Distance Two Surjective Labelling of Paths and Interval Graphs

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Graph labelling problem has been broadly studied for a long period for its applications, especially in frequency assignment in (mobile) communication system, X-ray crystallography, circuit design, etc. Nowadays, surjective $L(2, 1)$-labelling is a well-studied problem. Motivated from the $L(2, 1)$-labelling problem and the importance of surjective $L(2, 1)$-labelling problem, we consider surjective $L(2, 1)$-labelling ($SL21$-labelling) problems for paths and interval graphs. For any graph $G = (V, E)$, an $SL21$-labelling is a mapping $\phi: V \rightarrow \{1, 2, \ldots, n\}$ so that, for every pair of nodes $u$ and $v$, if $d(u, v) = 1$, then $|\phi(u) - \phi(v)| \geq 2$; and if $d(u, v) = 2$, then $|\phi(u) - \phi(v)| \geq 1$, and every label $1, 2, \ldots, n$ is used exactly once, where $d(u, v)$ represents the distance between the nodes $u$ and $v$, and $n$ is the number of nodes of graph $G$. In the present article, it is proved that any path $P_n$ can be surjectively $L(2, 1)$-labelled if $n \geq 4$, and it is also proved that any interval graph (IG) $G$ having $n$ nodes and degree $\Delta > 2$ can be surjectively $L(2, 1)$-labelled if $n = 3\Delta - 1$. Also, we have designed two efficient algorithms for surjective $L(2, 1)$-labelling of paths and interval graphs. The results regarding both paths and interval graphs are the first result for surjective $L(2, 1)$-labelling.

1. Introduction

The frequency assignment problem is bottomed from the problems of distance labelling of graph. In 1992, $L(2, 1)$-labelling was invented by Griggs and Yeh [1] in conjunction with channel assigning problem in a multihop radio network.

For any graph $G = (V, E)$, an $L(2, 1)$-labelling is a mapping $\phi: V(G) \rightarrow \{1, 2, \ldots, n\}$, so that $|\phi(u) - \phi(v)| \geq 2$ if $d(u, v) = 1$ and $|\phi(u) - \phi(v)| \geq 1$ if $d(u, v) = 2$. The span of $L(2, 1)$-labelling of $G$ is $\lambda_{L21}(G) = \max\{\phi(v): v \in V\}$. The $L(2, 1)$-labelling number $\lambda_{L21}(G)$ of $G$ is the smallest natural number $p$ so that $G$ has an $L(2, 1)$-labelling of span $p$.

A surjective $L(2, 1)$-labelling of $G = (V, E)$ is a mapping $\phi: V \rightarrow \{1, 2, \ldots, n\}$ so that $|\phi(u) - \phi(v)| \geq 2$ when $d(u, v) = 1$ and $|\phi(u) - \phi(v)| \geq 1$ when $d(u, v) = 2$, and it requires that each label, $1, 2, \ldots, n$, be used only once, where $n$ is the number of nodes of $G$. In Figure 1, we have shown an $L(2, 1)$-labelling of a path with 5 nodes and Figure 2 shows $SL21$-labelling of the same graph. In Figure 1, identical label is used several times but in Figure 2 the labels 1 to 5 are used only once. So, in $SL21$-labelling, there is a more complex task compared to $L(2, 1)$-labelling.

In 1994, Sakai has proved some results regarding distance two labelling of chordal graph. Later, in 2007, Bertossi and Bonuccelli have studied approximate $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring of trees and interval graphs. Amanathulla and Pal have studied a lot of problems regarding labelling of graphs, like $L(3, 2, 1)$-labelling problems on permutation graphs [2], $L(h_1, h_2, \ldots, h_m)$-labelling problems on interval graphs [3], $L(h_1, h_2, \ldots, h_m)$-labelling problems on circular-arc graphs [4], $L(1, 1, 1)$- and $L(1, 1, 1, 1)$-labelling problems of square of paths [5], and $L(3, 1, 1)$-labelling numbers of squares of paths, complete graphs, and complete bipartite graphs [6]. In 2019, Berhe...
Chang et al. [13] have showed and complete bipartite graph. NZ_hey have shown that subjective labelling for path, cycle, complete graph, caterpillar, graphs with randomly deleted edges [10] and Ranjini et al. published one paper regarding classes of infinite loaded composition of graphs and their Wiener indices. Hosamani et al. [9] have studied graphs with equal dominating and coindices of graphene sheet and C4C8(S) nanotubes and shown that any IG having can be surjectively labelled by $\lambda$. Hosamani et al. [9] have studied graphs with equal dominating and coindices of graphene sheet and C4C8(S) nanotubes and shown that any IG having can be surjectively labelled by $\lambda$. Hosamani et al. [9] have studied graphs with equal dominating and coindices of graphene sheet and C4C8(S) nanotubes and shown that any IG having can be surjectively labelled by $\lambda$.

