Research Article

Global Existence and Blow-Up for the Classical Solutions of the Long-Short Wave Equations with Viscosity

Jincheng Shi1 and Shengzhong Xiao2

1Department of Applied Mathematics, Guangzhou Huashang College, Guangzhou 510000, China
2Department of Mathematics, Guangdong AIB University, Guangzhou 510527, China

Correspondence should be addressed to Jincheng Shi; hning0818@163.com

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We are concerned with the global existence of classical solutions for a general model of viscosity long-short wave equations. Under suitable initial conditions, the existence of the global classical solutions for the viscosity long-short wave equations is proved. If it does not exist globally, the life span which is the largest time where the solutions exist is also obtained.

1. Introduction

In this paper, we studied the global well-posedness of the solutions for the long-short wave systems with viscosity which describes the coupling between nonlinear Schrödinger systems and the parabolical systems. In 1977, Benney [2] presented a general theory for deriving nonlinear partial differential equations in which both long and short wave solutions coexist and interact with each other nonlinearly.

\[
\begin{align*}
    iut + C_g \nabla \cdot v + \Delta u &= F(t, x, u, v), \\
    vt + C_f \nabla \cdot v + \mu \Delta v &= G(t, x, u, v),
\end{align*}
\]

where \( u(t, x) \) denotes the envelope of the short wave, \( v(t, x) \) is the amplitude of the long wave, the quantity \( C_f \) is the long wave speed, and \( C_g \) is the group velocity of the short waves. This system arises in the study of surface waves with both gravity and capillary models presented [8] and also in plasma physics [21].

In [20], Tsutsumi and Hatano studied the well-posedness of the Cauchy problems for one type of Benney equations

\[
\begin{align*}
    iu_t + \Delta u &= v u + |u|^2 u, \\
    v_t + f(v) &= |u|^2 v,
\end{align*}
\]

Page 1 of 16

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\begin{align*}
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    v_t + f(v) &= |u|^2 v,
\end{align*}
\]

and they got the well-posedness of the Cauchy problems of (2).

In [5, 6], the authors investigated the quasilinear Benney equations with the form,

\[
\begin{align*}
    iu_t + \Delta u &= v u + |u|^2 u, \\
    v_t + f(v) &= |u|^2 v,
\end{align*}
\]

In [5], they imposed the condition \( f(v) = av^2 - bv^3, a > 0, b > 0 \) and they established the existence of weak solutions for equation (3). In [6], they supposed that \( f \) is a polynomial real function, and they got the existence of local strong solutions for these versions of Benney equations. However, for the existence of the smooth solutions associated with an arbitrary flux-function \( f \) and the finite-time blow-up of these equations are still open. There are many papers that deal with the long-short wave equations, for example [1, 4, 14, 15, 17, 22, 23].

In the present paper, we extended the results to the viscosity long-short wave equations of the general form,

\[
\begin{align*}
    iu_t + \Delta u &= F(u, v), \\
    v_t - \eta \Delta v &= D_x(|u|^2 v) - D_x(f(v)), \\
    u(0, x) &= \epsilon \phi; v(0, x) = \epsilon g,
\end{align*}
\]
where \( x \in \mathbb{R}^n, t \geq 0, F = F(u, v), \) and \( f = f(v) \) are nonlinear functions satisfying hypotheses: in a neighborhood of \( \lambda = 0, \) say, for \( |\lambda| \leq 1, f(\lambda) = O(|\lambda|^{1+\alpha}), \) and \( F(\lambda) = O(|\lambda|^{\alpha}) \) where \( \alpha \) is an integer \( \geq 1, u(t, x) \in C, v = v(t, x) \in \mathbb{R}, i = \sqrt{-1}, \) and \( \eta \) is the viscosity coefficient.

Precisely, we are concerned with the viscosity long-short wave equations with the following form:

\[
\begin{align*}
\dot{u} + \Delta u &= v^2 u + |u|^2 u + u^2 v + v^3, \\
\dot{v} - \eta \Delta v &= D_x(|u|^2 v) - D_x(f(v)), \\
u(0, x) &= \epsilon^p; v(0, x) = \epsilon g.
\end{align*}
\] (5)

Our purpose is to estimate the life span of solutions which is expressed explicitly in \([11, 16].\) We call the maximal existence time of the solutions \((u, v)\) in the classical sense the life span (or blow-up time) of \((u, v)\). We state our main results as follows.

We state our main results as follows.

**Theorem 1.** Suppose that the nonlinear term \( f \) on the right-hand side of (5) satisfies (43) and

\[
\phi \in W^{2,1}(\mathbb{R}^n) \cap H^{5,1}(\mathbb{R}^n),
\]

\[
g \in W^{2,1}(\mathbb{R}^n) \cap H^{5,1}(\mathbb{R}^n).
\] (6)

Then, for any given integer \( S \geq n + 5, \) there exist positive constants \( e_0 \) and \( c_0 \) with \( c_0 e_0 \leq 1 \) such that, for any \( \epsilon \in (0, e_0], \) there exists a positive number \( T = T(\epsilon) \) such that Cauchy problem (5) admits on \([0, T(\epsilon)]\) a unique classical solution \( u \in X_{S, c, T(\epsilon)} \) where \( T(\epsilon) \) can be chosen as follows:

\[
T(\epsilon) = +\infty, \quad \text{if } n > 2,
\]

\[
T(\epsilon) = e^{4c_0^2} - 1, \quad \text{if } n = 2,
\]

\[
T(\epsilon) = be^{-2} - 1, \quad \text{if } n = 1,
\] (7)

where \( a, b \) are positive constants satisfying (148), (149), and (151). Moreover, with eventual modification on a set with zero measure in the variable \( t, \) we can obtain the results; for any finite \( T_0, \) with \( 0 < T_0 \leq T, \) we have

\[
u \in L^2(0, T_0; H^{5/2}(\mathbb{R}^n)) \cap C([0, T_0]; H^{3,1}(\mathbb{R}^n)),
\] (8)

\[
u_t \in L^2(0, T_0; H^{5/2}(\mathbb{R}^n)) \cap C([0, T_0]; H^{3,1}(\mathbb{R}^n)),
\] (9)

\[
u_t \in L^2(0, T_0; H^{5/2}(\mathbb{R}^n)) \cap C([0, T_0]; H^{3,1}(\mathbb{R}^n)),
\] (10)

Moreover, by the Sobolev embedding theorems (observing that \( S \geq n + 5, \) it easily follows from (8)–(11) that the solutions are classical solutions to the Cauchy problem (5).

**Remark 1.** In this paper, we took the viscosity coefficient \( \eta = 1 \) for simplicity. For the case when \( \eta \to 0, \) we will discuss it in another paper.

The remainder of this paper is as follows: Section 2 is devoted to giving some useful estimates which will be used in Section 3. In Section 3, we consider the Cauchy problem for \( n \)-dimensional nonlinear evolution equations and we get our main results. In Section 4, conclusion is given.

