Research Article

Maximum Reciprocal Degree Resistance Distance Index of Bicyclic Graphs

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Received 22 March 2021; Accepted 7 June 2021; Published 19 June 2021

Academic Editor: Junwei Wang

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The reciprocal degree resistance distance index of a connected graph G is defined as $RDR(G) = \sum_{\{u,v\} \subseteq V(G)} ((d_G(u) + d_G(v))/r_G(u, v))$, where $r_G(u, v)$ is the resistance distance between vertices u and v in G . Let $\mathcal{B}_n$ denote the set of bicyclic graphs without common edges and with n vertices. We study the graph with the maximum reciprocal degree resistance distance index among all graphs in $\mathcal{B}_n$ and characterize the corresponding extremal graph.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph of order n with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. For any vertex, $u \in V(G)$, denoted by $d_G(u)$ the degree of vertex u . The distance between vertices of u and v is defined as the length of the shortest path between them and denoted by $d_G(u, v)$. The resistance distance between vertices of u and v , denoted by $r_G(u, v)$, is the effective resistance between the two nodes of the electronic network obtained so that its nodes correspond to the vertices of G and each edge of G is replaced by a resistor of unit resistance.

Topological indices are numbers related to molecular structures, used as quantitative relationships between chemical structures and properties. Some of them are based on the distance of graph [1], the vertex degree [2], or the resistance distance. The first topological index was published by Wiener [3]. From then, many topological indices are defined, such as the Harary index [4], Kirchhoff index [5, 6], the first and the second Zagreb indices [7, 8], and degree distance [9, 10].

Recently, Alizadeh and Iranmanesh [11] proposed a new topological index, the reciprocal degree distance, which is defined as

$$RDD(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} \frac{d_G(u) + d_G(v)}{d_G(u, v)}. \quad (1)$$

Its lower and upper bounds were obtained in [12], and its chemical applications and mathematical properties have been well studied in [11, 13].

Especially, analogous to the reciprocal degree distance index $RDD(G)$, Cai et al. [14] introduced a new graph invariant based on both the vertex degree and the resistance, named the reciprocal degree resistance distance index, shown as follows:

$$RDR(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{d_G(u) + d_G(v)}{r_G(u, v)}. \quad (2)$$

A bicyclic (or unicyclic) graph is a connected graph of order n with $n + 1$ (or n) edges. Let $\mathcal{B}_n$ be the set of bicyclic graphs without common edges of order n . Tree, unicyclic graph, and bicyclic graph are three kinds of graphs with a simple structure; the reciprocal degree resistance distance index of tree is the same as the reciprocal degree distance, Cai et al. [14] determined the graph with the maximum reciprocal degree resistance distance index among all unicyclic graph and characterized the corresponding extremal

graph. In this paper, we determine the graph with the maximum reciprocal degree resistance distance index among all graphs in $\mathcal{B}_n$ and characterize the corresponding extremal graph.

This paper is organized as follows: in the second part, we give three types of transformation, edge-lifting transformation, cycle-lifting transformation, and cycle-shrinking transformation, to keep the reciprocal degree resistance distance index increasing. In the third part, we give the maximum reciprocal degree resistance distance index among all graphs in $\mathcal{B}_n$.

2. Preliminaries

Let $C_g = v_1 v_2, \dots, v_g v_1$ be the cycle graph with girth $g \geq 3$. For any two vertices $v_i, v_j \in V(C_g)$ with $i < j$, by Ohm's law, one has

$$r_{C_g}(v_i, v_j) = \frac{(j-i)(g+i-j)}{g}. \quad (3)$$

Lemma 1 (See [6]). *Let x be a cut vertex of a connected graph G and s and t be two vertices occurring in different components which arise upon the deletion of x . Then,*

$$r_G(s, t) = r_G(s, x) + r_G(x, t). \quad (4)$$

2.1. Edge-Lifting Transformation. Let G' be obtained from G by an edge-lifting transformation at $e = \{u_0, v_0\}$, see Figure 1. See [14] for specific description.

Lemma 2 (See [14]). *If G' can be obtained from G by an edge-lifting transformation, see Figure 1, then $RDR(G) < RDR(G')$.*

2.2. Cycle-Lifting Transformation. Let G_p^i be the star graph K_{1,s_i} ($1 \leq i \leq p$) with order $s_i + 1$ and G_q^j be the star graph K_{1,t_j} ($2 \leq j \leq q$) with order $t_j + 1$. Let $\mathcal{C}$ be a bicyclic graph with exactly two cycles, which are $C_p = v_1 v_2, \dots, v_p v_1$ and $C_q = v_1 u_2, \dots, u_q v_1$ ($v_i \neq u_j, 1 \leq i \leq p, 2 \leq j \leq q$), and G is obtained from $\mathcal{C}$ by identifying v_i with the center of G_p^i for all i ($1 \leq i \leq p$) and u_j with the center of G_q^j for all j ($2 \leq j \leq q$). Deleting all edges in G_p^i ($2 \leq i \leq p$), joining v_1 to all pendent vertices of G_p^i ($2 \leq i \leq p$), we obtained a new graph, denoted by G' , see Figure 2. This operation is called a cycle-lifting transformation of G with respect to C_p .

