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*Research Article*

# Distance Two Surjective Labelling of Paths and Interval Graphs

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Graph labelling problem has been broadly studied for a long period for its applications, especially in frequency assignment in (mobile) communication system, $X$-ray crystallography, circuit design, etc. Nowadays, surjective $L(2,1)$-labelling is a well-studied problem. Motivated from the $L(2,1)$-labelling problem and the importance of surjective $L(2,1)$-labelling problem, we consider surjective $L(2,1)$-labelling (SL21-labelling) problems for paths and interval graphs. For any graph $G = (V, E)$, an SL21-labelling is a mapping $\varphi: V \longrightarrow \{1, 2, \ldots, n\}$ so that, for every pair of nodes $u$ and $v$, if $d(u,v) = 1$, then $|\varphi(u) - \varphi(v)| \geq 2$; and if $d(u,v) = 2$, then $|\varphi(u) - \varphi(v)| \geq 1$, and every label $1, 2, \ldots, n$ is used exactly once, where $d(u,v)$ represents the distance between the nodes $u$ and $v$, and $n$ is the number of nodes of graph $G$. In the present article, it is proved that any path $P_n$ can be surjectively $L(2,1)$-labelled if $n \geq 4$, and it is also proved that any interval graph (IG) $G$ having $n$ nodes and degree $\Delta > 2$ can be surjectively $L(2,1)$-labelled if $n = 3\Delta - 1$. Also, we have designed two efficient algorithms for surjective $L(2,1)$-labelling of paths and interval graphs. The results regarding both paths and interval graphs are the first result for surjective $L(2,1)$-labelling.

## 1. Introduction

The frequency assignment problem is bottomed from the problems of distance labelling of graph. In 1992, $L(2,1)$-labelling was invented by Griggs and Yeh [1] in conjunction with channel assigning problem in a multihop radio network.

For any graph $G = (V, E)$, an $L(2,1)$-labelling is a mapping $\varphi: V(G) \longrightarrow \{1, 2, \ldots,\}$, so that $|\varphi(u) - \varphi(v)| \geq 2$ if $d(u,v) = 1$ and $|\varphi(u) - \varphi(v)| \geq 1$ if $d(u,v) = 2$. The span of $L(2,1)$-labelling of $G$ is $\max\{\varphi(v)\colon v \in V\}$. The $L(2,1)$-labelling number $\lambda_{2,1}(G)$ of $G$ is the smallest natural number $p$ so that $G$ has an $L(2,1)$-labelling of span $p$.

A surjective $L(2,1)$-labelling of $G = (V, E)$ is a mapping $\varphi: V \longrightarrow \{1, 2, \ldots, n\}$ so that $|\varphi(u) - \varphi(v)| \geq 2$ when $d(u,v) = 1$ and $|\varphi(u) - \varphi(v)| \geq 1$ when $d(u,v) = 2$, and it requires that each label, $\{1, 2, \ldots, n\}$, be used only once, where $n$ is the number of nodes of $G$.

In Figure 1, we have shown an $L(2,1)$-labelling of a path with 5 nodes and Figure 2 shows SL21-labelling of the same graph. In Figure 1, identical label is used several times but in Figure 2 the labels 1 to 5 are used only once. So, in SL21-labelling, there is a more complex task compared to $L(2,1)$-labelling.

In 1994, Sakai has proved some results regarding distance two labelling of chordal graph. Later, in 2007, Bertossi and Bonuccelli have studied approximate $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring of trees and interval graphs. Amanathulla and Pal have studied a lot of problems regarding labelling of graphs, like $L(3,2,1)$-labelling problems on permutation graphs [2], $L(h_1, h_2, \ldots, h_m)$-labelling problems on interval graphs [3], $L(h_1, h_2, \ldots, h_m)$-labelling problems on circular-arc graphs [4], $L(1,1,1)$- and $L(1,1,1,1)$-labelling problems of square of paths [5], and $L(3,1,1)$-labelling numbers of squares of paths, complete graphs, and complete bipartite graphs [6]. In 2019, Berhe

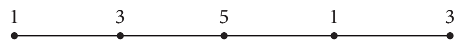


Figure 1: $L(2,1)$-labelling of a path with five nodes.

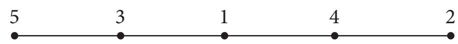


Figure 2: Surjective $L(2,1)$-labelling of a path with five nodes.

and Wang have studied computation of certain topological coindices of graphene sheet and C4C8(S) nanotubes and nanotorus [7]; also, Goyal et al. [8] have studied new composition of graphs and their Wiener indices. Hosamani et al. [9] have studied graphs with equal dominating and c-dominating energy. Very recently, B. M. Gurevich has published one paper regarding classes of infinite loaded graphs with randomly deleted edges [10] and Ranjini et al. have studied degree sequence of graph operator for some standard graphs [11]. For general graph, $\lambda_{2,1}(G) \geq \Delta + 1$. In 1992, Griggs et al. showed that $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$ and have proposed a conjecture [1].

In 1993, Jonas [12] has shown that $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta - 4$. Chang et al. [13] have showed $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$. Král' and Skrekovski [14] proved that $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 1$ and they further improved it to $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$ [15].

Different bounds for $\lambda_{h_1,h_2,\ldots,h_n}(G)$ were obtained for different classes of graphs. Some results regarding upper bound of $L(h_1, h_2, \ldots, h_n)$-labelling are shown in Table 1.

In [37], Lingscheit et al. investigated minimal and surjective labelling for path, cycle, complete graph, caterpillar, and complete bipartite graph. They have shown that $P_n$ can be surjectively $L(3,2,1)$-labelled when $n \geq 7$. Very recently, Amanathulla and Pal have studied SL21-labelling of cycle and circular-arc graph (CAG) and obtained good results for it [38].

$L(2,1)$-labelling of graphs is a rapidly studied problem for its applications in various fields, especially in channel assignment in radio network. In $L(2,1)$-labelling, although there is a light chance to overlap the frequencies in radio network, it cannot be neglected, but in SL21-labelling there is no chance to overlap the frequencies, as in this case the labels are distinct. For this reason, in the recent year, SL21-labelling of graph has become a well-studied problem due to its applications. This motivates us to consider SL21-labelling of paths and IGs. Recently, many researchers applied various related concepts on graphs in different aspects (see, e.g., [39–43]).

In the present article, it is shown that any path $P_n$ is surjectively labelled by $L(2,1)$-labelling if $n \geq 4$ and it also shown that any IG having $n$ nodes can be surjectively $L(2,1)$-labelled if $n = 3\Delta - 1$. Two polynomial time algorithms are also established to label a path and an IG by SL21-labelling.

The remainder of this article is organized as follows: in Section 2, some notations and preliminary definitions are given. In Section 3, SL21-labelling of path has been presented. In Section 4, SL21-labelling of IG is investigated. The last section presents concluding remarks.

