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Research Article

Robustness Analysis of Control Laws in Complex Dynamical Networks Evoked by Deviating Argument

Biwen Lia, Jingjing Huang, and Donglun Wang

College of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

Correspondence should be addressed to Biwen Lia; lbw20200320@163.com, Jingjing Huang; 1170823018@qq.com, and Donglun Wang; 2511393083@qq.com

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In recent years, robust performance of the system has been broadly studied as a trending topic among a vast array of scholars. This paper discusses the robustness of control laws for complex dynamic networks (CDNs) with a deviation argument. We design two categories of control laws (linear control law and nonlinear control law) for the undisturbed CDNs to achieve exponential synchronization. It is intractable to ascertain the range of the deviation function exactly. Hence, some corresponding sufficient criteria are put forward to ensure exponential synchronization of CDNs with deviation argument when control laws are not changed. By adopting the Gronwall–Bellman lemma and solving the transcendental equation, we can obtain the admissible upper limits of the deviating function, to keep the corresponding control laws. In comparison with previous research findings, robustness, deviating argument, and control laws are all considered in this study, which enhances the previous findings. Finally, two emulation examples verify the validity of the analysis.

1. Introduction

More recently, as a vital component of the nonlinear system, CDNs have been broadly investigated and have found many potential applications in various domains, including biology, sociology, physics, network science, engineering, automatic control, and so on [1–3]. In particular, the dynamical behavior of complex systems has flourished vigorously in the field of control engineering, which has aroused tremendous interest among scholars [4].

Synchronization is a fairly vital nonlinear phenomenon widely existing in nature. It has wide-ranging implications for numerous applications, such as secret communication, facial recognition, artificial intelligence, and associative memory. Recently, the synchronization of CDNs has been extensively studied, for instance, quasi-synchronization [5], cluster synchronization [6], master-slave synchronization [7], exponential synchronization [8, 9], and so on.

At present, the control law is indispensable in achieving the synchronization of CDNs. To guarantee synchronization, many effective synchronization control laws have been proposed. On the basis of existing literature [10–12], there are two main categories of control laws of the systems: linear control law and nonlinear control law. As we all know, linear control law is the basis of the control strategy, and the structure of the controller is relatively uncomplicated. Compared with linear control law, nonlinear control law has become an increasingly interesting research topic for researchers from diverse backgrounds. In [8], the exponential synchronization for a kind of CDN with stochastic perturbations and delays was well investigated by utilizing the time-delayed impulsive controller. In [4], based on the event-triggered strategy, quasi-synchronization of CDNs was obtained. In [12], the author addressed robust synchronization between fractional-order CDNs involving parameter uncertainty and applying the nonlinear control law. By means of a sliding mode control scheme, global asymptotical synchronization for CDNs was discussed in [13].

As a result of imperfect measurements as well as the finite switching frequency of amplifiers, delays are always inherent in real applications of machine learning, which can exacerbate performance and derail the stability of the models. Generally speaking, time delays are inevitable, and we can never be too concerned about them. Especially, deviation argument can have substantial consequences in the running process of the dynamical system model. As a matter of fact, numerous biomedical models are inscribed by differential equations with a deviation argument. In the process of motion, the deviation function alters the relevant deviation characteristics, so the differential equation with deviation argument unites delay and advanced equation. More precisely, with respect to the study of economics, biology, and physics, past and future events are crucial to current decisions. Therefore, it is of great significance to discuss the models of retarded and advanced alternating differential equations. A system with a deviation argument is a mixture of retarded and advanced systems. Sufficient conditions on the globally exponentially stable for recursive systems containing deviating functions were obtained by virtue of Lyapunov functions in [14]. The author utilized algebraic inequalities to present novel analytical findings from Mittag-Leffler stability about the fractional-order model involving deviating arguments in [15]. On the basis of the comparison principle, valuable theoretic results on the stability and robustness of interval fuzzy Cohen-Grossberg networks were presented in [16]. Additionally, a useful impulsive control law was considered to accomplish the synchronization in the presence of the deviating argument of CDN in [9]. In order to elaborate on the convenience and completeness of CDNs, we will analyze CDNs with deviating arguments here. Among the above studies, only one investigated the synchronization of CDNs with deviating arguments. At present, there are few research studies on robust synchronization control schemes for CDNs with deviating arguments. The objective of our research is to fill the gap.

To our knowledge, some work has been performed on the stabilization of dynamical systems equipped with deviating arguments [14, 17]. Some work has been conducted on stabilization and synchronization from the perspective of control [13, 18, 19]. Others have also studied research on complex dynamic networks with deviating arguments [9], but their purpose was to study stability rather than robustness. However, the robustness of control laws in complex dynamic networks is rarely studied by setting appropriate parameters for deviation argument.

Motivated by the above-mentioned discussion, we will find that the presence of deviating arguments will exacerbate difficulties in achieving synchronization of controlled complex dynamical networks (CDNs). In view of the literature [20, 21], it is worth considering the influence of the deviation argument on the control method. A new issue arises: can linear control law (nonlinear law) still be kept if a deviation function occurs in the system? As the deviation function involves information about the past and future, it is worthwhile to investigate: under the constraints of linear or nonlinear control laws, how much is the deviation argument intensity to allow CDNs to maintain exponential synchronization?

Therefore, the following is concretely scheduled for this paper. Some mathematical preliminaries are provided in Section 2. In view of the two categories of control laws, several crucial lemmas and theorems are further elaborated in Section 3. Two emulation examples are mentioned to validate the feasibility of the analytical results in Section 4. Finally, Section 5 of this paper carries a concise summary and outlook for the future research direction.

2. Preliminaries

2.1. Notation. On the basis of this paper, $\mathbb{N}$ represents the sets of natural numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ be made up of all n real vectors, all $n\times m$ real matrices, respectively. I_n is denoted as the n-order identity matrix. Moreover, 0_n refers to $n\times n$ zero matrix. $\|\cdot\|$ means the Euclidean vector norm or the induced matrix norm. Denote $E\otimes F$ as the Kronecker product of matrices E and F.

Generally, a graph $G=(\nu,\varepsilon,\widetilde{A})$ has three basic elements. $\nu=\{1,\dots,N\}$ signifies the set of nodes; $\varepsilon=\{e_{ij}\}$, $(i,j\in\nu)$ is the set of edges; and the coupling matrix $\widetilde{A}=(\mathrm{a_{ij}})_{\mathrm{N\times N}}$, where a_{ij} stands for coupling weight between i th CDN and node j th CDN. If the message is delivered from j th CDN to i th $(i\neq j)$ CDN, then $a_{ij}\neq 0$; otherwise, $a_{ij}=0$.

Provide two real-valued sequences $\{\rho_q\}$, $\{\eta_q\}$, $q\in\mathbb{N}$, such that $\rho_q<\rho_{q+1}$, $\rho_q\leq\eta_q\leq\rho_{q+1}$ for all $\rho_q\longrightarrow\infty$ as $q\longrightarrow\infty$.

