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Research Article

Single-Machine Scheduling Problems with the General Sum-of-Processing-Time and Position-Dependent Effect Function

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This paper considers the combination of the general sum-of-processing-time effect and position-dependent effect on a single machine. The actual processing time of a job is defined by functions of the sum of the normal processing times of the jobs processed and its position and control parameter in the sequence. We consider two monotonic effect functions: the nondecreasing function and the nonincreasing function. Our focus is the following objective functions, including the makespan, the sum of the completion time, the sum of the weighted completion time, and the maximum lateness. For the nonincreasing effect function, polynomial algorithm is presented for the makespan problem and the sum of completion time problem, respectively. The latter two objective functions can also be solved in polynomial time if the weight or due date and the normal processing time satisfy some agreeable relations. For the nondecreasing effect function, assume that the given parameter is zero. We also show that the makespan problem can remain polynomially solvable. For the sum of the total completion time problem and a_1 is the deteriorating rate of the jobs, there exists an optimal solution for $a_1 \geq M$; a V-shaped property with respect to the normal processing times is obtained for $0 < a_1 \leq 1$. Finally, we show that the sum of the weighted completion problem and the maximum lateness problem have polynomial-time solutions for $a_1 > M$ under some agreeable conditions, respectively.

1. Introduction

Recent years, position-effect and processing-time-dependent scheduling problems have been paid more attention. Significant contributions also are presented to solve these problems, including the following. Browne and Yechiali [1] gave some applications to concern the control of queues and communication systems, where there exists deterioration phenomenon in the process of awaiting processing. Kunnathur and Gupta [2] and Mosheiov [3] presented several real-life situations of deteriorating jobs, including the search for an object under worsening weather and performance of medical treatments under some deteriorated health conditions. We refer to the surveys [4] for detailed state-of-the-art reviews in this time-dependent scheduling, as well as for references to practical applications. Among most common rationales for deterioration, the authors often mention the loss of the processing quality of machinery over time and/or the decrease in the productivity of a human operator who gets tired. Cheng et al. [5] considered deteriorated-effect scheduling problems, where the actual processing time of a job means a function of the logarithm of the sum of the normal processing time of the jobs processed and the setup times are proportional to the actual processing times of the jobs processed. Yin et al. [6] addressed another deterioration model to minimize the makespan and the total completion time, where the actual processing time of a job depends on its starting time and its position. They showed that there exists optimal sequence based on the relationships between problem parameters, including the shortest processing time, longest processing time, or V-shaped with respect to the normal processing times. Rudek [7, 8] considered the general sum-of-processing time-based learning or aging effects and showed that the total weighted completion times' problem is strongly NP-hard, respectively. Gawiejnowicz [9] gave a detail review for four decades of time-dependent scheduling, including main results, new topics, etc. Jiang et al. [10] studied general truncated sum-of-actual processing-time-based effect on the single machine. The actual processing time of a job is affected by the sum-of-actual processing times of previous jobs and by a job-dependent truncation parameter. More recent papers considered deteriorating jobs: Li et al. [11], Liang et al. [12], Gawiejnowicz and Kurc [13], and Wang et al. [14].

Learning effects are divided into the following two types. (1) Position dependent: the actual processing time of job J_j depends on p_j and on its position in the sequence. (2) Cumulative: the actual processing time of job J_j depends on p_j and on the sum of normal processing times of jobs sequenced earlier. Biskup [15] and Cheng and Wang [16] were one of the pioneers who brought the concept of learning into the field of scheduling. Biskup [17] presented a detailed review for learning effect in 2008. Wang and Wang [18] investigated a general model with the agreeable position weight. The general models can cover the majority of existent sum-of-processing-time-based scheduling models. Luo [19] presented more general sum-of-processing-time-based scheduling models, which cover the normal processing time or the actual processing times. The distinctive proof technique is developed based on the adding-term operation, the subtracting-term operation, and the Lagrange mean value theorem. Lin [20] studied job-dependent learning effect and controllable processing time on the unrelated parallel machine. The three objective functions are considered, including the weighted sum of total completion time, total load, and total compression cost. Extensive surveys of different scheduling models can be found in Azadeh et al. [21], Pei et al. [22], and Tai [23]. More application of scheduling models, especially, many real-world problems have been explained by using mathematical models such as higher-order spectral analysis of stray flux signals for faults' detection in induction motors, vortex theory for two-dimensional Boussinesq equations, normal complex contact metric manifolds admitting a semisymmetric metric connection, urea injection and uniformity of ammonia distribution in the SCR system of diesel engine, and new complex and hyperbolic forms for Ablowitz-Kaup-Newell-Segur wave equation with fourth order can be found in the following papers: Iglesias Mart et al. [24], Sharifi and Reasi [25], Jiao and Zheng [26], and Eskita et al. [27].