### 2. Preliminary

First, we considered the following homogeneous equations:

\[
\begin{align*}
\dot{u} + \Delta u &= 0, \\
u(x, 0) &= \phi(x),
\end{align*}
\] (12)

\[
\begin{align*}
\dot{v} - \Delta v &= 0, \\
u(x, 0) &= g(x).
\end{align*}
\] (13)

It is well known that, by means of the Fourier transformation, the solutions to the Cauchy problem of equations (12) and (13) can be expressed in the following explicit form:

\[
u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} \phi(y)dy,
\] (14)

and

\[
u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} g(y)dy,
\] (15)

where \( y = (y_1, y_2, \ldots, y_n) \) and \( |x - y|^2 = \sum_{i=1}^{n} (x_i - y_i)^2. \)

For simplicity, we wrote (14) in the form,

\[
u(x, t) = S(t) \phi,
\] (16)

and (15) in the form,

\[
u(x, t) = R(t) g,
\] (17)

where

\[
S(t): \phi \longrightarrow u(t, \cdot),
\]

\[
R(t): g \longrightarrow \nu(t, \cdot).
\] (18)

Then, by Duhamel's principle, the solutions to the Cauchy problem for inhomogeneous heat equations

\[
\begin{align*}
\dot{u} + \Delta u &= F(t, x), \\
\dot{v} - \Delta v &= G(t, x),
\end{align*}
\] (19)

\[
\begin{align*}
u(x, 0) &= \phi(x), \\
u(x, 0) &= g(x),
\end{align*}
\]

can be denoted as

\[
u = S(t) \phi + \int_0^t S(t - \tau) F(\tau, \cdot) d\tau,
\] (20)
and
\[ v = R(t)g + \int_0^t R(t - \tau)G(\tau, \cdot)\,d\tau, \tag{21} \]
or precisely,
\[
u(x, t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\|x - y\|^2/4t} \phi(y)\,dy
\]
\[
\quad + \frac{1}{(4\pi t)^{d/2}} \int_0^t \int_{\mathbb{R}^d} e^{\|x - y\|^2/(t - \tau)} F(\tau, y)\,dy\,d\tau,
\]
\[
v(x, t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\|x - y\|^2/4t} g(y)\,dy
\]
\[
\quad + \frac{1}{(4\pi t)^{d/2}} \int_0^t \int_{\mathbb{R}^d} e^{\|x - y\|^2/(t - \tau)} G(\tau, y)\,dy\,d\tau.
\]

Now, we used the explicit expressions (14) and (15) to establish some decay estimates \( t \to \infty \) for solutions to Cauchy problem (12) and (13) for the n-dimensional homogeneous equations.

The following lemmas were employed by Li and Chen [16], but for completeness, we included them here without proving.

**Lemma 1.** Suppose that all norms appearing on the right-hand side below are bounded. For any integer \( N \geq 0 \), solutions (14) and (15) to the Cauchy problem (12) and (13) satisfy the following estimates:
\[
\| R(t)g \|_{W^{N,\infty}} \leq c_0 (1 + t)^{-N/2} \| g \|_{W^{N+1,1}(\mathbb{R}^d)},
\]
\[
\| R(t)g \|_{W^{N,1}} \leq \| g \|_{W^{N+1,1}(\mathbb{R}^d)}, \quad \forall t \geq 0,
\]
\[
\| S(t)\phi \|_{W^{N,\infty}} \leq c_0 (1 + t)^{-N/2} \| \phi \|_{W^{N+1,1}(\mathbb{R}^d)},
\]
and
\[
\| S(t)\phi \|_{H^N} = \| \phi \|_{H^N}(\mathbb{R}^d).
\]

**Lemma 2.** Suppose that \( \phi \in H^S(\mathbb{R}^d), F \in L^2(0, T; H^{S-1}) \), where \( S \geq 1 \) is an integer, then the Cauchy problem
\[
\begin{align*}
\Delta u - \Delta u &= F(t, x), \\
\tau = 0, \quad u &= \phi(x),
\end{align*}
\]
admits the following estimate:
\[
\int_0^t \sum_{|k| = 2} \left\| D_{x,k}^2 u(\tau, \cdot) \right\|^2_{H^{S-1}}\,d\tau \leq c_0 \left( \| \phi \|^2_{H^S} + \int_0^t \| F(\tau, \cdot) \|^2_{H^{S-1}}\,d\tau \right),
\]
where \( c_0 \) is a positive constant independent of \( t, k = (k_1, k_2, \ldots, k_n) \) is a multi-index, \( |k| = k_1 + k_2 + \cdots + k_n \), and \( D_x^k = \partial^{\|k\|} / \partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n} \).

**Lemma 3.** Under assumption \( 1/r = 1/p + 1/q \), if all norms appearing on the right-hand side below are bounded, then for any given integer \( S \geq 0 \), we have
\[
\| f \|_{W^{S,1}(\mathbb{R}^d)} \leq C(\| f \|_{L^p(\mathbb{R}^d)} + \| f \|_{W^{S+1,1}(\mathbb{R}^d)})
\]
\[
\| f \|_{L^q(\mathbb{R}^d)} \leq \| f \|_{W^{S,1}(\mathbb{R}^d)}.
\]

Furthermore, suppose that \( F = F(w) \) is a sufficiently smooth function of \( w = (w_1, w_2, \ldots, w_n) \), satisfying that if
\[
|w| \leq v_0,
\]
then
\[
F(w) = O(|w|^{1+\alpha}), \quad (\alpha \geq 1 \text{ integer}).
\]

For any given integer \( S \geq 0 \), if a vector function \( w = w(x) \) satisfies
\[
\| w \|_{L^\infty} \leq v_0,
\]
and such that all norms appearing on the right-hand side below are bounded, then
\[
\| F(w) \|_{W^{S,1}(\mathbb{R}^d)} \leq C_S \| w \|_{W^{S+1,1}(\mathbb{R}^d)} \prod_{i=1}^n \| w \|_{L^{p_i}(\mathbb{R}^d)}.
\]

**Lemma 4.** Suppose that \( F = F(w) \) satisfies the same assumptions as in Lemma 3. If vector functions \( w = w(x) \) and \( \overline{w} = \overline{w}(x) \) satisfy (30), respectively, and such that all norms appearing on the right-hand side below are bounded, then for any given integer \( S \geq 0 \), let
\[
\overline{w} = w - \overline{w}.
\]

We have
\[ \|F(\mathbf{u}) - F(\mathbf{v})\|_{W^{\alpha}(\mathbb{R}^n)} \leq C_5 \left\{ \|\mathbf{u}\|_{L^p(\mathbb{R}^n)}^p \|\mathbf{v}\|_{W^{\alpha^p}(\mathbb{R}^n)}^p + \|\mathbf{v}\|_{W^{\alpha}(\mathbb{R}^n)}^p \|\mathbf{u}\|_{L^p(\mathbb{R}^n)} \right\}^{\alpha - 1}, \]

where \( p, q \) and \( r \) satisfy \( 1/r = 1/p + 1/q \) and \( C_5 \) is a positive constant (depending on \( v_0 \)).

3. The Proof of Theorem 1

In this section, we shall discuss the Cauchy problem for \( n \)-dimensional nonlinear evolution equations:

\[
\begin{aligned}
&\{iu_t + \Delta u = v^2 u + |u|^2 u + u^2 v + v^3, \\
&\quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \\
&\quad v_t - \Delta v = D_x (|u|^2 v) - D_x (f(v)), \\
&\quad t = 0: u = \varepsilon \phi(x), \\
&\quad x = (x_1, x_2, \ldots, x_n), \\
&\quad t = 0: v = \varepsilon g(x),
\end{aligned}
\]

(39)

where \( D_x u = (u_{x_1}, \ldots, u_{x_n}) = (u_i; i = 1, \ldots, n) \), and \( \varepsilon > 0 \) is a small parameter.