2.2. Model. Consider a kind of CDN with a deviating argument consisting of N coupled nodes

$$\begin{aligned}\dot{y}_i(t)&=f\left(y_i(t)\right)+c\sum_{j=1}^{N}a_{ij}y_j(t)+c\sum_{j=1}^{N}b_{ij}y_j(\delta(t))+u_i(t),\\ y\left(t_0\right)&=y_0\in\mathbb{R}^n.\end{aligned}\tag{1}$$

where $y_i(t)=(y_{i1},\cdots y_{in})^T\in\mathbb{R}^n$ is the state vector of the i th node; the nonlinear $f\colon\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is called continuous vector-valued function; c stands for the coupling strength; $u_i(t)\in\mathbb{R}^n$ is defined as the control input vectors of node i; and a_{ij} and b_{ij} denote the (i,j)-th term of coupling matrix $\widetilde{A}$ and $\widetilde{B}$ satisfying $a_{ii}=-\sum_{j=1,j\neq i}^{N}a_{ij}$ and $b_{ii}=-\sum_{j=1,j\neq i}^{N}b_{ij}$, respectively. $\delta(t)$ is a deviating argument satisfying $\delta(t)=\eta_q\in[\rho_q,\rho_{q+1}]$, if $t\in(\rho_q,\rho_{q+1}]$.

Remark 1. Obviously, when $t\in(\rho_q,\rho_{q+1}]$, if $\delta(t)>t$, then the coupling term of system (1) is advanced, and if $\delta(t)<t$, then the coupling term of system (1) is retarded. Hence, system (1) is a hybrid CDN, which integrates alternately advanced and retarded argument.

The dynamical equation for the isolated node of CDN (1) is

$$\dot{s}(t)=f(s(t)),\tag{2}$$

where $s(t)$ allows to be defined by an arbitrary desired state, that is, the equilibrium point, the periodic orbit, and so on.

Remark 2. Further, in contrast to (1), in case of $\delta(t)=t$, the CDN (1) turns into the following CDN:

$$\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}x_j(t) + c\sum_{j=1}^{N} b_{ij}x_j(t) + u_i(t),$$
$$x(t_0) = x_0 \in \mathbb{R}^n. \tag{3}$$

Similarly,

$$\dot{s}(t) = f(s(t)), \tag{4}$$

Also referred to as the dynamical equation of the isolated node of CDN (3).

Define the synchronization error as $e_i(t) = x_i(t) - s(t)$, and subtracting (4) from (3) to obtain the following system:

$$\dot{e}_i(t) = f(e_i(t)) + c\sum_{j=1}^{N} a_{ij}e_j(t) + c\sum_{j=1}^{N} b_{ij}e_j(t) + u_i(t),$$
$$e(t_0) = e_0 \in \mathbb{R}^n, \tag{5}$$

where $f(e_i(t)) = f(x_i(t)) - f(s(t))$.

To acquire synchronization, the linear controller is arranged as

$$u_i(t) = \widetilde{W}e_i(t), \quad i = 1, \ldots, N, \tag{6}$$

where $\widetilde{W} \in \mathbb{R}^{n\times n}$ represents the feedback controller gain matrix.

Combining (5) and (6), one has that

$$\dot{e}_i(t) = f(e_i(t)) + c\sum_{j=1}^{N} a_{ij}e_j(t) + c\sum_{j=1}^{N} b_{ij}e_j(t) + \widetilde{W}e_i(t),$$
$$e(t_0) = e_0 \in \mathbb{R}^n. \tag{7}$$

With a view to simplifying writing, the following notations:

$e(t) = (e_1^T(t), \ldots e_N^T(t))^T$, $e(\delta(t)) = (e_1^T(\delta(t)), \ldots e_N^T(\delta(t)))^T$, $f(e(t)) = (f^T(e_1(t)), \ldots, f^T(e_N(t)))^T$, $\widetilde{A} = (a_{ij})_{N\times N}$, $\widetilde{B} = (b_{ij})_{N\times N}$, $A = I_n \otimes \widetilde{A}$, $B = I_n \otimes \widetilde{B}$, $W = I_n \otimes \widetilde{W}$, are put forward to describe the system. Incorporating these simplifications, the error system (7) is further characterized as a compact representation:

$$\dot{e}(t) = f(e(t)) + \mathrm{cAe}(t) + \mathrm{cBe}(t) + We(t),$$
$$e(t_0) = e_0 \in \mathbb{R}^n. \tag{8}$$

In the same way, we can define error $z_i(t) = y_i(t) - s(t)$, and the following error dynamic equation can be obtained by subtracting (2) from (1)

$$\dot{z}(t) = f(z(t)) + cAz(t) + cBz(\delta(t)) + Wz(t),$$
$$z(t_0) = z_0 \in \mathbb{R}^n. \tag{9}$$

As a starting point, we will provide some essential definitions and required assumptions for this paper.

Definition 1. If the error system (8) is exponential stability, then the CDN (3) with (4) is described as exponential synchronization, namely, for any initial value $e_0 \in \mathbb{R}^n$, there exist two scalars $\alpha > 0$ and $\beta > 0$ such that for any $t \in \mathbb{R}^+$, satisfying $\|e(t)\| \le \alpha\|e_0\|e^{-\beta(t-t_0)}$:

(D1) Assume that the existence of nonnegative scalar p, satisfying $\|f(m_i(t)) - f(n_i(t))\| \le p\|m_i(t) - n_i(t)\|$ and $f(0_n) = 0$ for any $m_i(t) \in \mathbb{R}^n$, $n_i(t) \in \mathbb{R}^n$

(D2) There is a positive constant ρ that has the property $\rho_{q+1} - \rho_q \le \rho$, for all $q \in \mathbb{N}$

(D3) $\rho(h_1 + 2h_2)\exp\{h_1\rho\} < 1$

3. Main Results

Under (D1), (D2), and (D3), system (9) has a unique state $z(t; t_0, z_0)$ on $t > t_0$ for any initial state (t_0, z_0). Obviously, $z = 0$ is the equilibrium point of system (9). Synchronization, as a highly representative subject of complex system, has been paid attention to and researched by many academics, for example [5–8, 10, 15]. For now, a fundamental point requires consideration. While keeping the original controller unchanged, if the deviation argument is added to the CDN, can synchronization of the system still hold? Obviously, synchronization is not established. How much is the deviation argument intensity in order to allow the CDNs (1) to maintain exponential synchronization when the control law is as valid as before? On the basis of this fact, we are going to investigate the robustness of the controller of CDNs (1) with a deviating argument when the control scheme of the CDNs (2) is fixed.

As we all know, among all kinds of control laws, the most efficient and concise control law is the linear feedback controller. What should be noted is that the linear control law is the foundation of the control system. Compared with the nonlinear controller, its structure is relatively uncomplicated. However, nonlinear control law has become an increasingly interesting topic of study for researchers from diverse backgrounds. Therefore, it is worthwhile to design two types of controllers for study: the linear feedback controller and the nonlinear feedback controller.

3.1. The Linear Feedback Controller. Before describing the principal theorem of this section, some important lemmas need to be presented here. The following assumptions are also required:

$$\alpha \exp(-\beta T) + 2h_2\alpha \exp\{(2h_1 + 6h_2)T\}/\beta < 1, \tag{10}$$

Remark 3. As can be inferred from Theorem 2 appearing in [6], the existence and uniqueness of the solution of system (1) is collectively ensured by (D1), (D2), and (D3).