Lemma 3. *If G' can be obtained from G by a cycle-lifting transformation, see Figure 2, and $\sum_{i=2}^p s_i > 0$, then $RDR(G) < RDR(G')$.*

Proof. Let G and G' be as shown in Figure 2. By the definition of $RDR(G)$,

$$\begin{aligned} RDR(G) = & \sum_{i=1}^p \sum_{x,y \in V(G_p^i) \setminus \{v_i\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \frac{d_G(v_i) + d_G(x)}{r_G(v_i, x)} + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{y \in V(C_p) \setminus \{v_i\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ & + \sum_{i,j=1, i \neq j}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{y \in V(G_p^j) \setminus \{v_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ & + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{j=2}^q \sum_{y \in V(G_q^j) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ & + \sum_{x,y \in V(C_p)} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{x \in V(C_p)} \sum_{y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{j=2}^q \sum_{x \in V(C_p)} \sum_{y \in V(G_p^i) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ & + \sum_{x,y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ & + \sum_{j=2}^q \sum_{x \in V(C_q) \setminus \{v_1\}} \sum_{y \in V(G_p^i) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{j=2}^q \sum_{x,y \in V(G_p^i) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ & + \sum_{i,j=2, i \neq j}^q \sum_{x \in V(G_q^i) \setminus \{u_i\}} \sum_{y \in V(G_q^j) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)}. \end{aligned} \quad (5)$$

FIGURE 1: The edge-lifting transformation.

FIGURE 2: The cycle-lifting transformation.

$$\begin{aligned}
 \text{RDR}(G') = & \sum_{i=1}^p \sum_{x,y \in V(G_p^i) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{y \in V(C_p) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
 & + \sum_{i,j=1 \neq j}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{y \in V(G_p^j) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
 & + \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{j=2}^q \sum_{y \in V(G_q^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
 & + \sum_{x,y \in V(C_p)} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{x \in V(C_p)} \sum_{y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{j=2}^q \sum_{x \in V(C_p)} \sum_{y \in V(G_p^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
 & + \sum_{x,y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
 & + \sum_{j=2}^q \sum_{x \in V(C_q) \setminus \{v_1\}} \sum_{y \in V(G_p^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{j=2}^q \sum_{x,y \in V(G_p^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
 & + \sum_{i,j=2 \neq j}^q \sum_{x \in V(G_q^i) \setminus \{u_i\}} \sum_{y \in V(G_q^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)}.
 \end{aligned} \tag{6}$$

- (1) Noting that $d_G(x) + d_G(y) = d_{G'}(x) + d_{G'}(y)$, $r_G(x, y) = r_{G'}(x, y)$ for $x, y \in V(G_p^i) \setminus \{v_1\}$ of G and $x, y \in V(G_p^i) \setminus \{v_1\}$ of G' , we have

$$\sum_{i=1}^p \sum_{x,y \in V(G_p^i) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{i=1}^p \sum_{x,y \in V(G_p^i) \setminus \{v_i\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = 0. \quad (7)$$

(2) Noting that $r_G(v_i, x) = r_{G'}(v_1, x) = 1$, $d_G(v_1) + d_G(x) = s_1 + 5$ for $x \in V(G_p^1) \setminus \{v_1\}$ and $d_G(v_i) + d_G(x) = s_i + 3$ for $x \in V(G_p^i) \setminus \{v_i\}$ ($2 \leq i \leq p$) of G , and

$d_{G'}(v_1) + d_{G'}(x) = s_1 + \dots + s_p + 5$ for $x \in V(G_p^i) \setminus \{v_1\}$ of G' , we have

$$\sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, y)} - \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \frac{d_G(v_i) + d_G(x)}{r_G(v_i, y)} = \sum_{i,j=1, i \neq j}^p s_i \cdot s_j + 2 \sum_{i=2}^p s_i. \quad (8)$$

(3) Noting that $d_G(x) + d_G(v_1) = s_1 + 5$, $r_G(x, v_1) = 1 + r_G(v_i, v_1)$ for $x \in V(G_p^i) \setminus \{v_i\}$ ($2 \leq i \leq p$) of G , $d_G(x) + d_G(y) = s_j + 3$, $r_G(x, y) = 1 + r_G(v_i, v_j)$, for $x \in V(G_p^i) \setminus \{v_i\}$ ($1 \leq i \leq p$), $y = v_j \in V(C_p) \setminus \{v_1, v_i\}$ of

G , and $d_{G'}(x) + d_{G'}(y) = 3$, $r_{G'}(x, y) = 1 + r_{G'}(v_1, y)$ for $x \in V(G_p^i) \setminus \{v_1\}$, $y \in V(C_p) \setminus \{v_1\}$ of G' , we have

$$\begin{aligned} & \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{y \in V(C_p) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{y \in V(C_p) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ &= - \left(\sum_{i,j=1, i \neq j}^p \frac{s_i \cdot s_j}{1 + r_G(v_i, v_j)} + 2 \sum_{i=2}^p \frac{s_i}{1 + r_G(v_i, v_1)} \right). \end{aligned} \quad (9)$$

