Research Article

Bounds of Degree-Based Molecular Descriptors for Generalized F -sum Graphs

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A molecular descriptor is a mathematical measure that associates a molecular graph with some real numbers and predicts the various biological, chemical, and structural properties of the underlying molecular graph. Wiener (1947) and Trinajstić and Gutman (1972) used molecular descriptors to find the boiling point of paraffin and total π -electron energy of the molecules, respectively. For molecular graphs, the general sum-connectivity and general Randić are well-studied fundamental topological indices (TIs) which are considered as degree-based molecular descriptors. In this paper, we obtain the bounds of the aforesaid TIs for the generalized F -sum graphs. The foresaid TIs are also obtained for some particular classes of the generalized F -sum graphs as the consequences of the obtained results. At the end, 3 D -graphical presentations are also included to illustrate the results for better understanding.

1. Introduction

A molecular descriptor called by the topological index (TI) is a function from the set of (molecular) graphs to the set of real numbers. TIs are studied as a subtopic of chemical graph theory to predict the chemical reactions, biological attributes, and physical features of the compounds in theoretical chemistry, toxicology, pharmaceutical industry, and environmental chemistry, see [1]. In addition, these TIs are also used to characterize the molecular structure with respect to quantitative structure activity and property relationships which are studied in the subject of cheminformatics, see [2]. TIs have been classified into different classes but degree-based TIs play a significant part in the theory of chemical structures or networks. Firstly, Wiener [3] used the TI to find boiling point of paraffin. Gutman and Trinajstić calculated total π -electron energy of the molecules by a TI that is recognized as the first

Zagreb index in the literature, see [4]. In 2009, Zhou and Trinajstić [5] suggested the sum-connectivity index, which was subsequently generalized in 2010 [6]. The Randić index was defined in 1975 [7]; later on the idea was subsequently extended to the generalized Randić connectivity index by Li and Gutman, see [8]. For more studies, we refer to [9–12].

In chemical graph theory, operations of graphs are frequently used to find the new families of graphs. Yan et al. [13] defined the four subdivision-related operations (S_1^1, S_1^2, S_1^3 , and S_1^4) on a molecular graph M and obtained the Wiener indices of the resultant graphs $S_1^1(M), S_1^2(M), S_1^3(M)$, and $S_1^4(M)$. After that, Eliasi and Taeri [14] defined the F -sum graph $M_1 +_{F_1} M_2$ with the help of Cartesian product of M_1 and $F(M_2)$, where $F_1 \in \{S_1^1, S_1^2, S_1^3, S_1^4\}$. Deng et al. [15] and Akhter and Imran [16] also computed the 1st and 2nd Zagreb and general sum-connectivity indices of the F -sum graphs, respectively. Recently, Liu et al. [17]

generalized these subdivision-related operations and defined the generalized F -sum graphs $M_1 +_{F_k} M_2$ for $F_k \in \{S_k^1, S_k^2, S_k^3, S_k^4\}$, where $k \geq 1$ is an integral value. They also computed the 1st and 2nd Zagreb indices for these newly obtained graphs. For further studies of F -sum and generalized F -sum graphs, see [18–27].

Now, we extend this study by computing the bounds (upper and lower) of the general sum-connectivity and general Randić indices for the generalized F -sum graphs. In the remaining paper, Section 2 consists of main results of bounds and Section 3 has conclusion and applications of the obtained results.

2. Preliminaries

Throughout, we consider undirected, connected, and simple graphs with $V(M) = \{a_i: 1 \leq i \leq n\}$ and $E(M) \subseteq V(M) \times V(M)$ as vertex and edge sets, respectively. In addition, $|V(M)| = n$ and $|E(M)| = m$ are the order and size of M . For a vertex $a \in V(M)$ and $d_M(a)$ as a degree of a in M , $\Delta_M = \max\{d_M(a): a \in V(M)\}$ and $\delta_M(a) = \min\{d_M(a): a \in V(M)\}$ are the maximum and minimum degrees of the graph M . The graphs $V(P_n) = \{a_i: 1 \leq i \leq n\}$ & $E(P_n) = \{a_i a_{i+1}: 1 \leq i \leq n-1\} \cup \{a_n a_1\}$ and $V(C_n) = \{a_i: 1 \leq i \leq n\}$ & $E(C_n) = \{a_i a_{i+1}: 1 \leq i \leq n-1\} \cup \{a_n a_1\}$ are called path (P_n), cycle (C_n), and complete (K_n), respectively. For further basic terminologies, see [28].

Definition 1. For a real number α and (molecular) graph M , the general sum-connectivity index (GSCI) and general Randić index (GRI) are

$$\begin{aligned}\chi_\alpha(M) &= \sum_{x_1, x_2 \in E(M)} [d_M(x_1) + d_M(x_2)]^\alpha, \\ R_\alpha(M) &= \sum_{x_1, x_2 \in E(M)} [d_M(x_1) d_M(x_2)]^\alpha.\end{aligned}\quad (1)$$

Moreover, $\alpha = 1, 2$ and then $\chi_\alpha(M)$ are known as the 1st Zagreb index and hyper-Zagreb index. If $\alpha = (-1/2)$, 1 , $(1/2)$, then $R_\alpha(M)$; it is known as Randić, 2nd Zagreb, and reciprocal Randić indices.

Liu [17] defined the following graphs using the generalized subdivision-related operations:

- (i) $S_k^1(M)$ graph is obtained by inserting k -vertices in each edge of M .
- (ii) $S_k^2(M)$ is obtained from $S_k^1(M)$ by joining the old vertices which are adjacent in M .
- (iii) $S_k^3(M)$ is obtained from $S_k^1(M)$ by joining the new vertices lying in an edge to the corresponding new vertices of the other edge, if these edges have some common vertexes in M .
- (iv) $S_k^4(M)$ is the union of $S_k^2(M)$ and $S_k^3(M)$ graphs.

For more details, see Figure 1 for $k = 4$.