Let

\[ \tilde{\lambda} = \lambda; (\lambda_i, i = 1, \ldots, n). \]

(42)

For the nonlinear term \( f \) in (39), we give the following hypotheses: in a neighborhood of \( \tilde{\lambda} = 0 \), say, for \( |\lambda| \leq 1 \), \( f = f(\lambda) \) is sufficiently smooth and

\[ f(\lambda) = O(|\lambda|^{1+}) \]

(43)

where \( \alpha \) is an integer \( \geq 1 \).

For any given integer \( S \) such that \( S \geq n + 5 \), and any positive numbers \( E, T \) \((0 < T \leq +\infty)\), we introduce the following set of functions:

\[ X_{S,E,T} = \{ (u, v) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n : v = v(t, x) \mid D_{S,T}(u, v) \leq E \}, \]

(44)

where

\[
D_{S,T}(u, v) = \sup_{0 \leq t \leq T} (1 + t)^{n/2} \|v\|_{W^{S-3, \alpha}(\mathbb{R}^n)}
\]

\[
+ \sup_{0 \leq t \leq T} \left\| v \right\|_{W^{S-1, \alpha}(\mathbb{R}^n)} + \left( \int_0^T \sum_{|\nabla|^2 \leq 2} \left\| D_x v \right\|_{H^{n-1}(\mathbb{R}^n)} \right)^{1/2}
\]

\[
+ \sup_{0 \leq t \leq T} (1 + t)^{n/2} \|u\|_{W^{S-3, \alpha}(\mathbb{R}^n)} + \sup_{0 \leq t \leq T} \|u\|_{H^n(\mathbb{R}^n)}.
\]

(45)

We now define a map

\[ (\mathbf{u}, \mathbf{v}) \rightarrow M(\mathbf{u}, \mathbf{v}) = (u, v), \]

(46)

where \((u, v)\) are the solutions of the linear equations

\[
\begin{aligned}
\{iu_t + \Delta u = \tilde{v}^2 u + |\tilde{u}|^2 \tilde{u} + \tilde{u}^2 \tilde{v} + \tilde{v}^3, \\
\quad \tilde{v}_t - \Delta \tilde{v} = D_x (|\tilde{u}|^2 \tilde{v}) - D_x (f(\tilde{v})), \\
\quad t = 0: u = \varepsilon \phi(x), \\
\quad t = 0: \tilde{v} = \varepsilon g(x).
\end{aligned}
\]

(47)

The next step is to prove that the map \( M \) is a strict contraction in the complete metric space \( X_{S,E,T} \).

Lemma 5. For any \((\tilde{u}, \tilde{v}) \in X_{S,E,T} \) (in which \( S \geq n + 5, E \leq 1 \)), \((u, v) = M(\tilde{u}, \tilde{v})\) satisfies

\[ D_{S,T}(u, v) \leq \tilde{c}_2 \left\{ \|v\|_{E^2} \left( 1 + \int_0^T (1 + \tau)^{-n/2} d\tau \right) \right\}. \]

(48)

Proof. From (21), we have

\[ v = \varepsilon R(t)g + \int_0^t R(t - \tau)D_x (f(\tilde{v})) d\tau \]

(49)

By (23), we have

\[ \|v(t, \cdot)\|_{W^{S-3, \alpha}} \leq \tilde{c} \left( 1 + t \right)^{-n/2} \|g\|_{W^{S-2, 1}} + \int_0^t \left( 1 + \tau \right)^{-n/2} \|D_x (f(\tilde{v}))\|_{W^{S-2, 1}} d\tau \]

(50)

Using (29) and (33), we get

\[ \|D_x (f(\tilde{v}))\|_{W^{S-2, 1}} \leq \tilde{c} \|\tilde{v}\|_{W^{S-1, 1}} \|\tilde{u}\|_{H^n(\mathbb{R}^n)} + \tilde{c} \|\tilde{u}\|_{H^{n-1}} \|\tilde{v}\|_{W^{S-1, 1}}. \]

(51)

A combination of (50)–(52) leads to

\[ \|v(t, \cdot)\|_{W^{S-3, \alpha}} \leq \tilde{c} \left( 1 + t \right)^{-n/2} \|g\|_{W^{S-2, 1}} + \int_0^t \left( 1 + \tau \right)^{-n/2} \|\tilde{v}\|_{W^{S-1, 1}} \|\tilde{u}\|_{H^n(\mathbb{R}^n)} + \|\tilde{u}\|_{H^{n-1}} \|\tilde{v}\|_{W^{S-1, 1}} d\tau. \]

(53)

Noting that \( S \geq n + 5 \), by the definition of \( X_{S,E,T} \), we have

\[ \|v\|_{W^{S-3, \alpha}} \leq \|v\|_{W^{S-3, \alpha}} \leq E (1 + t)^{-n/2}, \]

(54)

\[ \|\tilde{v}\|_{W^{S-3, \alpha}} \leq E, \]

(55)

\[ \|\tilde{u}\|_{H^{n-1}} \leq \|\tilde{u}\|_{H^n} \leq E, \]

(56)
\[ \|u\|_{H^{s,0}} \leq \|v\|_{W^{\alpha,3,0}} \leq E(1 + t)^{-n/2}. \]  

On combining (53)–(57), we get
\[ \|v(t, \cdot)\|_{W_{2,1}} \left( 1 + t \right)^{-n/2} + c \int_0^t (1 + t - \tau)^{-n/2} (1 + \tau)^{-b} d\tau \]
\[ + c \int_0^t (1 + t - \tau)^{-n/2} E \left( (1 + \tau)^{-n/2} + (1 + \tau)^{-n} \right) d\tau. \]

We point out the fact that, if \( b \geq a \geq 0 \), then we have
\[ \int_0^t (1 + t - \tau)^{-a} (1 + \tau)^{-b} d\tau \leq c(1 + t)^{-a} \int_0^t (1 + \tau)^{-b} d\tau. \]

The proof of (59) was shown in [16], but to complete it, we include here. Let
\[ I_1 = \int_0^{t/2} (1 + t - \tau)^{-a} (1 + t)^{-b} d\tau \leq \left( 1 + \frac{t}{2} \right)^{-a} \int_0^{t/2} (1 + t)^{-b} d\tau, \]
\[ I_2 = \int_0^t (1 + t - \tau)^{-a} (1 + \tau)^{-b} d\tau \leq \left( 1 + \frac{t}{2} \right)^{-a} \int_t^t (1 + t - \tau)^{-a} d\tau, \]
\[ \leq c(1 + \tau)^{-b} \int_0^t (1 + t - \tau)^{-a} d\tau \leq c(1 + \tau)^{-a} \int_0^{t/2} (1 + \tau)^{-a} d\tau \]
\[ \leq c(1 + \tau)^{-a} \int_0^{t/2} (1 + t)^{-b} d\tau. \]

The combination of (60) and (61) gives (59).

