Research Article

On the Existence of Solutions for a Class of Schrödinger–Kirchhoff-Type Equations with Sign-Changing Potential

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In this paper, we consider the following Kirchhoff problem

$$\begin{aligned}
-\left(a+b\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)\Delta u + \lambda V(x)u &= |u|^{p-2} u, \quad \text{in } \mathbb{R}^3 \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}$$

where $a, b > 0$ are constants, $\lambda$ is a positive parameter, and $4 < p < 6$. Under suitable assumptions on $V(x)$, the existence of nontrivial solution is obtained via variational methods. The potential $V(x)$ is allowed to be sign-changing.

1. Introduction and Main Results

In this paper, we consider the following Kirchhoff type problem:

$$
\begin{aligned}
\left\{-\left(a+b\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)\Delta u + \lambda V(x)u &= |u|^{p-2} u, \quad \text{in } \mathbb{R}^3 \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
$$

where $a, b > 0$ are constants, $\lambda$ is a positive parameter, $4 < p < 6$, and the potential $V$ satisfies the following conditions:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $V$ is bounded below

(V2) there exists a constant $c > 0$ such that the set $\{x \in \mathbb{R}^3 : V(x) \leq c\}$ is nonempty and $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq c\} < +\infty$, where $\text{meas}$ denote the Lebesgue measure in $\mathbb{R}^3$

This kind of assumptions was first introduced by Bartsch and Qiang Wang [1] in the study of the nonlinear Schrödinger equations and has attracted the attention of several researchers.

In recent years, the Kirchhoff problem on a bounded domain $\Omega \subset \mathbb{R}^N$

$$
\begin{aligned}
\left\{-\left(a+b\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)\Delta u + \lambda V(x)u &= |u|^{p-2} u, \quad \text{in } \Omega \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
$$

has been studied by many authors (see, for example, [2–8]). More recently, many researchers focused on the Kirchhoff problem defined on the whole space $\mathbb{R}^3$, i.e., the following problem:

$$
\begin{aligned}
\left\{-\left(a+b\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)\Delta u + \lambda V(x)u &= f(x, u), \quad \text{in } \mathbb{R}^3 \\
u &= 0,
\end{aligned}
$$

where $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a potential function and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. In [9], Wu studied (3) by using a symmetric Mountain Pass Theorem under the following assumptions about potential $V$.
\((V)\) \(V \in C(R^3, R)\), \(\inf V(x) \geq a_0 > 0\), where \(a_0 > 0\) is a constant. Moreover, for any \(M > 0\), \(\text{mes} \{x \in R^3: V(x) \leq M\} < +\infty\), where \(\text{mes}\) denotes the Lebesque measure in \(R^3\).

Under this condition, by Lemma 3.4 in [10], the embedding \(H^1(R^3) \hookrightarrow L^s(R^3)\) is compact for any \(s \in [2, 6]\). Hence, the corresponding results in [9] have been obtained by using the variational techniques in a standard way. In [11–13], the authors considered Kirchhoff type problem (3) with a steep potential well. Precisely, the potential function satisfies the following conditions besides (V).

\[(V_2): \int V(x) \geq 0 \text{ on } R^3, \quad \text{Lipschitz boundary and } V^{-1}(0) \text{ is a nonempty open set with locally as } b \text{ small and } \lambda \text{ large in the case where } f(x, u) \text{ behave like } |u|^{p-2}u \text{ with } 2 < p < 4. \quad \text{For more results about Kirchhoff type problems, we refer the reader to [14–18] and the references therein.} \]

Under the assumption of \((V_1)\), the potential \(V\) may change sign. The purpose of this paper is to consider the multiplicity of solutions for (1) in this case. To our best knowledge, there is no existence result of solutions for (1) with sign-changing potentials. Our main result as follows.

**Theorem 1.** Suppose that \((V_1)\) and \((V_2)\) and \(4 < p < 6\) hold. Then, system (1) possesses infinitely many distinct pairs of nontrivial solutions whenever \(\lambda > 0\) is sufficiently large.

\[
\int_{R^3} (|\nabla u|^2 + u^2) \, dx \leq \int_{R^3} |\nabla u|^2 \, dx + \int_{V_1}^{2/3} \left( \int_{V_1}^{1/3} \left( \int_{R^3 \setminus V} |u|^6 \, dx \right)^{1/3} + \frac{1}{c} \int_{R^3 \setminus V} V(x) u^2 \, dx \right) \leq \max \left\{ 1 + \frac{|V_1|}{2/3}, c^{-1} \right\} \int_{R^3} (|\nabla u|^2 + V(x) u^2) \, dx,
\]

which implies that the embedding \(E_\lambda \hookrightarrow H^1(R^3)\) is continuous. Here, \(S\) is the best constant for the embedding of \(D^{1,2}(R^3)\) in \(L^s(R^3)\). Combine with the continuity of the following embedding:

\[
H^1(R^3) \hookrightarrow L^s(R^3), \quad 2 \leq s \leq 6.
\]