Definition 2. Let M_1 and M_2 be two connected graphs, $F_k \in \{S_k^1, S_k^2, S_k^3, S_k^4\}$ and $F_k(M_1)$ be a graph (obtained after applying the operation F_k on M_1 with vertex set $V(F_k(M_1))$ and edge set $E(F_k(M_1))$). Then, the generalized F -sum graph $M_1 +_{F_k} M_2$ is a graph with vertex set $V(M_1 +_{F_k} M_2) = V(F_k(M_1)) \times V(M_2) = (V(M_1) \cup E(M_1)) \times V(M_2)$ in such a way that $(x_1, y_1), (x_2, y_2) \in V(M_1 +_{F_k} M_2)$ are adjacent if $[x_1 = x_2 \in V(M_1) \text{ and } (y_1, y_2) \in E(M_2)]$ or $[y_1 = y_2 \in V(M_2) \text{ and } (x_1, x_2) \in E(F_k(M_1))]$, where $k \geq 1$ is an integral value.

For more details, see Figures 2 and 3.

3. Main Results

In this section, we find out the sharp bounds of GSCI and GRI of generalized F -sum graphs.

Theorem 1. For a real number $\alpha > 0$ and counting number $k \geq 1$, the lower and upper bounds on the GSCI and GRI of generalized F -sum graph $(M_1 + S_k^1 M_2)$ are as follows:

- (a) $2^\alpha n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^\alpha + 2n_2 m_1 (\Delta_{M_1} + \Delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1) \leq \chi_\alpha(M_1 + S_k^1 M_2) \leq 2^\alpha n_1 m_2 (\delta_{M_1} + \delta_{M_2})^\alpha + 2n_2 m_1 (\delta_{M_1} + \delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1)$,
- (b) $n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (\Delta_{M_1} + \Delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1) \leq R_\alpha(M_1 + S_k^1 M_2) \leq n_1 m_2 (\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (\delta_{M_1} + \delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1)$,

where equalities hold if M_1 and M_2 are regular graphs.

Proof. (a) By the definition of GSCI, we have

FIGURE 1: (a) $M \cong P_4$, (b) $S_4(M)$, (c) $R_4(M)$, (d) $Q_4(M)$, and (e) $T_4(M)$.

FIGURE 2: (a) $M_1 \cong P_4$, (b) $M_2 \cong P_4$, (c) $M_{1+S_2}M_2$, and (d) $M_{1+R_2}M_2$.

FIGURE 3: (a) $M_{1+Q_2}M_2$. (b) $M_{1+T_2}M_2$.

$$\begin{aligned}
\chi_\alpha(M_1 + S_k^1 M_2) &= \sum_{(x_1, y_1) (x_2, y_2) \in E(M_1 + S_k^1 M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y_1) + d_{(M_1 + S_k^1 M_2)}(x_2, y_2) \right]^\alpha \\
&= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\
&\quad + \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(S_k^1(M_1))} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha \\
&= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\
&\quad + \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(S_k^1(M_1))} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha \\
&\quad \quad \quad x_1 \in V(M_1), x_2 \in V(S_k^1(M_1) - M_1) \\
&\quad + \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(S_k^1(M_1))} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha = \sum 1 + \sum 2 + \sum 3. \\
&\quad \quad \quad x_1, x_2 \in V(S_k^1(M_1) - M_1)
\end{aligned} \tag{2}$$

Consider

$$\begin{aligned}
\sum 1 &= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\
&= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[2d_{M_1}(x) + d_{M_2}(y_1) + d_{M_2}(y_2) \right]^\alpha \leq 2^\alpha n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^\alpha.
\end{aligned} \tag{3}$$

Since $|E(S_k^1(M_1))| = 2|E(M_1)|$ and $\Delta_{S_k^1(M_1)} = \Delta_{M_1}$, we have

$$\begin{aligned}
 \sum 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1 \in V(M_1), x_2 \in V(S_k^1(M_1) - M_1)}} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(G + S_k^1 M_2)}(x_2, y) \right]^\alpha, \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1 \in V(M_1), x_2 \in V(S_k^1(M_1) - M_1)}} \left(S_k^1(M_1) - V(M_1) \right) \left[d_{M_1}(x_1) + d_{M_2}(y) + d_{S_k^1 M_1}(x_2) \right]^\alpha \\
 &\leq 2n_2 m_1 (\Delta_{M_1} + \Delta_{M_2} + 2)^\alpha \\
 \sum 3 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1, x_2 \in V(S_k^1(M_1) - M_1)}} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1, x_2 \in V(S_k^1(M_1) - M_1)}} [2 + 2]^\alpha.
 \end{aligned} \tag{4}$$

Since in this case $|E(S_k^1(M_1) - E(S_1^1))| = (k-1)|E(M_1)|$, we have

Consequently,

$$\begin{aligned}
 &= (k-1)|E(M_1)| \sum_{y \in V(M_2)} (4)^\alpha \\
 &= 4^\alpha (k-1)|V(M_2)||E(M_1)| = 4^\alpha (k-1)n_2 m_1.
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 &2^\alpha n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^\alpha + 2n_2 m_1 (\Delta_{M_1} + \Delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1) \leq \chi_\alpha(M_1 + S_k^1 M_2) \\
 &\leq 2^\alpha n_1 m_2 (\delta_{M_1} + \delta_{M_2})^\alpha + 2n_2 m_1 (\delta_{M_1} + \delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1),
 \end{aligned} \tag{6}$$

where equalities hold if M_1 and M_2 are regular graphs. $\square$ *Proof.* (b) By the definition of GRI, we have

$$\begin{aligned}
 R_\alpha(M_1 + S_k^1 M_2) &= \sum_{(x_1, y_1)(x_2, y_2) \in E(M_1 + S_k^1 M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y_1) + d_{(M_1 + S_k^1 M_2)}(x_2, y_2) \right]^\alpha \\
 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\
 &\quad + \sum_{y \in V(M_2)} \sum_{x_1 x_2 \in E(S_k^1(M_1))} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha \\
 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\
 &\quad + \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1 \in V(M_1), x_2 \in V(S_k^1(M_1) - M_1)}} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha \\
 &\quad + \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1, x_2 \in V(S_k^1(M_1) - M_1)}} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha = \sum 1 + \sum 2 + \sum 3.
 \end{aligned} \tag{7}$$

Consider

$$\begin{aligned}
 \sum 1 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\
 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{M_1}(x) + d_{M_2}(y_1) \right] \left[d_{M_1}(x) + d_{M_2}(y_2) \right]^\alpha \\
 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{M_1}^2(x) + d_{M_1}(x) [d_{M_2}(y_1) + d_{M_2}(y_2)] + d_{M_2}(y_1) d_{M_2}(y_2) \right]^\alpha \\
 &\leq n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^{2\alpha}.
 \end{aligned} \tag{8}$$