Thus, from (58) and (59), we obtain
\[ \sup_{0 < t \leq T} (1 + t)^{n/2} \|v(t, \cdot)\|_{W_{2,1}} \leq c \left( \epsilon + E^{1+a} \int_0^t (1 + \tau)^{-n/2} d\tau + E^3 \int_0^t (1 + \tau)^{-n/2} d\tau \right). \]

By (49) and (24), we also have
\[ \|v\|_{H^{s,1}} \leq c \left( \epsilon \|g\|_{W^{s,1}} + \int_0^t \|D_x f (v)\|_{W^{s,1}} d\tau + \int_0^t \|D_x (\tilde{u} \tilde{v})\|_{W^{s,1}} d\tau \right). \]

From (29) and (33), we have
\[ \|D_x f (\tilde{v})\|_{W^{s,1}} \leq c \|\tilde{v}\|_{H^s}^2 \|v\|_{W^{s,0}}, \]
\[ \|D_x (\tilde{u} \tilde{v})\|_{W^{s,1}} \leq c \|\tilde{u}\|_{H^s}^2 \|v\|_{W^{s,0}} + c \|\tilde{u}\|_{W^{s,0}} \|v\|_{W^{s,1}}. \]

Moreover, in fact,
On combining (63)–(65) and (67) and the definition of $X_{S,E,T}$, we obtain

$$\sup_{0 \leq t \leq T} \|v\|_{W^{1,1}} \leq c \left[ \epsilon + E^{1/4} \left( 1 + \int_0^T (1 + \tau)^{-n/2} \, d\tau \right) + E^3 \int_0^T \left( (1 + \tau)^{-n/2} + (1 + \tau)^{-n} \right) \, d\tau \right].$$

(68)

By (28), we have

$$\int_0^T \left| \|D_x^k v\|_{H^{-1}}^2 \right| \, d\tau \leq c(\epsilon^2 \|g\|_{H^{-1}}^2)$$

(69)

$$+ \int_0^T \|D_x (f \nabla v) (\tau, \cdot)\|_{H^{-1}}^2 \, d\tau + \int_0^T \|D_x |\hat{u}|^2 \|_{H^{-1}}^2 \, d\tau.$$

Noting the hypotheses of (43), it follows from (33) that

$$\|D_x (f \nabla v)\|_{H^{-1}} \leq c \|f\|_{H^2} \|v\|_{W^{1,\infty}},$$

$$\|D_x |\hat{u}|^2\|_{H^{-1}} \leq c \|\hat{u}\|_{H^2} \|u\|_{W^{1,\infty}} + c \|v\|_{H^2} \|v\|_{W^{1,\infty}}.$$

(70)

Using (67), $\int_0^T (1 + \tau)^{-\left(1+2\alpha\right)n/2} \, d\tau \leq c$, and the definition of $X_{S,E,T}$, we obtain

$$\left( \int_0^T \sum_{k \geq 2} \|D_x^k v\|_{H^{-1}}^2 \, d\tau \right)^{1/2} \leq \left( \epsilon + E^{1/4} + E^3 \left( \int_0^T (1 + \tau)^{-\alpha} \, d\tau + 1 \right) \right).$$

(71)

We can also get

$$u = cS(t) + \int_0^T S(t - \tau) (\nabla^2 u + |\hat{u}|^2 \hat{u} + \hat{u}^2 \hat{v} + v^2) \, d\tau.$$  

(72)

By (25), we have

$$\|u\|_{W^{\infty,\infty}} \leq cE (1 + t)^{-n/2} \|u\|_{W^{2,1}}$$

$$+ c \int_0^T (1 + \tau)^{-n/2} \|\nabla^2 u\|_{W^{2,2}} \, d\tau$$

$$+ c \int_0^T (1 + \tau)^{-n/2} \|\hat{u}\|_{W^{2,2}} \, d\tau$$

$$+ c \int_0^T (1 + \tau)^{-n/2} \|\hat{v}\|_{W^{2,2}} \, d\tau$$

$$+ c \int_0^T (1 + \tau)^{-n/2} \|v\|_{W^{2,2}} \, d\tau.$$  

(73)

By (29) and (33), we have

$$\|\hat{u}\|_{H^{2-1/2}} \leq c \|\hat{u}\|_{W^{2,1}} \|\hat{u}\|_{L^2} + c \|\hat{u}\|_{L^2} \|\hat{u}\|_{W^{2,1}},$$

$$\leq c \|\hat{u}\|_{H^{2-1/2}} \|\hat{u}\|_{L^2} + c \|\hat{u}\|_{L^2} \|\hat{u}\|_{W^{2,1}}$$

(77)

$$\leq cE^3 (1 + t)^{-n/2} + cE^3 (1 + t)^{-n},$$

and

$$\|\hat{v}\|_{H^{2-1/2}} \leq c \|\hat{v}\|_{W^{2,1}} \|\hat{v}\|_{L^2} \leq cE^3 (1 + t)^{-n}.$$  

(78)
On combining (73)–(78), we obtain

\[
\|u\|_{W^{n-1,\infty}} \leq c\epsilon (1 + t)^{-n/2}\|\phi\|_{W^{n-1,1}} + cE^3(1 + t)^{-n/2}\int_0^t (1 + r)^{-3n/4} dr + \int_0^t (1 + r)^{-n/2} d + \left(\int_0^t (1 + r)^{-n} dr\right)^{1/2} + \int_0^t (1 + r)^{-n} dr.
\]

(79)

Thus,

\[
\|u\|_{W^{n-1,\infty}} \leq c\epsilon + E^3\left(1 + \int_0^t (1 + r)^{-n/2} dr\right).
\]

(80)

Finally, from (26) and (72), we have

\[
\|u\|_{H^1} \leq c\epsilon\|\phi\|_{H^1} + c\int_0^t \|\tilde{u}\|_{H^1} dr + c\int_0^t \|\tilde{\phi}\|_{H^1} \tilde{u} dr + c\int_0^t \|\tilde{\phi}\|_{L^2} \tilde{u} \tilde{\phi} dr + c\int_0^t \|\tilde{\phi}\|_{L^2} \tilde{u} \tilde{\phi} \tilde{\phi} dr,
\]

(81)

On combining (75) and (81), we obtain

\[
\|u\|_{H^1} \leq c\epsilon + E^3\left(1 + \int_0^t (1 + r)^{-n/2} dr\right).
\]

(82)

On combining the above discussions, we obtain

\[
D_{\delta,T}(u,v) \leq \tilde{c}^1\epsilon + E^3\left(1 + \int_0^t (1 + r)^{-n/2} dr\right).
\]

(83)

Lemma 6. For any \((\bar{u}, \bar{v}), (\tilde{u}, \tilde{v}) \in X_{S,E,T}\) (in which \(S \geq n + 5, E \leq 1\) and \(0 < T < +\infty\)), if \(u_{\tilde{v}}(\tau, \bar{v}) = M(\bar{u}, \bar{v}), (\tilde{u}, \tilde{v}) = M(\tilde{u}, \tilde{v})\) also satisfy \((\bar{u}, \bar{v}), (\tilde{u}, \tilde{v}) \in X_{S,E,T}\), then

\[
D_{\delta,T}(u^*, v^*) \leq c_2 E D_{S,T}(h, p) \left[1 + \int_0^t (1 + r)^{-n/2} dr\right],
\]

(84)

where \(u^* = \bar{u} - \tilde{u}, v^* = \bar{v} = \tilde{v}, h = \bar{u} - \tilde{u}, p = \bar{v} - \tilde{v}\).