Table 1: Different types of graphs and their upper bounds.

| Graphs | $L(h,k)$-labelling numbers |
|---|---|
| General graphs | $0 \leq \lambda_{0,1} \leq \Delta^2 - \Delta$ [16]<br>$\Delta \leq \lambda_{1,1} \leq \Delta^2$ [17]<br>$\Delta + 1 \leq \lambda_{2,1} \leq \Delta^2 + \Delta - 2$ [1, 15]<br>$\lambda_{3,2,1}(G) \leq \Delta^3 + 2\Delta$ [18]<br>$\lambda_{4,3,2,1}(G) \leq \Delta^3 + 2\Delta^2 + 6\Delta$ [19] |
| Paths | $\lambda_{0,1}(P_n) = 0$ or 1 [20]<br>$\lambda_{1,1}(P_n) = 1$ or 2 [21]<br>$\lambda_{2,1}(P_n) = 2$, 3 or 4 [1]<br>$\lambda_{3,2,1}(P_n) = 0$, 3, 5, 6 or 7 [22]<br>For $d \geq 4$, $\lambda_{d,2,1}(P_n) = 0$, $d$, $d+2$ or $d+4$ [22]<br>For $n \geq 2$ $\lambda_{4,3,2,1}(P_n) = 5$, 8, 9, 11 or 12 [23] |
| Cycles | For $n \geq 3$, $\lambda_{0,1}(C_n) = 1$ or 2 [16]<br>For $n \geq 3$, $\lambda_{1,1}(C_n) = 2$ or 3 [21]<br>For $n \geq 3$, $\lambda_{2,1}(C_n) = 4$ [1]<br>For, $n \geq 3$, $\lambda_{3,2,1}(C_n) = 6$, 7, 8 or 9 [22]<br>For, $n \geq 4$, $d \geq 5$, $\lambda_{d,2,1}(C_n) = d+4$, $d+6$, $2d+1$ or $2d$ [22]<br>For, $n \geq 3$, $\lambda_{4,3,2,1}(C_n) = 9$, 11, 14 or 13 [23] |
| Complete | $\lambda_{1,1}(k_n) = n - 1$ [24]<br>$\lambda_{d,2,1}(K_n) = d(n-1) + 1$ [22] |
| Complete bipartite | $\lambda_{1,1}(K_{m,n}) = m + n - 1$ [24]<br>$\lambda_{d,2,1}(K_{m,n}) = d + 2(m+n) - 3$ [22] |
| Planar | $\lambda_{1,1}(G) \leq \lceil (5/3)\Delta + 1 \rceil + 77$ [25]<br>$\lambda_{2,1}(G) \leq 2\Delta + 35$ [26]<br>$\lambda_{2,1}(G) \leq (5/3)\Delta + 95$ [25]<br>$\lambda_{h,k}(G) \leq k\lceil (5/3)\Delta \rceil + 18h + 77k - 18$ [25]<br>$\lambda_{3,2,1}(G) \leq 15(\Delta^2 - \Delta + 1)$ [27] |
| Interval | $\lambda_{2,1}(G) \leq \Delta + w$ [28]<br>$\lambda_{h,k}(G) \leq \max \cdot \{h, 2k\}\Delta$ [29]<br>$\lambda_{3,2,1}(G) \leq 6\Delta - 3$ [30]<br>$\lambda_{4,3,2,1}(G) \leq 10\Delta - 6$ [30] |
| Circular-arc | $\lambda_{0,1}(G) \leq \Delta$ [31]<br>$\lambda_{1,1}(G) \leq 2\Delta$ [31]<br>$\lambda_{h,k}(G) \leq \max \cdot \{h, 2k\}\Delta + h\omega$ [29]<br>$\lambda_{2,1}(G) \leq \Delta + 3w$ [32]<br>$\lambda_{3,2,1}(G) \leq 9\Delta - 6$ [33]<br>$\lambda_{4,3,2,1}(G) \leq 16\Delta - 12$ [33] |
| Permutation | $\lambda_{0,1}(G) \leq 2\Delta - 2$ [34]<br>$\lambda_{0,1}(G) \leq \Delta - 1$ [35]<br>$\lambda_{2,1}(G) \leq \max\{4\Delta - 2, 5\Delta - 8\}$ [36]<br>$\lambda_{2,1}(G) \leq 5\Delta - 2$ [34] |

## 2. Preliminaries and Notations

A path is a graph $G = (V, E)$, where $(v_j, v_{j+1}) \in E$, for all $1 \leq j \leq n - 1$, where $V = \{v_1, v_2, \ldots, v_n\}$, and it is denoted by $P_n$. Here, we consider IG (IG) which is not a path, so $\Delta > 2$, because if $\Delta = 2$, then it may be a path.

Let the set of intervals in real line be $I = \{I_1, I_2, \ldots, I_n\}$, where $I_k = [l_k, r_k]$, $k = 1, 2, \ldots, n$, $l_k$, and $r_k$ are the left and right endpoints of $I_k$. For any interval $I_k$, $k = 1, 2, \ldots, n$, we draw a node $v_k$, and two nodes $v_p$ and $v_q$ have joined by a line segment that if the corresponding intervals have common portion, then we obtain an IG [44]. Throughout the paper, an interval $I_k$ and a node $v_k$ are the same. An IG and its interval representation are shown in Figure 3.

*Notations.* For any IG $G$ with $n$ nodes and corresponding set of intervals $I = \{I_1, I_2, \ldots, I_n\}$, we define the following:

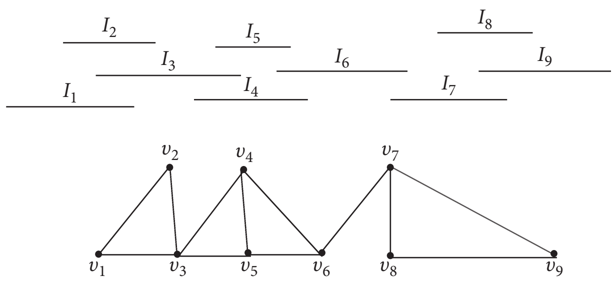


Figure 3: An interval representation and its corresponding IG.

(1) $L^s(I_k)$: the set of used SL21-labels which are used before labelling the interval $I_k$, for every interval $I_k \in I$.
(2) $L^i(I_k)$: the set of used SL21-labels at distance $i$ ($i = 1, 2$) from the interval $I_k$, before labelling $I_k$, for every $I_k \in I$.
(3) $L^{vl}(1, I_k)$: the set of valid labels for the interval $I_k$ before labelling $I_k$, satisfying the adjoining condition of $L(2,1)$-labelling, for every interval $I_k \in I$.
(4) $L^{vl}(2, I_k)$: the valid set of labels of the interval $I_k$ before labelling $I_k$, satisfying $L(2,1)$-labelling condition, for every interval $I_k \in I$.
(5) $L^{svl}(I_k)$: the set of valid SL21-labels of the interval $I_k$ before labelling $I_k$, for every interval $I_k \in I$.
(6) $\lambda_{2,1}(G)$: $L(2,1)$-labelling number of $G$.
(7) $f_j^s$: the SL21-label of the interval $I_j \in I$.
(8) $L^s$: the set of labels after completion of SL21-labelling of $G$.