Since $|E(S_k^1(M_1))| = 2|E(M_1)|$ and $\Delta_{S_k^1(M_1)} = \Delta_{M_1}$, we have

$$\begin{aligned}
 \sum 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1 \in V(M_1), x_2 \in V(S_k^1(M_1) - M_1)}} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha, \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1 \in V(M_1), x_2 \in V(S_k^1(M_1) - M_1)}} \left[[d_{M_1}(x_1) + d_{M_2}(y)] [d_{S_k^1(M_1)}(x_2)] \right]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1 \in V(M_1), x_2 \in V(S_k^1(M_1) - M_1)}} \left[(\Delta_{M_1} + \Delta_{M_2}) 2 \right]^\alpha \leq 2^{\alpha+1} n_2 m_1 (\Delta_{M_1} + \Delta_{M_2})^\alpha \\
 \sum 3 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1, x_2 \in V(S_k^1(M_1) - M_1)}} \left[d_{(M_1 + S_k^1 M_2)}(x_1, y) + d_{(M_1 + S_k^1 M_2)}(x_2, y) \right]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^1(M_1)) \\ x_1, x_2 \in V(S_k^1(M_1) - M_1)}} [2 \times 2]^\alpha.
 \end{aligned} \tag{9}$$

Since in this case $|E(S_k^1(M_1) - E(S_1^1))| = (k-1)|E(M_1)|$, we have $= (k-1)|E(M_1)| \sum_{y \in V(M_2)} (4)^\alpha = 4^\alpha (k-1) n_2 m_1$.

Consequently,

$$\begin{aligned}
 n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (\Delta_{M_1} + \Delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1) &\leq R_\alpha(M_1 + S_k^1 M_2) \\
 &\leq n_1 m_2 (\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (\delta_{M_1} + \delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1),
 \end{aligned} \tag{10}$$

where equalities hold if M_1 and M_2 are regular graphs. $\square$

Theorem 2. For a real number $\alpha > 0$ and counting number $k \geq 1$, the lower and upper bounds on the GSCI and GRI of the generalized F-sum graph $(M_1 + S_k^2 M_2)$ are as follows:

$$(a) \quad 2^\alpha (n_1 m_2 + n_2 m_1) (2\Delta_{M_1} + \Delta_{M_2})^\alpha + 2n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1) \leq \chi_\alpha(M_1 + S_k^1 M_2) \leq 2^\alpha$$

$$(n_1 m_2 + n_2 m_1) (2\delta_{M_1} + \delta_{M_2})^\alpha + 2n_2 m_1 (2\delta_{M_1} + \delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1),$$

$$(b) \quad (n_1 m_2 + n_2 m_1) (2\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1) \leq R_\alpha(M_1 + S_k^2 M_2) \leq (n_1 m_2 + n_2 m_1) (2\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (2\delta_{M_1} + \delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1),$$

where equalities hold if and only if M_1 and M_2 are regular graphs.

Proof. (a) By the definition of the GSCI, we have

$$\begin{aligned}\chi_\alpha(M_1 + S_k^2 M_2) &= \sum_{(x_1, y_1)(x_2, y_2) \in E(M_1 + S_k^2 M_2)} \left[d_{(M_1 + S_k^2 M_2)}(x_1, y_1) + d_{(M_1 + S_k^2 M_2)}(x_2, y_2) \right]^\alpha \\ &= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\ &\quad + \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(S_k(M_1))} \left[d_{(M_1 + S_k^2 M_2)}(x_1, y) + d_{(M_1 + S_k^2 M_2)}(x_2, y) \right]^\alpha = \sum 1 + \sum 2.\end{aligned}\tag{11}$$

Consider

$$\begin{aligned}\sum 1 &= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[d_{(M_1 + S_k^1 M_2)}(x, y_1) + d_{(M_1 + S_k^1 M_2)}(x, y_2) \right]^\alpha \\ &= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[4d_{M_1}(x) + d_{M_2}(y_1) + d_{M_2}(y_2) \right]^\alpha \leq 2^\alpha n_1 m_2 (2\Delta_{M_1} + \Delta_{M_2})^\alpha \\ \sum 2 &= \sum_{x \in V(M_1)} \sum_{x_1, x_2 \in V(M_1)} [d(x_1, y) + d(x_2, y)]^\alpha \\ &= \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(M_1)}} [d(x_1, y) + d(x_2, y)]^\alpha \\ &\quad + \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^2(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^2(M_1) - V(M_1))}} [d(x_1, y) + d(x_2, y)]^\alpha \\ &\quad + \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(S_k^2(M_1) - V(M_1))}} [d(x_1, y) + d(x_2, y)]^\alpha = \sum' 2 + \sum'' 2 + \sum''' 2.\end{aligned}\tag{12}$$

Consider for $x_1, x_2 \in V(M_1)$, we have $x_1, x_2 \in E(S_k^2(M_1))$ if $x_1, x_2 \in E(M_1)$; for $x_1 \in V(M_1)$, we obtain $d_{S_{1M_1}^2}(x_1) = 2d_{M_1}(x_1)$ and for $x_2 \in V(S_{kM_1}^2) - V(M_1)$, we have $d_{S_{kM_1}^2}(x_2) = 2$.
Now, consider

$$\begin{aligned}\sum' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(M_1) \\ x_1, x_2 \in V(M_1)}} [d(x_1, y) + d(x_2, y)]^\alpha = \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(M_1)} [d(x_1, y) + d(x_2, y)]^\alpha \\ &= \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(M_1)} [2d_{M_2}(y) + 2d_{M_1}(x_1) + 2d_{M_1}(x_2)]^\alpha \leq 2^\alpha n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2})^\alpha.\end{aligned}\tag{13}$$

Note that $|E(S_1^1(M_1))| = 2|E(M_1)|$ and if $x_1 \in V(M_1)$, then $d_{S_k^2}(M_1)x_1 = 2d_{M_1}(x_1)$, and if $x_2 \in V(S_k^2(M_1) - V(M_1))$, then $d_{S_k^2}(M_1)x_2 = 2$.