Lemma 1. *Let $z(t)$ stands for the current state of error dynamic (9) and conditions (D1), (D2), (D3), and (D4) are all met. So, the following inequality:*

$$\|z(\delta(t))\| \le \lambda\|z(t)\|. \tag{11}$$

It exists for any $t > 0$, where

$$\begin{aligned} &\lambda = 1/(1-u_1),\\ &u_1 = h_2\rho + h_1\rho(1+h_2\rho)\exp\{h_1\rho\},\\ &h_1 = p + c\|A\| + \|W\|,\\ &h_2 = c\|B\|. \end{aligned} \tag{12}$$

Proof. For the deviation term $\delta(t) = \eta_q$, define a set $\sigma = \{t/t > 0, \rho_q \leq t \leq \rho_{q+1}\}$, let $t \in \sigma$, $q \in \mathbb{N}$, and then one obtains

$$z(t) = z(\eta_q) + \int_{\eta_q}^{t} (f(z(s))) + cAz(s) + cBz(\eta_q) + Wz(s)ds. \tag{13}$$

Combining (D1) and (13), one has that

$$\begin{aligned} \|z(t)\| &\leq \|z(\eta_q)\| + \int_{\eta_q}^{t} \left(\|f(z(s))\| + c\|A\|\|z(s)\| + c\|B\|\|z(\eta_q)\| + \|W\|\|z(s)\|\right)ds\\ &= (1+h_2\rho)\|z(\eta_q)\| + \int_{\eta_q}^{t} h_1\|z(s)\|ds, \end{aligned} \tag{14}$$

where

$$\begin{aligned} &h_1 = p + c\|A\| + \|W\|,\\ &h_2 = c\|B\|. \end{aligned} \tag{15}$$

By virtue of the Gronwall–Bellman's inequality, (14) can evolve into

$$\|z(t)\| \leq \left[(1+h_2\rho)\|z(\eta_q)\|\right]\exp\{h_1\rho\}, \tag{16}$$

Otherwise, for $\rho_q \leq t \leq \rho_{q+1}$, similarly, we have

$$\begin{aligned} \|z(\eta_q)\| &\leq \|z(t)\| + \int_{\eta_q}^{t} \left(\|f(z(s))\| + c\|A\|\|z(s)\| + c\|B\|\|z(\eta_q)\| + \|W\|\|z(s)\|\right)ds\\ &\leq \|z(t)\| + h_2\rho\|z(\eta_q)\| + \int_{\eta_q}^{t} h_1\|z(s)\|ds \leq \|z(t)\| + h_2\rho\|z(\eta_q)\| + h_1\rho\left[(1+h_2\rho)\|z(\eta_q)\|\right]\exp\{h_1\rho\}\\ &\leq \|z(t)\| + [h_2\rho + h_1\rho(1+h_2\rho)\exp\{h_1\rho\}]\|z(\eta_q)\| \leq \|z(t)\| + u_1\|z(\eta_q)\|, \end{aligned} \tag{17}$$

where $u_1 = [h_2\rho + h_1\rho(1+h_2\rho)\exp\{h_1\rho\}]$, and h_1, h_2 are defined in (15).

Hence, by uniting the aforementioned equation with similar entries, we further get

$$(1-u_1)\|z(\eta_q)\| \leq \|z(t)\|. \tag{18}$$

Accordingly, when $\delta(t) = \eta_q, u_1 < 1$ for (D4), it follows that

$$\begin{aligned} \|z(\eta_q)\| &\leq (1-u_1)^{01}\|z(t)\|\\ &=: \lambda\|z(t)\|, \end{aligned} \tag{19}$$

where $\lambda = 1/(1-u_1)$. By this means, (11) is effective for $t > 0$. □

Remark 4. By virtue of Lemma 1, we establish the link from the deviating argument $z(\delta(t))$ to the state $z(t)$ and provide a strong base to prove the subsequent Theorem 1.

Next, we explore the influence of the deviation argument on the robustness of exponentially stable of error system (9).

Theorem 1. *If assumption (D1) (D2) (D3) (D4) (D5) hold, and error system (8) is exponential stability, then error system (9) is exponential stability, that is, system (1) is said to be exponential synchronization under the linear-type controller (6), if $\rho < \overline{\rho}$, where $\overline{\rho}$ is the only solution of the transcendental equation:*

$$k_2\exp\{2k_1T\} + \alpha\exp\{-\beta T\} = 1, \tag{20}$$

where $k_1 = h_1 + (2+\lambda)h_2$, $k_2 = h_2(1+\lambda)\alpha/\beta$, $\lambda = (1 - [h_2\rho + h_1\rho(1+h_2\rho)\exp\{h_1\rho\}])^{-1}$ and $T > (\ln\alpha)/\beta$. Here, in addition to h_1, h_2, λ and T, all of them are consistent with those defined in Lemma 1.Proof For convenience, $e(t) = e(t; t_0, e_0)$ and $z(t) = z(t; t_0, z_0)$ are expressed by $e(t)$ and $z(t)$, respectively. According to (8) and (9), as well as the initial value $e_0 = z_0$, one has

$$z(t) - e(t) = \int_{t_0}^{t} [(f(z(s)) - f(e(s))) + cA(z(s) - e(s)) + cB(z(\delta(t)) - e(s)) + W(z(s) - e(s))]ds. \tag{21}$$

Then,

$$\begin{aligned}\|z(t)-e(t)\| &= \left\|\int_{t_0}^{t}[(f(z(s))-f(e(s)))+cA(z(s)-e(s))+cB(z(\delta(t))-e(s))+W(z(s)-e(s))]\mathrm{d}s\right\| \\ &\le \int_{t_0}^{t}[\|f(z(s))-f(e(s))\|+c\|A\|\|z(s)-e(s)\|+c\|B\|\|z(\delta(t))-e(s)\|+\|W\|\|z(s)-e(s)\|]\mathrm{d}s.\end{aligned} \tag{22}$$

In view of (D1) and the norm inequality, for (22), one has

$$\begin{aligned}\|z(t)-e(t)\| &\le \int_{t_0}^{t}[p\|z(s)-e(s)\|+c\|A\|\|z(s)-e(s)\|+c\|B\|\|z(\delta(s))-e(s)\|+\|W\|\|z(s)-e(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(p+c\|A\|+\|W\|)\|z(s)-e(s)\|+c\|B\|\|z(\delta(s))-e(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(p+c\|A\|+\|W\|)\|z(s)-e(s)\|+c\|B\|\|z(\delta(s))-z(s)+z(s)-e(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(p+c\|A\|+\|W\|)\|z(s)-e(s)\|+c\|B\|\|z(\delta(s))-z(s)\|+c\|B\|\|z(s)-e(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(p+c\|A\|+c\|B\|+\|W\|)\|z(s)-e(s)\|+c\|B\|\|z(\delta(s))-z(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(h_1+h_2)\|z(s)-e(s)\|+h_2\|z(\delta(s))-z(s)\|]\mathrm{d}s.\end{aligned} \tag{23}$$

By Lemma 1, when $0 \le t_0 \le t$, then

$$\begin{aligned}\|z(t)-e(t)\| &\le \int_{t_0}^{t}[(h_1+h_2)\|z(s)-e(s)\|+h_2\|z(\delta(s))\|+h_2\|z(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(h_1+h_2)\|z(s)-e(s)\|+h_2(1+\lambda)\|z(s)\|]\mathrm{d}s \\ &= (h_1+h_2)\int_{t_0}^{t}\|z(s)-e(s)\|ds+h_2(1+\lambda)\int_{t_0}^{t}\|z(s)-e(s)+e(s)\|\mathrm{d}s \\ &\le (h_1+h_2)\int_{t_0}^{t}\|z(s)-e(s)\|ds+h_2(1+\lambda)\int_{t_0}^{t}\|z(s)-e(s)\|+\|e(s)\|\mathrm{d}s \\ &\le [h_1+(2+\lambda)h_2]\int_{t_0}^{t}\|z(s)-e(s)\|ds+h_2(1+\lambda)\int_{t_0}^{t}\|e(s)\|\mathrm{d}s.\end{aligned} \tag{24}$$