Motivated on the above discussion, the general sum-of-processing-time-based effect and position-dependent effect are provided. The job processing times are defined by functions of the sum of the normal processing times of jobs processed, its position and and control parameter in the sequence. Two monotonic effect functions are studied: nondecreasing function and nonincreasing function. Our four objective functions is the makespan, the sum of completion time, the sum of the weighted completion time, and the maximum lateness. Our contribution in this paper is listed as follows:

(i) The nonincreasing effect function.

(ii) The makespan problem and the sum of the completion time problem can be solved in polynomial time.

(iii) The total weighted completion times' problem and the lateness problem can also be solved in polynomial time if the weight or due date and the normal processing time satisfy some agreeable relations.

(iv) The nondecreasing effect function and the given parameter is zero.

(v) The makespan problem can remain polynomially solvable.

(vi) The sum of completion time problem for $a_1 \geq M\,(>1)$ can be optimally solved, where a_1 is the deteriorating rate of the jobs. Moreover, for the sum of the completion time problem with $0 < a_1 \leq 1$, a V-shaped property based on the normal processing times is obtained in an optimal sequence which satisfies some agreeable relations.

(vii) The sum of the weighted completion times' problem and the maximum lateness problem for $a_1 > M$ have polynomial-time solutions under some agreeable conditions.

The rest of the paper is organized as follows. In Section 2, we give the problem description. In Sections 3 and 4, we consider two different actual processing times. Our conclusion will be given in Section 5.

2. Problem Description

Single-machine scheduling problems can be normally narrated as follows: jobs' set.

(i) $J = \{J_1, J_2, \ldots, J_n\}$

(ii) p_j: the normal processing time of job J_j, $j = 1, 2, \ldots, n$

(iii) w_j: the weight of job J_j, $j = 1, 2, \ldots, n$

(iv) d_j: the due date of job J_j, $j = 1, 2, \ldots, n$

(v) p_{jr}: the actual processing time of the job J_j scheduled in the rth position in the sequence

(vi) $f(x, y)$: a bivariate continuous convex function on x and y $g(x)$ a continuous function on x with $g''(x) \leq 0$ $p_{[l]}$ the normal processing time of a job scheduled in the lth position in a job sequence

(vii) $C_j(\pi)$: the completion time of job J_j in job sequence π

(viii) $C_{\max} = \max\{C_j \mid j = 1, 2, \ldots, n\}$: the makespan

(ix) $\sum C_j$: the total completion

(x) $\sum w_j C_j$: the total weighted completion time

(xi) $L_{\max} = \max\{L_j = C_j - d_j \mid j = 1, 2, \ldots, n\}$: the maximum lateness

The proposed scheduling model is considered as follows:

$$p_{jr} = p_j \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right), r \right), \beta \right\}, \quad (1)$$

where $\beta\,(\geq 0)$ is a given control parameter. Moreover, assume that $g(x) \geq 0$ for $x \geq 0$, $f(x, 0) = 0$, and $f(x, 1) = x$.

Note that the bivariate function f is only continuous convex function. Next, we will consider two monotonic function on x: nondecreasing function and nonincreasing function. For the former, assume that $(\partial f/\partial x) \leq 0$, $(\partial^2 f/\partial x^2) \geq 0$ and $g'(x) \leq 0$. However, for the latter, we only consider the special case of the continuous convex function f, $g'(x) \geq 0$ and a given parameter $\beta = 0$.

3. The Nonincreasing Function $f(x, y)$ on x

This section will consider the nonincreasing function $f(x, y)$ on x and the nonincreasing function $g(x)$ on x. Four objective functions will be studied, including the makespan, the sum of the completion times, the sum of the weighted completion times, and the lateness.

Theorem 1. *Problem* $1\,|p_{jr} = p_j \max\{f(\sum_{l=1}^{r-1} g(p_{[l]}), r), \beta\}|\,C_{\max}$ *can be obtained as an optimal schedule by nondecreasing normal processing times (the shortest processing time (SPT) rule).*

Proof. The properties of the optimal solutions for some single-machine problems are proved by the pairwise job interchange technique. Let π and π' be two job schedules where the difference between sigma and σ' is a pairwise interchange of two adjacent jobs J_i and J_j, i.e., $\sigma = [\mathcal{S}_1, J_i, J_j, \mathcal{S}_2]$ and $\sigma' = [\mathcal{S}_1, J_j, J_i, \mathcal{S}_2]$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are partial sequences. Assume that t_0 denotes the completion time of the last job scheduled in $(r-1)$th position of $\mathcal{S}_1$. Under σ, the completion times of jobs J_i and J_j are

$$C_i(\sigma) = t_0 + p_i \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right), r \right), \beta \right\}, \quad (2)$$

$$C_j(\sigma) = t_0 + p_i \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right), r \right), \beta \right\} + p_j \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right) + g(p_i), r+1 \right), \beta \right\}. \quad (3)$$

Under σ', the completion times of jobs J_j and J_i are

$$C_j(\sigma') = t_0 + p_j \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right), r \right), \beta \right\}, \quad (4)$$

$$C_i(\sigma') = t_0 + p_j \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right), r \right), \beta \right\} + p_i \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right) + g(p_j), r+1 \right), \beta \right\}. \quad (5)$$