In 1992, Griggs et al. showed that $\lambda_1(G) \leq \Delta^2 + 2\Delta$ and have proposed a conjecture [1].

In 1993, Jonas [12] has shown that $\lambda_1(G) \leq \Delta^2 + 2\Delta - 4$. Chang et al. [13] have showed $\lambda_1(G) \leq \Delta + \Delta$. Kráľ and Skrekovski [14] proved that $\lambda_2(G) \leq \Delta^2 + \Delta - 1$ and they further improved it to $\lambda_2(G) \leq \Delta^2 + \Delta - 2$ [15].

Different bounds for $\lambda_{1,2,...,n}(G)$ were obtained for different classes of graphs. Some results regarding upper bound of $L(h_1,h_2,...,h_n)$-labelling are shown in Table 1.

In [37], Lingscheit et al. investigated minimal and surjective labelling for path, cycle, complete graph, caterpillar, and complete bipartite graph. They have showed that $P_n$ can be surjectively labelled when $n \geq 7$. Very recently, Amanathulla and Pal have studied SL21-labelling of cycle and circular-arc graph (CAG) and obtained good results for it [38].

$L(2,1)$-labelling of graphs is a rapidly studied problem for its applications in various fields, especially in channel assignment in radio network. In $L(2,1)$-labelling, although there is a light chance to overlap the frequencies in radio network, it cannot be neglected, but in $L(2,1)$-labelling there is no chance to overlap the frequencies, as in this case the labels are distinct. For this reason, in the recent year, SL21-labelling of graph has become a well-studied problem due to its applications. This motivates us to consider SL21-labelling of paths and IGs. Recently, many researchers applied various related concepts on graphs in different aspects (see, e.g., [39–43]).

In the present article, it is shown that any path $P_n$ is surjectively labelled by $L(2,1)$-labelling if $n \geq 4$ and it also showed that any IG having $n$ nodes can be surjectively $L(2,1)$-labelled if $n = 3\Delta - 1$. Two polynomial time algorithms are also established to label a path and an IG by SL21-labelling.

The remainder of this article is organized as follows: in Section 2, some notations and preliminary definitions are given. In Section 3, SL21-labelling of path has been presented. In Section 4, SL21-labelling of IG is investigated. The last section presents concluding remarks.

### Table 1: Different types of graphs and their upper bounds.

<table>
<thead>
<tr>
<th>Graphs</th>
<th>$L(h,k)$-labelling numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq \lambda_0 \leq \Delta^2 - \Delta$</td>
<td>[16]</td>
</tr>
<tr>
<td>$\Delta \leq \Delta^1 \leq \Delta^2$</td>
<td>[17]</td>
</tr>
</tbody>
</table>

| General graphs | $\Delta + 1 \leq \lambda_1 \leq \Delta^2 + \Delta - 2$ | [1, 15] |
|\lambda_{3,3,2} \leq \lambda_2 \leq \Delta^2 + 2\Delta + 6\Delta | [18] |
|\lambda_{4,3,2,1} \leq \lambda_2 \leq \Delta^2 + 2\Delta + 6\Delta | [19] |

| Paths | $\lambda_0(P_n) = 0$ or $1$ | [20] |
| $\lambda_1(P_n) = 1$ or $2$ | [21] |
| $\lambda_2(P_n) = 2$, $3$ or $4$ | [1] |
| For $d \geq 2$, $\lambda_{d,2}(P_n) = 0$, $d$, $d + 2$ or $d + 4$ | [22] |
| For $n \geq 2\lambda_{d,3,2,1}(P_n) = 5$, $8$, $9$, $11$ or $12$ | [23] |