Due to the error system (8) is exponential stability, according to Definition 1, on the interval $[t_0-\rho, t_0+\rho]$, it comes to the conclusion that

$$\|e(t)\| \le \alpha\|e_0\|e^{-\beta(t-t_0)}. \tag{25}$$

And then,

$$\frac{\int_{t_0}^{t}\|e(t)\|\mathrm{d}s \le \alpha\|e_0\|}{\beta}. \tag{26}$$

Furthermore,

$$\begin{aligned}\|z(t)-e(t)\| &\le [h_1+(2+\lambda)h_2]\int_{t_0}^{t}\|z(s)-e(s)\|\mathrm{d}s \\ &\quad + h_2(1+\lambda)\frac{\|e_0\|\alpha}{\beta} \\ &= k_1\int_{t_0}^{t}\|z(s)-e(s)\|\mathrm{d}s+k_2\|e_0\|,\end{aligned} \tag{27}$$

where

$$k_1 = h_1 + (2+\lambda)h_2,$$
$$k_2 = \alpha h_2 \frac{(1+\lambda)}{\beta}. \tag{28}$$

By the Gronwall–Bellman's inequality, when $t_0 + \rho \le t \le t_0 + 2T$, we can acquire

$$\|z(t) - e(t)\| \le k_2 \|e_0\| \exp\{2k_1 T\}. \tag{29}$$

Since $t_0 - \rho + T \le t \le t_0 - \rho + 2T$, from (27) and (29), one has

$$\begin{aligned} \|z(t)\| &= \|z(t) - e(t) + e(t)\| \le \|z(t) - e(t)\| + \|e(t)\| \\ &\le k_2 \|e_0\| \exp(2k_1 T) + \alpha \|e_0\| \exp(-\beta T) \\ &= \{k_2 \exp(2k_1 T) + \alpha \exp(-\beta T)\} \|e_0\| \\ &\le \widehat{h} \sup_{t_0-\rho \le t \le t_0-\rho+T} \|z(t)\|, \end{aligned} \tag{30}$$

where $\widehat{h} = k_2 \exp\{2k_1 T\} + \alpha \exp(-\beta T)$.

Denote

$$\begin{aligned} H(\lambda) &= k_2 \exp\{2k_1 T\} + \alpha \exp(-\beta T) \\ &= h_2 (1+\lambda)\alpha/\beta \exp\{2[h_1 + (2+\lambda)h_2]T\} \\ &\quad + \alpha \exp(-\beta T). \end{aligned} \tag{31}$$

By substituting $\lambda = 1$ into (31), we can readily get

$$H(1) = \alpha \exp(-\beta T) + 2h_2\alpha \exp\{(2h_1 + 6h_2)T\}/\beta < 1. \tag{32}$$

Clearly that, $H(+\infty) > 1$. In addition, $H(\lambda)$ is monotonically increasing for λ. Accordingly, there is only one $\overline{\lambda} \in (1, +\infty)$ meeting

$$H(\overline{\lambda}) = 1. \tag{33}$$

Denote

$$\Gamma(\rho) = h_2\rho + h_1\rho(1 + h_2\rho)\exp\{h_1\rho\}, \tag{34}$$

and identify $\widetilde{\rho}$ as the only one positive solution to $\Gamma(\rho) = 1$.

Apparently,

$$\lambda(\rho) = (1 - \Gamma(\rho))^{-1} \in (1, +\infty), \tag{35}$$

for $\rho \in (0, \widetilde{\rho})$. Furthermore, λ is monotonically increasing for ρ. In this sense, there is the only one positive $\overline{\rho} \in (0, \widetilde{\rho})$ to satisfy

$$\lambda = \overline{\lambda}, \tag{36}$$

and $\overline{\rho}$ is the only one positive solution to (20).

Consequently, for $\rho < \overline{\rho}$,

$$\widehat{h} = k_2 \exp\{2k_1 T\} + \alpha \exp(-\beta T) < 1. \tag{37}$$

Picking out $\xi = -(\ln\widehat{h})/T > 0$, and in consideration of (30), one has

$$\sup_{t_0-\rho+T \le t \le t_0-\rho+2T} \|z(t)\| \le \exp(-\xi T) \sup_{t_0-\rho \le t \le t_0-\rho+T} \|z(t)\|. \tag{38}$$

Considering the existence and uniqueness of the solution $z(t)$ of system (9), when $t \ge t_0 - \rho + (l-1)T$, it holds

$$z(t, t_0, z_0) = z(t, t_0 - \rho + (l-1)T, z(t_0 - \rho + (l-1)T, t_0, z_0)). \tag{39}$$

From (38) and (39), it follows that

$$\begin{aligned} &\sup_{t_0-\rho+lT \le t \le t_0-\rho+(l+1)T} \|z(t, t_0, z_0)\| \\ &= \sup_{t_0-\rho+(l-1)T+T \le t \le t_0-\rho+(l-1)T+2T} \|z(t; t_0 - \rho + (l-1)T, z(t_0 - \rho + (l-1)T; t_0, z_0))\| \\ &\le \exp(-\xi T) \sup_{t_0-\rho+(l-1)T \le t \le t_0-\rho+lT} \|z(t; t_0, z_0)\| \\ &\le \exp(-\xi lT) \sup_{t_0-\rho \le t \le t_0-\rho+T} \|z(t; t_0, z_0)\| \\ &= G \exp(-\xi lT), \end{aligned} \tag{40}$$

where $G = \sup_{t_0-\rho \le t \le t_0-\rho+T} \|z(t; t_0, z_0)\|$.

To go a step further, there is the only scalar $l \in \mathbb{N}$ so that $t_0 - \rho + lT \le t \le t_0 - \rho + (l+1)T$, and one can easily show

$$\begin{aligned} \|z(t; t_0, z_0)\| &\le G \exp(-\xi lT) \le G \exp\{-\xi(t - t_0) + \xi(T - \rho)\} \\ &\le G \exp(\xi T) \exp\{-\xi(t - t_0)\}. \end{aligned} \tag{41}$$

Based on Theorem 1, one can readily find that an error system (9) is exponentially stable, i.e., the system (1) can achieve exponential synchronization under a designed linear-type controller (6).

Remark 5. Theorem 1demonstrates that when an error system (8) is exponentially stable, the perturbed system (9) evoked by the deviation argument can still remain exponentially stable as long as the interval length of the deviating argument $\delta(t)$ is less than the estimated upper bound. So, system (1) involving deviating argument is still exponentially synchronous under a designed linear-type controller (6).

Remark 6. As can be seen in Figure 1, there is a relationship between the interval length of a deviating argument and time in the proof of Theorem 1.

3.2. The Nonlinear Feedback Controller. To acquire synchronization, the nonlinear controller is defined as

$$u_i(t) = -f(x_i(t)) + f(s(t)) + \widetilde{R}e_i(t), \quad i = 1, \ldots, N, \tag{42}$$

where $\widetilde{R} \in \mathbb{R}^{n\times n}$ represents the feedback controller gain matrix.

With the nonlinear control law (42), system (5) is reworded to

$$\begin{aligned} \dot{e}_i(t) &= c\sum_{j=1}^{N} a_{ij}e_j(t) + c\sum_{j=1}^{N} b_{ij}e_j(t) + \widetilde{R}e_i(t), \\ e(t_0) &= e_0 \in \mathbb{R}^n. \end{aligned} \tag{43}$$

That is,

$$\dot{e}(t) = \mathrm{cAe}(t) + \mathrm{cBe}(t) + \mathrm{Re}(t), \quad e(t_0) = e_0 \in \mathbb{R}^n, \tag{44}$$

where $R = I_n \otimes \widetilde{R}$. Here, in addition to R, all of them are consistent with previous notations in (8).