Note that $p_i \leq p_j$. Next, we will show that σ dominates σ'. Taking the difference between (4) and (14), it is obtained that

$$\begin{aligned} C_j(\sigma) - C_i(\sigma') = {} & \left(p_i - p_j\right) \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right), r \right), \beta \right\} \\ & + p_j \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right) + g(p_i), r+1 \right), \beta \right\} \\ & - p_i \max\left\{ f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right) + g(p_j), r+1 \right), \beta \right\}. \end{aligned} \quad (6)$$

Based on the monotonicity of function f and g, we have

$$f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right) + g(p_i), r+1 \right) \leq f\left(\sum_{l=1}^{r-1} g\left(p_{[l]}\right) + g(p_j), r+1 \right) \leq f\left(\sum_{l=1}^{r-1} g\left(p_{(l)}\right), r \right). \quad (7)$$

Next, the parameter β will be discussed by four cases as follows:

(1) $\beta \geq f(\sum_{l=1}^{r-1} g(p_{[l]}), r)$. Then, we have

$$C_j(\sigma) - C_i(\sigma') = \left(p_i - p_j\right)\beta + p_j\beta - p_i\beta = 0. \quad (8)$$

(2) $f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_j), r+1) \leq \beta \leq f(\sum_{l=1}^{r-1} g(p_{[l]}), r)$. Then, we have

$$\begin{aligned} C_j(\sigma) - C_i(\sigma') &= (p_i - p_j) f\left(\sum_{l=1}^{r-1} g(p_{[l]}), r\right) + p_j\beta - p_i\beta \\ &= (p_i - p_j)\left(f\left(\sum_{l=1}^{r-1} g(p_{[l]}), r\right) - \beta\right) \\ &\leq 0. \end{aligned} \tag{9}$$

(3) $f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_i), r+1) \leq \beta \leq f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_j), r+1)$. Then, we have

$$\begin{aligned} C_j(\sigma) - C_i(\sigma') &= (p_i - p_j) f\left(\sum_{l=1}^{r-1} g(p_{[l]}), r\right) + p_j\beta - p_i f\left(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_j), r+1\right) \\ &\leq (p_i - p_j)\left(f\left(\sum_{l=1}^{r-1} g(p_{[l]}), r\right) - \beta\right) \\ &\leq 0. \end{aligned} \tag{10}$$

(4) $\beta \leq f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_i), r+1)$. Then, we have

$$\begin{aligned} C_j(\sigma) - C_i(\sigma') &= (p_i - p_j) f\left(\sum_{l=1}^{r-1} g(p_{[l]}), r\right) + p_j f\left(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_i), r+1\right) \\ &\quad - p_i f\left(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_j), r+1\right) \\ &= p_i p_j \left[\frac{f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_j), r+1) - f(\sum_{l=1}^{r-1} g(p_{[l]}), r)}{p_j}\right. \\ &\quad \left. - \frac{f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_i), r+1) - f(\sum_{l=1}^{r-1} g(p_{[l]}), r)}{p_i}\right]. \end{aligned} \tag{11}$$

Let $\varphi(x) = (f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(x), r+1) - f(\sum_{l=1}^{r-1} g(p_{[l]}), r))/x$, $x > 0$. Then, we can obtain

$$\frac{d\varphi(x)}{dx} = \frac{(\partial f/\partial x) g'(x) x - f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(x), r+1) + f(\sum_{l=1}^{r-1} g(p_{[l]}), r)}{x^2}. \tag{12}$$

Let $\Phi(x) = (\partial f/\partial x)g'(x)x - f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(x), r+1) + f(\sum_{l=1}^{r-1} g(p_{[l]}), r)$, and we have

$$\begin{aligned}\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} &= \frac{\partial^2 f}{\partial x^2}(g'(x))^2 x + \frac{\partial f}{\partial x}g''(x)x + \frac{\partial f}{\partial x}g'(x) - \frac{\partial f}{\partial x}g'(x) \\ &= \frac{\partial^2 f}{\partial x^2}(g'(x))^2 x + \frac{\partial f}{\partial x}g''(x)x.\end{aligned} \tag{13}$$

Based on $(\partial^2 f/\partial x^2) \geq 0$, $(\partial f/\partial x) \leq 0$, and $g''(x) \leq 0$, we can obtain that $(\mathrm{d}\Phi(x)/\mathrm{d}x) \geq 0$, i.e., the function $\Phi(x)$ is an nondecreasing function for $\Phi(x) \geq \Phi(0) = f(\sum_{l=1}^{r-1} g(p_{[l]}), r) - f(\sum_{l=1}^{r-1} g(p_{[l]}) + g(0), r+1) \geq 0$. Thus, $\varphi(p_i) \leq \varphi(p_j)$ for $p_i \leq p_j$. Moreover, we have $C_j(\sigma) \leq C_i(\sigma')$.