| Cycles | For $n \geq 3$, $\lambda_1(C_n) = 1$ or $2$ | [16] |
| For $n \geq 3$, $\lambda_1(C_n) = 2$ or $3$ | [21] |
| For $n \geq 3$, $\lambda_1(C_n) = 4$ | [1] |
| For $n \geq 3$, $\lambda_2(C_n) = 6$, $7$, $8$ or $9$ | [22] |
| For $n \geq 4$, $d \geq 5$, $\lambda_{d,2,1}(C_n) = d + 4$, $d + 6$, $2d + 1$ or $2d$ | [22] |
| For $n \geq 3$, $\lambda_{d,3,2,1}(C_n) = 9$, $11$, $14$ or $13$ | [23] |

| Complete | $\lambda_{d,1}(K_n) = n - 1$ | [24] |
| $\lambda_{d,2,1}(K_n) = d(n - 1) + 1$ | [22] |

| Complete bipartite | $\lambda_{d,1}(K_{m,n}) = m + n - 1$ | [24] |
| $\lambda_{d,2,1}(K_{m,n}) = d + 2(m + n) - 3$ | [22] |

| Planar | $\lambda_1(G) \leq (5/3)\Delta + 1 + 77$ | [25] |
| $\lambda_2(G) \leq 2\Delta + 35$ | [26] |
| $\lambda_3(G) \leq (5/3)\Delta + 95$ | [25] |
| $\lambda_{h,k}(G) \leq (5/3)\Delta + 18 + 77k - 18$ | [25] |
| $\lambda_{d,1,3,2,1}(G) \leq 15(\Delta^2 - \Delta + 1)$ | [27] |

| Interval | $\lambda_1(G) \leq \Delta + w$ | [28] |
| $\lambda_1(G) \leq \max \{h, 2k\}\Delta$ | [29] |
| $\lambda_2(G) \leq 6\Delta - 3$ | [30] |
| $\lambda_{h,k}(G) \leq 10\Delta - 6$ | [30] |

| Circular-arc | $\lambda_1(G) \leq \Delta^2 - 3\Delta$ | [31] |
| $\lambda_1(G) \leq 2\Delta^2$ | [31] |
| $\lambda_{h,k}(G) \leq \max \{h, 2k\}\Delta + hw$ | [29] |
| $\lambda_1(G) \leq \Delta + 3w$ | [32] |
| $\lambda_2(G) \leq 9\Delta - 6$ | [33] |
| $\lambda_{h,k}(G) \leq 16\Delta - 12$ | [33] |

| Permutation | $\lambda_1(G) \leq 2\Delta - 2$ | [34] |
| $\lambda_2(G) \leq \Delta - 1$ | [35] |
| $\lambda_{h,k}(G) \leq \Delta \leq 2, 5\Delta - 8$ | [36] |
| $\lambda_2(G) \leq 5\Delta - 2$ | [34] |

### 2. Preliminaries and Notations

A path is a graph $G = (V, E)$, where $(v_1, v_2, \ldots, v_n) \in E$, for all $1 \leq j \leq n - 1$, where $V' = \{v_1, v_2, \ldots, v_n\}$, and it is denoted by $P_n$. Here, we consider IG (IG) which is not a path, so $\Delta > 2$, because if $\Delta = 2$, then it may be a path.

Let the set of intervals in real line be $I = [I_1, I_2, \ldots, I_n]$, where $I_k = [l_k, r_k]$, $k = 1, 2, \ldots, n$, and $l_k$ and $r_k$ are the left and right endpoints of $I_k$. For any interval $I_k$, $k = 1, 2, \ldots, n$, we draw a node $v_k$ and two nodes $v_p$ and $v_q$ have joined by a line segment that if the corresponding intervals have common portion, then we obtain an IG [44]. Throughout the paper, an interval $I_k$ and a node $v_k$ are the same. An IG and its interval representation are shown in Figure 3.