$$\begin{aligned}
 \sum'' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_1 \in V(M_1), x_2 \in V(S_k^2(M_1) - V(M_1))}} [d(x_1, y) + d(x_2, y)]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(M_1) \\ x_1 \in V(M_1), x_2 \in V(M_1) - V(M_1)}} [2d_{M_1}(x_1) + d_{S_k^2(M_1)}(x_2) + d_{M_2}(y)]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E((M_1)) \\ x_1 \in V(M_1), x_2 \in V((M_1) - V(M_1))}} [2d_{M_1}(x_1) + 2 + d_{M_2}(y)]^\alpha \leq 2n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2} + 2)^\alpha \quad (14)
 \end{aligned}$$

$$\begin{aligned}
 \sum''' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(S_k^2(M_1) - V(M_1))}} [d(x_1, y) + d(x_2, y)]^\alpha \\
 &= (k-1) \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(S_k^2(M_1) - V(M_1))}} [d_{S_k^2(M_1)}(x_1) + d_{S_k^2(M_1)}(x_2)]^\alpha = 4^\alpha (k-1) m_1 n_2.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 2^\alpha (n_1 m_2 + n_2 m_1) (2\Delta_{M_1} + \Delta_{M_2})^\alpha + 2n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1) &\leq \chi_\alpha \\
 (M_1 + S_k^2 M_2) &\leq 2^\alpha (n_1 m_2 + n_2 m_1) (2\delta_{M_1} + \delta_{M_2})^\alpha + 2n_2 m_1 (2\delta_{M_1} + \delta_{M_2} + 2)^\alpha + 4^\alpha n_2 m_1 (k-1). \quad (15)
 \end{aligned}$$

Equality holds if M_1 and M_2 are regular graphs. $\square$ *Proof.* (b) By the definition of GRI, we have

$$\begin{aligned}
 R_\alpha(M_1 + S_k^2 M_2) &= \sum_{(x_1, y_1)(x_2, y_2) \in E(M_1 + S_k^2 M_2)} [d_{(G + S_k^2 M_2)}(x_1, y_1) d_{(M_1 + S_k^2 M_2)}(x_2, y_2)]^\alpha, \\
 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} [d_{(M_1 + S_k^2 M_2)}(x, y_1) d_{(M_1 + S_k^2 M_2)}(x, y_2)]^\alpha \quad (16) \\
 &\quad + \sum_{y \in V(M_2)} \sum_{x_1 x_2 \in E(S_k^2(M_1))} [d_{(M_1 + S_k^2 M_2)}(x, y_1) d_{(M_1 + S_k^2 M_2)}(x_2, y)]^\alpha = \sum 1 + \sum 2.
 \end{aligned}$$

Consider

$$\begin{aligned}
 \sum 1 &= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[d_{(M_1 + S_k^2 M_2)}(x, y_1) d_{(M_1 + S_k^2 M_2)}(x, y_2) \right]^\alpha, \\
 &= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[[2d_{M_1}(x) + d_{M_2}(y_1)][2d_{M_1}(x) + d_{M_2}(y_2)] \right]^\alpha \\
 &= \sum_{x \in V(M_1)} \sum_{y_1, y_2 \in E(M_2)} \left[[4d_{M_1}^2(x) + 2d_{M_1}(x_1)[d_{M_2}(y_1) + d_{M_2}(y_2)] + d_{M_2}(y_1)d_{M_2}(y_2)] \right]^\alpha \\
 &\leq n_1 m_2 (2\Delta_{M_1} + \Delta_{M_2})^{2\alpha}, \\
 \sum 2 &= \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(S_k^2(M_1))} [d(x_1, y) d(x_2, y)]^\alpha \tag{17} \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha + \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^2(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^2(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha \\
 &\quad + \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(S_k^2(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha = \sum 2 + \sum 2 + \sum 2.
 \end{aligned}$$

Consider for $x_1, x_2 \in V(M_1)$, we have $x_1, x_2 \in E(S_k^2(M_1))$ if $x_1, x_2 \in E(M_1)$; for $x_1 \in V(M_1)$, we obtain $d_{S_k^2(M_1)}(x_1) =$

$2d_{M_1}(x_1)$ and for $x_2 \in V(S_k^2(M_1)) - V(M_1)$, we have $d_{S_k^2(M_1)}(x_2) = 2$. Now, consider

$$\begin{aligned}
 \sum' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha = \sum_{y \in V(M_1)} \sum_{x_1, x_2 \in E(M_1)} [d(x_1, y) d(x_2, y)]^\alpha, \\
 &= \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(M_1)} \left[[d_{S_k^2(M_1)}(x_1) + d_{M_2}(y)] [d_{S_k^2(M_1)}(x_2) + d_{M_2}(y)] \right]^\alpha \tag{18} \\
 &= \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(M_1)} \left[4d_{(M_1)}(x_1) d_{(M_1)}(x_2) + 2d_{(M_1)}(x_1) d_{M_2}(y) + 2d_{(M_1)}(x_2) d_{M_2}(y) + d_{M_2}^2(y) \right]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(M_1)} \left[4\Delta_{M_1} \Delta_{M_1} + 2\Delta_{M_2} [\Delta_{M_1} + \Delta_{M_1}] + \Delta_{M_2}^2 \right]^\alpha \leq n_2 m_1 [2\Delta_{M_1} + \Delta_{M_2}]^{2\alpha}.
 \end{aligned}$$

Note that $|E(S_1^2(M_1))| = 2|E(M_1)|$ and if $x_1 \in V(M_1)$, then $d_{S_k^2(M_1)}(x_1) = 2d_{M_1}(x_1)$, and if $x_2 \in V(S_k^2(M_1)) - V(M_1)$, then $d_{S_k^2(M_1)}(x_2) = 2$.

$$\begin{aligned}
 \sum'' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_1 \in V(M_1) x_2 \in V(S_k^2(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha, \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^2(M_1)) - V(M_1)}} \left[\left[d_{S_k^2(M_1)}(x_1) + d_{M_2}(y) \right] \left[d_{S_k^2(M_1)}(x_2) \right] \right]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^2(M_1)) - V(M_1)}} \left[\left[2d_{M_1}(x_1) + d_{M_2}(y) \right] [2] \right]^\alpha \leq 2^{\alpha+1} n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2})^\alpha
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 \sum''' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_2 \in V(S_k^2(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha, \\
 &= (k-1) \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^2(M_1)) \\ x_1, x_2 \in V(S_k^2(M_1)) - V(M_1)}} \left[d_{S_k^2(M_1)}(x_1) d_{S_k^2(M_1)}(x_2) \right]^\alpha = 4^\alpha (k-1) m_1 n_2.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (n_1 m_2 + n_2 m_1) (2\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1) &\leq R_\alpha \\
 (M_1 + S_k^3 M_2) &\leq (n_1 m_2 + n_2 m_1) (2\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^{\alpha+1} n_2 m_1 (2\delta_{M_1} + \delta_{M_2})^\alpha + 4^\alpha n_2 m_1 (k-1).
 \end{aligned} \tag{20}$$