Note that the completion times of the job J_h if scheduled in $(r+2)$th in the job sequence σ and σ', respectively, are denoted as follows:

$$C_h(\sigma) = C_j(\sigma) + p_h \max\left\{ f\left(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_i) + g(p_j), r+2 \right), \beta \right\}, \tag{14}$$

$$C_h(\sigma') = C_i(\sigma') + p_h \max\left\{ f\left(\sum_{l=1}^{r-1} g(p_{[l]}) + g(p_i) + g(p_j), r+2 \right), \beta \right\}. \tag{15}$$

From $C_j(\sigma) \leq C_i(\sigma')$, we have $C_h(\sigma) \leq C_h(\sigma')$, i.e, the starting time of the first job J_h in partial job sequence $\mathcal{S}_2$ of job sequence σ is earlier than job sequence σ'. Therefore, we have $C_k(\sigma) \leq C_k(\sigma')$ for job J_k in the partial sequence $\mathcal{S}_2$. Hence, the optimality of the SPT rule can be showed by repeating this argument for the proposed scheduling problem.

Note that $C_i(\sigma) - C_j(\sigma') = (p_i - p_j)\max\{f(\sum_{l=1}^{r-1} g(p_{[l]}), r), \beta\} \leq 0$ by the equations (2) and (4). Then, the following theorem will be presented.

Theorem 2. *Problem* $1 | p_{jr} = p_j \max\{f(\sum_{l=1}^{r-1} g(p_{[l]}), r), \beta\} | \sum C_j$ *can be solved by the SPT rule.*

For the total weighted completion time and the maximum lateness, we only show that this two problems can be solved in polynomial time under some special agreeable relations, respectively.

Theorem 3. *For the problem* $1 | p_{jr} = p_j \max\{f(\sum_{l=1}^{r-1} g(p_{[l]}), r), \beta\} | \sum w_j C_j$, *an optimal schedule can be obtained by the weighted smallest processing time, i.e., the WSPT rule, if the jobs have agreeable weights, i.e.,* $p_j \leq p_k$ *implies* $w_k \leq w_j$ *for all jobs* J_j *and* J_k.

Proof. Similar to the same notations in the proof of Theorem 1. Let the jobs J_i and J_j satisfy the agreeable relation, i.e., $(p_i/w_i) \leq (p_j/w_j)$ which implies $p_i \leq p_j$ and $w_i \geq w_j$. Next, we will show that $\sum w_j C_j(\sigma) \leq \sum w_j C_j(\sigma')$.

Since partial job sequence S_1 in job sequence σ and σ' has the same job position, then the completion time of job J_h of partial job sequence S_1 is the equal, i.e., $C_h(\sigma) = C_h(\sigma')$, $h = 1, 2, \ldots, r-1$.

From Theorems 1 and 2, we have

$$\begin{aligned}&w_i C_i(\sigma) + w_j C_j(\sigma) - w_i C_i(\sigma') - w_j C_j(\sigma') \\ &= w_i(C_i(\sigma) - C_i(\sigma')) + w_j(C_j(\sigma) - C_j(\sigma')) \\ &\leq {}_j\left[C_i(\sigma) - C_i(\sigma') + C_j(\sigma) - C_j(\sigma')\right] \leq 0.\end{aligned} \tag{16}$$

Additionally, we have $C_h(\sigma) \leq C_h(\sigma')$ for $J_h \in S_2$ by Theorem 1. □

Theorem 4. *For the problem* $1 | p_{jr} = p_j \max\{f(\sum_{l=1}^{r-1} g(p_{[l]}), r), \beta\} | \sum w_j C_j$, *an optimal schedule can be obtained by the earliest due date, i.e., the EDD rule, if the jobs have agreeable weights, i.e.,* $p_j \leq p_k$ *implies* $d_j \leq d_k$ *for all jobs* J_j *and* J_k.

Proof. Using the same notations of Theorem 1, we will show that $L_{\max}(\sigma_i) \leq L_{\max}(\sigma')$ based on the agreeable relation, i.e., $p_i \leq p_j$ and $d_i \leq d_j$. From Theorems 1 and 2, we have

$$\begin{aligned}L_j(\sigma_i) &= C_j(\sigma) - d_j \leq C_i(\sigma') - d_j \leq C_i(\sigma') - d_i = L_i(\sigma'), \\ L_i(\sigma_i) &= C_i(\sigma) - d_i \leq C_i(\sigma') - d_i = L_i(\sigma').\end{aligned} \tag{17}$$

Moreover, we can obtain $L_{\max}(\sigma_i) \leq L_{\max}(\sigma')$. Hence, interchanging the job position will not increase the value of $L_{\max}$.

4. The Nondecreasing Function $f(x, y)$ on x

In this section, the special case of the nondecreasing function $f(x, y)$ on x will be given, and $\beta = 0$. Firstly, some notations are defined as follows: a_1 and a_2 denote the deteriorating or learning rate and the learning rate, respectively. M_1, M_2, and M are three given positive numbers, where $M_1 = (1 + \sum_{l=1}^{l=n} \beta_l \ln p_l)/\beta_1$, $M_2 = \max_j \ln p_j$, and $M = 1 + M_1 + M_2$.

