share ∞ CM, Remark 5 (ii) leads to

$$N_{(2}(r,f)+N_{(2}(r,L(z,f))=2N_2(r,\infty)=S(r,f), \tag{22}$$

where $N_{(2}(r,f)$ denotes the counting function of the multiple poles of f in $|z|<r$, and each point in $N_{(2}(r,f)$ is counted according to its multiplicity as a pole of f. $N_{(2}(r,L(z,f))$ is similar with $N_{(2}(r,f)$. Hence,

$$N(r,L(z,f)-f)\le \overline{N}_{(1,1)}(r,\infty)+S(r,f), \tag{23}$$

where $\overline{N}_{(1,1)}(r,\infty)$ is the counting function of the common simple poles of f and g. By the definition of $L(z,f)$ and Lemma 4, we deduce

$$\begin{aligned} m(r,L(z,f)-f)&=m\left(r,\frac{L(z,f)-f}{f}f\right)\\ &\le m(r,f)+m\left(r,\frac{L(z,f)}{f}\right)+O(1)\\ &\le m(r,f)+S(r,f). \end{aligned} \tag{24}$$

From (23), (24), Remark 5 (ii), and the assumption that f and $L(z,f)$ share a_1,a_2,a_3 IM and ∞ CM, we deduce

$$\begin{aligned} 2T(r,f)&=\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+\overline{N}\left(r,\frac{1}{f-a_3}\right)+\overline{N}(r,f)+S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{L(z,f)-f}\right)+\overline{N}_{(1,1)}(r,\infty)+S(r,f)\\ &\le T\left(r,\frac{1}{L(z,f)-f}\right)+\overline{N}_{(1,1)}(r,\infty)+S(r,f)\\ &=m(r,L(z,f)-f)+N(r,L(z,f)-f)+\overline{N}_{(1,1)}(r,\infty)+S(r,f)\\ &\le m(r,L(z,f)-f)+2\overline{N}_{(1,1)}(r,\infty)+S(r,f)\\ &\le T(r,f)+\overline{N}_{(1,1)}(r,\infty)+S(r,f). \end{aligned} \tag{25}$$

Hence, we get

$$\lim_{\substack{z\to\infty\\ z\in I}}\frac{N(r,f)}{T(r,f)}=1, \tag{26}$$

where $I\subset(0,+\infty)$ with logarithmic measure $\operatorname{logmes} I=+\infty$. In view of Lemma 5, we deduce that f and $L(z,f)$ share a_1,a_2,a_3 and ∞ CM. Then, by Lemma 6, we obtain that f is a Möbius transformation of $L(z,f)$. We set

$$f=\frac{AL(z,f)+B}{CL(z,f)+D}, \tag{27}$$

where A,B,C, and D are constants and $A\,D-BC\neq 0$. We consider the following two cases.

Case 1. Suppose that $C=0$. Then, from (27), we get

$$f=A_1L(z,f)+B_1, \tag{28}$$

where $A_1(\neq 0)$ and B_1 are two constants, and $A_1=(A/D)$, and $B_1=(B/D)$. If two of $\{a_1,a_2,a_3\}$ are not the Picard exceptional value of f, we get $f\equiv L(z,f)$, which is a contradiction, so two of $\{a_1,a_2,a_3\}$ are Picard exceptional values of f. Without loss of generality, we set a_1 and a_2 as Picard exceptional values of f, and by the assumption of Theorem 2, we get a_1 and a_2 as Picard exceptional values of $L(z,f)$. By rewriting (28), we obtain

$$f-a_1=A_1\left(L(z,f)-\frac{a_1-B_1}{A_1}\right), \tag{29}$$

and noting $f\neq a_1$, we deduce $((a_1-B_1)/A_1)=a_1$ or a_2. Similarly, we get

$$f-a_2=A_1\left(L(z,f)-\frac{a_2-B_1}{A_1}\right), \tag{30}$$

and $((a_2-B_1)/A_1)=a_1$ or a_2.

$((a_1-B_1)/A_1)=a_1$ and $((a_2-B_1)/A_1)=a_1$ imply that $a_1=a_2$, which is impossible. By $((a_1-B_1)/A_1)=a_1$ and $((a_2-B_1)/A_1)=a_2$, we obtain $A_1=1$ and $B_1=0$, that is, $f\equiv L(z,f)$, which is a contradiction to the assumption.