Equality holds if M_1 and M_2 are regular graphs. $\square$

Theorem 3. For a real number $\alpha > 0$ and counting number $k \geq 1$, the lower and upper bounds on the GSCI and GRI of the generalized F-sum graph $(M_1 + S_k^3 M_2)$ are as follows:

$$\begin{aligned}
 (a) \quad & 2^\alpha n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^\alpha + 2n_2 m_1 (3\Delta_{M_1} + \Delta_{M_2})^\alpha + \\
 & 2(k-1)n_1 [Z_1(M_1)]^\alpha + 4^\alpha \Delta_{M_1}^\alpha n_2 ((1/2)Z_1(M_1) + m_1)^\alpha \leq \chi_\alpha(M_1 + S_k^3 M_2) \leq 2^\alpha n_1 m_2 (\delta_{M_1} + \delta_{M_2})^\alpha + \\
 & 2n_2 m_1 (3\delta_{M_1} + \delta_{M_2})^\alpha + 2(k-1)n_1 [Z_1(M_1)]^\alpha + 4^\alpha \delta_{M_1}^\alpha n_2 ((1/2)Z_1(M_1) + m_1)^\alpha,
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad & n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (\Delta_{M_1}^2 + \Delta_{M_1} \Delta_{M_2})^\alpha + \\
 & 2(k-1)n_2 Z_2(M_1)^\alpha + 4^\alpha \Delta_{M_1}^{2\alpha} n_2 \leq R_\alpha(M_1 + S_k^3 M_2) \leq \\
 & n_1 m_2 (\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (\delta_{M_1}^2 + \delta_{M_1} \delta_{M_2})^\alpha + \\
 & 2(k-1)n_2 Z_2(M_1)^\alpha + 4^\alpha \delta_{M_1}^{2\alpha} n_2,
 \end{aligned}$$

where equalities hold if and only if M_1 and M_2 are regular graphs.

Proof. (a) By the definition of the GSCI, we have

$$\begin{aligned}
 \chi_\alpha(M_1 + S_k^3 M_2) &= \sum_{(x_1, y_1) (x_2, y_2) \in E(M_1 + S_k^3 M_2)} \left[d_{(M_1 + S_k^3 M_2)}(x_1, y_1) + d_{(M_1 + S_k^3 M_2)}(x_2, y_2) \right]^\alpha \\
 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} [d(x, y_1) + d(x, y_2)]^\alpha + \sum_{y \in V(M_2)} \sum_{x_1 x_2 \in E(S_k^3(M_2))} [d(x, y_1) + d(x, y_2)]^\alpha \\
 &= \sum 1 + \sum 2.
 \end{aligned} \tag{21}$$

Now,

$$\begin{aligned}
 \sum 1 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} [d(x, y_1) + d(x, y_2)]^\alpha \\
 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} [2d_{M_1}(x) + d_{M_2}(y_1) + d_{M_2}(y_2)]^\alpha \leq 2^\alpha n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^\alpha \\
 \sum 2 &= \sum_{y \in V(M_2)} \sum_{x_1 x_2 \in E(S_k^3(M_1))} [d(x_1, y) + d(x_2, y)]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) + d(x_2, y)]^\alpha \\
 &\quad + \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) + d(x_2, y)]^\alpha = ' \sum 2 + '' \sum 2.
 \end{aligned} \tag{22}$$

Now,

$$\begin{aligned}
 \sum 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) + d(x_2, y)]^\alpha, \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_2)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d_{S_k^3(M_1)}(x_1) + d_{M_2}(y) + d_{S_k^3(M_1)}(x_2)]^\alpha \\
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_2)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d_{S_k^3(M_1)}(x_1) + d_{M_2}(y) + d_{S_k^3(M_1)}(x_2)]^\alpha.
 \end{aligned} \tag{23}$$

Note that $d_{S^3(M_1)}(x_2) = d_{M_1}(w_i) + d_{M_1}(w_j)$ for $a_2 \in V(S^3(M_1)) - V(M_1)$, where a_2 is the vertex inserted into the edge $w_i w_j$ of M_1 . Then, we have

$$\begin{aligned}
 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_2)) \\ x_1 \in V(M_1) x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d_{S_k^3(M_1)}(x_1) + d_{M_2}(y) + d_{M_1}(w_i) + d_{M_1}(w_j)]^\alpha \\
 &\leq 2n_2 m_1 (3\Delta_{M_1} + \Delta_{M_2})^\alpha
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 '' \sum 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1, x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) + d(x_2, y)].
 \end{aligned}$$

We break this sum into two parts $\sum 2 = \sum 3 + \sum 4$, where $\sum 3$ belong the $S_k^3(M)$ edges and $\sum 4$ belong the $S_k^3(M_1)$.

$$\begin{aligned}
\sum 3 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 x_2 \in V(S_k^3(M_1)) - V(M_1)}} \left[d_{S_k^3(M_1)}(x_1) + d_{S_k^3(M_1)}(x_2) \right]^\alpha, \\
&= 2(k-1) \sum_{y \in V(M_2)} \sum_{w_i w_j \in E(M)} \left[d_{M_1}(w_i) + d_{M_1}(w_j) \right]^\alpha \leq 2(k-1)n_1 [Z_1(M_1)]^\alpha \\
\sum 4 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 x_2 \in V(S_k^3(M_1)) - V(M_1)}} \left[d(x_1, y) + d(x_2, y) \right]^\alpha, \\
&= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 x_2 \in V(S_k^3(M_1)) - V(M_1)}} \left[d_{S_k^3(M_1)}(x_1) + d_{S_k^3(M_1)}(x_2) \right]^\alpha.
\end{aligned} \tag{25}$$