The special sum-of-processing-time model can be described as follows:

$$p_{jr} = p_j \left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} \right)^{a_1} q_r^{a_2}, \tag{18}$$

where $a_1 \in \{(0, 1], (M, +\infty)\}$. $0 \leq q_1 \leq \cdots \leq q_n$, $0 \leq \beta_1 \leq \cdots \leq \beta_n$, $a_2 < 0$ and $\ln p_{[l]} \geq 1$.

Next, some useful lemmas will be given. Based on $x = \ln p_i$ and $\lambda = p_j/p_i$, the proofs of some lemmas can be obtained by differentiation.

Lemma 1. *$1 - \lambda + \lambda(1 + cx)^{a_1} q^{a_2} - (1 + cx + c \ln \lambda)^{a_1} q^{a_2} \le 0$, for $\lambda \ge 1$, $a_1 \ge M$, $a_2 < 0$, $c \ge (1/M_1)$, $x \ge 1$, and $q > 1$.*

Lemma 2. *$1 - \lambda + \lambda(1 + cx)^{a_1} q^{a_2} - (1 + cx + c \ln \lambda)^{a_1} q^{a_2} \ge 0$, for $\lambda \ge 1$, $0 < a_1 \le 1$, $a_2 < 0$, $c \ge (1/M_1)$, $x \ge 1$, and $q > 1$.*

Lemma 3. *$1 - \lambda + \lambda_2\lambda(1 + cx)^{a_1} q^{a_2} - \lambda_1(1 + cx + c \ln \lambda)^{a_1} q^{a_2} \le 0$, for $\lambda \ge 1$, $a_1 \ge M$, $a_2 < 0$, $\lambda_1 \ge \lambda_2 > 0$, $c \ge (1/M_1)$, $x \ge 1$, and $q > 1$.*

Similar to the notations of Theorem 1, we will give the following results. Under σ, the completion times of jobs J_i and J_j are

$$C_i(\sigma) = t_0 + p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2}, \quad (19)$$

$$\begin{aligned} C_j(\sigma) = t_0 + p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2} \\ + p_j\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_i\right)^{a_1} q_{r+1}^{a_2}. \end{aligned} \quad (20)$$

Under σ', the completion times of jobs J_j and J_i are

$$C_j(\sigma') = t_0 + p_j\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2}, \quad (21)$$

$$\begin{aligned} C_i(\sigma') = t_0 + p_j\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2} \\ + p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_j\right)^{a_1} q_{r+1}^{a_2}. \end{aligned} \quad (22)$$

Theorem 5. *For the problem $1 | p_{jr} = p_j(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]})^{a_1} q_r^{a_2} | C_{\max}$,*

(1) $a_1 \ge M$, SPT rule is optimal

(2) $0 < a_1 \le 1$, SPT rule is optimal

Proof. Note that $p_i \le p_j$. We will show that $C_j(\sigma) \le C_i(\sigma')$. Taking the difference between (20) and (22), it is obtained that

$$\begin{aligned} C_j(\sigma) - C_i(\sigma') = (p_i - p_j)\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2} \\ + p_j\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_i\right)^{a_1} q_{r+1}^{a_2} \\ - p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_j\right)^{a_1} q_{r+1}^{a_2}. \end{aligned} \quad (23)$$

(1) By substituting $\lambda = p_j/p_i$, $c = \beta_r/1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}$, $q = q_{r+1}/q_r$, and $x = \ln p_i$ into equation (23), it is simplified to

$$C_j(\sigma) - C_i(\sigma') = p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2} \left[1 - \lambda + \lambda(1 + cx)^{a_1} q^{a_2} - (1 + cx + c \ln \lambda)^{a_1} q^{a_2}\right]. \quad (24)$$

Since $q_r \le q_{r+1}$ and Lemma 1, then $C_j(\sigma) \le C_i(\sigma')$. This means that the completion times of the jobs processed before jobs J_j and J_i is not change by interchange. Furthermore, $C_j(\sigma) \le C_i(\sigma')$ implies that the staring times of the jobs processed after jobs cannot be decreased by interchanging σ and σ'.

(2) From case 2 and Lemma 2, the result can be easily obtained. □

Theorem 6. *For the problem $1 | p_{jr} = p_j(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]})^{a_1} q_r^{a_2} | \sum C_j$, if $a_1 \ge M$, then an optimal schedule can be obtained by the SPT rule.*

Proof. Suppose that $p_i \le p_j$. To show that σ dominates σ', it suffices to show that $\sum C_j(\sigma) \le \sum C_j(\sigma')$. Taking the difference between (19) and (21), it is obtained that $C_i(\sigma) - C_j(\sigma') = (p_i - p_j)(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]})^{a_1} q_r^{a_2} \le 0$. Stem from case 1 and case 2 of Theorem 6, we have $C_j(\sigma) \le C_i(\sigma')$ and $C_i(\sigma) \le C_j(\sigma')$. The completion times of the jobs processed before jobs J_j and J_i are not affected by interchange. Furthermore, $C_j(\sigma') \ge C_i(\sigma)$ implies that the staring times of the jobs processed after jobs J_j and J_i cannot decrease by interchange σ and σ'. Hence, $\sum C_j(\sigma) \le \sum C_j(\sigma')$.