(4) Noting that $d_G(x) + d_G(y) = d_{G'}(x) + d_{G'}(y)$, $r_G(x, y) = r_G(x, v_i) + r_G(v_i, v_j) + r_G(v_j, y) > r_{G'}(x, v_1) + r_{G'}(v_1, y) = r_{G'}(x, y)$ for $x \in V(G_p^i) \setminus \{v_i\}$, $y \in V(G_p^j) \setminus \{v_j\}$ ($1 \leq i, j \leq p, i \neq j$) of G and $x \in V(G_p^i) \setminus \{v_1\}$, $y \in V(G_p^j) \setminus \{v_1\}$ ($1 \leq i, j \leq p, i \neq j$) of G' , similarly for $x \in V(G_p^i) \setminus \{v_i\}$ ($1 \leq i \leq p$), $y \in V(C_q) \setminus \{v_1\}$ and $x \in V(G_p^i) \setminus \{v_1\}$ ($1 \leq i \leq p$), $y \in$

$V(C_q) \setminus \{v_1\}$ of G and G' , respectively, and for $x \in V(G_p^i) \setminus \{v_i\}$ ($1 \leq i \leq p$), $y \in V(G_q^j) \setminus \{u_j\}$ ($2 \leq j \leq q$) and $x \in V(G_p^i) \setminus \{v_1\}$ ($1 \leq i \leq p$), $y \in V(G_q^j) \setminus \{u_j\}$ ($2 \leq j \leq q$) of G and G' , respectively, we have the same conclusions; thus,

$$\begin{aligned} & \left(\sum_{i,j=1, i \neq j}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{y \in V(G_p^j) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{i,j=1, i \neq j}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{y \in V(G_p^j) \setminus \{v_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \right) \\ &+ \left(\sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \right) \\ &+ \left(\sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_1\}} \sum_{j=2}^q \sum_{y \in V(G_q^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{i=1}^p \sum_{x \in V(G_p^i) \setminus \{v_i\}} \sum_{j=2}^q \sum_{y \in V(G_q^j) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \right) > 0 \end{aligned} \quad (10)$$

presents.

(5) We note that $d_G(v_1) = s_1 + 4$, $d_G(v_i) = s_i + 2$ ($2 \leq i \leq p$) for any $v_i \in V(C_p)$ of G and $d_{G'}(v_1) = s_1 + \dots + s_p + 4$, $d_{G'}(v_i) = 2$ ($2 \leq i \leq p$) for

any $v_i \in V(C_p)$ of G' . Let $r_G(v_i, v_j) = r_{G'}(v_i, v_j) = r_t$, when $|j - i| = t$; then, $r_t = r_{p-t}$. Thus, we have

$$\sum_{x, y \in V(C_p)} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{i=2}^p \frac{d_G(v_1) + d_G(v_i)}{r_G(v_1, v_i)} + \sum_{i=3}^p \frac{d_G(v_2) + d_G(v_i)}{r_G(v_2, v_i)} + \dots + \frac{d_G(v_{p-1}) + d_G(v_p)}{r_G(v_{p-1}, v_p)}, \quad (11)$$

and then, we have

$$\begin{aligned} \sum_{i=2}^p \frac{d_G(v_1) + d_G(v_i)}{r_G(v_1, v_i)} &= \frac{s_1 + s_2 + 6}{r_1} + \frac{s_1 + s_3 + 6}{r_2} + \dots + \frac{s_1 + s_p + 6}{r_{p-1}}, \\ \sum_{i=3}^p \frac{d_G(v_2) + d_G(v_i)}{r_G(v_2, v_i)} &= \frac{s_2 + s_3 + 4}{r_1} + \dots + \frac{s_2 + s_p + 4}{r_{p-2}}, \\ &\dots \\ \frac{d_G(v_{p-1}) + d_G(v_p)}{r_G(v_{p-1}, v_p)} &= \frac{s_{p-1} + s_p + 4}{r_1}. \end{aligned} \quad (12)$$

Thus,

$$\begin{aligned} \sum_{x, y \in V(C_p)} \frac{d_G(x) + d_G(y)}{r_G(x, y)} &= \frac{(s_1 + \dots + s_{p-1}) + (s_2 + \dots + s_p) + 4(p-2) + 6}{r_1} \\ &+ \frac{(s_1 + \dots + s_{p-2}) + (s_3 + \dots + s_p) + 4(p-3) + 6}{r_2} + \dots + \frac{s_1 + s_p + 6}{r_{p-1}}. \end{aligned} \quad (13)$$

Similarly,

$$\sum_{x, y \in V(C_p)} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} = \frac{(s_1 + \dots + s_p) + 4(p-2) + 6}{r_1} + \frac{(s_1 + \dots + s_p) + 4(p-3) + 6}{r_2} + \dots + \frac{s_1 + \dots + s_p + 6}{r_{p-1}}. \quad (14)$$

Then,

$$\sum_{x, y \in V(C_p)} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x, y \in V(C_p)} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \frac{s_2 + \dots + s_{p-1}}{r_1} + \frac{s_3 + \dots + s_{p-2}}{r_2} + \dots + \frac{-(s_2 + \dots + s_{p-1})}{r_{p-1}} = 0. \quad (15)$$