Notations. For any IG $G$ with $n$ nodes and corresponding set of intervals $I = [I_1, I_2, \ldots, I_n]$, we define the following:
From the above result, it is concluded that $P_n$ can be SL21-labelled for $n = 1, 2, 3$. \hfill \square

**Theorem 2.** The minimum path that can be labelled by SL21-labelling is $P_4$.

**Proof.** From Theorem 1, we have $\lambda_{2,1}(P_2) = 3$ and $\lambda_{2,1}(P_3) = 4$, so, for $n < 4$, a path $P_n$ cannot be labelled by SL21-labelling. The labelling pattern $\{3, 1, 4, 2\}$ of path $P_4$ (see Figure 5) shows that $P_4$ can be labelled by SL21-labelling. Hence, $P_4$ is the minimum path that can be labelled by SL21-labelling (Figure 6). \hfill \square

For this path, the node $V = \{v_1, v_2, \ldots, v_{22}\}$. Here, $n > 4$, so this path can be surjectively labelled by $L(2, 1)$-labelling. $f^*_\ell$ is the SL21-label of the node $v_k$ for $k = 1, 2, \ldots, 22$. According to Algorithm 1, we rearrange the nodes as follows:

$v_2 = v_3$, $v_3 = v_5$, $v_4 = v_7$, $v_5 = v_9$, $v_6 = v_11$, $v_7 = v_{13}$, $v_8 = v_{15}$, $v_9 = v_{17}$, $v_{10} = v_{19}$, $v_{11} = v_{21}$, $v_{12} = v_{14}$, $v_{13} = v_{16}$, $v_{14} = v_8$, $v_{15} = v_{10}$, $v_{16} = v_{12}$, $v_{17} = v_{14}$, $v_{18} = v_{16}$, $v_{19} = v_{18}$, $v_{20} = v_{20}$, $v_{21} = v_{22}$, $v_{22} = v_2$, and $v_1$ remains unchanged.

Now, node $v_k$ is labelled by $k$; that is, $f^*_\ell = k$ for each $k = 1, 2, \ldots, 22$. After completion of surjective $L(2, 1)$-labelling of $P_{22}$, the node and the label of the corresponding node are shown in Figure 7(b).

**4. Surjective $L(2, 1)$-Labelling of IGs**

Here, some lemmas that we have used to develop the proposed algorithm are presented.

**Lemma 1.** For any IG $G$, $|L^2(I_k)| \leq \Delta - 1$, for $I_k \in I$.

**Proof.** Let $G$ be an IG having $n$ nodes. The labelling of $G$ starts from the leftmost interval. Let node $v_k$ be corresponding to the interval $I_k$ of the IG $G$. Suppose that in a stage the intervals $I_1, I_2, \ldots, I_{k-1}$ (for some $k = 2, 3, 4, \ldots, n$) are previously labelled by SL21-labelling and the remaining intervals are unlabelled.

Let $|L^2(I_k)| = p$. This means that the number of distinct SL21-labelling for labelling distance two intervals from the interval $I_k$ before labelling $I_k$ is $p$. Since the degree of the IG $G$ is $\Delta$, there exists an interval $I_a$ (see Figure 8) and those are adjoining to $\Delta$ intervals at most. In Figure 8, $I_a$ is adjoining to $I_k, I_{k+1}, I_{k+2}, I_{k+3}$. Among the intervals, some intervals $(I_k, I_{k+1}, I_{k+2}, I_{k+3})$ in Figure 8) are of distance two apart from $I_k$ and among the intervals there is an interval $(I_k$ in Figure 8) whose distance is not two from $I_k$. Hence, $p \leq \Delta - 1$; that is, $|L^2(I_k)| \leq \Delta - 1$. \hfill \square

**Observation 1.** For any IG $G$, $L^1(I_k) \subseteq L^1(I_k)$, for any interval $I_k$, $i = 1, 2$.

**Observation 2.** For any IG $G$, $|L^1(I_k)| \leq \Delta$, for any interval $I_k \in I$. 

3. **Surjective $L(2, 1)$-Labelling of Paths**

In this portion, we have presented SL21-labelling of path and have shown that any path $P_n$ is surjectively $L(2, 1)$-labelled if $n \geq 4$. Also, we have presented a greedy algorithm to label a path by SL21-labelling.