In the same way, error system with mixed coupling terms containing deviation arguments is reformulated as

$$\dot{z}(t) = cAz(t) + cBz(\delta(t)) + Rz(t), \quad z(t_0) = z_0 \in \mathbb{R}^n, \tag{45}$$

where $R = I_n \otimes \widetilde{R}$.

To validate Theorem 2 more expeditiously, we will introduce Lemma 2. In this subsection, the following assumptions are required:

(i) (D6) $\rho(\varepsilon_1 + 2\varepsilon_2)\exp\{\varepsilon_1\rho\} < 1$

(ii) (D7) $\varepsilon_2\rho + \varepsilon_1\rho(1 + \varepsilon_2\rho)\exp\{\varepsilon_1\rho\} < 1$

(D8) The following inequality characterize the parameters of the system (44)

$$\frac{\alpha\ \exp(-\beta T) + 2\varepsilon_2\alpha\ \exp\{(2\varepsilon_1 + 6\varepsilon_2)T\}}{\beta < 1}, \tag{46}$$

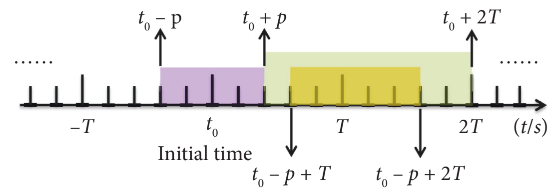

Figure 1: Relationship between interval length of deviating argument and time.

where $T > (\ln\alpha)/\beta$, $\varepsilon_1 = c\|A\| + \|R\|$, $\varepsilon_2 = c\|B\|$.

Lemma 2. *Let $z(t)$ stands for the current state of error system (45) and conditions (D1) (D2) (D6) (D7) are all met. So, we have the following inequality:*

$$\|z(\delta(t))\| \le \kappa\|z(t)\|, \tag{47}$$

This exists for any $t > 0$, where

$$\begin{aligned} \kappa &= \frac{1}{1 - \upsilon_1}, \\ \upsilon_1 &= [\varepsilon_2\rho + \varepsilon_1\rho(1 + \varepsilon_2\rho)\exp\{\varepsilon_1\rho\}], \\ \varepsilon_1 &= c\|A\| + \|R\|, \\ \varepsilon_2 &= c\|B\|. \end{aligned} \tag{48}$$

Proof. For the deviation term $\delta(t) = \eta_q$, define a set $\sigma = \{t > 0, \rho_q \le t \le \rho_{q+1}\}$, let $t \in \sigma$, and $q \in \mathbb{N}$, and then, we have

$$z(t) = z(\eta_q) + \int_{\eta_q}^{t} \left(cAz(s) + cBz(\eta_q) + Rz(s)\right)ds. \tag{49}$$

Combining (D1), we obtain

$$\begin{aligned} \|z(t)\| &\le \left\|z(\eta_q)\right\| + \int_{\eta_q}^{t} \left(c\|A\|z(s) + c\|B\|\left\|z(\eta_q)\right\| + \|R\|\|z(s)\|\right)\mathrm{d}s \\ &\le \left\|z(\eta_q)\right\| + \int_{\eta_q}^{t} (c\|A\| + \|R\|)\|z(s)\|ds + \int_{\eta_q}^{t} c\|B\|\left\|z(\eta_q)\right\|\mathrm{d}s \\ &= (1 + \varepsilon_2\rho)\left\|z(\eta_q)\right\| + \int_{\eta_q}^{t} \varepsilon_1\|z(s)\|\mathrm{d}s, \end{aligned} \tag{50}$$

where

$$\begin{aligned} \varepsilon_1 &= c\|A\| + \|R\|, \\ \varepsilon_2 &= c\|B\|. \end{aligned} \tag{51}$$

Therefore, for (50), using the Gronwall–Bellman's inequality, it yields that

$$\|z(t)\| \le \left[(1 + \varepsilon_2\rho)\left\|z(\eta_q)\right\|\right]\exp\{\varepsilon_1\rho\}. \tag{52}$$

Otherwise, for $\rho_q \le t \le \rho_{q+1}$, similarly, it follows that

$$\begin{aligned}\left\|z\left(\eta_q\right)\right\| &\le \|z(t)\| + \int_{\eta_q}^{t}\left(c\|A\|z(s) + c\|B\|\left\|z\left(\eta_q\right)\right\| + \|R\|\|z(s)\|\right)\mathrm{d}s \\ &\le \|z(t)\| + \varepsilon_2\rho\left\|z\left(\eta_q\right)\right\| + \int_{\eta_q}^{t}\varepsilon_1\|z(s)\|\mathrm{d}s \le \|z(t)\| + \varepsilon_2\rho\left\|z\left(\eta_q\right)\right\| + \varepsilon_1\rho\left[\left(1+\varepsilon_2\rho\right)\left\|z\left(\eta_q\right)\right\|\right]\exp\{\varepsilon_1\rho\} \\ &\le \|z(t)\| + \left[\varepsilon_2\rho + \varepsilon_1\rho\left(1+\varepsilon_2\rho\right)\exp\{\varepsilon_1\rho\}\right]\left\|z\left(\eta_q\right)\right\| \le \|z(t)\| + \upsilon_1\left\|z\left(\eta_q\right)\right\|,\end{aligned} \tag{53}$$

where $\upsilon_1 = [\varepsilon_2\rho + \varepsilon_1\rho(1+\varepsilon_2\rho)\exp\{\varepsilon_1\rho\}]$ and ε_1 and ε_2 are defined in (51).

Hence, by uniting the aforementioned equation with similar entries, we can further require

$$\left(1-\upsilon_1\right)\left\|z\left(\eta_q\right)\right\| \le \|z(t)\|. \tag{54}$$

Accordingly, when $\delta(t) = \eta_q$, $\upsilon_1 < 1$ for (D7), it further derives

$$\begin{aligned}\left\|z\left(\eta_q\right)\right\| &\le \left(1-\upsilon_1\right)^{01}\|z(t)\| \\ &=: \kappa\|z(t)\|,\end{aligned} \tag{55}$$

where $\kappa = 1/(1-u_1)^{-1}$. By this means, (47) is available for $t \ge 0$. □

Theorem 2. *If assumption (D1) (D2) (D6) (D7) (D8) hold, and error system (44) is exponentially stable, then, error system (45) is exponentially stable, that is, system (1) is said to be exponential synchronization under the nonlinear-type controller (42), if $\rho < \overline{\rho}$, where $\overline{\rho}$ is the only solution of the transcendental equation:*

$$\Lambda_2 \exp\{2\Lambda_1 T\} + \alpha\exp\{-\beta T\} = 1, \tag{56}$$

where $\Lambda_2 = \alpha\varepsilon_2(1+\kappa)/\beta$, $\kappa = (1 - [\varepsilon_2\rho + \varepsilon_1\rho(1+\varepsilon_2\rho)\exp\{\varepsilon_1\rho\}])^{-1}$ and $T > (\ln\alpha)/\beta$. Here, in addition to $\varepsilon_1, \varepsilon_2, \kappa$ and T, all of them are consistent with those defined in Lemma 2.