Though we want to give an polynomial time algorithm for $0 < a \le 1$, we can present the following example to show that there does not exist an polynomial time algorithm: □

Example 1. $n = 3$, $p_1 = 5$, $p_2 = 4$, and $p_3 = 6$. The deterioration index $a_1 = 1$, the learning index $a_2 = -1$, $\beta_1 = 0.01$, $\beta_2 = 0.02$, and $\beta_3 = 0.1$. The SPT sequence is $[J_2, J_1, J_3]$, $\sum C_j(SPT) = 9.68$. The LPT sequence is $[J_3, J_1, J_2]$, $\sum C_j(\text{SPT}) = 10.08$. Obviously, the optimal sequence is $[J_2, J_3, J_1]$, $\sum C_j(\text{SPT}) = 9.053$.

From Example 1, we know that the SPT rule or LPT rule cannot give an optimal solution for the proposed problem if $0 < a \le 1$. It remains an open problem. Now, we will present that problem $1 | p_{jr} = p_j(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]})^{a_1} q_r^{a_2}, \ 0 < a \le 1 | \sum C_j$, has an important property, i.e., V-shaped normal job processing times.

Definition 1. A schedule is V-shaped normal job processing times if jobs, processed before some job with the smallest p_j, are arranged in descending order, but in ascending order if placed after it.

Theorem 7. *For the problem $1 | p_{jr} = p_j(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]})^{a_1} q_r^{a_2}, 0 < a_1 \le 1 | \sum C_j$, an optimal schedule exists, which is V-shaped normal job processing times.*

Proof. Consider a schedule Σ with three consecutive jobs, J_i, J_j, and J_k, i.e., $\Sigma = [\mathcal{T}_1, J_i, J_j, J_k, \mathcal{T}_2]$ such that $p_j > p_i$ and $p_j > p_k$. Let Σ_1 (Σ_2) be the schedule obtained from Σ by interchanging J_i and J_j (J_j and J_k), i.e., $\Sigma_1 = [\mathcal{T}, J_j, J_i, J_k, \mathcal{T}_2]$ ($\Sigma_2 = [\mathcal{T}_1, J_i, J_k, J_j, \mathcal{T}_2]$). Furthermore, let τ_0 denote the completion time of the last job in $\mathfrak{R}_1$, and there are $r-1$ jobs in $\mathcal{T}_1$. Then, the contribution of the three jobs to the total completion time is

$$\Delta(\Sigma) = 3\tau_0 + 3p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2} + 2p_j\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_i\right)^{a_1} q_{r+1}^{a_2} + p_k\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_i + \beta_{r+1} \ln p_j\right)^{a_1} q_{r+2}^{a_2}. \quad (25)$$

Similar expressions are easily obtained for Σ_1 and Σ_2:

$$\begin{aligned}\Delta(\Sigma_1) &= 3\tau_0 + 3p_j\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2} + 2p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_j\right)^{a_1} q_{r+1}^{a_2} \\ &\quad + p_k\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_j + \beta_{r+1} \ln p_i\right)^{a_1} q_{r+2}^{a_2}, \\ \Delta(\Sigma_2) &= 3\tau_0 + 3p_i\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2} + 2p_k\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_i\right)^{a_1} q_{r+1}^{a_2} \\ &\quad + p_j\left(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]} + \beta_r \ln p_i + \beta_{r+1} \ln p_k\right)^{a_1} q_{r+2}^{a_2}.\end{aligned} \quad (26)$$

It follows that

$$\begin{aligned}\Delta(\Sigma)-\Delta(\Sigma_1) = {} & 3\left(p_i-p_j\right)\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}\right)^{a_1}q_r^{a_2}\\ & +2p_j\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i\right)^{a_1}q_{r+1}^{a_2}\\ & -2p_i\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_j\right)^{a_1}q_{r+1}^{a_2}\\ & +p_kq_{r+2}^{a_2}\left[\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i+\beta_{r+1}\ln p_j\right)^{a_1}\right.\\ & \left.-\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_j+\beta_{r+1}\ln p_i\right)^{a_1}\right],\end{aligned} \tag{27}$$

$$\begin{aligned}\Delta(\Sigma)-\Delta(\Sigma_2) = {} & 2\left(p_j-p_k\right)\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i\right)^{a_1}q_{r+1}^{a_2}\\ & +p_k\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i+\beta_{r+1}\ln p_j\right)^{a_1}q_{r+2}^{a_2}\\ & -p_j\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i+\beta_{r+1}\ln p_k\right)^{a_1}q_{r+2}^{a_2}.\end{aligned} \tag{28}$$