(6) It is noted that $d_G(v_1) = s_1 + 4$, $d_G(v_i) = s_i + 2$ ($2 \leq i \leq p$), $d_G(u_j) = t_j + 2$ ($2 \leq j \leq q$) for $v_i \in V(C_p)$ and $u_j \in V(C_q) \setminus \{v_1\}$ of G . Similarly, It is noted that $d_{G'}(v_1) = \sum_{i=1}^p s_i + 4$, $d_{G'}(v_i) = 2$ ($2 \leq i \leq p$), $d_{G'}(u_j) = t_j + 2$ ($2 \leq j \leq q$), for $v_i \in V(C_p)$ and

$u_j \in V(C_q) \setminus \{v_1\}$ of G' . Let $r_G(v_i, v_1) = r_p^t$, when $i - 1 = t$ ($2 \leq i \leq p, 1 \leq t \leq p - 1$). Let $r_G(v_1, u_j) = r_q^s$, when $j - 1 = s$ ($2 \leq j \leq q, 1 \leq s \leq q - 1$). Then, $r_G(x, y) = r_G(v_i, v_1) + r_G(v_1, u_i) = r_p^t + r_q^s = r_{G'}(x, y)$ for $x \in V(C_p)$ and $y \in V(C_q) \setminus \{v_1\}$. Thus, we have

$$\sum_{x \in V(C_p), y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{i=2}^q \frac{d_G(v_1) + d_G(u_i)}{r_G(v_1, u_i)} + \sum_{i=2}^q \frac{d_G(v_2) + d_G(u_i)}{r_G(v_2, u_i)} + \dots + \sum_{i=2}^q \frac{d_G(v_p) + d_G(u_i)}{r_G(v_p, u_i)}, \quad (16)$$

and then, we have

$$\begin{aligned} \sum_{i=2}^q \frac{d_G(v_1) + d_G(u_i)}{r_G(v_1, u_i)} &= \frac{s_1 + t_2 + 6}{r_q^1} + \frac{s_1 + t_3 + 6}{r_q^2} + \dots + \frac{s_1 + t_q + 6}{r_q^{q-1}}, \\ \sum_{i=2}^q \frac{d_G(v_2) + d_G(u_i)}{r_G(v_2, u_i)} &= \frac{s_2 + t_2 + 4}{r_p^1 + r_q^1} + \frac{s_2 + t_3 + 4}{r_p^1 + r_q^2} + \dots + \frac{s_2 + t_q + 4}{r_p^1 + r_q^{q-1}}, \end{aligned} \quad (17)$$

...

$$\sum_{i=2}^q \frac{d_G(v_p) + d_G(u_i)}{r_G(v_p, u_i)} = \frac{s_p + t_2 + 4}{r_p^{p-1} + r_q^1} + \frac{s_p + t_3 + 4}{r_p^{p-1} + r_q^2} + \dots + \frac{s_p + t_q + 4}{r_p^{p-1} + r_q^{q-1}}.$$

Similarly,

$$\sum_{x \in V(C_p), y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} = \sum_{i=2}^q \frac{d_{G'}(v_1) + d_{G'}(u_i)}{r_{G'}(v_1, u_i)} + \sum_{i=2}^q \frac{d_{G'}(v_2) + d_{G'}(u_i)}{r_{G'}(v_2, u_i)} + \dots + \sum_{i=2}^q \frac{d_{G'}(v_p) + d_{G'}(u_i)}{r_{G'}(v_p, u_i)}, \quad (18)$$

and then, we have

$$\begin{aligned} \sum_{i=2}^q \frac{d_{G'}(v_1) + d_{G'}(u_i)}{r_{G'}(v_1, u_i)} &= \frac{s_1 + \dots + s_p + t_2 + 6}{r_q^1} + \frac{s_1 + \dots + s_p + t_3 + 6}{r_q^2} + \dots + \frac{s_1 + \dots + s_p + t_q + 6}{r_q^{q-1}}, \\ \sum_{i=2}^q \frac{d_{G'}(v_2) + d_{G'}(u_i)}{r_{G'}(v_2, u_i)} &= \frac{t_2 + 4}{r_p^1 + r_q^1} + \frac{t_3 + 4}{r_p^1 + r_q^2} + \dots + \frac{t_q + 4}{r_p^1 + r_q^{q-1}}, \end{aligned} \quad (19)$$

...

$$\sum_{i=2}^q \frac{d_{G'}(v_p) + d_{G'}(u_i)}{r_{G'}(v_p, u_i)} = \frac{t_2 + 4}{r_p^{p-1} + r_q^1} + \frac{t_3 + 4}{r_p^{p-1} + r_q^2} + \dots + \frac{t_q + 4}{r_p^{p-1} + r_q^{q-1}}.$$

Thus,

$$\sum_{x \in V(C_p) y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} = - \sum_{x \in V(C_p) y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{i=2}^p \sum_{j=1}^{q-1} \frac{s_i}{r_q^j} - \sum_{i=2}^p \sum_{j=1}^{q-1} \frac{s_i}{r_p^{i-1} + r_q^j} > 0. \quad (20)$$