Proof. For convenience, $e(t) = e(t; t_0, e_0)$ and $z(t) = z(t; t_0, z_0)$ are expressed by $e(t)$ and $z(t)$, respectively. According to (44) and (45), as well as the initial value $e_0 = z_0$, one has

$$\begin{aligned}z(t) - e(t) = \int_{t_0}^{t} &[cA(z(s) - e(s)) + cB(z(\delta(t)) \\ &- e(s)) + R(z(s) - e(s))]\mathrm{d}s.\end{aligned} \tag{57}$$

Then,

$$\begin{aligned}\|z(t) - e(t)\| &= \left\|\int_{t_0}^{t}[cA(z(s)-e(s)) + cB(z(\delta(t)) - e(s)) + R(z(s)-e(s))]\mathrm{d}s\right\| \\ &\le \int_{t_0}^{t}[c\|A\|\|z(s)-e(s)\| + c\|B\|\|z(\delta(t)) - e(s)\| + \|R\|\|z(s)-e(s)\|]\mathrm{d}s.\end{aligned} \tag{58}$$

In view of the norm inequality, for (58), one has

$$\begin{aligned}\|z(t) - e(t)\| &\le \int_{t_0}^{t}[c\|A\|\|z(s)-e(s)\| + c\|B\|\|z(\delta(s)) - e(s)\| + \|R\|\|z(s)-e(s)\|]\mathrm{d}s \\ &= \int_{t_0}^{t}[(c\|A\| + \|R\|)\|z(s)-e(s)\| + c\|B\|\|z(\delta(s)) - e(s)\|]\mathrm{d}s \\ &= \int_{t_0}^{t}[(c\|A\| + \|R\|)\|z(s)-e(s)\| + c\|B\|\|z(\delta(s)) - z(s) + z(s) - e(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(c\|A\| + \|R\|)\|z(s)-e(s)\| + c\|B\|\|z(\delta(s)) - z(s)\| + c\|B\|\|z(s) - e(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(c\|A\| + c\|B\| + \|R\|)\|z(s)-e(s)\| + c\|B\|\|z(\delta(s)) - z(s)\|]\mathrm{d}s \\ &\le \int_{t_0}^{t}[(\varepsilon_1 + \varepsilon_2)\|z(s)-e(s)\| + \varepsilon_2\|z(\delta(s)) - z(s)\|]\mathrm{d}s.\end{aligned} \tag{59}$$

By Lemma 2, when $0 \le t_0 \le t$, then

$$\begin{aligned}\|z(t)-e(t)\| &\le \int_{t_0}^{t}\left[(\varepsilon_1+\varepsilon_2)\|z(s)-e(s)\|+\varepsilon_2\|z(\delta(s))\|+\varepsilon_2\|z(s)\|\right]\mathrm{d}s\\ &\le \int_{t_0}^{t}\left[(\varepsilon_1+\varepsilon_2)\|z(s)-e(s)\|+\varepsilon_2(1+\kappa)\|z(s)\|\right]\mathrm{d}s\\ &= (\varepsilon_1+\varepsilon_2)\int_{t_0}^{t}\|z(s)-e(s)\|\mathrm{d}s+\varepsilon_2(1+\kappa)\int_{t_0}^{t}\|z(s)-e(s)+e(s)\|\mathrm{d}s\\ &\le (\varepsilon_1+\varepsilon_2)\int_{t_0}^{t}\|z(s)-e(s)\|\mathrm{d}s+\varepsilon_2(1+\kappa)\int_{t_0}^{t}\|z(s)-e(s)\|+\|e(s)\|\mathrm{d}s\\ &\le \left[\varepsilon_1+(2+\kappa)\varepsilon_2\right]\int_{t_0}^{t}\|z(s)-e(s)\|\mathrm{d}s+\varepsilon_2(1+\kappa)\int_{t_0}^{t}\|e(s)\|\mathrm{d}s.\end{aligned} \tag{60}$$

Due to CDN, (44) is exponential stability, and according to Definition 1, on the interval $[t_0-\rho, t_0+\rho]$, it comes to the conclusion that

$$\|e(t)\| \le \alpha\|e_0\|e^{-\beta(t-t_0)}. \tag{61}$$

And then,

$$\int_{t_0}^{t}\|e(t)\|\mathrm{d}s \le \alpha\|e_0\|/\beta. \tag{62}$$

Furthermore,

$$\begin{aligned}\|z(t)-e(t)\| &\le \left[\varepsilon_1+(2+\kappa)\varepsilon_2\right]\int_{t_0}^{t}\|z(s)-e(s)\|\mathrm{d}s\\ &\quad+\frac{\left(\alpha\varepsilon_2(1+\kappa)\|e_0\|\right)}{\beta}\\ &= \Lambda_1\int_{t_0}^{t}\|z(s)-e(s)\|\mathrm{d}s+\Lambda_2\|e_0\|,\end{aligned} \tag{63}$$

where

$$\begin{aligned}\Lambda_1 &= \varepsilon_1+(2+\kappa)\varepsilon_2,\\ \Lambda_2 &= \alpha\varepsilon_2(1+\kappa)/\beta.\end{aligned} \tag{64}$$

By the Gronwall–Bellman's inequality, when $t_0+\rho \le t \le t_0+2T$, we can expediently acquire

$$\|z(t)-e(t)\| \le \Lambda_2\|e_0\|\exp\{2\Lambda_1 T\}. \tag{65}$$

Since $t_0-\rho+T \le t \le t_0-\rho+2T$, from (65) and (66), further we derive

$$\begin{aligned}\|z(t)\| &= \|z(t)-e(t)+e(t)\|\\ &\le \|z(t)-e(t)\|+\|e(t)\|\\ &\le \Lambda_2\|e_0\|\exp(2\Lambda_1 T)+\alpha\|e_0\|\exp(-\beta T)\\ &= \{\Lambda_2\exp(2\Lambda_1 T)+\alpha\exp(-\beta T)\}\|e_0\|\\ &\le \hat{\imath}\sup_{t_0-\rho\le t\le t_0-\rho+T}\|z(t)\|,\end{aligned} \tag{66}$$

where $\hat{\imath} = \Lambda_2\exp\{2\Lambda_1 T\}+\alpha\exp(-\beta T)$.

Denote

$$\begin{aligned}\Psi(\kappa) &= \Lambda_2\exp\{2\Lambda_1 T\}+\alpha\exp(-\beta T)\\ &= \alpha\varepsilon_2(1+\kappa)\exp\frac{\{2[\varepsilon_1+(2+\kappa)\varepsilon_2]T\}}{\beta}+\alpha\exp(-\beta T).\end{aligned} \tag{67}$$

By substituting $\kappa = 1$ into (74), we can see that

$$\Psi(1) = \alpha\exp(-\beta T)+2\varepsilon_2\alpha\exp\frac{\{(2\varepsilon_1+6\varepsilon_2)T\}}{\beta<1}. \tag{68}$$

Clearly that, $\Psi(+\infty) > 1$. In addition, $\Psi(\kappa)$ is strictly monotonously increasing for κ. Accordingly, there is the only one $\overline{\kappa}\in(1,+\infty)$ make

$$\Psi(\overline{\kappa}) = 1. \tag{69}$$

Denote

$$\mathfrak{I}(\rho) = \varepsilon_2\rho+\varepsilon_1\rho(1+\varepsilon_2\rho)\exp\{\varepsilon_1\rho\}, \tag{70}$$

and identify $\tilde{\rho}$ as the only one positive solution to $\mathfrak{I}(\rho) = 1$.

Apparently,

$$\kappa = (1-\mathfrak{I}(\rho))^{-1} \in (1,+\infty), \tag{71}$$

for $\rho \in (0,\tilde{\rho})$. Furthermore, κ is increase strictly monotonically for ρ. In this sense, there is the only one positive scalar $\overline{\rho}\in(0,\tilde{\rho})$ satisfy

$$\kappa = \overline{\kappa}, \tag{72}$$

and $\overline{\rho}$ is the only one positive solution for (56).