**Theorem 1.** For $P_n$,

$$\lambda_{2,1}(P_n) = \begin{cases} 1, & \text{if } n = 1, \\ 3, & \text{if } n = 2, \\ 4, & \text{if } n = 3. \end{cases} \quad (1)$$

**Proof.** Let $P_n$ be a path having $n$ nodes.

Case 1: $n = 1$.

This result holds trivially.

Case 2: $n = 2$.

The labels used are 1 and 3 and hence $\lambda_{2,1}(P_2) = 3$.

Case 3: $n = 3$.

There are two possible cases shown in Figures 4(a) and 4(b). The labelling sequences are $\{3, 1, 4\}$ and $\{1, 4, 2\}$. 

From the above result, it is concluded that $P_n$ can be SL21-labelled for $n = 1, 2, 3$. \hfill \square
Input: The nodes of the path $P_n$ $(n > 6)$, $V = \{v_1, v_2, \ldots, v_n\}$.
Output: The SL21-label of the path $P_n$.

Step 1: Rearrange the intervals as follows:

Case I: $n$ is odd
- $v_n = v_2$;
- $v_{i+1} = v_{2i+1}$, for $i = 1, 2, \ldots, \left(\frac{n-1}{2}\right)$;
- $v_{i+\left(\frac{n-1}{2}\right)} = v_{2i}$, for $i = 2, 3, \ldots, \left(\frac{n-1}{2}\right)$;
- $v_1$ remains same;

Case II: $n$ is even
- $v_n = v_2$;
- $v_{i+1} = v_{2i+1}$, for $i = 1, 2, \ldots, \left(\frac{n}{2}\right) - 1$;
- $v_{i+\left(\frac{n}{2}\right)-1} = v_{2i}$, for $i = 2, 3, \ldots, \left(\frac{n}{2}\right)$;
- $v_1$ remains same;

Step 2: Label the node $v_i$ by $i$, i.e., $f_i^r = i$, for $i = 1, 2, \ldots, n$

end AMPSL21.

Algorithm 1: AMPSPL21.

Figure 4: A path with three nodes and their surjective labels.

Figure 5: A path $P_4$ labelled by SL21-labelling.

Figure 6: (a) A path with of 14 nodes; (b) the path after rearrangement of the nodes.

Figure 7: A path labeled by SL21-labelling.

Figure 8: A set of intervals.
Theorem 4. Any IGG with $n$ nodes is surjectively labelled if $n = 3\Delta - 1$.

Proof. Since $G$ has $n$ nodes, let $I = \{I_1, I_2, \ldots, I_k\}$. Since we want to label the intervals of an IG by SL21-labelling, every label is used exactly once and the labels must be in $\{1, 2, \ldots, n\}$. So,

$$\lambda_{2,1}(G) \leq 2|L^1(I_k)| + |L^2(I_k)| \leq 2\Delta + (\Delta - 1), \text{ by Lemma 1} \tag{2}$$

$$\leq 3\Delta - 1.$$  

Again, since $G$ has $n$ nodes, to label the whole graph by SL21-labelling, $n$ distinct labels must be required. Also, since $\lambda_{2,1}(G) \leq 3\Delta - 1$, in the extreme unfavourable cases $3\Delta - 1$ labels are required to label graph $G$ by $L(2, 1)$-labelling. Again, in SL21-labelling, the highest label is equal to $n$. Hence, an IGG is surjectively labelled using $L(2, 1)$-labelling if $n = 3\Delta - 1$.

If $n \neq 3\Delta - 1$, then the IG may or may not be labelled by SL21-labelling, because in the worst case $3\Delta - 1$ labels are required to label the IG, which is not equal to $n$. This contradicts the condition that the used label must belong to $\{1, 2, \ldots, n\}$ and the highest label must be equal to $n$ for SL21-labelling. $\square$

4.1. Algorithm for Surjective $L(2, 1)$-Labelling of IGs. In this part, two algorithms are designed: one is to compute $L^d(k, I)$ and the other is to compute SL21-label for an IG (Algorithm 2).