Proof. By the definition of \(M\), we have

\[
\begin{align*}
\frac{d}{dt} \tilde{v}^* - \Delta \tilde{v}^* &= \tilde{v}^* \tilde{u} - \tilde{v}^* \bar{u} + \tilde{v} \tilde{u} - \tilde{v} \bar{u} + \tilde{u} \tilde{v} - \bar{u} \bar{v} + \tilde{v}^3 - \tilde{v}^3, \\
\frac{d}{dt} \tilde{v}^* - \Delta \tilde{v}^* &= D_\chi(\tilde{u}^2 \tilde{v}) - D_\chi(\tilde{u}^2 \bar{v}) - D_\chi(f(\bar{v}) + D_\chi(f(\tilde{v}))).
\end{align*}
\]

(85)

\[
u^*(x, 0) = v^*(x, 0) = 0.
\]

(86)

Similar to (53), we have

\[
\|v^*\|_{W^{n-2,\infty}} \leq c\int_0^t (1 + t - r)^{-n/2}\|D_\chi(f(\bar{v})) - D_\chi(f(\tilde{v}))\|_{W^{n-2,1}} dr + c\int_0^t (1 + t - r)^{-n/2}\|D_\chi(f(\bar{v})) - D_\chi(f(\tilde{v}))\|_{W^{n-2,1}} dr.
\]

(87)
Noting that \( S \geq n + 5 \), and the definition of \( X_{S,E,T} \) and using the hypotheses (43) and (38) (in which we take \( r = 1, p = +\infty, q = 1 \)), we get

\[
\|D_x (f(\bar{v})) - D_x (f(\bar{v}))\|_{W^{5-1,1}} \leq c \|p\|_{W^{5-1,1}} (\|\bar{v}\|_{W^{7,1}} + \|\bar{v}\|_{W^{1,\infty}})^{\alpha}
\]

\[
+ c \|p\|_{W^{5-1,1}} (\|\bar{v}\|_{W^{5-1,1}} + \|\bar{v}\|_{W^{7,1}} + \|\bar{v}\|_{W^{1,\infty}})^{\alpha-1}
\]

\[
\leq c \|p\|_{W^{5-1,1}} (\|\bar{v}\|_{W^{5-n,1,\infty}} + \|\bar{v}\|_{W^{5-n,1} + \|\bar{v}\|_{W^{7,1}} + \|\bar{v}\|_{W^{1,\infty}}})^{\alpha}
\]

\[
+ c \|p\|_{W^{5-n,1}} (\|\bar{v}\|_{W^{5-n,1,\infty}} + \|\bar{v}\|_{W^{5-n,1}} + \|\bar{v}\|_{W^{7,1}} + \|\bar{v}\|_{W^{1,\infty}})^{\alpha-1}
\]

\[
\|D_x (|\bar{u}|^2 \bar{v}) - D_x (|\bar{u}|^2 \bar{v})\|_{W^{5-1,1}} = \|D_x ((|\bar{u}|^2 - |\bar{u}|^2) \bar{v} + |\bar{u}|^2 (\bar{v} - \bar{v}))\|_{W^{5-1,1}}
\]

For we have

\[
\|D_x (|\bar{u}|^2 (\bar{v} - \bar{v}))\|_{W^{5-1,1}} \leq c \||\bar{u}|^2 \||\bar{u}|^2 \|_{W^{5-1,1}} + \|\bar{u}\|_{W^{5,1,1}} \|\bar{u}\|_{W^{5,1,1}},
\]

\[
\leq c \||\bar{u}|^2 \|_{W^{5,1,1}} \|\bar{u}\|_{W^{5,1,1}} + \|\bar{u}\|_{W^{5,1,1}} \|\bar{u}\|_{W^{5,1,1}} \|\bar{u}\|_{W^{5,1,1}}
\]

\[
\leq c (1 + t)^{-n/2} E^2 D_{S,T} (h, p) + c E^2 (1 + t)^{-n/2} D_{S,T} (h, p),
\]

\[(89)\]

Thus, we can get

\[
\|D_x (|\bar{u}|^2 \bar{v}) - D_x (|\bar{u}|^2 \bar{v})\|_{W^{5-1,1}} \leq c (1 + t)^{-n/2} E^2 D_{S,T} (h, p)
\]

\[
+ c E^2 (1 + t)^{-n/2} D_{S,T} (h, p).
\]

\[(91)\]

Putting (88) and (91) into (87), and using the definition of \( X_{S,E,T} \) and (59), we obtain

\[
\sup_{0 < t \leq T} (1 + t)^{n/2} \|v^*\|_{W^{5-n,1,\infty}} \leq c E^2 D_{S,T} (h, p) \int_0^t (1 + \tau)^{-n/2} d\tau + c E^2 D_{S,T} (h, p) \int_0^t (1 + \tau)^{-n/2} d\tau.
\]

\[(92)\]

Next, similar to (63), we have

\[
\|v^*\|_{W^{5,1,1}} \leq \int_0^t \|D_x (f(\bar{v})) - D_x (f(\bar{v}))\|_{W^{5,1,1}} d\tau
\]

\[
+ \int_0^t \|D_x (|\bar{u}|^2 \bar{v} - D_x (|\bar{u}|^2 \bar{v})\|_{W^{5,1,1}} d\tau.
\]

\[(93)\]

Using (38) (in which we take \( r = 1, p = q = 2 \), we have

\[
\|D_x (f(\bar{v})) - D_x (f(\bar{v}))\|_{W^{5,1,1}} \leq c \|p\|_{H^1} (\|\bar{v}\|_{H^1} + \|\bar{v}\|_{H^1}) + \|p\|_{H^1} (\|\bar{v}\|_{H^1} + \|\bar{v}\|_{H^1}) (\|\bar{v}\|_{W^{7,1}} + \|\bar{v}\|_{W^{1,\infty}})^{\alpha-1}
\]

\[(94)\]
By the definition of $\| \cdot \|_{\mathcal{H}^s}$ and the definition of $D_{S,T}(h, p)$, we get
\[
\| p \|_{\mathcal{H}^s}^2 \leq \| p \|_{\mathcal{H}^s}^2 = \| \sum_{|k|=2} D_k^s p \|_{\mathcal{H}^s}^2 \leq c \| p \|_{W^{s+3,\infty}} \| p \|_{W^{s+1}^1}
+ \sum_{|k|=2} \| D_k^s p \|_{\mathcal{H}^s}^2 \leq c (1 + t)^{-n/2} D_{S,T}^2(h, p).
\]

Moreover, we have
\[
\| p \|_{\mathcal{H}^s}^2 \leq c \| p \|_{W^{1,\infty}} \| p \|_{W^{1,1}} \leq c (1 + t)^{-n/2} D_{S,T}^2(h, p).
\]