Since x_1 is the vertex inserted into the edge $w_i w_j$ of M_1 and a_2 is the vertex inserted into the edge $w_j w_k$ of M_1 ,

$$\begin{aligned}
&= \sum_{y \in V(M_2)} \sum_{\substack{w_i w_j \in E(M_1) \\ w_j w_k \in E(M_1)}} \left[d_{M_1}(w_i) + d_{M_1}(w_j) + d_{M_1}(w_j) + d_{M_1}(w_k) \right]^\alpha \\
&= \sum_{y \in V(M_2)} \sum_{\substack{w_i w_j \in E(M_1) \\ w_j w_k \in E(M_1)}} \left[d_{M_1}(w_i) + d_{M_1}(w_k + 2d_{M_1}(w_j)) \right]^\alpha \leq 4^\alpha k \Delta_{M_1}^\alpha n_2 \left(\frac{1}{2} Z_1(M_1) + m_1 \right)^\alpha.
\end{aligned} \tag{26}$$

Consequently, we have

$$\begin{aligned}
&2^\alpha n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^\alpha + 2n_2 m_1 (3\Delta_{M_1} + \Delta_{M_2})^\alpha + 2(k-1)n_1 [Z_1(M_1)]^\alpha \\
&+ 4^\alpha \Delta_{M_1}^\alpha n_2 \left(\frac{1}{2} Z_1(M_1) + m_1 \right)^\alpha \leq \chi_\alpha(M_1 + S_k^3 M_2) \leq 2^\alpha n_1 m_2 (\delta_{M_1} + \delta_{M_1})^\alpha + 2n_2 m_1 (3\delta_{M_1} + \delta_{M_2})^\alpha \\
&+ 2(k-1)n_1 [(Z_1(M_1))]^\alpha + 4^\alpha \delta_{M_1}^\alpha n_2 \left(\frac{1}{2} Z_1(M_1) + m_1 \right)^\alpha.
\end{aligned} \tag{27}$$

Proof. (b) By the definition of the GRI, we have

$$\begin{aligned}
R_\alpha(M_1 + S_k^3 M_2) &= \sum_{(x_1, y_1) (x_2, y_2) \in E(M_1 + S_k^3 M_2)} \left[d_{(M_1 + S_k^3 M_2)}(x_1, y_1) d_{(M_1 + S_k^3 M_2)}(x_2, y_2) \right]^\alpha, \\
&= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} [d(x, y_1) d(x, y_2)]^\alpha + \sum_{y \in V(M_2)} \sum_{x_1 x_2 \in E(S_k^3(M_2))} [d(x_1, y) d(x_2, y)]^\alpha = \sum 1 + \sum 2.
\end{aligned} \tag{28}$$

Now,

$$\begin{aligned}
\sum 1 &= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{(M_1 + S_k^3 M_2)}(x, y_1) d_{(M_1 + S_k^3 M_2)}(x, y_2) \right]^\alpha, \\
&= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{M_1}(x) + d_{M_2}(y_1) \right] \left[d_{M_1}(x) + d_{M_2}(y_2) \right]^\alpha \\
&= \sum_{x \in V(M_1)} \sum_{y_1 y_2 \in E(M_2)} \left[d_{M_1}^2(x) + d_{M_1}(x_1) \left[d_{M_2}(y_1) + d_{M_2}(y_2) \right] + d_{M_2}(y_1) d_{M_2}(y_2) \right]^\alpha \\
&\leq n_1 m_2 \left(\Delta_{M_1}^2 + \Delta_{M_1} (2\Delta_{M_2}) + \Delta_{M_2}^2 \right)^\alpha \leq n_1 m_2 \left(\Delta_{M_1} + \Delta_{M_2} \right)^{2\alpha}, \\
\sum 2 &= \sum_{y \in V(M_2)} \sum_{x_1, x_2 \in E(S_k^3(M_1))} [d(x_1, y) d(x_2, y)]^\alpha, \\
&= \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^3(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha \\
&\quad + \sum_{y \in V(M_2)} \sum_{\substack{x_1, x_2 \in E(S_k^3(M_1)) \\ x_1 x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha = \sum' 2 + \sum 2.
\end{aligned} \tag{29}$$

Now,

$$\begin{aligned}
\sum' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)]^\alpha, \\
&= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 \in V(M_1) \\ x_2 \in V(S_k^3(M_1)) - V(M_1)}} \left[\left[d_{S_k^3(M_1)}(x_1) + d_{M_2}(y) \right] \left[d_{S_k^3(M_1)}(x_2) \right] \right]^\alpha.
\end{aligned} \tag{30}$$

Note that $d_{S_k^3(M_1)}(x_2) = d_{M_1}(w_i) + d_{M_1}(w_j)$ for $x_2 \in V(S_k^3(M_1)) - V(M_1)$, where x_2 is the vertex inserted into the edge $w_i w_j$ of M_1 . Then, we have

$$\begin{aligned}
&= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 \in V(M_1) x_2 \in V(S_k^3(M_1)) - V(M_1)}} \left[d_{(M_1)}(x_1) + d_{M_2}(y) \right] \left[d_{M_1}(w_i) + d_{M_1}(w_j) \right]^\alpha \\
&\leq 2^\alpha n_2 m_1 \left(\Delta_{M_1}^2 + \Delta_{M_1} \Delta_{M_2} \right)^\alpha \\
\sum'' 2 &= \sum_{y \in V(M_2)} \sum_{\substack{x_1 x_2 \in E(S_k^3(M_1)) \\ x_1 x_2 \in V(S_k^3(M_1)) - V(M_1)}} [d(x_1, y) d(x_2, y)].
\end{aligned} \tag{31}$$

We break this sum into two parts $\sum'' 2 = \sum 3 + \sum 4$, where $\sum 3$ belong the $S_k^3(M)$ edges and $\sum 4$ belong the $S_k^3(M_1)$.

$$\begin{aligned}
 \sum 3 &= \sum_{y \in V(M_2)} \sum_{x_1 x_2 \in E(S_k^3(M_1))} \sum_{x_1, x_2 \in V(S_k^3(M_1)) - V(M_1)} \left[d_{S_k^3(M_1)}(x_1) d_{S_k^3(M_1)}(x_2) \right]^\alpha, \\
 &= 2(k-1) \sum_{y \in V(M_2)} \sum_{w_i w_j \in E(M)} \left[d_{M_1}(w_i) d_{M_1}(w_j) \right]^\alpha \leq 2(k-1) n_2 Z_2(M_1)^\alpha \\
 \sum 4 &= \sum_{y \in V(M_2)} \sum_{x_1 x_2 \in E(S_k^3(M_1))} \sum_{x_1, x_2 \in V(S_k^3(M_1)) - V(M_1)} [d(x_1, y) d(x_2, y)]^\alpha, \\
 &= \left[d_{S_k^3(M_1)}(x_1) d_{S_k^3(M_1)}(x_2) \right]^\alpha \left[d_{S_k^3(M_1)}(x_1) d_{S_k^3(M_1)}(x_2) \right]^\alpha.
 \end{aligned} \tag{32}$$