Lemma 2. $L^d(p, I)$ for $p = 1, 2$ is correctly computed by Algorithm 2 and the time complexity of the above algorithm is $O(\Delta^2)$.

Proof. According to Lemma 2, for each element $i \in L^d(1, I)$, $L^d(2, I)$ differs from $I_k$ by at least 2 for each $I_k \subseteq L^d(1, I)$. Therefore, $|I - I_k| \geq 2$ for all $I_k \subseteq L^d(1, I)$, for all $I_k \subseteq L^d(1, I)$. So, Algorithm 2 correctly computes the set $L^d(1, I)$ for each $I_k \subseteq I$, $k = 1, 2, \ldots, n$. Again, each element $l_{k}$ of $L^d(2, I)$ differs from $I_k$ by at least 1 for each $I_k \subseteq L^d(1, I)$. Therefore, $|I - I_k| \geq 2$ for all $I_k \subseteq L^d(2, I)$, for all $I_k \subseteq L^d(2, I)$. So, Algorithm 2 correctly computes the set $L^d(p, I)$ for each $k = 1, 2$. As $|L^d|$ is the cardinality of the set of labels $L^d$, $|L^d(I_k)| \leq |L^d|$ for $i = 1, 2$ and $I_k \subseteq I$, and also $r \leq 3\Delta + 1$, where $r = \max\{L^d(I_k)\} + 2$. So, $L^d(1, I_k)$ is computed by using at most $(\Delta^2 + 1)|L^d|$ times, that is, using $O(\Delta^3)$ times. Similarly, $L^d(2, I_k)$ is computed using at most $(\Delta^2 + 1)|L^d|$ times, that is, using $O(\Delta^3)$ times. Since $|L^d| \leq 3\Delta + 1$, the iterative time for algorithm SLKVL is $O(\Delta^3)$. $\square$

Lemma 3. For each IGG, $L^d(1, I)$ is the nonempty largest set satisfying distance one condition of $L(2, 1)$-labelling, $l \leq r$, for every $l \in L^d(1, I)$, and $r = \max\{L^d(I_k)\} + 2$, for any $I_k \subseteq I$.

Proof. Since $r = \max\{L^d(I_k)\} + 2$ and $L^d(I_k) \subseteq L^d(I_k)$ (by Observation 1), $\max\{L^d(I_k)\}$, for every $l \in L^d(1, I_k)$, so, $L^d(I_k)$ is a nonempty set. Also, let $B$ be an arbitrary set of labels, which satisfies distance one condition of $L(2, 1)$-labelling, $l \leq r$, for all $l \in B$, and $r = \max\{L^d(I_k)\} + 2$. Then, for $b \in B$, $\max\{L^d(I_k)\} + 2 = 2$. Since, $b \in B \Rightarrow b \in L^d(1, I_k)$. Then, $B \subseteq L^d(1, I_k)$. Since $B$ is arbitrary, $L^d(1, I_k)$ is the largest nonempty set of labels which satisfies distance one condition of $L(2, 1)$-labelling, $l \leq r$, for every $l \in L^d(1, I_k)$, and $r = \max\{L^d(I_k)\} + 2$, for any $I_k \subseteq I$. $\square$

Lemma 4. For any IGG, $L^d(2, I_k)$ is the nonempty largest set satisfying $L(2, 1)$-labelling condition, $l \leq r$, for every $l \in L^d(1, I_k)$, $r = \max\{L^d(I_k)\} + 2$, and $I_k \subseteq I$.

Proof. Since $r = \max\{L^d(I_k)\} + 2$ and $L^d(I_k) \subseteq L^d(I_k)$, for $i = 1, 2$ (by Observation 1), $\max\{L^d(I_k)\} + 2 = 2$. That is, $\max\{L^d(I_k)\} + 2$ for all $I_k \subseteq L^d(I_k)$, and $\max\{L^d(I_k)\} + 2$ for all $I_k \subseteq L^d(I_k)$. Hence, $r$ is the valid $L(2, 1)$-label of $I_k$; therefore, $r = \max\{L^d(I_k)\}$. This shows that $L^d(2, I_k)$ is a nonempty set. Also, let $B$ be an arbitrary set of labels which satisfies $L(2, 1)$-labelling conditions, $l \leq r$ for every $l \in B$, and $r = \max\{L^d(I_k)\} + 2$. Then, for $b \in B$, $\max\{L^d(I_k)\} + 2 = 2$. Since, $b \in B \Rightarrow b \in L^d(2, I_k)$. So, $B \subseteq L^d(2, I_k)$. Since $B$ is arbitrary, $L^d(2, I_k)$ is the largest nonempty set of labels which satisfies $L(2, 1)$-labelling, $l \leq r$ for every $l \in L^d(2, I_k)$, and $r = \max\{L^d(I_k)\} + 2$, for any $I_k \subseteq I$. $\square$

