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Research Article

On the Existence of Solutions for a Class of Schrödinger–Kirchhoff-Type Equations with Sign-Changing Potential

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In this paper, we consider the following Kirchhoff problem $\begin{cases} -(a+b\int_{R^3}|\nabla u|^2 \mathrm{d}x)\Delta u + \lambda V(x)u = |u|^{p-2}u, \text{ in } R^3 \\ u \in H^1(R^3) \end{cases}$ where $a, b > 0$ are constants, λ is a positive parameter, and $4 < p < 6$. Under suitable assumptions on $V(x)$, the existence of nontrivial solution is obtained via variational methods. The potential $V(x)$ is allowed to be sign-changing.

1. Introduction and Main Results

In this paper, we consider the following Kirchhoff type problem:

$$\begin{cases} -\left(a+b\int_{R^3}|\nabla u|^2 \mathrm{d}x\right)\Delta u + \lambda V(x)u = |u|^{p-2}u, & \text{in } R^3 \\ u \in H^1\left(R^3\right), \end{cases} \tag{1}$$

where $a, b > 0$ are constants, λ is a positive parameter, $4 < p < 6$, and the potential V satisfies the following conditions:

(V_1) $V \in C(R^3, R)$ and V is bounded below

(V_2) there exists a constant $c > 0$ such that the set $\{x \in R^3\colon V(x) \le c\}$ is nonempty and $\text{meas}\{x \in R^3\colon V(x) \le c\} < +\infty$, where meas denote the Lebesgue measure in R^3

This kind of assumptions was first introduced by Bartsch and Qiang Wang [1] in the study of the nonlinear Schrödinger equations and has attracted the attention of several researchers.

In recent years, the Kirchhoff problem on a bounded domain $\Omega \subset R^N$

$$\begin{cases} -\left(a+b\int_{R^3}|\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

has been studied by many authors (see, for example, [2–8]). More recently, many researchers focused on the Kirchhoff problem defined on the whole space R^3, i.e., the following problem:

$$\begin{cases} -\left(a+b\int_{R^3}|\nabla u|^2 \mathrm{d}x\right)\Delta u + \lambda V(x)u = f(x,u), & \text{in } R^3 \\ u \in H^1\left(R^3\right), \end{cases} \tag{3}$$

where $V\colon R^3 \longrightarrow R$ is a potential function and $f \in C(R^3 \times R, R)$. In [9], Wu studied (3) by using a symmetric Mountain Pass Theorem under the following assumptions about potential V

(V) $V \in C(R^3, R)$, $\inf_{x\in R^3} V(x) \geq a_0 > 0$, where $a_0 > 0$ is a constant. Moreover, for any $M > 0$, $\text{meas}\{x \in R^3: V(x) \leq M\} < +\infty$, where meas denotes the Lebesgue measure in R^3.

Under this condition, by Lemma 3.4 in [10], the embedding $H^1(R^3) \hookrightarrow L^s R^3$ is compact for any $s \in [2, 6)$. Hence, the corresponding results in [9] have been obtained by using the variational techniques in a standard way. In [11–13], the authors considered Kirchhoff type problem (3) with a steep potential well. Precisely, the potential function satisfies the following conditions besides (V_2):

(V_3) $V \in C(R^3, R)$ and $V \geq 0$ on R^3

(V_4) $\Omega = \text{int}\, V^{-1}(0)$ is a nonempty open set with locally Lipschitz boundary and $\overline{\Omega} = V^{-1}(0)$

By using this conditions, Sun and Wu [11] considered (3) in the case where the nonlinearity $f(x, s)$ is asymptotically k-linear $(k = 1, 2, 4)$ with respect to s at infinity. Du et al. [12] studied (3) when $f(x, u)$ behaves like $|u|^{p-2}u$ with $4 < p < 6$ and proved the existence and asymptotic behavior of ground state solutions. Zhang and Du [13] investigated the existence and asymptotic behavior of positive solutions for (3) by combining the truncation technique and the parameter-dependent compactness lemma for b small and λ large in the case where $f(x, u)$ behave like $|u|^{p-2}u$ with $2 < p < 4$. For more results about Kirchhoff type problems, we refer the reader to [14–18] and the references therein.

Under the assumption of (V_1), the potential V may change sign. The purpose of this paper is to consider the multiplicity of solutions for (1) in this case. To our best knowledge, there is no existence result of solutions for (1) with sign-changing potentials. Our main result as follows.