## 3. Surjective $L(2,1)$-Labelling of Paths

In this portion, we have presented SL21-labelling of path and have showed that any path $P_n$ is surjectively $L(2,1)$-labelled if $n \geq 4$. Also, we have presented a greedy algorithm to label a path by SL21-labelling.

**Theorem 1.** *For $P_n$,*

$$\lambda_{2,1}(P_n) = \begin{cases} 1, & \text{if } n = 1, \\ 3, & \text{if } n = 2, \\ 4, & \text{if } n = 3. \end{cases} \qquad (1)$$

*Proof.* Let $P_n$ be a path having $n$ nodes.

Case 1: $n = 1$.

This result holds trivially.

Case 2: $n = 2$.

The labels used are 1 and 3 and hence $\lambda_{2,1}(P_2) = 3$.

Case 3: $n = 3$.

There are two possible cases shown in Figures 4(a) and 4(b). The labelling sequences are $\{3,1,4\}$ and $\{1,4,2\}$.

From the above result, it is concluded that $P_n$ can be SL21-labelled for $n = 1, 2, 3$. □

**Theorem 2.** *The minimum path that can be labelled by SL21-labelling is $P_4$.*

*Proof.* From Theorem 1, we have $\lambda_{2,1}(P_2) = 3$ and $\lambda_{2,1}(P_3) = 4$, so, for $n < 4$, a path $P_n$ cannot be labelled by SL21-labelling. The labelling pattern $\{3,1,4,2\}$ of path $P_4$ (see Figure 5) shows that $P_4$ can be labelled by SL21-labelling. Hence, $P_4$ is the minimum path that can be labelled by SL21-labelling (Figure 6). □

For this path, the node $V = \{v_1, v_2, \ldots, v_{22}\}$. Here, $n > 4$, so this path can be surjectively labelled by $L(2,1)$-labelling. $f_k^s$ is the SL21-label of the node $v_k$, for $k = 1, 2, \ldots, 22$.

According to Algorithm 1, we rearrange the nodes as follows:
$v_2 = v_3$, $v_3 = v_5$, $v_4 = v_7$, $v_5 = v_9$, $v_6 = v_{11}$, $v_7 = v_{13}$, $v_8 = v_{15}$, $v_9 = v_{17}$, $v_{10} = v_{19}$, $v_{11} = v_{21}$, $v_{12} = v_4$, $v_{13} = v_6$, $v_{14} = v_8$, $v_{15} = v_{10}$, $v_{16} = v_{12}$, $v_{17} = v_{14}$, $v_{18} = v_{16}$, $v_{19} = v_{18}$, $v_{20} = v_{20}$, $v_{21} = v_{22}$, $v_{22} = v_2$, and $v_1$ remains unchanged.

Now, node $v_k$ is labelled by $k$; that is, $f_k^s = k$, for each $k = 1, 2, \ldots, 22$. After completion of surjective $L(2,1)$-labelling of $P_{22}$, the node and the label of the corresponding node are shown in Figure 7(b).

## 4. Surjective $L(2, 1)$-Labelling of IGs

Here, some lemmas that we have used to develop the proposed algorithm are presented.

**Lemma 1.** *For any IG $G$, $|L^2(I_k)| \leq \Delta - 1$, for $I_k \in I$.*

*Proof.* Let $G$ be an IG having $n$ nodes. The labelling of $G$ starts from the leftmost interval. Let node $v_k$ be corresponding to the interval $I_k$ of the IG $G$. Suppose that in a stage the intervals $I_1, I_2, \ldots, I_{k-1}$ (for some $k = 2, 3, 4, \ldots, n$) are previously labelled by SL21-labelling and the remaining intervals are unlabelled.

Let $|L^2(I_k)| = p$. This means that the number of distinct SL21-labels used for labelling distance two intervals from the interval $I_k$ before labelling $I_k$ is $p$. Since the degree of the IG $G$ is $\Delta$, there exists an interval $I_\alpha$ (see Figure 8) and those are adjoining to $\Delta$ intervals at most. In Figure 8, $I_\alpha$ is adjoining to $I_k, I_\beta, I_{k_{21}}, I_{k_{22}}$. Among the intervals, some intervals ($I_\beta, I_{k_{21}}, I_{k_{22}}$ in Figure 8) are of distance two apart from $I_k$ and among the intervals there is an interval ($I_k$ in Figure 8) whose distance is not two from $I_k$. Hence, $p \leq \Delta - 1$; that is, $|L^2(I_k)| \leq \Delta - 1$. □

*Observation 1.* For any IG $G$, $L^i(I_k) \subseteq L^s(I_k)$, for any interval $I_k$, $i = 1, 2$.

*Observation 2.* For any IG $G$, $|L^1(I_k)| \leq \Delta$, for any interval $I_k \in I$.

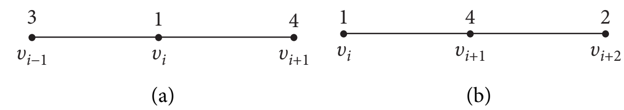


Figure 4: A path with three nodes and their surjective labels.


Figure 5: A path $P_4$ labelled by SL21-labelling.

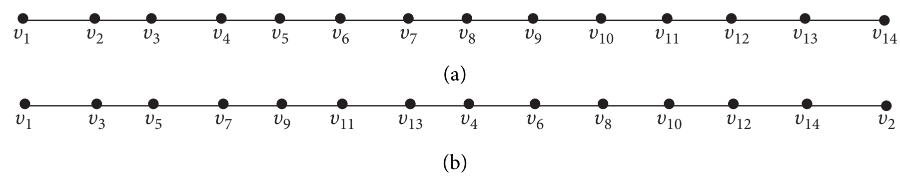


Figure 6: (a) A path with of 14 nodes; (b) the path after rearrangement of the nodes.