Theorem 5. Algorithm 3 correctly labels an IG by SL21-labelling, where $n = 3\Delta - 1$.

Proof. Let $G$ be an IG with $n$ nodes such that $n = 3\Delta - 1$. We rearranged the nodes so that no two consecutive intervals are adjacent to each other. After rearrangement of the intervals, let $I = \{I_1, I_2, \ldots, I_k\}$ and let $f_1 = 1$ and $L^d(I) = \{1\}$.

We consider circumstances in which the intervals $I_1, I_2, \ldots, I_k$ are already labelled for $k = 2, 3, \ldots, n$ and the remaining intervals are not labelled. In this stage, our aim is to label $I_k$ by $L^d(1, I_k)$. Now, $L^d(2, I_k)$ is the nonempty largest set of labels satisfying $L(2, 1)$-labelling, $l \leq r$ for every $l \in L^d(2, I_k)$, and $r = \max\{L^d(I_k)\} + 2$ for any $I_k \subseteq I$ (by Lemma 4).

Again, $L^d(1, I_k) = L^d(2, I_k) - L^d(I_k)$, so $L^d(1, I_k)$ is the nonempty largest set satisfying SL21-labelling, as the label in $L^d(1, I_k)$ was not used previously to label any interval and also satisfies $L(2, 1)$-labelling. Therefore, $f_k = q$, where $q = \min\{L^d(2, I_k)\}$. Since $L^d(2, I_k)$ is the largest set of labels satisfying SL21-labelling, $q$ is the least surjective label of $I_k$. Since $L^d(I_k) \subseteq L^d(1, I_k)$, the interval $I_k$ is labelled by using only the labels from $\{1, 2, \ldots, n\}$ which have not been used earlier to label any interval. Since $I_k$ is arbitrary, any IG is surjectively labelled by $L(2, 1)$-labelling by Algorithm 3. $\square$
Lemma 2 we see that by algorithm AMPSL21 one can compute the sets $L^v(I_k)$ for $k = 2, 3, \ldots, n$.

\medskip

\textbf{Algorithm 2: AMPKVL.}

\medskip

\textbf{Algorithm 3: AMPSL21.}

\medskip

\textbf{Algorithm 4: AMPKL.}

\medskip

\textbf{Theorem 6.} The running time of Algorithm 3 is $O(n\Delta^3)$, where $n = 3\Delta - 1$. 

\medskip

Proof. According to Algorithm 3, the SL21-label of the interval $I_k$ is computed. Now by Lemma 2 we see that by algorithm AMPSL21 one can compute the set $L^v(I_k)$ for $k = 2, 3, \ldots, n$. By Lemma 2 we see that the running time of Algorithm 3 is $O(n\Delta^3)$, that is, $O(n\Delta^3)$. 

4.1.1. Illustration of Algorithm AMPSL21. We take an IG having 14 nodes (see Figure 9) and label that graph by Algorithm 3. The graph after completion of surjective $L(2, 1)$-labelling is given in Figure 10.

For the above graph, the set of intervals $I = \{I_1, I_2, \ldots, I_{14}\}$ and $\Delta = 5$. Here, $3\Delta - 1 = 14 = n$, so this IG can be
surjectively labelled by $L(2,1)$-labelling. $f^k_j$ is the SL21-label of the interval $I_k$, for $k = 1, 2, \ldots, 14$.

According to Algorithm 3, at first, we rearrange the intervals as follows:

$I_2 = I_3, I_5 = I_6, I_8 = I_9, I_10 = I_11, I_12 = I_{13}, I_{14}$ remain the same.