Theorem 1. *Suppose that (V_1) and (V_2) and $4 < p < 6$ hold. Then, system (1) possesses infinitely many distinct pairs of nontrivial solutions whenever $\lambda > 0$ is sufficiently large.*

2. Preliminaries

As a matter of convenience, without loss of generality, we may assume that $a = 1$ and $b = 1$. Consequently, we are dealing with the Kirchhoff type problem as

$$\begin{cases} -\left(1 + \int_{R^3} |\nabla u|^2 dx\right)\Delta u + \lambda V(x)u = |u|^{p-2}u, & \text{in } R^3 \\ u \in H^1(R^3). \end{cases} \tag{4}$$

Let

$$H^1(R^3) = \{u \in L^2(R^3): \nabla u \in L^2(R^3)\}, \tag{5}$$

be the usual Sobolev space with the standard inner product and norm as follows:

$$\begin{aligned} (u, v) &= \int_{R^3} [\nabla u \nabla v + uv]\mathrm{d}x, \\ \|u\| &= \left(\int_{R^3} \left(|\nabla u|^2 + u^2\right)\mathrm{d}x\right)^{1/2}. \end{aligned} \tag{6}$$

In our problem, we work in the space defined by

$$E_\lambda = \left\{u \in H^1(R^3): \int_{R^3} \left(|\nabla u|^2 + \lambda V^+(x)u^2\right)\mathrm{d}x < +\infty\right\}, \tag{7}$$

with the inner product and the norm as follows:

$$\begin{aligned} \langle u, v\rangle_{E_\lambda} &= \int_{R^3} (\nabla u \cdot \nabla v + \lambda V^+(x)uv)\mathrm{d}x, \\ \|u\|_{E_\lambda} &= \langle u, u\rangle_{E_\lambda}^{1/2}, \end{aligned} \tag{8}$$

where $V^{\pm}(x) = \max\{\pm V(x), 0\}$ and $V(x) = V^+(x) - V^-(x)$. It follows from the conditions (V_1) and (V_2) and the Hölder and Sobolev inequalities that

$$\begin{aligned} \int_{R^3} \left(|\nabla u|^2 + u^2\right)\mathrm{d}x &\leq \int_{R^3} |\nabla u|^2 \mathrm{d}x + |V_c|^{2/3}\left(\int_{V_c} |u|^6 \mathrm{d}x\right)^{1/3} + \frac{1}{c}\int_{R^3/V_c} V(x)u^2 \mathrm{d}x \\ &\leq \max\left\{1 + |V_c|^{2/3}S^{-1}, c^{-1}\right\}\int_{R^3} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x, \end{aligned} \tag{9}$$

which implies that the embedding $E_\lambda \hookrightarrow H^1(R^3)$ is continuous. Here, S is the best constant for the embedding of $D^{1,2}(R^3)$ in $L^6(R^3)$. Combine with the continuity of the following embedding:

$$H^1(R^3) \hookrightarrow L^s(R^3),\ 2 \leq s \leq 6. \tag{10}$$

There is a constant $a_s > 0$ such that

$$\|u\|_s \leq a_s\|u\|_{E_\lambda},\ \forall u \in E_\lambda. \tag{11}$$

As a consequence, the functional $I_\lambda: E_\lambda \longrightarrow R$ given by

$$I_\lambda(u) = \left(\frac{1}{2}\right)\int_{R^3} \left(|\nabla u|^2 + \lambda V(x)u^2\right)\mathrm{d}x + \left(\frac{1}{4}\right)\left(\int_{R^3} |\nabla u|^2\right)^2 - \left(\frac{1}{p}\right)\int_{R^3} |u|^p \mathrm{d}x. \tag{12}$$

is well defined, and it is of class C^1 with derivative

$$< I_\lambda'(u), v> = \int_{R^3} \nabla u \nabla v dx + \int_{R^3} \lambda V(x) uv dx + \int_{R^3} |\nabla u|^2 dx \int_{R^3} \nabla u \nabla v dx - \int_{R^3} |u|^{p-2} uv dx. \quad (13)$$

for all $u, v \in E_\lambda$. As in [19], let

$$F_\lambda = \{u \in E_\lambda : suppu \subset V^{-1}([0,\infty))\}, \quad (14)$$

and denote the orthogonal complement of F_λ in E_λ by $F_\lambda^\perp$. Consider the eigenvalue problem

$$-\Delta u + V^+(x) u = \mu V^-(x) u, \quad u \in F_\lambda^\perp. \quad (15)$$

In view of (V_1) and (V_2), the quadratic form $u \mapsto \int_{R^3} V^-(x) u^2 dx$ is weakly continuous. We have the following proposition.