There is a constant \(a_0 > 0\) such that

\[
\|u\|_s \leq a_0 \|u\|_{E_\lambda} \quad \forall u \in E_\lambda.
\]

As a consequence, the functional \(I_\lambda: E_\lambda \to R\) given by

\[
I_\lambda(u) = \frac{1}{2} \int_{R^3} (|\nabla u|^2 + \lambda V(x) u^2) \, dx + \frac{1}{p} \left( \int_{R^3} |\nabla u|^2 \right)^{2/p} - \left( \frac{1}{p} \right) \int_{R^3} |u|^p \, dx.
\]

is well defined, and it is of class \(C^1\) with derivative

2. Preliminaries

As a matter of convenience, without loss of generality, we may assume that \(a = 1\) and \(b = 1\). Consequently, we are dealing with the Kirchhoff type problem as

\[
\begin{cases}
- \left( 1 + \int_{R^3} |\nabla u|^2 \, dx \right) \Delta u + \lambda V(x) u = |u|^{p-2} u, & \text{in } R^3 \\
 u \in H^1(R^3).
\end{cases}
\]

Let

\[
H^1(R^3) = \left\{ u \in L^2(R^3) : \forall \lambda, u \in L^p(R^3) \right\},
\]

be the usual Sobolev space with the standard inner product and norm as follows:

\[
\langle u, v \rangle = \int_{R^3} \nabla u \cdot \nabla v + \lambda V^+(x) u v \, dx,
\]

\[
\|u\| = \left( \int_{R^3} (|\nabla u|^2 + \lambda V^+(x) u^2) \, dx \right)^{1/2}.
\]

In our problem, we work in the space defined by

\[
E_\lambda = \left\{ u \in H^1(R^3) : \int_{R^3} (|\nabla u|^2 + \lambda V^+(x) u^2) \, dx < +
\]

with the inner product and the norm as follows:

\[
\langle u, v \rangle_{E_\lambda} = \int_{R^3} \nabla u \cdot \nabla v + \lambda V^+(x) u v \, dx,
\]

\[
\|u\|_{E_\lambda} = \left( \int_{R^3} (|\nabla u|^2 + \lambda V^+(x) u^2) \, dx \right)^{1/2},
\]

where \(V^+(x) = \max\{ \pm V(x), 0\} \) and \(V(x) = V^+(x) - V^-(x)\). It follows from the conditions \((V_1)\) and \((V_2)\) and the Hölder and Sobolev inequalities that
<I_{\lambda}(u), v> = \int_{R^{n}} \nabla u \nabla v dx + \int_{R^{n}} \lambda V(x) uv dx + \int_{R^{n}} |V u|^2 dx \int_{R^{n}} \nabla u \nabla v dx - \int_{R^{n}} |u|^{p-2} uv dx.

(13)

\[ E_{\lambda} = E_{\lambda}^{1} \oplus F_{\lambda} \oplus E_{\lambda}^{-1}. \]

(17)

Moreover, dim $E_{\lambda}^1 < + \infty$ for every fixed $\lambda > 0$.

To complete the proof of our theorem, we need the following results.

**Theorem 2** (see Theorem 9.12 in [20]). Let $E$ be an infinite dimensional Banach space, and let $I \in C^1(E,R)$ be even, satisfying (PS) condition and $I(0) = 0$. If $E = V \oplus X$, where $V$ is finite dimensional and $I$ satisfies the following:

$(i_1)$ there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_{\rho}(0)} \geq \alpha$

$(i_2)$ for each finite dimensional subspace $E \subset E$, there is an $R = R(E)$ such that $I \leq 0$ on $E/B_{\rho}(0)$

then $I$ possesses an unbounded sequence of critical values.

### 3. Proof of Main Results

**Lemma 1.** Suppose that $4 < p < 6$ and $(V_1)$ and $(V_2)$ hold. Then, there exist $\alpha, \rho > 0$ such that $I_{\lambda}(u) \geq \alpha$ for all $u \in E_{\lambda}$ with $\|u\|_{\lambda} = \rho$.