Since x_1 is the vertex inserted into the edge $w_i w_j$ of M_1 and a_2 is the vertex inserted into the edge $w_j w_k$ of M_1 ,

$$\begin{aligned}
 &= \sum_{y \in V(M_2)} \sum_{\substack{w_i w_j \in E(M_1) \\ w_j w_k \in E(M_1)}} [d_{M_1}(w_i) + d_{M_1}(w_j)] \\
 &\quad \cdot [d_{M_1}(w_j) + d_{M_1}(w_k)]^\alpha \leq 4^\alpha \Delta_{M_1}^{2\alpha} n_2.
 \end{aligned} \tag{33}$$

Consequently, we have

$$\begin{aligned}
 n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (\Delta_{M_1}^2 + \Delta_{M_1} \Delta_{M_2})^\alpha + 2(k-1) n_2 Z_2(M_1)^\alpha + 4^\alpha \Delta_{M_1}^{2\alpha} n_2 &\leq R_\alpha \\
 (M_1 + S_k^3 M_2) \leq n_1 m_2 (\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (\delta_{M_1}^2 + \delta_{M_1} \delta_{M_2})^\alpha + 2(k-1) n_2 Z_2(M_1)^\alpha + 4^\alpha \delta_{M_1}^{2\alpha} n_2.
 \end{aligned} \tag{34}$$

Theorem 4. For a real number $\alpha > 0$ and counting number $k \geq 1$, the lower and upper bounds on the GSCI and GRI of the generalized F -sum graph $(M_1 + S_k^1 M_2)$ are

- $2^\alpha n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2})^\alpha + 2n_2 m_1 (3\Delta_{M_1} + \Delta_{M_2})^\alpha + 2(k-1) n_1 [Z_1(M_1)]^\alpha + 4^\alpha \Delta_{M_1}^{2\alpha} n_2$
 $n_2 ((1/2)Z_1(M_1) + m_1)^\alpha \leq \chi_\alpha(M_1 + S_k^1 M_2) \leq 2^\alpha n_1 m_2 (\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (2\delta_{M_1} + \delta_{M_2})^\alpha,$
- $n_1 m_2 (\Delta_{M_1} + \Delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (2\Delta_{M_1} + \Delta_{M_2})^\alpha + 2^\alpha n_2 m_1 (\Delta_{M_1}^2 + \Delta_{M_1} \Delta_{M_2})^\alpha + 2(k-1) n_2 [Z_2(M_1)]^\alpha + 4^\alpha \Delta_{M_1}^{2\alpha} n_2 \leq R_\alpha$
 $(M_1 + S_k^1 M_2) \leq n_1 m_2 (\delta_{M_1} + \delta_{M_2})^{2\alpha} + 2^\alpha n_2 m_1 (2\delta_{M_1} + \delta_{M_2})^\alpha + 2(k-1) n_2 Z_2(M_1)^\alpha + 4^\alpha \delta_{M_1}^{2\alpha} n_2,$

where r is the number of common neighbors of x_1 and x_2 in M_1 .

Proof. The proof follows by Theorem 2 and Theorem 3. $\square$

4. Applications

For a real number $\alpha > 0$, the lower and upper bounds on GSCI and GRI of the generalized F -sum graphs obtained by the particular classes of graphs as the consequences of the obtained results are given in Figures 4–11.

Example 1. For $M_1 = P_n$, $M_2 = P_m$, and $k = 4$, we have

- $mn(6.4^\alpha) + m4^\alpha - 4^\alpha n \leq \chi_\alpha(P_n + S_4^1 P_m) \leq mn(8^\alpha + 2.6^\alpha + 3.4^\alpha) - m(2.6^\alpha - 3.4^\alpha) - 8^\alpha n,$ and
- $mn(2^{4\alpha} + 2^{3\alpha+1} + 3.4^\alpha) - m(2^{3\alpha+1} - 3.4^\alpha) - 2^{4\alpha} n \leq R_\alpha(P_n + S_4^1 P_m) \leq mn(2^{2\alpha} + 2^{2\alpha+1} + 3.4^\alpha) - m(2^{2\alpha+1} - 3.4^\alpha) - 22\alpha n.$

The graphical representation of Example 1(a) is depicted in Figure 4, the lower bounds are represented by the blue graph and the upper bounds are represented by the red graph. The graphical representation of Example 1(b) is depicted in Figure 5, the lower bounds are represented by the green graph and the upper bounds are represented by the blue graph.

FIGURE 4: Graph for $(\alpha = 2) mn(6 \times 4^2) + 4^2m - 4^2n \leq \chi_\alpha(P_n + S_4^1 P_m) \leq mn(8^2 + 2 \times 6^2 + 3 \times 4^2) - m(2 \times 6^2 - 3 \times 4^2) - 8^2n$.

FIGURE 6: Graph for $(\alpha = 2) mn(2.12^2 + 2.8^2 + 3.4^2) - m(12^2 + 2.8^2 + 3.4^2) - 12^2n \leq \chi_\alpha(P_n + S_4^2 P_m) \leq mn(2.6^2 + 2.5^2 + 3.4^2) - m(6^2 + 2.5^2 + 3.4^2) - 6^2n$.

FIGURE 5: Graph for $(\alpha = 2) mn(2^{4^2} + 2^{3^2+1} + 3.4^2) - m(2^{3^2+1} - 3.4^2) - 2^{4^2}n \leq R_\alpha(P_n + S_4^3 P_m) \leq mn(2^{2^2} + 2^{2^2+1} + 3.4^2) - m(2^{2^2+1} - 3.4^2) - 2^{2^2}n$.