(7) For $x \in V(C_p), y \in V(G_p^j) \setminus \{u_j\}$ of G and G' , respectively, simulating the abovementioned calculation method of (6), we get the similar conclusion,

$$\sum_{j=2}^q \sum_{x \in V(C_p) y \in V(G_p^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{j=2}^q \sum_{x \in V(C_p) y \in V(G_p^j) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{i=2}^p \sum_{j=2}^q \frac{s_i \cdot t_j}{r_q^{j-1} + 1} - \sum_{i=2}^p \sum_{j=2}^q \frac{s_i \cdot t_j}{r_p^{i-1} + r_q^{j-1} + 1} > 0. \quad (21)$$

(8) Obviously,

$$\begin{aligned} & \sum_{x, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{j=2}^q \sum_{x \in V(C_q) \setminus \{v_1\} y \in V(G_p^j) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\ & + \sum_{j=2}^q \sum_{x, y \in V(G_p^j) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{i, j=2i \neq j} \sum_{x \in V(G_q^i) \setminus \{u_i\} y \in V(G_q^j) \setminus \{u_j\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)}, \\ & = \sum_{x, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{j=2}^q \sum_{x \in V(C_q) \setminus \{v_1\} y \in V(G_p^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\ & + \sum_{j=2}^q \sum_{x, y \in V(G_p^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{i, j=2i \neq j} \sum_{x \in V(G_q^i) \setminus \{u_i\} y \in V(G_q^j) \setminus \{u_j\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)}. \end{aligned} \quad (22)$$

Thus, combining (1)–(8), we get $RDR(G') - RDR(G) > 0$, and the results are as follows.

Corollary 1. *If G' can be obtained from G by a cycle-lifting transformation, see Figure 3, and $\sum_{j=2}^q t_j > 0$, then $RDR(G) < RDR(G')$.*

2.3. Cycle-Shrinking Transformation. Let $G = S_n^{p,q}$, which is obtained by coalescing C_p and C_q at the vertex v_1 and attaching $n - p - q + 1$ pendent edges to the vertex v_1 . We note that $p, q \geq 3, p + q \leq n + 1$. Deleting the edges $v_2 v_3, \dots, v_{p-1} v_p$ and adding the edges $v_3 v_1, v_4 v_1, \dots,$

$v_{p-1} v_1, v_2 v_p$, we get $G' = S_n^{3,q}$. This operation is called cycle-shrinking transformation. It is denoted by $(w(w'), \text{resp})$, the set of pendent vertices of $(G(G'), \text{resp})$. Let $|w| = k \geq 0$; w' can be partitioned into two subsets: one has k vertices, which is naturally corresponding to w (also denoted by w), and another has $p - 3$ vertices, denoted by $\tilde{w}$.

Lemma 4. *Let $G = S_n^{p,q}$ ($p \geq 4, p + q \leq n + 1$); if G' can be obtained from G by a cycle-shrinking transformation, see Figure 4, then $RDR(G) < RDR(G')$.*

Proof. Let G and G' be as shown in Figure 4. By the definition of $RDR(G)$,

FIGURE 3: The cycle-lifting transformation.

FIGURE 4: The cycle-shrinking transformation.

$$\begin{aligned}
 \text{RDR}(G) = & \sum_{x,y \in w} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{x \in w} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, y)} + \sum_{x \in w, y \in \{v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{x \in w, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\
 & + \sum_{x \in w, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{x \in \{v_2, v_p\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, y)} + \sum_{x \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, y)} \\
 & + \sum_{x \in V(C_q) \setminus \{v_1\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} + \frac{d_G(v_2) + d_G(v_p)}{r_G(v_2, v_p)} + \sum_{x \in \{v_2, v_p\}, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\
 & + \sum_{x \in \{v_2, v_p\}, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \\
 & + \sum_{x, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{x \in V(C_p) \setminus \{v_1, v_2, v_p\}, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} + \sum_{x, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)},
 \end{aligned}$$

$$\begin{aligned}
\text{RDR}(G') = & \sum_{x,y \in w} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{x \in w} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} + \sum_{x \in w, y \in \{v_2, v_p\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{x \in w, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
& + \sum_{x \in w, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{x \in \{v_2, v_p\}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} + \sum_{x \in \tilde{w}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} \\
& + \sum_{x \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} + \frac{d_{G'}(v_2) + d_{G'}(v_p)}{r_{G'}(v_2, v_p)} \\
& + \sum_{x \in \{v_2, v_p\}, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{x \in \{v_2, v_p\}, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{x, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} \\
& + \sum_{x \in \tilde{w}, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} + \sum_{x, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)}. \tag{23}
\end{aligned}$$

(1) Noting that $d_G(x) + d_G(y) = d_{G'}(x) + d_{G'}(y)$, $r_G(x, y) = r_{G'}(x, y)$ for any $x, y \in w$ of G and G' , respectively, we have

$$\sum_{x, y \in w} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{x, y \in w} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)}. \tag{24}$$

(2) Noting that $d_G(v_1) + d_G(x) = k + 5$, $d_{G'}(v_1) + d_{G'}(x) = k + p + 2$, $r_G(v_1, x) = r_{G'}(v_1, x) = 1$ for $x \in w$ in G and G' , respectively, we have

$$\begin{aligned}
& \sum_{x \in w} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} - \sum_{x \in w} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \\
& = k(k + p + 2) - k(k + 5) = k(p - 3). \tag{25}
\end{aligned}$$