**Proof.** By Proposition 1, for each fixed $\lambda > \Lambda$, there exists a positive integer $n_{j}$ such that $\mu_{j}(\lambda) \leq 1$ for $j < n_{1}$ and $\mu_{j}(\lambda) > 1$ for $j \geq n_{3}$. Thus, for any $u = u_{1} + u_{2} \in E_{\lambda}^{1} \oplus F_{\lambda}$, we have

\[ I_{\lambda}(u) = \left( \frac{1}{2} \right) \|u\|_{E_{\lambda}}^2 - \left( \frac{1}{2} \right) \int_{R^{n}} \lambda V(x) u^{2} dx + \left( \frac{1}{4} \right) \left( \int_{R^{n}} |V u|^2 dx \right)^{2} - \left( \frac{1}{p} \right) \int_{R^{n}} |u|^{p} dx \]

\[ \geq \left( \frac{1}{2} \right) \left( 1 - \left( \frac{1}{\mu_{n_{1}}(\lambda)} \right) \right) \|u\|_{E_{\lambda}}^2 - \left( \frac{1}{p} \right) \|u\|_{p}^p \]

\[ \geq \left[ \left( \frac{1}{2} \right) \left( 1 - \left( \frac{1}{\mu_{n_{1}}(\lambda)} \right) \right) \right] \|u\|_{E_{\lambda}}^2 - \|u\|_{p}^p \]

\[ \geq \left[ \left( \frac{1}{2} \right) \left( 1 - \left( \frac{1}{\mu_{n_{1}}(\lambda)} \right) \right) \right] \|u\|_{E_{\lambda}}^2 - C \|u\|_{p}^{p-2} \]

for all $u \in B_{\rho}(0)$, where $B_{\rho}(0) = \{ u \in E_{\lambda}^{1} \oplus F_{\lambda}; \|u\|_{p} < \rho \}$.

Since $p > 2$, the conclusion follows by choosing $\rho$ sufficiently small. \(\square\)

**Lemma 2.** Suppose that $4 < p < 6$ and $(V_1)$ and $(V_2)$ hold. Then, there is a large $r > 0$ such that $I(u) < 0$ on $E/B_{r}(0)$.

**Proof.** Since all norms are equivalent in a finite dimensional space, there are constants $C_{p} > 0$ and $C > 0$ such that

\[ \|u\|_{D^{1,2}(\Omega)} \leq C \|u\|_{E_{\lambda}}, \]

\[ \|u\|_{p} \geq C_{p} \|u\|_{E_{\lambda}}, \quad \forall u \in E \subset E_{\lambda}. \]

(19)

where $\|u\|_{D^{1,2}(\Omega)} = \int_{\Omega} |Vu|^2 dx$. Hence, for all $u \in E$,
I_\lambda(u) = \left(\frac{1}{2}\right)\|u\|_{E_1}^2 - \left(\frac{1}{2}\right)\int_{R^3} \lambda V^- (x) u^2 \, dx + \left(\frac{1}{4}\right) \left(\int_{R^3} |\nabla u|^2 \, dx\right)^2 - \left(\frac{1}{p}\right) \int_{R^3} |u|^p \, dx
\leq \left(\frac{1}{2}\right)\|u\|_{E_1}^2 + \left(\frac{C}{4}\right)\|u\|^4_{E_1} - \left(\frac{1}{p}\right)\|u\|^p_{E_1}
\leq \left(\frac{1}{2}\right)\|u\|_{E_1}^2 + \left(\frac{C}{4}\right)\|u\|^4_{E_1} - \left(\frac{C_\mu}{p}\right)\|u\|^p_{E_1}.
(20)

Since \( p > 4 \), consequently, there is a large \( r > 0 \) such that
\( I(u) < 0 \) on \( \bar{E}/B_r(0) \).

Lemma 3. Let \( 4 < p < 6 \) and \((V_1)\) and \((V_2)\) be satisfied. Then, there exists \( \Lambda > 0 \) such that, for each \( c \in R \), \( I_\lambda \) satisfies the \((PS)\) condition for all \( \lambda \geq \Lambda \).

Proof. Let \( \{u_n\} \) be a \((PS)_c\) sequence, that is, \( I_\lambda(u_n) \to c \) and \( I'_\lambda(u_n) \rightharpoonup 0 \). If \( \{u_n\} \) is unbounded in \( E_\lambda \), up to a subsequence, we can assume that

\[
o(1) = \left(\frac{1}{\|u_n\|_{E_1}}\right) \left( I_\lambda(u_n) - \frac{1}{p} \langle I'_\lambda(u_n), u_n \rangle \right)
= \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \int_{R^3} V^- (x) \omega_n^2 \, dx + \left(\frac{1}{4} - \frac{1}{p}\right) \left(\frac{1}{\|u_n\|_{E_1}}\right) \left(\int_{R^3} |\nabla u_n|^2 \, dx\right)^2
\geq \frac{1}{2} - \frac{1}{p} + o(1),
\]
a contradiction. If \( w \neq 0 \), then the set \( \Omega = \{ x \in R^3 : \omega(x) \neq 0 \} \) has positive Lebesgue measure. For \( x \in \Omega \), one has