Thus,

$$\hat{\imath} = \Lambda_2\exp\{2\Lambda_1 T\}+\alpha\exp(-\beta T) < 1, \tag{73}$$

for $\rho < \overline{\rho}$.

Picking out $g = -(\ln\hat{\imath})/T > 0$, and by (67), one gets

$$\sup_{t_0-\rho+T\le t\le t_0-\rho+2T}\|z(t)\| \le \exp(-gT)\sup_{t_0-\rho\le t\le t_0-\rho+T}\|z(t)\|. \tag{74}$$

Considering the existence and uniqueness of solution $z(t)$ of the system (45), when $t \geq t_0 - \rho + (l-1)T$, it holds

$$z(t, t_0, z_0) = z(t, t_0 - \rho + (l-1)T, z(t_0 - \rho + (l-1)T, t_0, z_0)). \tag{75}$$

From (75) and (76), we have

$$\begin{aligned}
&\sup_{t_0-\rho+lT \leq t \leq t_0-\rho+(l+1)T} \left\| z(t, t_0, z_0) \right\| \\
&= \sup_{t_0-\rho+(l-1)T+T \leq t \leq t_0-\rho+(l-1)T+2T} \left\| z(t; t_0 - \rho + (l-1)T, z(t_0 - \rho + (l-1)T; t_0, z_0)) \right\| \\
&\leq \exp(-gT) \sup_{t_0-\rho+(l-1)T \leq t \leq t_0-\rho+lT} \left\| z(t; t_0, z_0) \right\| \\
&\leq \exp(-glT) \sup_{t_0-\rho \leq t \leq t_0-\rho+T} \left\| z(t; t_0, z_0) \right\| \\
&= \mathrm{G}\exp(-glT),
\end{aligned} \tag{76}$$

where $G = \sup_{t_0-\rho \leq t \leq t_0-\rho+T} \| z(t; t_0, z_0) \|$.

Furthermore, there is only scalar $l \in \mathbb{N}$, so that $t_0 - \rho + lT \leq t \leq t_0 - \rho + (l+1)T$, and one can easily show that

$$\begin{aligned}
\left\| z(t; t_0, z_0) \right\| &\leq G\exp(-glT) \leq G\exp\{-g(t - t_0) + g(T - \rho)\} \\
&\leq G\exp(gT)\exp\{-g(t - t_0)\}.
\end{aligned} \tag{77}$$

By virtue of Theorem 2, one can readily deduce that error system (45) is exponentially stable, i.e., system (1) can achieve exponential synchronization under a designed nonlinear-type controller (42). □

Remark 7. Theorem 2 clearly indicates that when error system (42) is exponentially stable, the corresponding perturbed error system (45) evoked by a deviation argument can still remain exponentially stable as long as the interval length of the deviating argument $\delta(t)$ is less than the estimated upper bound. Furthermore, system (1) involving a deviating argument is still exponentially synchronous under a designed nonlinear-type controller (42).

4. Simulations

Two illustrative examples will be enumerated to show the validity of conclusions obtained above in this section.

Example 1. Here considering a CDN with linear control law, which consists of two nodes:

$$\begin{cases}
\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{2} a_{ij}x_j(t) + u_i(t), \\
x(t_0) = x_0 \in \mathbb{R}^n, \\
\dot{s}(t) = f(s(t)), \\
u_i(t) = \widetilde{W}e_i(t).
\end{cases} \tag{78}$$

In the case of deviation argument, system (78) turns into

$$\begin{cases}
\dot{x}_i(t) = f(x_i(t)) + c\sum_{j=1}^{2} a_{ij}x_j(t) + c\sum_{j=1}^{2} b_{ij}x_j(\delta(t)) + u_i(t), \\
x(t_0) = x_0 \in \mathbb{R}^n, \\
\dot{s}(t) = f(s(t)), \\
u_i(t) = \widetilde{W}e_i(t),
\end{cases} \tag{79}$$

where $i \in 1, 2$, and $x_i(t) = (x_{i1}(t), x_{i2}(t))^T \in \mathbb{R}^2$ is the state vectors of $i-$ th nodes for the CDN. let $e_i(t) = x_i(t) - s(t)$, one can see that error system (78) and error system (79) turns into, respectively,

$$\dot{e}(t) = F(e(t)) + cAe(t) + We(t), \quad e(t_0) = e_0 \in \mathbb{R}^n, \tag{80}$$

$$\begin{aligned}
&\dot{e}(t) = F(e(t)) + cAe(t) + cBe(\delta(t)) + We(t), \\
&e(t_0) = e_0 \in \mathbb{R}^n.
\end{aligned} \tag{81}$$

Let coupling matrices $A = \begin{pmatrix} -5 & 5 \\ 2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} -0.001 & 0.001 \\ 0.002 & -0.002 \end{pmatrix}$, and $K = \begin{pmatrix} -5.5 & 0 \\ 0 & -5.3 \end{pmatrix}$. The coupling strength is designed as $c = 0.1$. The activation function is $f(\cdot) = \tanh(\cdot)$. Two nodes and isolated nodes of the initial value can be designed as $x_1 = (-1.1, 1.2)^T$, $x_2 = (1.7, -1.4)^T$, and $s(t) = (0.2, 0.1)^T$.

As shown in Figure 2, error system (80) is exponentially stable when $\alpha = 1.1$ and $\beta = 0.8$.

Fix two consequences: $\{\rho_q\} = \{q/20\}$, $\{\eta_q\} = \{2q + 1/40\}$, and $q \in \mathbb{N}$. Let $T = 0.2 \geq (\ln 1.1)/0.8 = 0.1191$, $T = 0.2 \geq (\ln 1.1)/0.8 = 0.1191$; then by calculation, we can obtain $h_1 = 7.2616$ and $h_2 = 3.1623 \times 10^{-4}$.

Based on inequality (D4), it is expedient to calculate $\rho = 0.0781$. According to (30), we can obtain $\lambda = 6.8815$ and $\overline{\rho} = 0.0705$.

Selecting $\delta(t) = \rho = 1/20 < \overline{\rho} = 0.0705$, by conducting simple calculations, we can find

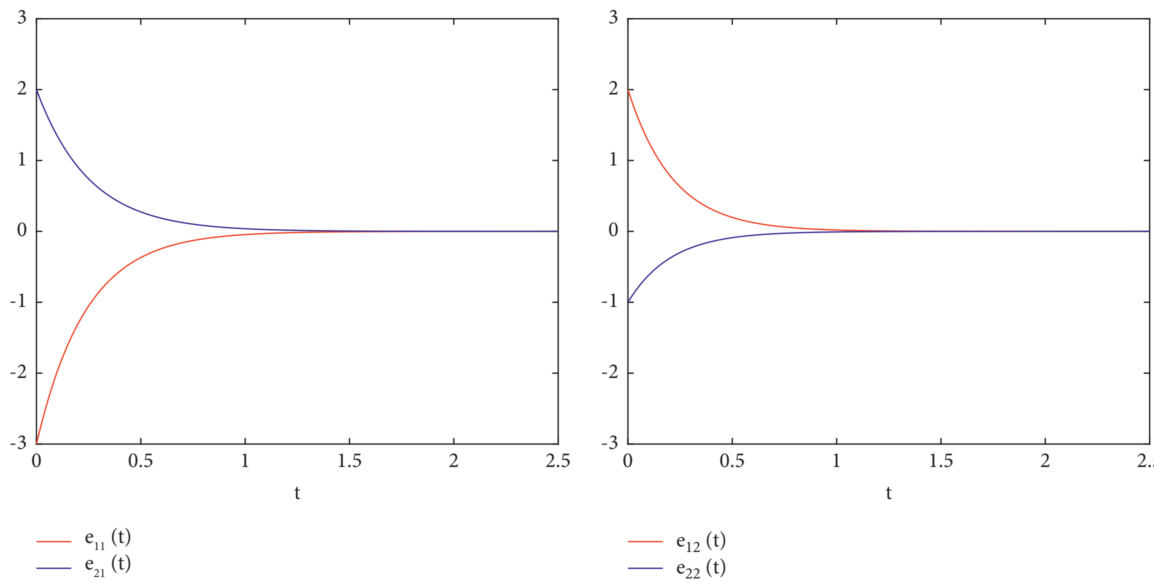

Figure 2: The convergent behavior of the error system under a nonlinear control law.