Proposition 1 (see Lemma 2.1 in [19]). *Suppose (V_1) and (V_2) and $V^- \neq 0$ hold. Then, for each fixed j,*

$(i)\mu_j(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow +\infty$

$(ii)\mu_j(\lambda)$ is a non-increasing continuous function of λ where $\mu_j(\lambda) = \inf_{dimM \geq j, M \subset F\perp/\lambda} sup\{\|u\|_{E_\lambda}^2 : u \in M, \int_{R^3} \lambda V^-(x) u^2 dx = 1\} (j = 1, 2, 3, \ldots)$ is sequence of positive eigenvalues of problem (P) satisfying $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots \leq \mu_j(\lambda) \longrightarrow \infty$ as $j \longrightarrow \infty$ and the corresponding eigenfunctions $\{e_j(\lambda)\}_{j=1}^\infty$.

Let

$$\begin{aligned} E_\lambda^- &= \text{span}\{e_j(\lambda) : \mu_j(\lambda) \leq 1\}, \\ E_\lambda^+ &= \text{span}\{e_j(\lambda) : \mu_j(\lambda) > 1\}. \end{aligned} \quad (16)$$

Then,

$$E_\lambda = E_\lambda^- \oplus F_\lambda \oplus E_\lambda^-. \quad (17)$$

Moreover, dim $E_\lambda^- < +\infty$ for every fixed $\lambda > 0$.

To complete the proof of our theorem, we need the following results.

Theorem 2 (see Theorem 9.12 in [20]). *Let E be an infinite dimensional Banach space, and let $I \in C^1(E, R)$ be even, satisfying (PS) condition and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional and I satisfies the following:*

(I_1) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$

(I_2) for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $R = R(\widetilde{E})$ such that $I \leq 0$ on $\widetilde{E}/B_{R(\widetilde{E})}$

then I possesses an unbounded sequence of critical values.

3. Proof of Main Results

Lemma 1. *Suppose that $4 < p < 6$ and (V_1) and (V_2) hold. Then, there exist $\alpha, \rho > 0$ such that $I_\lambda(u) \geq \alpha$ for all $u \in E_\lambda$ with $\|u\|_\lambda = \rho$.*

Proof. By Proposition 1, for each fixed $\lambda > \Lambda$, there exists a positive integer n_λ such that $\mu_j(\lambda) \leq 1$ for $j < n_\lambda$ and $\mu_j(\lambda) > 1$ for $j \geq n_\lambda$. Thus, for any $u = u_1 + u_2 \in E_\lambda^+ \oplus F_\lambda$, we have

$$\begin{aligned} I_\lambda(u) &= \left(\frac{1}{2}\right)\|u\|_{E_\lambda}^2 - \left(\frac{1}{2}\right)\int_{R^3} \lambda V^-(x) u^2 dx + \left(\frac{1}{4}\right)\left(\int_{R^3} |\nabla u|^2 dx\right)^2 - \left(\frac{1}{p}\right)\int_{R^3} |u|^p dx \\ &\geq \left(\frac{1}{2}\right)\int_{R^3} \left(|\nabla u_1|^2 + \lambda V(x) u_1^2\right) dx + \left(\frac{1}{2}\right)\|u\|_{2E_\lambda}^2 - \left(\frac{1}{p}\right)\int_{R^3} |u|^p dx \\ &\geq \left(\frac{1}{2}\right)\left(1 - \left(\frac{1}{\mu_{n_\lambda}(\lambda)}\right)\right)\|u\|_{E_\lambda}^2 - \left(\frac{1}{p}\right)\|u\|_p^p \\ &\geq \left[\left(\frac{1}{2}\right)\left(1 - \left(\frac{1}{\mu_{n_\lambda}(\lambda)}\right)\right)\right]\|u\|_{E_\lambda}^2 - C\|u\|_{E_\lambda}^p \\ &\geq \left[\left(\frac{1}{2}\right)\left(1 - \left(\frac{1}{\mu_{n_\lambda}(\lambda)}\right)\right) - C\|u\|_{E_\lambda}^{p-2}\right]\|u\|_{E_\lambda}^2, \end{aligned} \quad (18)$$

for all $u \in \overline{B_\rho(0)}$, where $B_\rho(0) = \{u \in E_\lambda^+ \oplus F_\lambda : \|u\|_{E_\lambda} < \rho\}$. Since $p > 2$, the conclusion follows by choosing ρ sufficiently small. □

Lemma 2. *Suppose that $4 < p < 6$ and (V_1) and (V_2) hold. Then, there is a large $r > 0$ such that $I(u) < 0$ on $\widetilde{E}/B_r(0)$.*