**Input:** The nodes of the path $P_n$ $(n > 6)$, $V = \{v_1, v_2, \ldots, v_n\}$.
**Output:** The SL21-label of the path $P_n$.
**Step 1:** Rearrange the intervals as follows:
**Case I:** $n$ is odd
$v_n = v_2$;
$v_{i+1} = v_{2i+1}$, for $i = 1, 2, \ldots, ((n-1)/2)$;
$v_{i+((n-1)/2)} = v_{2i}$, for $i = 2, 3, \ldots, ((n-1)/2)$;
$v_1$ remains same;
**Case II:** $n$ is even
$v_n = v_2$;
$v_{i+1} = v_{2i+1}$, for $i = 1, 2, \ldots, (n/2) - 1$;
$v_{i+(n/2)-1} = v_{2i}$, for $i = 2, 3, \ldots, (n/2)$;
$v_1$ remain same;
**Step 2:** Label the node $v_i$ by $i$, i.e., $f_i^s = i$, for $i = 1, 2, \ldots, n$
**end** AMPSL21.

Algorithm 1: AMPSPL21.

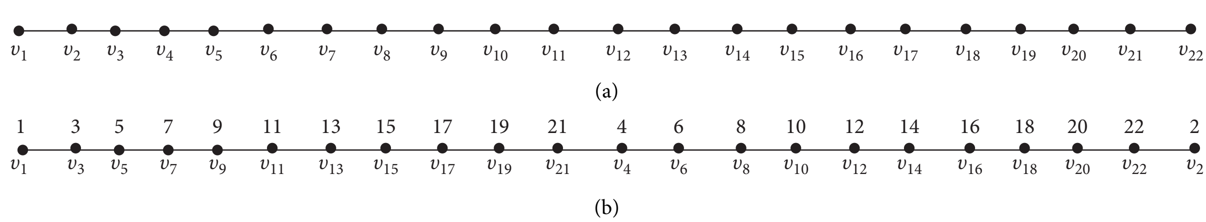


Figure 7: A path labeled by SL21-labelling.

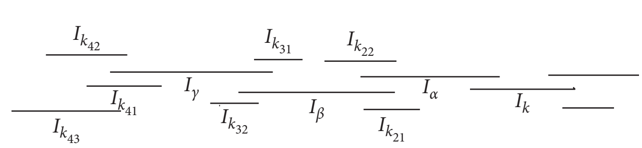


Figure 8: A set of intervals.

**Theorem 4.** *Any IG $G$ with $n$ nodes is surjectively $L(2,1)$-labelled if $n = 3\Delta - 1$.*

*Proof.* Since $G$ has $n$ nodes, let $I = \{I_1, I_2, \ldots, I_n\}$. Since we want to label the intervals of an IG by SL21-labelling, every label is used exactly once and the labels must be in $\{1, 2, \ldots, n\}$. So,

$$\begin{aligned}\lambda_{2,1}(G) &\leq 2\left|L^1(I_k)\right| + \left|L^2(I_k)\right| \\ &\leq 2\Delta + (\Delta - 1), \text{ [by Lemma1]} \\ &\leq 3\Delta - 1.\end{aligned} \tag{2}$$

Again, since $G$ has $n$ nodes, to label the whole graph by SL21-labelling, $n$ distinct labels must be required. Also, since $\lambda_{2,1}(G) \leq 3\Delta - 1$, in the extreme unfavorable cases $3\Delta - 1$ labels are required to label graph $G$ by $L(2,1)$-labelling. Again, in SL21-labelling, the highest label is equal to $n$. Hence, an IG $G$ is surjectively labelled using $L(2,1)$-labelling if $n = 3\Delta - 1$.

If $n \neq 3\Delta - 1$, then the IG may or may not be labelled by SL21-labelling, because in the worst case $3\Delta - 1$ labels are required to label the IG, which is not equal to $n$. This contradicts the condition that the used label must belong to $\{1, 2, \ldots, n\}$ and the highest label must be equal to $n$ for SL21-labelling. □

*4.1. Algorithm for Surjective L(2, 1)-Labelling of IGs.* In this part, two algorithms are designed: one is to compute $L^{\mathrm{vl}}(k, I_j)$ and the other is to compute SL21-label for an IG (Algorithm 2).

**Lemma 2.** *$L^{vl}(p, I_k)$ for $p = 1, 2$ is correctly computed by Algorithm 2 and the time complexity of the above algorithm is $O(\Delta^2)$.*

*Proof.* According to Algorithm 2, each element $i \in L^{\mathrm{vl}}(1, I_j)$ differs from $l_r$ by at least 2 for each $l_r \in L^1(I_k)$. Therefore, $|i - l_r| \geq 2$ for all $i \in L^{\mathrm{vl}}(1, I_k)$ and for all $l_r \in L^1(I_k)$. So, Algorithm 2 correctly computes the set $L^{\mathrm{vl}}(1, I_k)$ for each $I_k \in I$, $k = 2, 3, \ldots, n$. Again, each element $l_\alpha$ of $L^{\mathrm{vl}}(2, I_k)$ differs from $l_\beta$ by at least 1 for each $l_\beta \in L^2(I_k)$. Therefore, $|l_m - p_n| \geq 2$ for all $l_m \in L^{\mathrm{vl}}(2, I_k)$ and for all $p_n \in L^1(I_k)$, and $|l_m - p_n| \geq 1$ for all $l_m \in L^{\mathrm{vl}}(2, I_k)$ and for all $p_n \in L^2(I_k)$. Therefore, Algorithm 2 correctly computes $L^{\mathrm{vl}}(p, I_k)$ for every $k = 1, 2$. As $|L^s|$ is the cardinality of the set of labels $L^s$, $|L^i(I_k)| \leq |L^s|$ for $i = 1, 2$ and $I_k \in I$, and also $r \leq 3\Delta + 1$, where $r = \max\{L^s(I_k)\} + 2$. So, $L^{\mathrm{vl}}(1, I_k)$ is computed by using at most $(3\Delta + 1)|L^s|$ times, that is, using $O(\Delta|L^s|)$ times. Again, $|L^{\mathrm{vl}}(2, I_n)| \leq (3\Delta + 1)$, so, $L^{\mathrm{vl}}(2, I_k)$ is computed using at most $(3\Delta + 1)|L^s|$ times, that is, using $O(\Delta|L^s|)$ times. Since $|L^s| \leq 3\Delta + 1$, the iterative time for algorithm SLKVL is $O(\Delta^2)$. □

**Lemma 3.** *For each IG $G$, $L^{vl}(1, I_k)$ is the nonempty largest set satisfying distance one condition of $L(2,1)$-labelling, $l \leq r$, for every $l \in L^{vl}(1, I_k)$, and $r = \max\{L^s(I_k)\} + 2$, for any $I_k \in I$.*