Since $\beta_r\le\beta_{r+1}$ and $p_i<p_j$, then we have $(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i+\beta_{r+1}\ln p_j)^{a_1}>(1+\sum_l=1^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_j+\beta_{r+1}\ln p_i)^{a_1}$. Next, let $\lambda=p_j/p_i$, $c=\beta_r/1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}$, $x=\ln p_i$ and $q=q_{r+1}/q_r$. From equation (27), we have

$$\begin{aligned}\Delta(\Sigma)-\Delta(\Sigma_1) = {} & \left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}\right)^{a_1}q_r^{a_2}\left[3(1-\lambda)+2\lambda(1+cx)^{a_1}q^{a_2}\right.\\ & \left.-2(1+cx+c\ln\lambda)^{a_1}q^{a_2}\right]\\ & +p_kq_{r+2}^{a_2}\left[\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i+\beta_{r+1}\ln p_j\right)^{a_1}\right.\\ & \left.-\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_j+\beta_{r+1}\ln p_i\right)^{a_1}\right].\end{aligned} \tag{29}$$

Let $u=p_j/p_k$, $d=(\beta_{r+1})/(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i)$, $y=\ln p_k$, and $t=q_{r+2}/q_{r+1}$. From equation (27), we have

$$\begin{aligned}\Delta(\Sigma)-\Delta(\Sigma_2) = {} & p_k\left(1+\sum_{l=1}^{r-1}\beta_l\ln p_{[l]}+\beta_r\ln p_i\right)^{a_1}q_{r+1}^{a_2}\\ & \left[2(u-1)+(1+\mathrm{d}y+\mathrm{d}\ln u)^{a_1}t^{a_2}-u(1+\mathrm{d}y)^{a_1}t^{a_2}\right].\end{aligned} \tag{30}$$

Now, let $\Delta(\Sigma) - \Delta(\Sigma_1)$ be negative. Based on the above equations and $(1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]} + \beta_r \ln p_i + \beta_{r+1} \ln p_j)^{a_1} > (1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]} + \beta_r \ln p_j + \beta_{r+1} \ln p_i)^{a_1}$, we have

$$\begin{aligned}
&3(1-\lambda) + 2\lambda(1+cx)^{a_1}q^{a_2} - 2(1+cx+c\ln\lambda)^{a_1}q^{a_2} < 0\\
&\Rightarrow 2(1-\lambda) + 2\lambda(1+cx)^{a_1}q^{a_2} - (1+cx+c\ln\lambda)^{a_1}q^{a_2} + (1-\lambda) + 2\lambda(1+cx)^{a_1}q^{a_2}\\
&\quad - (1+cx+c\ln\lambda)^{a_1}q^{a_2} < 0\\
&\Rightarrow 2(1-\lambda) + 2\lambda(1+cx)^{a_1}q^{a_2} - (1+cx+c\ln\lambda)^{a_1}q^{a_2} < 0\,(\text{from Lemma 3})\\
&\Rightarrow 2(u-1) + (1+\mathrm{d}y+\mathrm{d}\ln u)^{a_1}t^{a_2} - u(1+\mathrm{d}y)^{a_1}t^{a_2} > 0.
\end{aligned} \tag{31}$$

Hence, we have $\Delta(\Sigma) - \Delta(\Sigma_2) > 0$.

Now, let $\Delta(\Sigma) - \Delta(\Sigma_2)$ be negative. Based on the above equations, we have

$$\begin{aligned}
&2(u-1) + (1+\mathrm{d}y+\mathrm{d}\ln u)^{a_1}t^{a_2} - u(1+\mathrm{d}y)^{a_1}t^{a_2} < 0\\
&\Rightarrow 2(u-1) + (1+\mathrm{d}y+\mathrm{d}\ln u)^{a_1}t^{a_2}\\
&\quad - u(1+\mathrm{d}y)^{a_1}t^{a_2} + (u-1) + (1+\mathrm{d}y+\mathrm{d}\ln u)^{a_1}t^{a_2} - u(1+\mathrm{d}y)^{a_1}t^{a_2} < 0\,(\text{from Lemma 3})\\
&\Rightarrow 3(u-1) + 2(1+\mathrm{d}y+\mathrm{d}\ln u)^{a_1}t^{a_2} - 2u(1+\mathrm{d}y)^{a_1}t^{a_2} < 0\\
&\Rightarrow 3(1-\lambda) + 2\lambda(1+cx)^{a_1}q^{a_2} - 2(1+cx+c\ln\lambda)^{a_1}q^{a_2} > 0.
\end{aligned} \tag{32}$$

Since $(1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]} + \beta_r \ln p_i + \beta_{r+1} \ln p_j)^{a_1} > (1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]} + \beta_r \ln p_j + \beta_{r+1} \ln p_i)^{a_1}$, we have $\Delta(\Sigma) - \Delta(\Sigma_1) > 0$.

We conclude that an optimal schedule exists, which is V-shaped normal job processing times.