Proof. Since all norms are equivalent in a finite dimensional space, there are constants $C_p > 0$ and $C > 0$ such that

$$\begin{aligned} \|u\|_{D^{1,2}(R^3)} &\leq C\|u\|_{E_\lambda}, \\ \|u\|_p &\geq C_p\|u\|_{E_\lambda}, \ \forall u \in \widetilde{E} \subset E_\lambda. \end{aligned} \quad (19)$$

where $\|u\|_{D^{1,2}(R^3)}^2 = \int_{R^3} |\nabla u|^2 dx$. Hence, for all $u \in \widetilde{E}$,

$$
\begin{aligned}
I_\lambda(u) &= \left(\frac{1}{2}\right)\|u\|_{E_\lambda}^2 - \left(\frac{1}{2}\right)\int_{R^3}\lambda V^-(x)u^2dx + \left(\frac{1}{4}\right)\left(\int_{R^3}|\nabla u|^2dx\right)^2 - \left(\frac{1}{p}\right)\int_{R^3}|u|^pdx \\
&\le \left(\frac{1}{2}\right)\|u\|_{E_\lambda}^2 + \left(\frac{C}{4}\right)\|u\|_{E_\lambda}^4 - \left(\frac{1}{p}\right)\|u\|_p^p \\
&\le \left(\frac{1}{2}\right)\|u\|_{E_\lambda}^2 + \left(\frac{C}{4}\right)\|u\|_{E_\lambda}^4 - \left(\frac{C_p}{p}\right)\|u\|_{E_\lambda}^p.
\end{aligned} \tag{20}
$$

Since $p > 4$, consequently, there is a large $r > 0$ such that $I(u) < 0$ on $\widetilde{E}/B_r(0)$. □

Lemma 3. *Let* $4 < p < 6$ *and* (V_1) *and* (V_2) *be satisfied. Then, there exists* $\Lambda > 0$ *such that, for each* $c \in R$, I_λ *satisfies the* $(PS)_c$ *condition for all* $\lambda \ge \Lambda$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence, that is, $I_\lambda(u_n) \longrightarrow c$ and $I_\lambda'(u_n) \longrightarrow 0$. If $\{u_n\}$ is unbounded in E_λ, up to a subsequence, we can assume that

$$
\begin{aligned}
\|u_n\|_{E_\lambda} &\longrightarrow +\infty, \\
I_\lambda(u_n) &\longrightarrow c, \\
\|I_\lambda'(u_n)\| &\longrightarrow 0,
\end{aligned} \tag{21}
$$

as $n \longrightarrow \infty$, after passing to a subsequence. Set $w_n = u_n/\|u_n\|_{E_\lambda}$, we can assume that $w_n \rightharpoonup w$ in E_λ and $w_n(x) \longrightarrow w(x)$ a.e. $x \in R^3$.

If $w = 0$, since $u \mapsto \int_{R^3} V^-(x)u^2dx \;\in$ is weakly continuous, we have

$$
\begin{aligned}
o(1) &= \left(\frac{1}{\|u_n\|_{E_\lambda}^2}\right)\left(I_\lambda(u_n) - \frac{1}{p}\langle I_\lambda'(u_n), u_n\rangle\right) \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) - \left(\frac{1}{2} - \frac{1}{p}\right)\int_{R^3}\lambda V^-(x)w_n^2dx + \left(\frac{1}{4} - \frac{1}{p}\right)\left(\frac{1}{\|u_n\|_{E_\lambda}^2}\right)\left(\int_{R^3}|\nabla u_n|^2dx\right)^2 \\
&\ge \frac{1}{2} - \frac{1}{p} + o(1),
\end{aligned} \tag{22}
$$

a contradiction. If $w \ne 0$, then the set $\Omega = \{x \in R^3\colon\ \omega(x) \ne 0\}$ has positive Lebesgue measure. For $x \in \Omega$, one has $|u_n(x)| \longrightarrow \infty$ as $n \longrightarrow \infty$; Fatou's lemma shows that $\int_\Omega |u_n|^{p-4}w_n^4dx \longrightarrow \infty$ as $n \longrightarrow \infty$. Thus, by (9), we obtain

$$
\begin{aligned}
1/p\int_{R^3}|u_n|^{p-4}w_n^4dx &= 1/2\|u_n\|_{E_\lambda}^2 - 1/2\|u_n\|_{E_\lambda}^2\int_{R^3}\lambda V^-(x)u_n^4dx + 1/4\|u_n\|_{E_\lambda}^4\left(\int_{R^3}|\nabla u_n|^2dx\right)^2 + o(1) \\
&\le 1/2\|u_n\|_{E_\lambda}^2 + 1/4\|u_n\|_{E_\lambda}^4\left(\int_{R^3}|\nabla u_n|^2dx\right)^2 + o(1) \\
&\le C/4 + o(1).
\end{aligned} \tag{23}
$$