From $((a_1 - B_1)/A_1) = a_2$ and $((a_2 - B_1)/A_1) = a_2$, we have $a_1 = a_2$ which is also impossible. Combining $((a_1 - B_1)/A_1) = a_2$ and $((a_2 - B_1)/A_1) = a_1$, we get $A_1 = -1$ and $B_1 = a_1 + a_2$, and then

$$f + L(z, f) = a_1 + a_2. \tag{31}$$

Since $f(z) \neq a_1, a_2$, we set

$$\frac{f - a_1}{f - a_2} = e^{\gamma(z)}, \tag{32}$$

where $\gamma(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, $a_0, a_1, \ldots, a_m (\neq 0)$, are constants, and $0 < m \leq \rho(f) < \infty$. By a calculation, for $j = 0, 1, 2, \ldots, n$, we deduce that

$$\begin{aligned} \gamma\left(z + d_j\right) = \gamma(z) + a_m C_m^1 d_j z^{m-1} \\ + \left(a_m C_m^1 d_j^2 + a_{m-1} C_{m-1}^1 d_j\right) z^{m-2} + \cdots. \end{aligned} \tag{33}$$

Set $a_m = |a_m| e^{i\theta_m}$; then, when $z = re^{-i(\theta_m/m)} \longrightarrow \infty$, we obtain

$$\begin{aligned} \operatorname{Re}\gamma(z) \sim |a_m| r^m, \\ e^{\gamma(z)} \longrightarrow \infty, \end{aligned} \tag{34}$$

and while $z = re^{-i(\theta_m/m) + (\pi/m)} \longrightarrow \infty$, we know

$$\begin{aligned} \operatorname{Re}\gamma(z) \sim -|a_m| r^m, \\ e^{\gamma(z)} \longrightarrow 0. \end{aligned} \tag{35}$$

Equation (32) implies

$$f = \frac{a_1 - a_2 e^{\gamma(z)}}{1 - e^{\gamma(z)}}. \tag{36}$$

So,

$$L(z, f) = \sum_{j=0}^{n} \alpha_j \frac{a_1 - a_2 e^{\gamma\left(z + d_j\right)}}{1 - e^{\gamma\left(z + d_j\right)}}. \tag{37}$$

Substituting (36) and (37) to (31), we get

$$\frac{a_1 - a_2 e^{\gamma(z)}}{1 - e^{\gamma(z)}} + \sum_{j=0}^{n} \alpha_j \frac{a_1 - a_2 e^{\gamma\left(z + d_j\right)}}{1 - e^{\gamma\left(z + d_j\right)}} = a_1 + a_2. \tag{38}$$

When $z = re^{-i(\theta_m/m)} \longrightarrow \infty$, from (33), (34), and (38), we know

$$a_2 + a_2 \sum_{j=0}^{n} \alpha_j = a_1 + a_2, \tag{39}$$

and when $z = re^{-i(\theta_m/m) + (\pi/m)} \longrightarrow \infty$, we get

$$a_1 + a_1 \sum_{j=0}^{n} \alpha_j = a_1 + a_2. \tag{40}$$

Combining (39) and (40), we deduce that

$$\begin{aligned} \sum_{j=0}^{n} \alpha_j + 1 = 0, \\ a_2 = -a_1, \\ B_1 = 0, \\ a_3 = 0. \end{aligned} \tag{41}$$

Then, by (31), we obtain

$$f + L(z, f) = 0. \tag{42}$$

Substituting (41) to (36) and (37), we have

$$f = a_1 \frac{1 + e^{\gamma(z)}}{1 - e^{\gamma(z)}}, \tag{43}$$

$$L(z, f) = a_1 \sum_{j=0}^{n} \alpha_j \frac{1 + e^{\gamma\left(z + d_j\right)}}{1 - e^{\gamma\left(z + d_j\right)}}, \tag{44}$$

$$\sum_{j=0}^{n} \alpha_j = -1. \tag{45}$$

Substituting (43) and (44) to (42), we deduce

$$\frac{1 + e^{\gamma(z)}}{1 - e^{\gamma(z)}} + \sum_{j=0}^{n} \alpha_j \frac{1 + e^{\gamma\left(z + d_j\right)}}{1 - e^{\gamma\left(z + d_j\right)}} = 0. \tag{46}$$

Denoting $\beta_{n+1} = 1$, $d_{n+1} = 0$, and $\beta_j = \alpha_j$, where $j = 0, 1, \ldots, n$, then

$$\sum_{j=0}^{n+1} \beta_j \frac{1 + e^{\gamma\left(z + d_j\right)}}{1 - e^{\gamma\left(z + d_j\right)}} = 0, \tag{47}$$

where $\sum_{j=0}^{n+1} \beta_j = 0$. If there exist some j, $0 \leq j \leq n$, such that $d_j = 0$, we denote $\beta_{n+1} = 1 + d_j$ and $\beta_j = 0$; then, we get a similar equation with the above one.

So,

$$\sum_{j=0}^{n+1} \beta_j \left(1 + e^{\gamma\left(z + d_j\right)}\right) \prod_{i \neq j} \left(1 - e^{\gamma\left(z + d_j\right)}\right) = 0. \tag{48}$$

Then, by a calculation, we get

$$\sum_{j=0}^{n+1} \beta_j e^{\gamma\left(z + d_j\right)} + \sum_{k=2}^{n} (-1)^{k-1} \sum_{i_1, i_2, \ldots, i_k} \left(\beta_{i_1} + \cdots + \beta_{i_k}\right) e^{\gamma\left(z + d_{i_1}\right) + \cdots + \gamma\left(z + d_{i_k}\right)} = 0, \tag{49}$$

where $\Sigma_{i_1,i_2,\ldots,i_k}$ denotes the sum for any k numbers in $\{0,1,\ldots,n+1\}$. Applying Lemma 8, we get that there exits nonnegative integer i, j and $i \neq j$ such that $\gamma(z+d_i)-\gamma(z+d_j)$ is a constant. From (33), we know that $\gamma(z)$ is a polynomial of $\deg\gamma = 1$. We set

$$\gamma(z) = az + b, \tag{50}$$

where $a \neq 0$, b are constants. We substitute into (43) and get

$$f(z) = a_1 \frac{1+e^{az+b}}{1-e^{az+b}}, \tag{51}$$

where $a \neq 0$, b are constants.