Thus, we see
\[
\| p \|_{\mathcal{H}^s} \leq c (1 + t)^{-n/4} D_{S,T}(h, p) + c \left( \sum_{|k|=2} \| D_k^s p \|_{\mathcal{H}^s}^2 \right)^{1/2},
\]

(97)

\[
\| p \|_{\mathcal{H}^s} \leq c (1 + t)^{-n/4} D_{S,T}(h, p).
\]

(98)

Similarly, we have
\[
\| \bar{V} \|_{\mathcal{H}^s} + \| \bar{V} \|_{\mathcal{H}^s} \leq c (1 + t)^{-n/4} E + c \left( \sum_{|k|=2} \| D_k^s \bar{V} \|_{\mathcal{H}^s}^2 + \sum_{|k|=2} \| D_k^s \bar{V} \|_{\mathcal{H}^s}^2 \right)^{1/2},
\]

(99)

\[
\| \bar{V} \|_{\mathcal{H}^s} + \| \bar{V} \|_{\mathcal{H}^s} \leq c (1 + t)^{-n/4} E.
\]

(100)

Besides, we have
\[
\| \bar{V} \|_{W^{s,\infty}} + \| \bar{V} \|_{W^{1,\infty}} \leq c E (1 + t)^{-n/2}.
\]

(101)

On combining (94)–(101), we obtain
\[
\int_0^t \| D_x (f(\bar{V})) - D_x (f(\bar{V})) \|_{W^{s,1}} \, dr \leq c E^2 \int_0^t (1 + r)^{-n/2} \, dr D_{S,T}^2(h, p)
+ c E^2 \int_0^t (1 + r)^{-n(a-1)/2} \left( \sum_{|k|=2} \| D_k^s p \|_{\mathcal{H}^s}^2 \right)^{1/2} \, dr
+ c E^2 D_{S,T}^2(h, p) \int_0^t (1 + r)^{-n/2} \, dr.
\]

(102)

Noting that $\alpha$ is an integer $\geq 1$, we have
\[
\alpha - 1/2 \geq \alpha/2.
\]

(103)

\[
\int_0^t \| D_x (f(\bar{V})) - D_x (f(\bar{V})) \|_{W^{s,1}} \, dr \leq c E^2 D_{S,T}^2(h, p) \left( 1 + \int_0^t (1 + r)^{-n/2} \, dr \right).
\]

(104)

Similar to (91), we can get
\[
\| D_x \left( |\bar{u}|^2 \bar{V} \right) - D_x \left( |\bar{u}|^2 \bar{V} \right) \|_{W^{s,1}} \leq c (1 + t)^{-n/2} E^2 D_{S,T}^2(h, p)
+ c E^2 (1 + t)^{-n} D_{S,T}^2(h, p).
\]

(105)

On combining (93), (104), and (105), we get
\[
\sup_{0 \leq s \leq T} \| V \|_{W^{s,1}} \leq c E^2 D_{S,T}^2(h, p) \left( \int_0^t (1 + r)^{-n/2} \, dr + 1 \right)
+ c E^2 D_{S,T}^2(h, p) \int_0^t (1 + r)^{-n/2} \, dr.
\]

(106)
\[ \left\| D_x (f (\tilde{v})) - D_x (f (\tilde{v})) \right\|_{H^{s-1}} \leq c \left\| P \right\|_{H^s} \left( \left\| \nabla \tilde{v} \right\|_{L^1} + \left\| \nabla \tilde{v} \right\|_{L^\infty} \right) + \left\| P \right\|_{W^{1,\infty} \left( \left\| \nabla \right\|_{H^s} + \left\| \tilde{v} \right\|_{H^s} \right)} \left( \left\| \nabla \tilde{v} \right\|_{L^1} + \left\| \nabla \tilde{v} \right\|_{L^\infty} \right)^{\alpha - 1}. \]  

(108)

Still using (97), (99), and (101) and notifying that it follows that

\[ \left\| P \right\|_{W^{1,\infty}} \leq c D_{S,T} (h, p) (1 + t)^{-n/2}. \]  

(109)

\[
\begin{align*}
\int_0^T \left\| D_x (f (\tilde{v})) - D_x (f (\tilde{v})) \right\|^2_{H^{s-1}}, dr & \leq c E^2 \int_0^T \left( 1 + r \right)^{-1 + 2n/2} dr D_{S,T}^2 (h, p) + c E^2 \int_0^T \left( 1 + r \right)^{-n} \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} dr \\
+ c E^2 (\alpha - 1) D_{S,T} (h, p) \int_0^T \left( 1 + r \right)^{-n} \left( \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} + \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} \right) dr.
\end{align*}
\]

(110)

For we know that

\[
\begin{align*}
\left\| D_x (\tilde{u}^2 \tilde{v}) - D_x (\tilde{u}^2 \tilde{v}) \right\|_{H^{s-1}} & = \left\| D_x \left( \left[ (\tilde{u}^2 - |\tilde{u}|^2) \tilde{v} + |\tilde{u}|^2 (\tilde{v} - \tilde{v}) \right] \right\|_{H^{s-1}} \\
& \leq c \left\| \left[ (\tilde{u}^2 + |\tilde{u}|^2) \tilde{v} + |\tilde{u}|^2 \tilde{P} \right] \right\|_{H^{s-1}} \\
& \leq c \left( \left\| \tilde{u}^2 \right\|_{H^{s}} + \left\| \tilde{u} \right\|_{H^{s}} \right) \left\| \tilde{v} \right\|_{L^\infty} + c \left( \left\| \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u} \right\|_{L^\infty} \right) \left( \left\| \tilde{v} \right\|_{H^{s}} + \left\| \tilde{v} \right\|_{L^\infty} \right) \\
& + c \left( \left\| \tilde{u}^2 \right\|_{L^\infty} + \left\| \tilde{u} \right\|_{L^\infty} \right) \left( \left\| \tilde{v} \right\|_{H^{s}} + \left\| \tilde{v} \right\|_{L^\infty} \right) + c \left| \tilde{u} \right| \left\| \tilde{P} \right\|_{H^{s}} + c \left| \tilde{u} \right| \left\| \tilde{P} \right\|_{L^\infty} \\
& \leq c E^2 (1 + t)^{-n} D_{S,T} (h, p) + c E (1 + t)^{-n} D_{S,T} (h, p) \left\| \tilde{v} \right\|_{H^{s}} + c E^2 (1 + t)^{-n} \left\| \tilde{P} \right\|_{H^{s}}.
\end{align*}
\]

(111)

From (67) and (97), we obtain

\[
\begin{align*}
\left\| D_x (\tilde{u}^2 \tilde{v}) - D_x (\tilde{u}^2 \tilde{v}) \right\|_{H^{s-1}} & \leq c E^2 (1 + t)^{-n} D_{S,T} (h, p) + c E^2 (1 + t)^{-5n/4} D_{S,T} (h, p) \\
& + c E (1 + t)^{-n} D_{S,T} (h, p) \left( \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} \right)^{1/2} + c E^2 (1 + t)^{-n} \left( \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} \right)^{1/2}.
\end{align*}
\]

(112)

Thus,

\[
\begin{align*}
\int_0^T \left\| D_x (\tilde{u}^2 \tilde{v}) - D_x (\tilde{u}^2 \tilde{v}) \right\|^2_{H^{s-1}}, dr & \leq c E^4 \int_0^T \left( 1 + r \right)^{-2n} dr + c E^2 D_{S,T}^2 (h, p) \int_0^T \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} dr + c E^4 \int_0^T \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} dr, \\
& \leq c E^4 D_{S,T}^2 (h, p),
\end{align*}
\]

(113)

in deriving (111), we have used \( \int_0^T (1 + r)^{-2n} dr \leq c \) and \( (1 + t)^{-1} \leq 1 \).