(3) Noting that $d_G(x) + d_G(y) = d_{G'}(x) + d_{G'}(y) = 3$, $r_G(x, y) = ((p - 1)/p) + 1 \geq r_{G'}(x, y) = (2/3) + 1$ for any $x \in w, y \in \{v_2, v_p\}$ in G and $x \in w, y \in \{v_2, v_p\}$ in G' , we have

$$\sum_{x \in w, y \in \{v_2, v_p\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in w, y \in \{v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} > 0. \tag{26}$$

(4) Noting that $d_G(x) + d_G(y) = 3, r_G(x, y) = r_G(v_i, v_1) + 1 > 2$ ($3 \leq i \leq p - 1$) for $x \in w, y \in V(C_p) \setminus \{v_1, v_2, v_p\}$ and $d_{G'}(x) + d_{G'}(y) = 2, r_{G'}(x, y) = 2$ for $x \in w, y \in \tilde{w}$, we have

$$\sum_{x \in w, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in w, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \geq k(p - 3) - \frac{3k(p - 3)}{2} = -\frac{k(p - 3)}{2}. \tag{27}$$

Thus, combining (2) and (4), we have

$$\begin{aligned}
& \left(\sum_{x \in w} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} - \sum_{x \in w} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \right) + \left(\sum_{x \in w, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in w, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \right) \\
& \geq \frac{k(p - 3)}{2} \geq 0. \tag{28}
\end{aligned}$$

(5) Noting that $d_G(x) + d_G(y) = d_{G'}(x) + d_{G'}(y)$, $r_G(x, y) = r_{G'}(x, y)$ for any $x \in w, y \in V(C_q) \setminus \{v_1\}$ of G and G' , respectively, we have

$$\sum_{x \in w, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = \sum_{x \in w, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)}. \quad (29)$$

(6) Noting that $d_G(v_1) + d_G(x) = k + 6$, $r_G(v_1, x) = ((p-1)/p)$ and $d_{G'}(v_1) + d_{G'}(x) = k + p + 3$, $r_{G'}(v_1, x) = (2/3)$ for $x \in \{v_2, v_p\}$ of G and G' , respectively, we have

$$\begin{aligned} & \sum_{x \in \{v_2, v_p\}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} - \sum_{x \in \{v_2, v_p\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \\ &= 3(k + p + 3) - \frac{2p(k + 6)}{p - 1} = \frac{(k + 3p + 3)(p - 3)}{p - 1}. \end{aligned} \quad (30)$$

(7) Noting that $d_G(v_1) + d_G(x) = k + 6$, $r_G(v_1, x) = r_G(v_1, v_i) > 1$ ($3 \leq i \leq p-1$) for $x \in V(C_p) \setminus \{v_1, v_2, v_p\}$ of G and $d_{G'}(v_1) + d_{G'}(x) = k + p + 2$, $r_{G'}(v_1, x) = 1$ for $x \in \tilde{w}$ of G' , we have

$$\begin{aligned} & \sum_{x \in \tilde{w}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} - \sum_{x \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \\ & \geq (p-3)(k + p + 2) - (p-3)(k + 6) = (p-3)(p-4). \end{aligned} \quad (31)$$

(8) Noting that $d_G(v_1) + d_G(x) = k + 6$, $d_{G'}(v_1) + d_{G'}(x) = k + p + 3$, $r_G(v_1, x) = r_{G'}(v_1, x)$ for $x \in V(C_q) \setminus \{v_1\}$ of G and G' , respectively, we have

$$\begin{aligned} & \sum_{x \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} - \sum_{x \in V(C_q) \setminus \{v_1\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \\ &= (p-3) \sum_{j=2}^q \frac{1}{r_G(v_1, u_j)}. \end{aligned} \quad (32)$$

(9) Noting that $d_G(v_2) + d_G(v_p) = 4$, $r_G(v_2, v_p) = ((2(p-2))/p)$ and $d_{G'}(v_2) + d_{G'}(v_p) = 4$, $r_{G'}(v_2, v_p) = (2/3)$, we have

$$\frac{d_{G'}(v_2) + d_{G'}(v_p)}{r_{G'}(v_2, v_p)} - \frac{d_G(v_2) + d_G(v_p)}{r_G(v_2, v_p)} = 6 - \frac{2p}{p-2} = \frac{4p-12}{p-2} > 0. \quad (33)$$

(10) Noting that $d_G(x) + d_G(y) = 4$ for $x \in \{v_2, v_p\}$, $y \in V(C_p) \setminus \{v_1, v_2, v_p\}$ of G and $d_{G'}(x) + d_{G'}(y) = 3$, $r_{G'}(x, y) = r_{G'}(v_2, v_1) + 1 = (2/3) + 1 = (5/3)$ for $x \in \{v_2, v_p\}$, $y \in \tilde{w}$ of G' , by the symmetry between v_2 and v_p , we have

$$\begin{aligned} & \sum_{x \in \{v_2, v_p\}, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in \{v_2, v_p\}, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} = 2 \sum_{i=3}^{p-1} \frac{3}{r_{G'}(v_2, v_1) + 1} - 2 \sum_{i=3}^{p-1} \frac{4}{r_G(v_2, v_i)}, \\ &= \frac{18(p-3)}{5} - \left(\frac{8p}{p-1} + 2 \sum_{i=4}^{p-1} \frac{4}{r_G(v_2, v_i)} \right). \end{aligned} \quad (34)$$