Case 2. Suppose that $C \neq 0$. By (27), we get

$$f = \frac{A_1 L(z,f) + B_1}{L(z,f) + D_1}, \tag{52}$$

where $A_1 = (A/C)$, $B_1 = (B/C)$, $D_1 = (D/C)$, and $A_1 D_1 - B_1 \neq 0$.

Since f and $L(z,f)$ share ∞ CM, from (52), we know ∞ is a Picard exceptional value of f and $L(z,f)$. If f can take every one of $\{a_1,a_2,a_3\}$, we get at least two of $\{a_1,a_2,a_3\}$ are equal, which is a contradiction, so one of $\{a_1,a_2,a_3\}$ is a Picard exceptional value of f and $L(z,f)$. Without loss of generality, we set a_3 is a Picard exceptional value of f; by assumption, a_3 is also a Picard exceptional value of $L(z,f)$. Noting that ∞ is Picard exceptional values of $L(z,f)$ and they share ∞ CM, by (52), we have A_1 is a Picard exceptional value of f, so $A_1 = a_3$. Since ∞ is Picard exceptional values of f and they share ∞ CM; by (52), we see that $-D_1$ is a Picard exceptional value of $L(z,f)$. Then, we have $-D_1 = a_3$. Thus, (52) turns to be

$$f = \frac{a_3 L(z,f) + B_1}{L(z,f) - a_3}, \tag{53}$$

where a_3 and B_1 are not all zero.

Noting that f and $L(z,f)$ share a_1 and a_2 CM and neither of them is a Picard exceptional value of f and $L(z,f)$, from (53), we get

$$\begin{aligned} a_1 &= \frac{a_3 a_1 + B_1}{a_1 - a_3}, \\ a_2 &= \frac{a_3 a_2 + B_1}{a_2 - a_3}. \end{aligned} \tag{54}$$

So, $a_3 = ((a_1+a_2)/2)$ and $B_1 = -a_1 a_2$. Then, by (53), we have

$$f = \frac{((a_1+a_2)/2)L(z,f) - a_1 a_2}{L(z,f) - ((a_1+a_2)/2)}. \tag{55}$$

By rewriting it, we obtain

$$fL(z,f) - \frac{a_1+a_2}{2} f - \frac{a_1+a_2}{2} L(z,f) = -a_1 a_2. \tag{56}$$

Noting that both ∞ and a_3 are Picard exceptional values of f and $L(z,f)$ and $a_3 = ((a_1+a_2)/2)$, we set

$$f = \frac{a_1+a_2}{2} + e^{\beta(z)}, \tag{57}$$

where $\beta(z) = b_p z^p + b_{p-1} z^{p-1} + \cdots + b_1 z + b_0$, $b_0, b_1, \ldots, b_p (\neq 0)$, are constants, and $p = \rho(f) > 0$. Then, by the definition of $L(z,f)$, we get

$$\begin{aligned} L(z,f) &= \sum_{j=0}^{n} \alpha_j f\left(z+d_j\right) \\ &= \sum_{j=0}^{n} \alpha_j \left(\frac{a_1+a_2}{2} + e^{\beta\left(z+d_j\right)}\right) \\ &= \alpha_0\left(\frac{a_1+a_2}{2} + e^{\beta\left(z+d_0\right)}\right) + \alpha_1\left(\frac{a_1+a_2}{2} + e^{\beta\left(z+d_1\right)}\right) \\ &\quad + \cdots + \alpha_n\left(\frac{a_1+a_2}{2} + e^{\beta\left(z+d_n\right)}\right). \end{aligned} \tag{58}$$

Substituting (57) and (58) to the above equation, we have

$$\left(\alpha_0 + \alpha_1 + \cdots + \alpha_n - 1\right)\frac{a_1+a_2}{2} e^{\beta(z)} + \alpha_0 e^{\beta\left(z+d_0\right)+\beta(z)} + \alpha_1 e^{\beta\left(z+d_1\right)+\beta(z)} + \cdots + \alpha_n e^{\beta\left(z+d_n\right)+\beta(z)} = \frac{\left(a_1-a_2\right)^2}{4}. \tag{59}$$

Then, using Lemma 7, we get a contradiction. Thus, we complete the proof of Theorem 2.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The study was supported by the National Natural Science Foundation of China (Grant no. 11871314) and the Scientific Research Project of Shanxi Datong University (Grant no. 2020K19).

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