On combining (107), (111), and (112) and using \( (1 + t)^{-1} \leq 1 \), we obtain

\[
\left( \int_0^T \sum |k| = 2 \left\| D_k x \right\|^2_{H^{s-1}} dr \right)^{1/2} \leq c E^2 D_{S,T} (h, p) + c E^2 D_{S,T} (h, p).
\]

(114)

Similar to (73), we obtain
Using (99), we obtain
\[
\|u^0 - \bar{u}\|_{W^{s-3,\infty}} \leq c \int_0^t (1 + t - \tau)^{-n/2} \left( \|\bar{u}^2 \bar{u} - |\bar{u}|^2 \bar{u}\|_{W^{s-2,1}} + \|\bar{u}^2 \bar{u} - \bar{u}^2 \|_{W^{s-2,1}} \right) d\tau \\
+ c \int_0^t (1 + t - \tau)^{-n/2} \left( \|\bar{v}^2 \bar{u} - \bar{v}^2 \bar{u}\|_{W^{s-2,1}} + \|\bar{v}^3 - \bar{v}^3\|_{W^{s-2,1}} \right) d\tau.
\]  
(115)

We can obtain
\[
\|u^0 - \bar{u}\|_{W^{s-3,\infty}} \leq cE^2 (1 + t)^{-n/2} D_{\delta T}(h, p) + cE^2 (1 + t)^{-n/2} D_{\delta T}(h, p).
\]  
(118)

We have from (38) that
\[
\|\bar{u}^2 \bar{u} - |\bar{u}|^2 \bar{u}\|_{W^{s-2,1}} + \|\bar{u}^2 \|_{W^{s-2,1}} + \|\bar{u}^2 \|_{W^{s-2,1}} \leq cE^2 D_{\delta T}(h, p) (1 + t)^{-n/2},
\]  
(116)

\[
\|\bar{u}^3 \bar{u} - \bar{v}^3 \bar{u}\|_{W^{s-2,1}} \leq c\|\bar{u}^2 \|_{W^{s-2,1}} + \|\bar{u}^2 \|_{W^{s-2,1}} + \|\bar{u}^2 \|_{W^{s-2,1}},
\]  
(119)

For we have
\[
\|\bar{v}^2 p\|_{W^{s-2,1}} \leq c (\|\bar{v}^2 \|_{W^{s-2,1}} + \|\bar{v}^2 \|_{W^{s-2,1}} + \|\bar{v}^2 \|_{W^{s-2,1}} + \|\bar{v}^2 \|_{W^{s-2,1}}).
\]  
(120)

Using (99), we obtain
\[
\|\bar{v}^2 \|_{W^{s-2,1}} \|\bar{v}^2 \|_{W^{s-2,1}} \leq cE^2 (1 + t)^{-n/2} D_{\delta T}(h, p) \leq cE^2 (1 + t)^{-n/2} D_{\delta T}(h, p) + cE (1 + t)^{-n/2} \sum_{|k| = 2} \|D_{\delta T}^k \|_{W^{s-1,1}}^2 D_{\delta T}(h, p).
\]  
(121)

So, we have
\[
\int_0^t (1 + t - \tau)^{-n/2} \|\bar{v}^2 \|_{W^{s-2,1}} \|\bar{v}^2 \|_{W^{s-2,1}} \|p\|_{L^1} d\tau \leq cE^2 (1 + t)^{-n/2} \int_0^t (1 + t)^{-n/2} \int_0^t (1 + t - \tau)^{-3n/4} d\tau D_{\delta T}(h, p) \\
+ cE \left( \int_0^t (1 + t - \tau)^{-n} (1 + t)^{-n} d\tau \right)^{1/2} \left( \sum_{|k| = 2} \|D_{\delta T}^k \|_{W^{s-1,1}}^2 \right)^{1/2} D_{\delta T}(h, p),
\]  
(122)

Following the same procedure as (122), we obtain
\[
\int_0^t (1 + t - \tau)^{-n/2} p \|p\|_{W^{n/2-1}} \|\nabla p\|_{L^n} \, d\tau \leq c E^2 (1 + t)^{-n/2} \int_0^t (1 + t)^{-3n/4} \, dt D_{S,T} (h, p) + c E^2 (1 + t)^{-n/2} \left[ 1 + \int_0^t (1 + t)^{-n} \, d\tau \right].
\]

(123)

We now begin to deal with \( \int_0^t (1 + t - \tau)^{-n/2} \|\nabla h\|_{W^{n/2-1}} \, d\tau \).

We can easily get

\[
\|\nabla h\|_{W^{n/2-1}} \leq c (1 + t)^{-n/4} E c \left( \sum_{|k| = 2} \|D_x^2 \nabla h\|_{L^n} \right)^{1/2}.
\]

(124)

\[
\|\nabla^2 h\|_{W^{n/2-1}} \leq c \|\nabla h\|_{W^{n/2-1}} \|\nabla^2 h\|_{L^n} + c \|\nabla^2 h\|_{H^{n/2}}^2,
\]

\[
\leq c E (1 + t)^{-n/2} D_{S,T} (h, p) \|\nabla h\|_{H^{n/2}} + c (1 + t)^{-n/2} D_{S,T} (h, p) \|\nabla^2 h\|_{H^{n/2}}^2
\]

\[
\leq c E^2 (1 + t)^{-n/2} D_{S,T} (h, p) + c (1 + t)^{-n/2} D_{S,T} (h, p) \left[ E^2 (1 + t)^{-n/2} + \sum_{|k| = 2} \|D_x^2 \nabla h\|_{H^{n/2}}^2 \right].
\]

(125)

Now, we need to seek a new method to give a bound for

\[
\int_0^t (1 + t - \tau)^{-n/2} (1 + \tau)^{-n/2} D_{S,T} (h, p) \sum_{|k| = 2} \|D_x^2 \nabla h\|_{H^{n/2}}^2 \, d\tau.
\]

(126)

For we know that,

\[
m(\tau) = (1 + t - \tau)^{-n/2} (1 + \tau)^{-n/2},
\]

(127)

is a continuous differentiable function in \([0, t]\), thus we have

\[
m'(\tau) = \frac{n}{2} (1 + t - \tau)^{-n/2-1} (1 + \tau)^{-n/2-1} (2\tau - t).
\]

(128)

We can easily get when \( \tau = t/2 \), we have \( m'(\tau) = 0 \).

\[
\int_0^t (1 + t - \tau)^{-n/2} (1 + \tau)^{-n/2} D_{S,T} (h, p) \sum_{|k| = 2} \|D_x^2 \nabla h\|_{H^{n/2}}^2 \, d\tau \leq E^2 (1 + t)^{-n/2} D_{S,T} (h, p).
\]

(131)