Since $r_G(v_2, v_i) \geq r_G(v_2, v_4) = ((2(p-2))/p)$ ($4 \leq i \leq p-1$), if $p \geq 12$, then $((2(p-2))/p) \geq (5/3)$, and

$$2 \sum_{i=4}^{p-1} \frac{4}{r_G(v_2, v_i)} \leq \frac{24(p-4)}{5}. \quad (35)$$

Thus,

$$\sum_{x \in \{v_2, v_p\}, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in \{v_2, v_p\}, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \geq \frac{18(p-3)}{5} - \left(\frac{8p}{p-1} + \frac{24(p-4)}{5} \right). \quad (36)$$

(11) Noting that $d_G(x) + d_G(y) = 4$ for $x \in \{v_2, v_p\}$, $y \in V(C_q) \setminus \{v_1\}$ of G , $d_{G'}(x) + d_{G'}(y) = 4$ for $x \in \{v_2, v_p\}$, $y \in V(C_q) \setminus \{v_1\}$ of G' , and $r_G(x, y) = r_G(v_2, v_1) + r_G(v_1, u_i) = ((p-1)/p) + r_G(v_1, u_i) \geq$

$r_{G'}(x, y) = r_{G'}(v_2, v_1) + r_{G'}(v_1, u_i) = (2/3) + r_{G'}(v_1, u_i)$ ($2 \leq i \leq q$), by symmetry between v_2 and v_p , we have

$$\sum_{x \in \{v_2, v_p\}, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in \{v_2, v_p\}, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \geq 0. \quad (37)$$

(12) Noting that $d_G(x) + d_G(y) = 4$, $r_G(x, y) = r_G(v_i, v_j)$ ($3 \leq i < j \leq p-1$) for $x, y \in V(C_p) \setminus \{v_1, v_2, v_p\}$ of

G and $d_{G'}(x) + d_{G'}(y) = 2$, $r_{G'}(x, y) = 2$ for $x, y \in \tilde{w}$ of G' , we have

$$\begin{aligned} \sum_{x, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} &= \frac{(p-3)(p-4)}{2} - \sum_{i, j=3i < j}^{p-1} \frac{4}{r_G(v_i, v_j)}, \\ \sum_{i, j=3i < j}^{p-1} \frac{4}{r_G(v_i, v_j)} &= \sum_{i=4}^p \frac{d_G(v_3) + d_G(v_i)}{r_G(v_3, v_i)} + \sum_{i=5}^p \frac{d_G(v_4) + d_G(v_i)}{r_G(v_4, v_i)} + \dots + \frac{d_G(v_{p-2}) + d_G(v_{p-1})}{r_G(v_{p-2}, v_{p-1})}. \end{aligned} \quad (38)$$

When $j - i = t$ ($i < j$), we denote $r_G(v_i, v_j) = r_t$, and then, $r_t = r_{p-t} = ((t(p-t))/p)$; thus,

$$\begin{aligned} \sum_{i=4}^p \frac{d_G(v_3) + d_G(v_i)}{r_G(v_3, v_i)} &= \frac{4}{r_1} + \frac{4}{r_2} + \dots + \frac{4}{r_{p-4}}, \\ \sum_{i=5}^p \frac{d_G(v_4) + d_G(v_i)}{r_G(v_4, v_i)} &= \frac{4}{r_1} + \dots + \frac{4}{r_{p-5}}, \\ &\dots \end{aligned} \quad (39)$$

$$\frac{d_G(v_{p-2}) + d_G(v_{p-1})}{r_G(v_{p-2}, v_{p-1})} = \frac{4}{r_1}.$$

Thus,

$$\sum_{i, j=3i < j}^{p-1} \frac{4}{r_G(v_i, v_j)} = \frac{4(p-4)}{r_1} + \frac{4(p-5)}{r_2} + \dots + \frac{4}{r_{p-4}}. \quad (40)$$

Since $r_1 = ((p-1)/p)$, $r_t \geq r_2 = ((2(p-2))/p)$ ($2 \leq t \leq p-4$). If $p \geq 12$, then $((2(p-2))/p) \geq (5/3)$, and

$$\sum_{i, j=3i < j}^{p-1} \frac{4}{r_G(v_i, v_j)} \leq \frac{4p(p-4)}{p-1} + \frac{6(p-4)(p-5)}{5}. \quad (41)$$

Thus,

$$\sum_{x, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \geq \frac{(p-3)(p-4)}{2} - \left(\frac{4p(p-4)}{p-1} + \frac{6(p-4)(p-5)}{5} \right). \quad (42)$$

When $p \geq 12$, combining (6), (7), (10), and (12), we have

$$\begin{aligned}
& \left(\sum_{x \in \{v_2, v_p\}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} - \sum_{x \in \{v_2, v_p\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \right) + \left(\sum_{x \in \tilde{w}} \frac{d_{G'}(v_1) + d_{G'}(x)}{r_{G'}(v_1, x)} - \sum_{x \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \right) \\
& + \left(\sum_{x \in \{v_2, v_p\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in \{v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \right) \\
& + \left(\sum_{x, y \in \tilde{w}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x, y \in V(C_p) \setminus \{v_1, v_2, v_p\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \right) \\
& \geq \frac{(k+3p+3)(p-3)}{p-1} + \frac{3(p-3)(p-4)}{2} + \frac{18(p-3)}{5} - \frac{4p(p-2)}{p-1} - \frac{6(p-1)(p-4)}{5} \\
& = \frac{3p^3 - 22p^2 + 53p - 114 + k(10p - 30)}{10(p-1)}.
\end{aligned} \tag{43}$$

(i) Let $f(p) = 3p^3 - 22p^2 + 53p - 114$, $p \in [12, +\infty)$; this function is strictly increasing in this interval; thus, $f(p) \geq f(12) = 2538 > 0$.