*Proof.* Since $r = \max\{L^s(I_k)\} + 2$ and $L^1(I_k) \subseteq L^s(I_k)$ (by Observation 1), $|r - l_i| \geq 2$ for every $l_i \in L^1(I_k)$. Therefore, $r \in L^{\mathrm{vl}}(1, I_k)$, so, $L^{\mathrm{vl}}(1, I_k)$ is a nonempty set. Also, let $B$ be an arbitrary set of labels, which satisfies distance one condition of $L(2,1)$-labelling, $l \leq r$, for all $l \in B$, and $r = \max\{L^s(I_k)\} + 2$. Then, for $b \in B$, $|b - l_i| \geq 2$ for any $l_i \in L^1(I_k)$. Thus, $b \in L^{\mathrm{vl}}(1, I_k)$. So, $b \in B \Rightarrow b \in L^{\mathrm{vl}}(1, I_k)$. Then, $B \subseteq L^{\mathrm{vl}}(1, I_k)$. Since $B$ is arbitrary, $L^{\mathrm{vl}}(1, I_k)$ is the largest nonempty set of labels which satisfies distance one condition of $L(2,1)$-labelling, $l \leq r$, for every $l \in L^{\mathrm{vl}}(1, I_k)$, and $r = \max\{L^s(I_k)\} + 2$, for any $I_k \in I$. □

**Lemma 4.** *For any IG $G$, $L^{vl}(2, I_k)$ is the nonempty largest set satisfying $L(2,1)$-labelling condition, $l \leq r$, for every $l \in L^{vl}(1, I_k)$, $r = \max\{L^s(I_k)\} + 2$, and $I_k \in I$.*

*Proof.* Since $r = \max\{L^s(I_k)\} + 2$ and $L^i(I_k) \subseteq L^s(I_k)$, for $i = 1, 2$ (by Observation 1), $|r - l_p| \geq 2$ for $l_p \in L^i(I_k)$, $i = 1, 2$; that is, $|r - l_p| \geq 2$ for all $l_p \in L^1(I_k)$ and $|r - l_p| \geq 1$ for all $l_p \in L^2(I_k)$. Hence, $r$ is the valid $L(2,1)$-label of $I_k$; therefore, $r \in L^{\mathrm{vl}}(2, I_k)$. This shows that $L^{\mathrm{vl}}(2, I_k)$ is a nonempty set. Also, let $B$ be an arbitrary set of labels which satisfies $L(2,1)$-labelling conditions, $l \leq r$ for every $l \in B$, and $r = \max\{L^s(I_k)\} + 2$. Then, for $b \in B$, $|b - l_p| \geq 2$ for $l_p \in L^1(I_k)$ and $|b - l_q| \geq 1$ for any $l_q \in L^2(I_k)$ . Thus, $b \in L^{\mathrm{vl}}(2, I_k)$. Thus, $b \in B \Rightarrow b \in L^{\mathrm{vl}}(2, I_K)$. So, $B \subseteq L^{\mathrm{vl}}(2, I_k)$. Since $B$ is arbitrary, $L^{\mathrm{vl}}(2, I_k)$ is the largest nonempty set of labels which satisfies $L(2,1)$-labelling, $l \leq r$ for every $l \in L^{\mathrm{vl}}(2, I_k)$, and $r = \max\{L^s(I_k)\} + 2$, for any $I_k \in I$ (Algorithm 3). □

**Theorem 5.** *Algorithm 3 correctly labels an IG by SL21-labelling, where $n = 3\Delta - 1$.*

*Proof.* Let $G$ be an IG with $n$ nodes such that $n = 3\Delta - 1$. We rearranged the nodes so that no two consecutive intervals are adjacent to each other. After rearrangement of the intervals, let $I = \{I_1, I_2, \ldots, I_n\}$ and let $f_1^s = 1$ and $L^s(I_2) = \{1\}$.

We consider circumstances in which the intervals $I_1, I_2, \ldots, I_{k-1}$ are already labelled for $k = 2, 3, \ldots, n$ and the remaining intervals are not labelled. In this stage, our aim is to label $I_k$ by SL21-labelling. Now, $L^{\mathrm{vl}}(2, I_k)$ is the nonempty largest set of labels satisfying $L(2,1)$-labelling, $l \leq r$ for any $l \in L^{\mathrm{vl}}(2, I_k)$, and $r = \max\{L^s(I_k)\} + 2$ for every $I_k \in I$ (by Lemma 4).

Again, $L^{\mathrm{svl}}(I_k) = L^{\mathrm{vl}}(2, I_k) - L^s(I_k)$, so $L^{\mathrm{svl}}(I_k)$ is the nonempty largest set satisfying SL21-labelling, as the label in $L^{\mathrm{svl}}(I_k)$ was not used previously to label any interval and also satisfies $L(2,1)$-labelling. Therefore, $f_k^s = q$, where $q = \min\{L^{\mathrm{svl}}(2, I_k)\}$. Since $L^{\mathrm{svl}}(I_k)$ is the largest set of labels satisfying SL21-labelling, $q$ is the least surjective label of $I_k$. Since $\lambda_{2,1}(G) \leq 3\Delta - 1$, the label of $I_k$ must be less than or equal to $3\Delta - 1$. Again, since $n = 3\Delta - 1$, the interval $I_k$ is labelled by using only the labels from $\{1, 2, \ldots, n\}$ which have not been used earlier to label any interval. Since $I_k$ is arbitrary, any IG is surjectively labelled by $L(2,1)$-labelling by Algorithm 3. □

**Input:** $I_k$, $L^1(I_k), L^2(I_k)$ and $L^s(I_k)$, $k = 2, 3, \ldots, n$.
**Output:** $L^{\mathrm{vl}}(p, I_k)$ for $p = 1, 2$; $k = 2, 3, \ldots, n$.
**Step 1:** for $i = 1$ to $r$, where $r = \max\{L^s(I_k)\} + 2$
  for $j = 1$ to $|L^1(I_k)|$
  let $l_j$ be the $j$ th element of $L^1(I_k)$
  if $|i - l_j| \geq 2$, then add $i$ to the set $L^{\mathrm{vl}}(1, I_k)$;
  end for;
end for;
**Step 2:** for $m = 1$ to $|L^{\mathrm{vl}}(1, I_k)|$
  for $n = 1$ to $|L^2(I_k)|$
    $l_m$ and $p_n$ be the elements of $|L^{\mathrm{vl}}(1, I_k)|$ and $|L^2(I_k)|$ respectively;
    if $|l_m - p_n| \geq 1$, then add $l_m$ to the set $L^{\mathrm{vl}}(2, I_k)$;
  end for;
end for;
**end** AMPKVL.

Algorithm 2: AMPKVL.