On combining (125)–(132), we obtain

\[
\begin{align*}
\int_0^t (1 + t - \tau)^{-n/2} \|\nabla h\|_{W^{n/2-1}} \, d\tau & \leq c E^2 (1 + t)^{-n/2} \\
\int_0^t (1 + \tau)^{-n/2} \, dt D_{S,T} (h, p) & + c E^2 (1 + t)^{-n/2} \\
\int_0^t (1 + \tau)^{-n} \, d\tau D_{S,T} (h, p).
\end{align*}
\]

(132)

Using (38), we obtain

\[
\int_0^t (1 + t - \tau)^{-n/2} \|\nabla h\|_{W^{n/2-1}} \, d\tau \leq c E^2 (1 + t)^{-n/2}
\]

\[
\int_0^t (1 + \tau)^{-n/2} \, dt D_{S,T} (h, p) + c E^2 (1 + t)^{-n/2}
\]

\[
\int_0^t (1 + \tau)^{-n} \, d\tau D_{S,T} (h, p).
\]
\[
\|\vec{v}^3 - \vec{v}^2\|_{W^{3,\infty}} \leq c \left( \|p\|_{L^\infty} (\|\vec{v}\|_{W^{3,\infty}} + \|\vec{p}\|_{W^{1,\infty}}) + c \|p\|_{W^{3,1}} (\|\vec{v}\|_{L^\infty} + \|\vec{p}\|_{L^\infty}) \right) (\|\vec{v}\|_{L^\infty} + \|\vec{p}\|_{L^\infty}), \]
(133)

On combining (116)–(134), we obtain
\[
\sup_{0 \leq t \leq T} (1 + t)^{-n/2} \|u\|_{W^{3,\infty}} \leq c E^2 D_{ST} (h, p) \left( 1 + \int_0^t (1 + r)^{-n/2} dr \right).
(134)
\]

Finally, applying (26), we have
\[
\|u\|_{H^3} \leq c \left( \int_0^t \|\vec{v}^2 \vec{u} - \vec{v}^2 \vec{u}\|_{H^3} \, dr + \int_0^t \|\vec{u}^2 \vec{u} - \vec{u}^2 \vec{u}\|_{H^3} \, dr + \int_0^t \|\vec{v}^2 \vec{v} - \vec{v}^2 \vec{v}\|_{H^3} \, dr + \int_0^t \|\vec{v}^3 - \vec{v}^3\|_{H^3} \, dr \right).
(135)
\]

From (38) (taking \( r = 2, p = \infty, q = 2 \), we get
\[
\|\vec{u}^2 \vec{u} - \vec{u}^2 \vec{u}\|_{H^3} \leq c (\|h\|_{L^\infty} (\|\vec{u}\|_{H^3} + \|\vec{u}\|_{H^3}) + \|h\|_{H^3} (\|\vec{u}\|_{L^\infty} + \|\vec{u}\|_{L^\infty})) (\|\vec{u}\|_{H^3} + \|\vec{u}\|_{H^3}),
(136)
\]

Thus, we can easily get
\[
\int_0^t \|\vec{u}^2 \vec{u} - \vec{u}^2 \vec{u}\|_{H^3} \, dr \leq c \int_0^t (1 + r)^{-n} E^2 \, dr D_{ST} (h, p).
(137)
\]

Similar to (119), we also have
\[
\|\vec{v}^3 - \vec{v}^3\|_{H^3} \leq c (\|\vec{v} + \vec{v}\|_{H^3} + \|\vec{v}\|_{H^3})^{1/2},
(138)
\]

\[
\|[\vec{v}^2]_{H^3}\|_{H^3} \leq c \|p\|_{L^\infty} \|\vec{u}\|_{H^3} + c \|\vec{u}\|_{L^\infty} \|\vec{u}\|_{H^3} \|\vec{u}\|_{L^\infty} + c \|\vec{u}\|_{H^3} \|\vec{u}\|_{L^\infty} \|\vec{u}\|_{L^\infty},
\]

\[
\leq c (1 + t)^{-n/2} E^2 (1 + t)^{-n/2} D_{ST} (h, p) + c (1 + t)^{-n/2} E^2 (1 + t)^{-n/2} D^2_{ST} (h, p) \left( \sum_{|k|=2} \|D^k_{\vec{x}}\|_{H^{n/2}}^2 \right)^{1/2}.
(139)
\]

We can get
\[
\int_0^t \|\vec{p}\|_{H^3} \, dr \leq c \int_0^t (1 + r)^{-n} E^2 D_{ST} (h, p) \, dr
+ c \int_0^t (1 + r)^{-3n/4} E^2 D_{ST} (h, p) \, dr + c E^2 D_{ST} (h, p).
(140)
\]

We need only to give a bound for \( \int_0^t \|\vec{v}^3\|_{H^3} \, dr \).
\[ \|v_t^2\|_{H^s} \leq c \left( 1 + t \right)^{-n/2} \left( \sum_{|k|=0}^{2} \|D^2_{x} v^2_{H^{k+1}}\right)^{1/2} E (1 + t)^{-n} D_{s,T} (h, p) + E^2 (1 + t)^{-n} D_{s,T} (h, p) \] \]  

We conclude that and that for any \( \varepsilon \in (0, \varepsilon_0) \) such that for any \( 0 < \varepsilon \leq \varepsilon_0 \), (145) holds in this case, we get the global solution.

\[ \int_0^T (1 + t)^{-n/2} dt \leq c, \quad \forall T > 0. \]  

(i) We can choose \( T(\varepsilon) = +\infty \), and let \( \varepsilon_0 \) be so small that for any \( 0 < \varepsilon \leq \varepsilon_0 \), (145) holds. In this case, we get the global solution.

(2) In the case that \( n = 2 \), since

\[ \int_0^T (1 + t)^{-1} dt = \ln (1 + T), \]  

(i) we can choose \( T(\varepsilon) = 2ae^{-\varepsilon} - 1 \), where \( a \) is a positive constant that satisfies

\[ 2ae^2 \leq 1. \]  

(ii) Thus, if \( \varepsilon_0 \) is so small that

\[ 2ae^2 \varepsilon_0 \leq 1, \]  

(145) holds. In that case, we get the so-called almost global solutions.

(3) In the case that \( n = 1 \), since

\[ \int_0^T (1 + t)^{-1/2} dt = \frac{1}{1 - n/2} \left[ (1 + T)^{1/2} - 1 \right] \leq 2 (1 + T)^{1/2}, \]  

we can choose \( T(\varepsilon) = be^{-\varepsilon} - 1 \), where \( b \) is a positive constant that satisfies

\[ 4b^{1/2} e^2 \varepsilon_0 \leq 1. \]  

Thus, we still get (145), provided (153) holds. On combining the above discussions, we get the desired Theorem 1.

4. Conclusion

In the present paper, the global existence and blow-up for the classical solutions of the long-short wave equations with
viscosity are studied. Using the method proposed in this paper, similar results can be obtained for other wave equations with different nonlinear terms. Our method is also valid for the weak solution. For the case when \( \eta \to 0 \), our method is no longer applicable. There may be difficulty in obtaining the energy estimates. We must seek new methods to overcome this difficulty. We think it is interesting. We will discuss it in another paper.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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