Thus,

$$\frac{3p^3 - 22p^2 + 53p - 114 + k(10p - 30)}{10(p-1)} > 0. \tag{44}$$

(13) Noting that $d_G(x) + d_G(y) = 4$, $r_G(x, y) = r_G(v_i, v_1) + r_G(v_1, u_j) \geq 1 + r_G(v_1, u_j)$ ($3 \leq i \leq p-1, 2 \leq j \leq q$) for $x \in V(C_p) \setminus \{v_1, v_2, v_p\}$, $y \in V(C_q) \setminus \{v_1\}$ of G and $d_{G'}(x) + d_{G'}(y) = 3$, $r_{G'}(x, y) = 1 + r_G(v_1, u_j)$ ($2 \leq j \leq q$) for $x, y \in \tilde{w}$ of G , we have

$$\begin{aligned}
& \sum_{x \in \tilde{w}, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in V(C_p) \setminus \{v_1, v_2, v_p\}, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)}, \\
& = (p-3) \sum_{j=2}^q \frac{3}{1 + r_G(v_1, u_j)} - \sum_{i=3}^{p-1} \sum_{j=2}^q \frac{4}{r_G(v_i, v_1) + r_G(v_1, u_j)} \\
& \geq -(p-3) \sum_{j=2}^q \frac{1}{1 + r_G(v_1, u_j)}.
\end{aligned} \tag{45}$$

Thus, combining (8) and (13), we have

$$\begin{aligned}
& \left(\sum_{x \in V(C_q) \setminus \{v_1\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} - \sum_{x \in V(C_q) \setminus \{v_1\}} \frac{d_G(v_1) + d_G(x)}{r_G(v_1, x)} \right) + \left(\sum_{x \in \tilde{w}, y \in V(C_q) \setminus \{v_1\}} \frac{d_{G'}(x) + d_{G'}(y)}{r_{G'}(x, y)} - \sum_{x \in V(C_p) \setminus \{v_1, v_2, v_p\}, y \in V(C_q) \setminus \{v_1\}} \frac{d_G(x) + d_G(y)}{r_G(x, y)} \right), \\
& = (p-3) \sum_{j=2}^q \frac{1}{r_G(v_1, u_j)} + \left(-(p-3) \sum_{j=2}^q \frac{1}{1 + r_G(v_1, u_j)} \right) > 0.
\end{aligned} \tag{46}$$

FIGURE 5: The cycle-shrinking transformation.

So, if $p \geq 12$, we get $RDR(G') - RDR(G) > 0$, and the results are as follows.

If $4 \leq p \leq 11$, $B_n(G_0) = \{S_n^{4,q}, S_n^{5,q}, S_n^{6,q}, S_n^{7,q}, S_n^{8,q}, S_n^{9,q}, S_n^{10,q}, S_n^{11,q}\}$. By comparison of direct calculation, we can obtain $RDR(S_n^{3,q}) > RDR(S_n^{p,q})$.

The proof is completed.

Corollary 2. If G' can be obtained from G by a cycle-shrinking transformation, see Figure 5, and $q \geq 4$, then $RDR(G) < RDR(G')$.

3. Maximum Reciprocal Degree Resistance Distance Index of Bicyclic Graphs

Let $G \in \mathcal{B}_n$; applying a sequence of edge-lifting transformations, cycle-lifting transformations, and cycle-shrinking transformations to G , we can obtain a new graph G' , denoted by $S_n^{3,3}$, see Figure 5. The next theorem discusses the maximum reciprocal degree resistance distance index in $\mathcal{B}_n$.

Theorem 1. Let $G \in \mathcal{B}_n$; then, $RDR(G) \leq RDR(S_n^{3,3})$, with equality holding if and only if $G \cong S_n^{3,3}$.

Proof. Let $G \in \mathcal{B}_n$ such that the $RDR(G)$ is as big as possible. By Lemma 2–4, Corollary 2, we have $RDR(G) \leq RDR(S_n^{3,3})$, and equality holds if and only if $G \cong S_n^{3,3}$.

4. Conclusions

In this paper, we characterize the maximum reciprocal degree resistance distance index among all bicyclic graphs without common edges of order n by using three types of transformation which keep the reciprocal degree resistance distance index increasing: edge-lifting transformation, cycle-lifting transformation, and cycle-shrinking transformation.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was jointly supported by the National Natural Science Foundation of China (11871077), the Natural Science Foundation of Anhui Province (1808085MA04 and 1908085MC62), and the Natural Science Foundation of Department of Education of Anhui Province (KJ2017A362).

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