$f^1_1 = 1$ and $L^1(I_2) = \{1\}$ are also initialized.

**Iteration 1:** For $k = 2$,

$L^1(I_2) = \{1\}, L^2(I_2) = \phi.$

$L^3(I_2) = \{1, 2, 3\}, L^4(I_2) = \{1, 2, 3\}$.

So, $L^v(I_2) = L^3(I_2) - L^4(I_2) = \{1, 2, 3\} - \{1\} = \{2, 3\}$.

Therefore, $f^2_2 = \min\{L^v(I_2)\} = 2$.

$L^v(I_3) = L^v(I_2) \cup \{f^2_2\} = \{1\} \cup \{2\} = \{1, 2\}$.

**Iteration 2:** For $k = 3$,

$L^1(I_3) = \{1\}, L^2(I_3) = \phi.$

$L^3(I_3) = \{3, 4\}, L^4(I_3) = \{3, 4\}$.

So, $L^v(I_3) = L^3(I_3) - L^4(I_3) = \{3, 4\} - \{3, 4\} = \{3, 4\}$.

Therefore, $f^3_3 = \min\{L^v(I_3)\} = 3$.

$L^v(I_4) = L^v(I_3) \cup \{f^3_3\} = \{1, 2, 3\}$.

**Iteration 3:** For $k = 4$,

$L^1(I_4) = \phi, L^2(I_4) = \phi.$

$L^3(I_4) = \{1, 2, 3, 4, 5\}, L^4(I_4) = \{1, 2, 3, 4, 5\}$.

So, $L^v(I_4) = L^3(I_4) - L^4(I_4) = \{1, 2, 3, 4, 5\} - \{1, 2, 3, 4, 5\} = \{\}$.

Therefore, $f^4_4 = \min\{L^v(I_4)\} = 4$.

$L^v(I_5) = L^v(I_4) \cup \{f^4_4\} = \{1, 2, 3, 4\}$.

**Iteration 4:** For $k = 5$,

$L^1(I_5) = \{1, 3\}, L^2(I_5) = \phi.$

$L^3(I_5) = \{5, 6\}, L^4(I_5) = \{5, 6\}$.

So, $L^v(I_5) = L^3(I_5) - L^4(I_5) = \{5, 6\} - \{5, 6\} = \{\}$.

Therefore, $f^5_5 = \min\{L^v(I_5)\} = 5$.

$L^v(I_6) = L^v(I_5) \cup \{f^5_5\} = \{1, 2, 3, 4, 5\}$.
Iteration 5: For $k = 6$,

$L^1(I_6) = \{4\}, L^2(I_6) = \phi.$

$L^1(I_6) = \{1, 2, 6, 7\}, L^2(I_6) = \{1, 2, 6, 7\}.$

So, $L^{10}(I_6) = L^{11}(I_6) = L^{12}(I_6) = \{1, 2, 6, 7\} - \{1, 2, 3, 4, 5\} = \{6, 7\}.$

Therefore, $f_6^1 = \min[L^{10}(I_6)] = 6.$

$L^2(I_6) = L^1(I_6) \cup \{f_6^2\} = \{1, 2, 3, 4, 5, 6\}.$

In this way, $f_7^2 = 7$, $f_8^3 = 8$, $f_9^3 = 9$, $f_{10}^4 = 10$, $f_{11}^4 = 11$, $f_{12}^4 = 12$, $f_{13}^4 = 13$, and, finally, $f_{14}^4 = 14$.

The nodes and the corresponding labels are shown in Table 2.

5. Conclusion

In $L(2,1)$-labelling, although there is a light chance to overlap the frequencies in radio network, it cannot be neglected, but in $SL21$-labelling there is no chance to overlap the frequencies, as in this case the labels are distinct. The results about $SL21$-labelling are clearly welcome. In the future, we can extend this work to other classes of intersection graph. So, there is a scope for the new research to study surjective labelling of permutation graph, trapezoid graph, and so forth.

Appendix

Here, an algorithm to compute $A-B$ is presented, where $A$ and $B$ are subsets of $\{1, 2, \ldots, 4\Delta-2\}$ (Algorithm 4).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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