This is a contradiction. This implies $\{u_n\}$ is bounded in E_λ. We assume that $\|u_n\|_{E_\lambda} \le T$. Passing to a subsequence if necessary, we can assume that there exists $u \in E_\lambda$ and $A \in R$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } E_\lambda,\\ &\int_{R^3} |\nabla u|^2 \mathrm{d}x \longrightarrow A^2,\\ &\int_{R^3} |\nabla u|^2 \mathrm{d}x \le A^2. \end{aligned} \tag{24}$$

Then, $I_\lambda'(u_n) \longrightarrow 0$ implies that

$$\int_{R^3} \nabla u \nabla v \mathrm{d}x + \int_{R^3} \lambda V(x) u v \mathrm{d}x + A^2 \int_{R^3} \nabla u \nabla v \mathrm{d}x - \int_{R^3} |u|^{p-2} u v \mathrm{d}x = 0, \quad \forall v \in E_\lambda. \tag{25}$$

Taking $v = u$ in (25), we obtain

$$\int_{R^3} |\nabla u|^2 \mathrm{d}x + \int_{R^3} \lambda V(x) u^2 \mathrm{d}x + A^2 \int_{R^3} |\nabla u|^2 \mathrm{d}x - \int_{R^3} |u|^p \mathrm{d}x = 0, \quad \forall v \in E_\lambda. \tag{26}$$

Let $v_n := u_n - u$. It follows from (V_1) and (V_2) that

$$\|v_n\|_2^2 = \int_{V(x)\ge c} v_n^2 \mathrm{d}x + \int_{V(x)<c} v_n^2 \mathrm{d}x \le \left(\frac{1}{\lambda c}\right) \|v_n\|_{E_\lambda}^2 + o(1). \tag{27}$$

Moreover, Let $0 < \alpha < \min\{6 - p/2, 1\}, 2 < p < 6$. Then, $2 < 2(p-\alpha)/2 - \alpha < 6$. By Sobolev inequalities and Hölder inequality, one has

$$\begin{aligned} \|v_n\|_p^p &= \int_{R^3} |v_n|^\alpha |v_n|^{p-\alpha} \mathrm{d}x\\ &\le \left(\int_{R^3} |v_n|^2 \mathrm{d}x\right)^{\alpha/2} \left(\int_{R^3} |v_n|^{2(p-\alpha)/2-\alpha} \mathrm{d}x\right)^{2-\alpha/2}\\ &= |v_n|_2^\alpha |v_n|_{2(p-\alpha)/2-\alpha}^{p-\alpha}\\ &\le C(\lambda c)^{-(\alpha/2)} |v_n|_{E_\lambda}^p + o(1), \end{aligned} \tag{28}$$

we know

$$\begin{aligned} o(1) &= \langle I_\lambda'(u_n), u_n \rangle\\ &= \|u_n\|_{E_\lambda}^2 - \int_{R^3} \lambda V^-(x) u_n^2 \mathrm{d}x + \|\nabla u_n\|_2^4 - \|u_n\|_p^p - \|u\|_{E_\lambda}^2\\ &\quad + \int_{R^3} \lambda V^-(x) u^2 \mathrm{d}x - A^2 \|\nabla u_n\|_2^2 + \|u\|_p^p\\ &= \|v_n\|_{E_\lambda}^2 - \|v_n\|_p^p + A^4 - A^2 \|\nabla u\|_2^2 + o(1)\\ &\ge \|v_n\|_{E_\lambda}^2 - \|v_n\|_p^{p-2} \|v_n\|_p^2 + o(1)\\ &\ge \left[1 - \left(2a_p T\right)^{p-2} C^{p-2/p} (\lambda c)^{-\alpha(p-2)/2p}\right] \|v_n\|_{E_\lambda}^2 + o(1). \end{aligned} \tag{29}$$

Letting $\Lambda > 0$ be so large that the term in the brackets above is positive when $\lambda \ge \Lambda$, we get $v_n \longrightarrow 0$ in E_λ. Since $v_n = u_n - u$ and $v_n \longrightarrow 0$, it follows that $u_n \longrightarrow u$ in E_λ. This completes the proof. □

Proof of Theorem 1. Obviously, $I(0) = 0$. Furthermore, I is even. The conclusion follows from Lemmas 1–3 and Theorem 1. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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