**Input:** The intervals of an IG $I = \{I_1, I_2, \ldots, I_n\}$ and $L^{\mathrm{vl}}(p, I_k)$ for $k = 2, 3, \ldots, n$ and $p = 1, 2$ where $n = 3\Delta - 1$.
**Output:** $f_k^s$, the SL21-label of the interval $I_k$, $k = 1, 2, \ldots, n$.
**Step 1:** Rearrange the intervals as follows:
  **Case I:** $n$ is odd
  $I_n = I_2$;
  $I_{i+1} = I_{2i+1}$, for $i = 1, 2, \ldots, ((n-1)/2)$;
  $I_{i+((n-1)/2)} = I_{2i}$, for $i = 2, 3, \ldots, ((n-1)/2)$;
  $I_1$ remains same;
  **Case II:** $n$ is even
  $I_n = I_2$;
  $I_{i+1} = I_{2i+1}$, for $i = 1, 2, \ldots, (n/2) - 1$;
  $I_{i+(n/2)-1} = I_{2i}$, for $i = 2, 3, \ldots, (n/2)$;
  $I_1$ is unchanged;
**Step 2:** (Initialization)
  $f_1^s = 1$;
  $L^s(I_2) = \{1\}$;
**Step 3:** for $k = 2$ to $n - 1$
  $L^{\mathrm{svl}}(I_k) = L^{\mathrm{vl}}(2, I_k) - L^s(I_k)$;
  $f_k^s = \min\{L^{\mathrm{svl}}(I_k)\}$;
  $L^s(I_{k+1}) = L^s(I_k) \bigcup \{f_k^s\}$;
end for;
**Step 4:** $L^{\mathrm{svl}}(I_n) = L^{\mathrm{vl}}(2, I_n) - L^s(I_n)$;
  $f_n^s = \min\{L^{\mathrm{svl}}(I_n)\}$;
**Step 5:** $L^s = L^s(I_n) \bigcup \{f_n^s\}$;
**end** AMPSL21.

Algorithm 3: AMPSL21.

**Theorem 6.** *The running time of Algorithm 3 is $O(n\Delta^3)$, where $n = 3\Delta - 1$.*

*Proof.* According to Algorithm 3, the SL21-label of $I_k$, that is, $f_k^s$, is computed if the set $L^{\mathrm{svl}}(I_k)$ is computed. Now by Lemma 2 we see that by algorithm AMPSL21 one can compute the set $L^{\mathrm{vl}}(p, I_k)$, $p = 1, 2$, by using $O(\Delta^2)$ time. Again, by Appendix, algorithm $A$ **diff** $B$, $L^{\mathrm{vl}}(2, I_k) - L^s(I_k)$ have been computed in $O(\Delta)$ time, for any $k$. Therefore, the time needed for computing $L^{\mathrm{svl}}(I_k)$ is $O(\Delta^3)$. Since we need to find $L^{\mathrm{svl}}(2, I_k)$ for $k = 2, 3, \ldots, n$, the time complexity for Algorithm 3 is $O((n-1)\Delta^3)$, that is, $O(n\Delta^3)$. □

*4.1.1. Illustration of Algorithm AMPSL21.* We take an IG having 14 nodes (see Figure 9) and label that graph by Algorithm 3. The graph after completion of surjective $L(2,1)$-labelling is given in Figure 10.

For the above graph, the set of intervals $I = \{I_1, I_2, \ldots, I_{14}\}$ and $\Delta = 5$. Here, $3\Delta - 1 = 14 = n$, so this IG can be

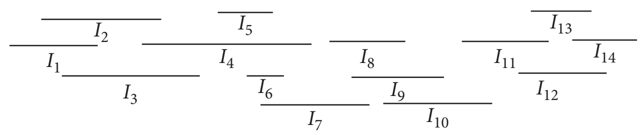


Figure 9: AnIG.

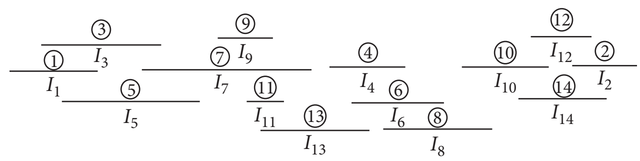


Figure 10: An IG labelled by SL21-labelling.

Table 2: Nodes and their corresponding surjective L(2,1)-labels.

| Nodes | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Surjective $L(2,1)$-labels | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

**Input:** $A = \{A_1, A_2, \ldots, A_p\}$ **and** $B = \{B_1, B_2, \ldots, B_q\}$.
**Output:** $A - B$.
**Step 1.** for $i = 1$ to $4\Delta - 2$
set $a_i = 0, b_i = 0$;//where $a_i$ and $b_i$ are the variables corresponding to $A_i$ and $B_i$ respectively.//
for $j = 1$ to $p$
if $A_j = k$ then $a_k = 1$;
**Step 2.** for $j = 1$ to $q$
if $B_j = k$ then $b_k = 1$;
**Step 3.** for $i = 1$ to $4\Delta - 2$
$c_i = a_i - b_i$;
if $c_i = 1$ then put $i$ to the set $A - B$;
stop.

Algorithm 4: $A$ diff $B$.

surjectively labelled by $L(2,1)$-labelling. $f_k^s$ is the SL21-label of the interval $I_k$, for $k = 1, 2, \ldots, 14$.

According to Algorithm 3, at first, we rearrange the intervals as follows:

$I_2 = I_3$, $I_3 = I_5$, $I_4 = I_7$, $I_5 = I_9$, $I_6 = I_{11}$, $I_7 = I_{13}$, $I_8 = I_4$, $I_9 = I_6$, $I_{10} = I_8$, $I_{11} = I_{10}$, $I_{12} = I_{12}$, $I_{13} = I_{14}$, $I_{14} = I_2$, and $I_1$ remains the same.

$f_1^s = 1$ and $L^s(I_2) = \{1\}$ are also initialized.

**Iteration 1:** For $k = 2$,

$L^1(I_2) = \{1\}$, $L^2(I_2) = \phi$.
$L^{\text{vl}}(1, I_2) = \{1, 2, 3\}$, $L^{\text{vl}}(2, I_2) = \{1, 2, 3\}$.
So, $L^{\text{svl}}(I_2) = L^{\text{vl}}(2, I_2) - L^s(I_2) = \{1, 2, 3\} - \{1\} = \{2, 3\}$.
Therefore, $f_2^s = \min\{L^{\text{svl}}(I_2)\} = 2$.
$L^s(I_3) = L^s(I_2) \cup \{f_2^s\} = \{1\} \cup \{2\} = \{1, 2\}$.

**Iteration 2:** For $k = 3$,

$L^1(I_3) = \{1\}$, $L^2(I_3) = \phi$.
$L^{\text{vl}}(1, I_3) = \{3, 4\}$, $L^{\text{vl}}(2, I_3) = \{3, 4\}$.
So, $L^{\text{svl}}(I_3) = L^{\text{vl}}(2, I_3) - L^s(I_3) = \{3, 4\} - \{1, 2\} = \{3, 4\}$.
Therefore, $f_3^s = \min\{L^{\text{svl}}(I_3)\} = 3$.
$L^s(I_4) = L^s(I_3) \cup \{f_3^s\} = \{1, 2, 3\}$.