FIGURE 7: Graph for $(\alpha = 2) mn(2.6^{2^2} + 3^2.2^{2^2+1} + 3.4^2) - m(6^{2^2} + 3^2.2^{2^2+1} + 3.4^2) - 6^{2^2}n \leq R_\alpha(P_n + S_4^4 P_m) \leq mn(2.3^{2^2} + 3^2.2^{2^2+1} + 3.4^2) - m(3^{2^2} + 3^2.2^{2^2+1} + 3.4^2) - 3^{2 \times 2}n$.

Example 2. For $M_1 = P_n$, $M_2 = P_m$, and $k = 4$, we have

- (a) $mn(2.12^\alpha + 2.8^\alpha + 3.4^\alpha) - m(12^\alpha + 2.8^\alpha + 3.4^\alpha) - 12^\alpha n \leq \chi_\alpha(P_n + S_4^2 P_m) \leq mn(2.6^\alpha + 2.5^\alpha + 3.4^\alpha) - m(6^\alpha + 2.5^\alpha + 3.4^\alpha) - 6^\alpha n$,
- (b) $mn(2.6^{2\alpha} + 3^\alpha.2^{2\alpha+1} + 3.4^\alpha) - m(6^{2\alpha} + 3^\alpha.2^{2\alpha+1} + 3.4^\alpha) - 6^{2\alpha}n \leq R_\alpha(P_n + S_4^2 P_m) \leq mn(2.3^{2\alpha} + 3^\alpha.2^{2\alpha+1} + 3.4^\alpha) - m(3^{2\alpha} + 3^\alpha.2^{2\alpha+1} + 3.4^\alpha) - 3^{2\alpha}n$.

The graphical representation of Example 2(a) is depicted in Figure 6. The lower bounds are represented by the Niagara Azure graph and the upper bounds are represented by the gold graph. The graphical representation of Example 2(b) is depicted in Figure 7. The lower bounds are represented by the pink graph and the upper bounds are represented by the yellow graph.

Example 3. For $M_1 = P_n$, $M_2 = P_m$, and $k = 4$, we have

- (a) $8^\alpha(nm - n) + 2.8^\alpha(mn - m) + 6n(4n - 6)^\alpha + 8^\alpha m(3n - 4)^\alpha \leq \chi_\alpha(P_n + S_4^3 P_m) \leq 4^\alpha(nm - n) + 2.4^\alpha(mn - m) + 6n(4n - 6)^\alpha + 4^\alpha m(3n - 4)^\alpha$,
- (b) $16^\alpha(nm - n) + 16^\alpha(mn - m) + 6m(4m - 6)^\alpha + 16^\alpha \leq R_\alpha(P_n + S_4^3 P_m) \leq 4^\alpha(nm - n) + 4^\alpha(mn - m) + 6m(4m - 6)^\alpha + 4^\alpha$.

The graphical representation of Example 3(a) is depicted in Figure 8. The lower bounds are represented by the blue graph and the upper bounds are represented by the green graph. The graphical representation of Example 3(b) is depicted in Figure 9. The lower bounds are represented by the yellow graph and the upper bounds are represented by the gray graph.

FIGURE 8: Graph for $(\alpha = 2) 8^2 (nm - n) + 2.8^2 (mn - m) + 6n (4n - 6)^2 + 8^2 m (3n - 4)^2 \leq \chi_\alpha (P_n + S_4^3 P_m) \leq 4^2 (nm - n) + 2.4^2 (mn - m) + 6n (4n - 6)^2 + 4^2 m (3n - 4)^2$.

FIGURE 9: Graph for $(\alpha = 2) 16^2 (nm - n) + 16^2 (mn - m) + 6m (4m - 6)^2 + 16^2 \leq R_\alpha (P_n + S_4^3 P_m) \leq 4^2 (nm - n) + 4 (mn - m) + 6m (4m - 6)^2 + 4^2$.

Example 4. For $M_1 = P_n$, $M_2 = P_m$, and $k = 4$, we have

- (a) $8^\alpha (nm - n) + 12^\alpha (mn - m) + 2.8^\alpha (mn - m) + 6n (4n - 6)^\alpha + 8^\alpha m (3n - 4)^\alpha \leq \chi_\alpha (P_n + S_4^4 P_m) \leq 4^\alpha (nm - n) + 6^\alpha (mn - m) + 2.4^\alpha (mn - m) + 6n (4n - 6)^\alpha + 4^\alpha m (3n - 4)^\alpha$,
- (b) $2^{4\alpha} (nm - n) + 2^{2\alpha} .3^\alpha (mn - m) + 24^\alpha (mn - m) + 6m (4m - 6)^\alpha + 24^\alpha m \leq R_\alpha (P_n + S_4^4 P_m) \leq 2^{2\alpha} (nm - n) + 6\alpha (mn - m) + 22^\alpha (mn - m) + 6m (4m - 6)^\alpha + 4^\alpha m$.

The graphical representation of Example 4(a) is depicted in Figure 10. The lower bounds are represented by the blue graph and the upper bounds are represented by the green graph. The graphical representation of Example 4(b) is depicted in Figure 11. The lower bounds are represented by

FIGURE 10: Graph for $\alpha = 2 8^2 (nm - n) + 12^2 (mn - m) + 2.8^2 (mn - m) + 6n (4n - 6)^2 + 8^2 m (3n - 4)^2 \leq \chi_\alpha (P_n + S_4^4 P_m) \leq 4^2 (nm - n) + 6^2 (mn - m) + 2.4^2 (mn - m) + 6n (4n - 6)^2 + 4^2 m (3n - 4)^2$.

FIGURE 11: Graph for $\alpha = 2 2^{4^2} (nm - n) + 2^{2^2} .3^2 (mn - m) + 24^2 (mn - m) + 6m (4m - 6)^2 + 24^2 m \leq R_\alpha (P_n + S_4^4 P_m) \leq 2^{2^2} (nm - n) + 6^2 (mn - m) + 22^2 (mn - m) + 6m (4m - 6)^2 + 4^2 m$.

the yellow graph and the upper bounds are represented by the blue graph.

5. Conclusion

In this paper, the lower and upper bounds of the general sum-connectivity and general Randić indices of the generalized F -sum graphs (F_k -sum graphs) are computed in terms of the order, size, maximum, and/or minimum degree and Zagreb indices of the factor graphs, where the F_k -sum graphs are obtained with the help of four generalized subdivision-related operations and the Cartesian product of graphs. However, the problem is still open for other types of product of graphs.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors have no conflicts of interest.

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