\[
|u_n(x)| \to \infty \text{ as } n \to \infty; \text{ Fatou's lemma shows that } \int_{\Omega} |u_n|^{p-4} \omega_n^2 \, dx \to \infty \text{ as } n \to \infty. \text{ Thus, by (9), we obtain }
\]

\[
1/p \int_{R^3} |u_n|^{p-4} \omega_n^2 \, dx = 1/2 \|u_n\|^2_{E_1} - 1/4 \|u_n\|^2_{E_1} \int_{R^3} \lambda V^- (x) u_n^2 \, dx + 1/4 \|u_n\|^4_{E_1} \left(\int_{R^3} |\nabla u_n|^2 \, dx\right)^2 + o(1)
\leq 1/2 \|u_n\|^2_{E_1} + 1/4 \|u_n\|^4_{E_1} \left(\int_{R^3} |\nabla u_n|^2 \, dx\right)^2 + o(1)
\leq C/4 + o(1).
\]

This is a contradiction. This implies \( \{u_n\} \) is bounded in \( E_\lambda \). We assume that \( \|u_n\|_{E_1} \leq T \). Passing to a subsequence if necessary, we can assume that there exists \( u \in E_\lambda \) and \( A \in R \) such that

\[
\|u_n\|_{E_1} \to + \infty,
I_\lambda(u_n) \to c,
\|I'_\lambda(u_n)\| \to 0,
\]
as \( n \to \infty \), after passing to a subsequence. Set \( w_n = u_n/\|u_n\|_{E_1} \), we can assume that \( w_n \to w \) in \( E_1 \) and \( w_n(x) \to w(x) \) a.e. \( x \in R^3 \).

If \( w = 0 \), since \( w \to \int_{R^3} V^- (x) u^2 \, dx \) is weakly continuous, we have

\[
|u_n(x)| \to \infty \text{ as } n \to \infty; \text{ Fatou's lemma shows that } \int_{\Omega} |u_n|^{p-4} \omega_n^2 \, dx \to \infty \text{ as } n \to \infty. \text{ Thus, by (9), we obtain }
\]
Let $u_n = u - u$. It follows from $(V_1)$ and $(V_2)$ that
\[
\|v_n\|^2 = \int_{V(x) \subset E} v_n^2 \, dx + \int_{V(x) \subset \mathbb{R}^N} v_n^2 \, dx \leq \left( \frac{1}{\lambda c} \right) \|v_n\|^2 + o(1).
\]

Moreover, Let $0 < \lambda < \min\{6 - p/2, 1\}, 2 < p < 6$. Then, $2 < (p - a)/2 - a < 6$. By Sobolev inequalities and Hölder inequality, one has
\[
\|v_n\|^p = \int_{R^2} |v_n|^p \, dx \leq \left( \int_{R^2} |v_n|^2 \, dx \right)^{a/2} \left( \int_{R^2} |v_n|^{p-2} \, dx \right)^{(2-a)/2} \leq C(\lambda c)^{-\alpha/2} \|v_n\|^p + o(1),
\]

we know
\[
o(1) = \langle I'_1(u_n), u_n \rangle = \|u_n\|^2 - \int_{R^2} \lambda V(x) u_n^2 \, dx + \|u_n\|^p - \|u\|^p + \int_{R^2} \lambda V(x) u_n^2 \, dx + A^2 \|u_n\|^2 + o(1) \geq \|u_n\|^2 - \|u\|^p + o(1) \geq \|u_n\|^2 - \|u\|^p + o(1). \]

Letting $\Lambda > 0$ be so large that the term in the brackets above is positive when $\lambda \geq \Lambda$, we get $v_n \to 0$ in $E_1$. Since $v_n = u_n - u$ and $v_n \to 0$, it follows that $u_n \to u$ in $E_1$. This completes the proof.

Then, $I'_1(u_n) \to 0$ implies that
\[
i I_0 \implies 0 \implies 0.
\]

Taking $v = u$ in (25), we obtain
\[
\int_{R^2} |u|^2 \, dx + \int_{R^2} \lambda V(x) u^2 \, dx + A^2 \int_{R^2} |u|^2 \, dx - \int_{R^2} |u|^{p-2} \, dx = 0, \quad \forall v \in E_1.
\]

Proof of Theorem 1. Obviously, $I(0) = 0$. Furthermore, $I$ is even. The conclusion follows from Lemmas 1–3 and Theorem 1.}

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