**Iteration 3:** For $k = 4$,

$L^1(I_4) = \phi$, $L^2(I_4) = \phi$.
$L^{\text{vl}}(1, I_4) = \{1, 2, 3, 4, 5\}$, $L^{\text{vl}}(2, I_4) = \{1, 2, 3, 4, 5\}$.
So, $L^{\text{svl}}(I_4) = L^{\text{vl}}(2, I_4) - L^s(I_4) = \{1, 2, 3, 4, 5\} - \{1, 2, 3\} = \{4, 5\}$.
Therefore, $f_4^s = \min\{L^{\text{svl}}(I_4)\} = 4$.
$L^s(I_5) = L^s(I_4) \cup \{f_4^s\} = \{1, 2, 3, 4\}$.

**Iteration 4:** For $k = 5$,

$L^1(I_5) = \{1, 3\}$, $L^2(I_5) = \phi$.
$L^{\text{vl}}(1, I_5) = \{5, 6\}$, $L^{\text{vl}}(2, I_5) = \{5, 6\}$.
So, $L^{\text{svl}}(I_5) = L^{\text{vl}}(2, I_5) - L^s(I_5) = \{5, 6\} - \{1, 2, 3, 4\} = \{5, 6\}$.
Therefore, $f_5^s = \min\{L^{\text{svl}}(I_5)\} = 5$.
$L^s(I_6) = L^s(I_5) \cup \{f_5^s\} = \{1, 2, 3, 4, 5\}$.

**Iteration 5:** For $k = 6$,

$L^1(I_6) = \{4\}$, $L^2(I_6) = \phi$.
$L^{\mathrm{vl}}(1, I_6) = \{1, 2, 6, 7\}$, $L^{\mathrm{vl}}(2, I_6) = \{1, 2, 6, 7\}$.
So, $L^{\mathrm{svl}}(I_6) = L^{\mathrm{vl}}(2, I_6) - L^s(I_6) = \{1, 2, 6, 7\} - \{1, 2, 3, 4, 5\} = \{6, 7\}$.
Therefore, $f_6^s = \min\{L^{\mathrm{svl}}(I_6)\} = 6$.
$L^s(I_7) = L^s(I_6) \cup \{f_6^s\} = \{1, 2, 3, 4, 5, 6\}$.

In this way, $f_7^s = 7$, $f_8^s = 8$, $f_9^s = 9$, $f_{10}^s = 10$, $f_{11}^s = 11$, $f_{12}^s = 12$, $f_{13}^s = 13$, and, finally, $f_{14}^s = 14$.

The nodes and their corresponding labels are shown in Table 2.

## 5. Conclusion

In $L(2,1)$-labelling, although there is a light chance to overlap the frequencies in radio network, it cannot be neglected, but in SL21-labelling there is no chance to overlap the frequencies, as in this case the labels are distinct. So, the results about SL21-labelling are clearly welcome. In the entire article, we have proved that a path $P_n$ can be surjectively $L(2,1)$-labelled if $n \geq 4$. An algorithm is also presented to label a path of $n$ nodes, for $n \geq 7$. Also, we have showed that an IG $G$ having $n$ nodes can be surjectively labelled by $L(2,1)$-labelling if $n = 3\Delta - 1$. Also, we have presented an $O(n\Delta^3)$ time algorithm to label an IG by SL21-labelling. If $n \neq 3\Delta - 1$, then the IG may or may not be labelled by SL21-labelling. In the future, we can extend this work to other classes of intersection graph. So, there is a scope for the new research to study surjective labelling of permutation graph, trapezoid graph, and so forth.

## Appendix

Here, an algorithm to compute $A - B$ is presented, where $A$ and $B$ are subsets of $\{1, 2, \ldots, 4\Delta - 2\}$ (Algorithm 4).

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## References

[1] J. R. Griggs and R. K. Yeh, "Labelling graphs with a condition at distance 2," *SIAM Journal on Discrete Mathematics*, vol. 5, no. 4, pp. 586–595, 1992.

[2] S. Amanathulla and M. Pal, "$L$(3,2,1)-labelling problems on permutation graphs," *Transylvanian Review*, vol. 25, no. 14, pp. 3939–3953, 2017.

[3] S. Amanathulla and M. Pal, "$L(h_1, h_2, \ldots, h_m)$-labelling problems on interval graphs," *International Journal of Control Theory and Applications*, vol. 10, no. 1, pp. 467–479, 2017.

[4] S. Amanathulla and M. Pal, "$L(h_1, h_2, \ldots, h_m)$labeling problems on circular-arc graphs," *Far East Journal of Mathematical Sciences (FJMS)*, vol. 102, no. 6, pp. 1279–1300, 2017.

[5] S. Amanathulla and M. Pal, "$L$(1, 1, 1)- and $L$(1, 1, 1, 1)-labeling problems of square of path, complete graph and complete bipartite graph," *Far East Journal of Mathematical Sciences (FJMS)*, vol. 106, no. 2, pp. 515–527, 2018.

[6] S. Amanathulla, M. Pal, and S. Sahoo, "$L$ (3, 1, 1)-labeling numbers of square of paths, complete graphs and complete bipartite graphs," *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 2, pp. 1917–1925, 2019.

[7] M. Berhe and C. Wang, "Computation of certain topological coindices of graphene sheet and C4C8(S) nanotubes and nanotorus," *Applied Mathematics and Nonlinear Sciences*, vol. 4, no. 2, pp. 455–468, 2019.

[8] S. Goyal, P. Garg, and V. N. Mishra, "New composition of graphs and their Wiener Indices," *Applied Mathematics and Non Linear Sciences*, vol. 4, no. 1, pp. 175–180, 2019.

[9] S. M. Hosamani, V. B. Awati, and R. M. Honmore, "On graphs with equal dominating and c-dominating energy," *Applied Mathematics and Nonlinear Sciences*, vol. 4, no. 2, pp. 503–512, 2019.

[10] B. M. Gurevich, "On classes of infinite loaded graphs with randomly deleted edges," *Applied Mathematics and Nonlinear Sciences*, vol. 5, no. 2, pp. 257–260, 2020.

[11] P. S. Ranjini, V. Lokesha, and S. Kumar, "Degree sequence of graph operator for some standard graphs," *Applied Mathematics and Non Linear Sciences*, vol. 5, no. 2, pp. 99–108, 2020.

[12] K. Jonas, *Graph coloring analogues with a condition at distance two: l(2,1)-labellings and listλ-labellings*, Ph.D. Thesis, University of South Carolina, Columbia, SC, USA, 1993.

[13] G. J. Chang and D. Kuo, "The $L$(2,1)-labeling problem on graphs," *SIAM Journal on Discrete Mathematics*, vol. 9, no. 2, pp. 309–316, 1996.