For $a_1 \geq M$, we will present polynomial-time solutions under some agreeable condition to minimize the total weighted completion times and the maximum lateness, respectively. □

Theorem 8. *Problem* $1|p_{jr} = p_j(1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]})^{a_1} q_r^{a_2} | \sum w_j C_j$ *can be obtained as an optimal solution by the nondecreasing order of* p_j/w_j *if the processing times and the weights are agreeable, i.e.,* $p_i \leq p_j \Rightarrow w_i \geq w_j$, *for all the jobs* J_i *and* J_j.

Proof. Suppose that $(p_j/p_i) \geq (w_j/w_i) \geq 1$. Since $p_i \leq p_j$. Thus, we will show that σ dominates σ'. From (19)–(22), we have

$$\begin{aligned}
\sum w_j C_j(\sigma) - \sum w_j C_j(\sigma') &= (w_i + w_j)(p_i - p_j)\left(1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2}\\
&\quad + w_j p_j \left(1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]} + \beta_r \ln p_i\right)^{a_1} q_{r+1}^{a_2}\\
&\quad - w_i p_i \left(1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]} + \beta_r \ln p_j\right)^{a_1} q_{r+1}^{a_2}.
\end{aligned} \tag{33}$$

By substituting $\lambda_1 = w_i/(w_i + w_j)$, $\lambda_2 = w_j/(w_i + w_j)$, $\lambda = p_j/p_i$, $c = \beta_r/1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]}$, $q = q_{r+1}/q_r$, and $x = \ln p_i$ into equation (33), it is simplified to

$$\begin{aligned}
\sum w_j C_j(\sigma) - \sum w_j C_j(\sigma') &= (w_i + w_j) p_i \left(1 + \sum_{l=1}^{r-1}\beta_l \ln p_{[l]}\right)^{a_1} q_r^{a_2}\\
&\quad [1 - \lambda = +\lambda_2\lambda(1+cx)^{a_1}q^{a_1} - \lambda_1(1+cx+c\ln\lambda)^{a_1}q^{a_1}].
\end{aligned} \tag{34}$$

Since $q_r \leq q_{r+1}$ and Lemma 3, we have $\sum w_j C_j(\sigma) - \sum w_j C_j(\sigma') \leq 0$. □

Theorem 9. *Problem* $1|p_{jr} = p_j(1 + \sum_{l=1}^{r-1} \beta_l \ln p_{[l]})^{a_1} q_r^{a_2}$, $a_1 \geq M|L_{\max}$ *can be solved optimally by nondecreasing order of* d_j *if the job processing times and the due dates are agreeable.*

Proof. By definition and equations (19)–(22), the lateness of jobs J_i and J_j in σ and jobs J_j and J_i in σ' is, respectively,

$$\begin{aligned} L_i(\sigma) &= C_i(\sigma) - d_i, \\ L_j(\sigma) &= C_j(\sigma) - d_j, \\ L_j(\sigma') &= C_j(\sigma) - d_j, \\ L_i(\sigma') &= C_i(\sigma') - d_i. \end{aligned} \tag{35}$$

Suppose that $d_i \leq d_j$, which implies $p_i \leq p_j$. Interchanging jobs J_i and J_j has no impact on the maximum lateness of the jobs in subsequence $\mathcal{S}_1$, and the maximum lateness of the jobs in subsequence $\mathcal{S}_2$ of σ cannot be larger than that of the jobs in S_2 of σ'.

Since $p_i \leq p_j$, from Theorems 5 and 6,

$$\begin{aligned} L_i(\sigma') - L_i(\sigma) &= C_i(\sigma') - C_i(\sigma) > 0, \\ L_i(\sigma') - L_j(\sigma) &= C_i(\sigma') - d_i - C_j(\sigma) + d_j > 0. \end{aligned} \tag{36}$$

Thus, repeating this job interchange argument for all the jobs not sequenced in the EDD rule completes the proof of the last theorem. □

5. Conclusion

The main contribution of this paper is that the machine scheduling problems with general sum-of-processing-time-based and position-dependent effect function are provided. Two monotonic effect functions, nondecreasing function and nonincreasing function, are considered. The objective functions are to minimize the makespan, the total completion time, the total weighted completion time, and the maximum lateness.

The nonincreasing effect function:

(1) The makespan problem and the sum of the total completion time problem can be solved in polynomial time, respectively
(2) The sum of the weighted completion time problem can also be solved in polynomial time if the weight and the normal processing time are under agreeable relations
(3) Maximum lateness problem can also be solved in polynomial time if the due date and the normal processing time are under agreeable relations

The nondecreasing effect function

(1) $a_1 \in \{(0,1], [M, +\infty)\}$, and the makespan problem can be optimally solved
(2) $a_1 \in [M, +\infty)$, and the sum of the completion time problem can be optimally solved
(3) $0 < a_1 \leq 1$, t, and the optimal sequence has a V-shaped property with respect to the normal processing times
(4) $a_1 > M$, and the total weighted completion time and the maximum lateness have polynomial-time solutions under some agreeable conditions, respectively

It is suggested that, for future research to investigate this open problems, the sum-of-processing-time-based deteriorating jobs and learning effect should be considered in the context of other scheduling problems or more sophisticated and efficient heuristic algorithms should be proposed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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