[14] D. Král' and R. Skrekovski, "A Theorem about the channel assignment problem," *SIAM Journal on Discrete Mathematics*, vol. 16, no. 3, pp. 426–437, 2003.

[15] D. Goncalves, "On the $L$($d$, 1)-labellinng of graphs," *Discrete Mathematics*, vol. 308, pp. 1405–1414, 2008.

[16] A. A. Bertossi and M. A. Bonuccelli, "Code assignment for hidden terminal interference avoidance in multihop packet radio networks," *IEEE/ACM Transactions on Networking*, vol. 3, no. 4, pp. 441–449, 1995.

[17] R. K. Yeh, *Labeling graphs with a condition at distance two*, Ph.D Thesis, University of South Carolina, Columbia, C, USA, 1990.

[18] M. L. Chia, D. Qua, H. Liao, C. Yang, and R. K. Yea, "$L$(3,2,1) -labelling of graphs," *Taiwanese Journal of Mathematics*, vol. 15, no. 6, pp. 2439–2457, 2011.

[19] J. Clipperton, "$L$(4,3,2,1) -labelling of simple graphs," *Applied Mathematics Science*, vol. 1, no. 2, pp. 95–102, 2011.

[20] T. Makansi, "Transmitter-oriented code assignment for multihop packet radio," *IEEE Transactions on Communications*, vol. 35, no. 12, pp. 1379–1382, 1987.

[21] R. Battiti, A. A. Bertossi, and M. A. Bonuccelli, "Assigning codes in wireless networks: bounds and scaling properties," *Wireless Networks*, vol. 5, no. 3, pp. 195–209, 1999.

[22] J. Clipperton, "$L$(d,2,1)-labelling of simple graphs," *Math Journal*, vol. 9, pp. 1–11, 2008.

[23] S. Atta and P. R. S. Mahapatra, "$L$(4,3,2,1)-labelling for simple graphs," *Informatation Systems Design and Intelligent Applications*, vol. 339, pp. 89–105, 2015.

[24] J. P. Georges and D. W. Mauro, "Generalized vertex labelling with a condition at distance two," *Congres. Numer.*vol. 109, pp. 141–159, 1995.

[25] M. Molloy and M. R. Salavatipour, "A Bound on the chromatic number of the square of a planar graph," *Journal of Combinatorial Theory, Series B*, vol. 94, no. 2, pp. 189–213, 2005.

[26] J. van den Heuvel and S. McGuinness, "Coloring the square of a planar graph," *Journal of Graph Theory*, vol. 42, no. 2, pp. 110–124, 2003.

[27] J. Lui and Z. Shao, "The L(3,2,1) -labelling problem on graphs," *Mathematics Applicata*, vol. 17, no. 4, pp. 596–602, 2004.

[28] S. Paul, M. Pal, and A. Pal, "$L$(2, 1)-labelling of interval graph," *Journal of Applied Mathematics and Computing*, vol. 49, no. 1, pp. 419–432, 2015.

[29] T. Calamoneri, S. Caminiti, R. Petreschi, and S. Olariu, "On the *L(h,k)*-labeling of co-comparability graphs and circular-arc graphs," *Networks*, vol. 53, no. 1, pp. 27–34, 2009.

[30] S. Amanathulla and M. Pal, "$L$(3, 2, 1)- and $L$(4, 3, 2, 1)-labeling problems on interval graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 14, no. 3, pp. 205–215, 2017.

[31] S. Amanathulla and M. Pal, "$L$(0,1)- and $L$(1,1)-labelling problems on circular-arc graphs," *International Journal of Soft Computing*, vol. 11, no. 6, pp. 343–350, 2016.

[32] S. Paul, M. Pal, and A. Pal, "$L$(2, 1)-labelling of circular-arc graph," *Annals of Pure and Applied Mathematics*, vol. 5, no. 2, pp. 208–219, 2014.

[33] S. Amanathulla and M. Pal, "$L$(3,2,1)- and $L$(4,3,2,1)-labelling problems on circular-arc graphs," *International Journal of Control Theory and Applications*, vol. 9, no. 34, pp. 869–884, 2016.

[34] H. L. Bodlaender, T. Kloks, R. B. Tan, and J. V. Leeuwen, "Approximations for -colorings of graphs," *The Computer Journal*, vol. 47, no. 2, pp. 193–204, 2004.

[35] S. Paul, M. Pal, and A. Pal, "$L$(0, 1)-labelling of permutation graphs," *Journal of Mathematical Modelling and Algorithms in Operations Research*, vol. 14, no. 4, pp. 469–479, 2015.

[36] S. Paul, M. Pal, and A. Pal, "$L$(2, 1)-labeling of permutation and bipartite permutation graphs," *Mathematics in Computer Science*, vol. 9, no. 1, pp. 113–123, 2014.

[37] M. Lingscheit, K. Ruff, and J. Ward, "$L(d, j, s)$-minimal and surjective graph labelling–," *Rose Hulman Undergragute Mathematics Journal*, vol. 10, no. 2, pp. 1–27, 2009.

[38] S. Amanathulla and M. Pal, "Surjective $L$ (2, 1)-labeling of cycles and circular-arc graphs," *Journal of Intelligent & Fuzzy Systems*, vol. 35, no. 1, pp. 739–748, 2018.

[39] H. Rashmanlou, G. Muhiuddin, S. Amanathulla, F. Mofidnakhaei, and M. Pal, "A study on cubic graphs with novel application," *Journal of Intelligent & Fuzzy Systems*, vol. 40, no. 1, pp. 89–101, 2021.

[40] S. S. Hussain, R. J. Hussain, and G. Muhiuddin, "Neutrosophic vague line graphs," *Neutrosophic Sets and Systems*, vol. 36, pp. 121–130, 2020.

[41] G. Muhiuddin, M. Mohseni Takallo, Y. B. Jun, and R. A. Borzooei, "Cubic graphs and their application to a traffic flow problem," *International Journal of Computational Intelligence Systems*, vol. 13, no. 1, pp. 1265–1280, 2020.

[42] T. Pramanik, G. Muhiuddin, A. M. Alanazi, and M. Pal, "An extension of fuzzy competition graph and its uses in manufacturing industries," *Mathematics*, vol. 8, no. 6, p. 1008, 2020.

[43] S. Samanta, G. Muhiuddin, A. M. Alanazi, and K. Das, "Mathematical approach on representation of competitions: competition cluster hypergraphs," *Mathematical Problems in Engineering*, vol. 2020, Article ID 2517415, 10 pages, 2020.

[44] Sk. Amanathulla and M. Pal, "$L$(h,k)-labelling problems on intersection graphs," in *Handbook of Research on Advanced Applications of Graph Theory in Modern Society*, pp. 437–468, IGI Global, Hershey, PA, USA, 2020.