$$\frac{1}{20}\times\left(7.2616+2\times 3.1623\times 10^{-4}\right)\times \exp\left(7.2616\times\frac{1}{20}\right)=0.5221<1, \tag{82}$$

and (D3) is satisfied.

It is obvious that all the requirements appearing are each fulfilled in Theorem 1. In view of Theorem 1 and Definition 1, we are able to deduce that error system (81) is exponential stability, that is, system (79) is exponential synchronization. As shown in Figure 3, the simulated findings closely match the theory.

Example 2. We discuss a CDN, consisting of two nodes, with nonlinear control law whose dynamics are described as

$$\begin{cases} \dot{x}_i(t)=f\left(x_i(t)\right)+c\sum_{j=1}^{2}a_{ij}x_j(t)+w_i(t),\\ x\left(t_0\right)=x_0\in\mathbb{R}^n,\\ \dot{s}(t)=f\left(s(t)\right),\\ u_i(t)=-\mathrm{f}\left(\mathrm{x_i}(\mathrm{t})\right)+\mathrm{f}\left(\mathrm{s}(\mathrm{t})\right)+\mathrm{e}\widetilde{R}_{\mathrm{i}}(\mathrm{t}). \end{cases} \tag{83}$$

Subsequently, in the presence of deviating argument, system (83) becomes

$$\begin{cases} \dot{x}_i(t)=f\left(x_i(t)\right)+c\sum_{j=1}^{2}a_{ij}x_j(t)+c\sum_{j=1}^{2}b_{ij}x_j(\delta(t))+w_i(t),\\ x\left(t_0\right)=x_0\in\mathbb{R}^n,\\ \dot{s}(t)=f\left(s(t)\right),\\ w_i(t)=-f\left(x_i(t)\right)+f\left(s(t)\right)+\widetilde{R}e_i(t), \end{cases}, \tag{84}$$

where $i\in 1,2$, and $x_i(t)=\left(x_{i1}(t),x_{i2}(t)\right)^T\in\mathbb{R}^2$, is the state vectors.

Let $e_i(t)=x_i(t)-s(t)$, one can see that error system (83) and error system (84) are simplified as, respectively,

$$\dot{e}(t)=cAe(t)+\mathrm{Re}(t),\quad e\left(t_0\right)=e_0\in\mathbb{R}^n, \tag{85}$$

$$\dot{e}(t)=cAe(t)+cBe(\delta(t))+\mathrm{Re}(t),\quad e\left(t_0\right)=e_0\in\mathbb{R}^n. \tag{86}$$

Let coupling matrices $A=\begin{pmatrix}-2 & 2\\ 4 & -4\end{pmatrix}$, $B=\begin{pmatrix}-0.003 & 0.003\\ 0.002 & -0.002\end{pmatrix}$, and $K=\begin{pmatrix}-4.1 & 0\\ 0 & -4.7\end{pmatrix}$. The coupling strength is designated as $c=0.05$ for $i=1,2$. The activation function is $f(\cdot)=\tanh(\cdot)$. Two nodes and isolated nodes of the initial value can be designed as $x_1=(-2,2)^T$, $x_2=(3,-1)^T$, and $s(t)=(1,0)^T$.

As shown in Figure 4, the error system (85) is exponential stability when $\alpha=1.1$ and $\beta=0.5$.

Fix two consequences: $\left\{\rho_q\right\}=\{q/20\}$, $\left\{\eta_q\right\}=\{2q+1/40\}$, $q\in\mathbb{N}$. Let $T=0.4\geq(\ln 1.1)/0.5=0.1906$, then by calculation, we can obtain $h_1=5.0162$, $h_2=2.5495\times 10^{-4}$.

Based on inequality (D7), it is expedient to calculate $\rho=0.1131$. According to (56), we can obtain $\lambda=2.2011$, $\overline{\rho}=0.0748$.

Selecting $\delta(t)=\rho=1/20<\overline{\rho}=0.0748$, by conducting simple calculations, we can find

$$\frac{1}{20}\times\left(5.0162+2\times 2.5495\times 10^{-4}\right)\times \exp\left(5.0162\times\frac{1}{20}\right)=0.3223<1, \tag{87}$$

and (D6) is satisfied.

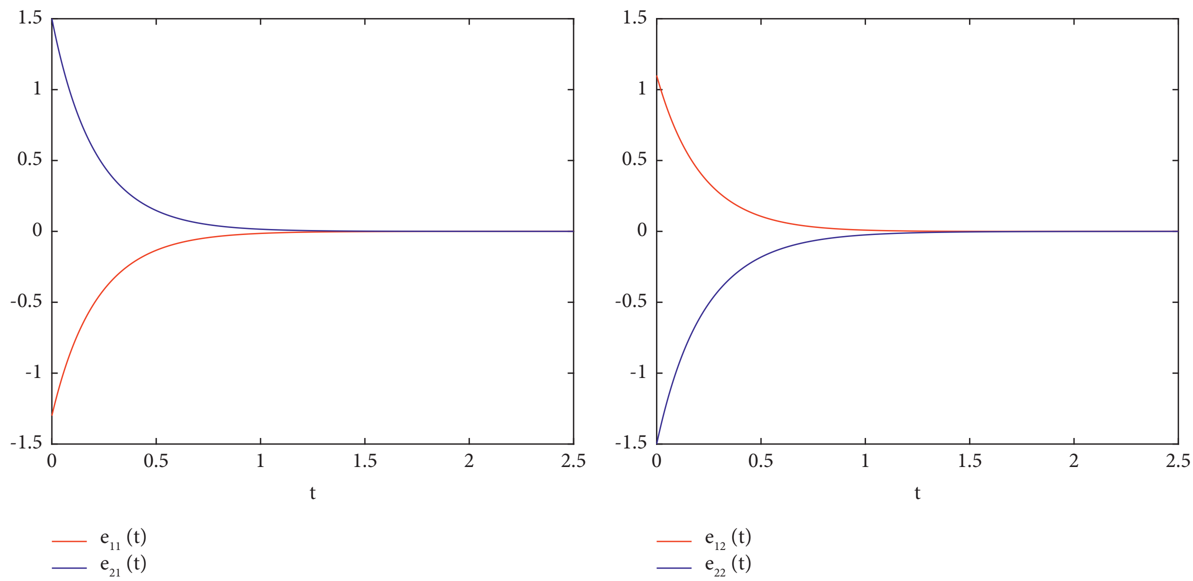

Figure 3: The convergent behavior of error system (80) under a linear control law.

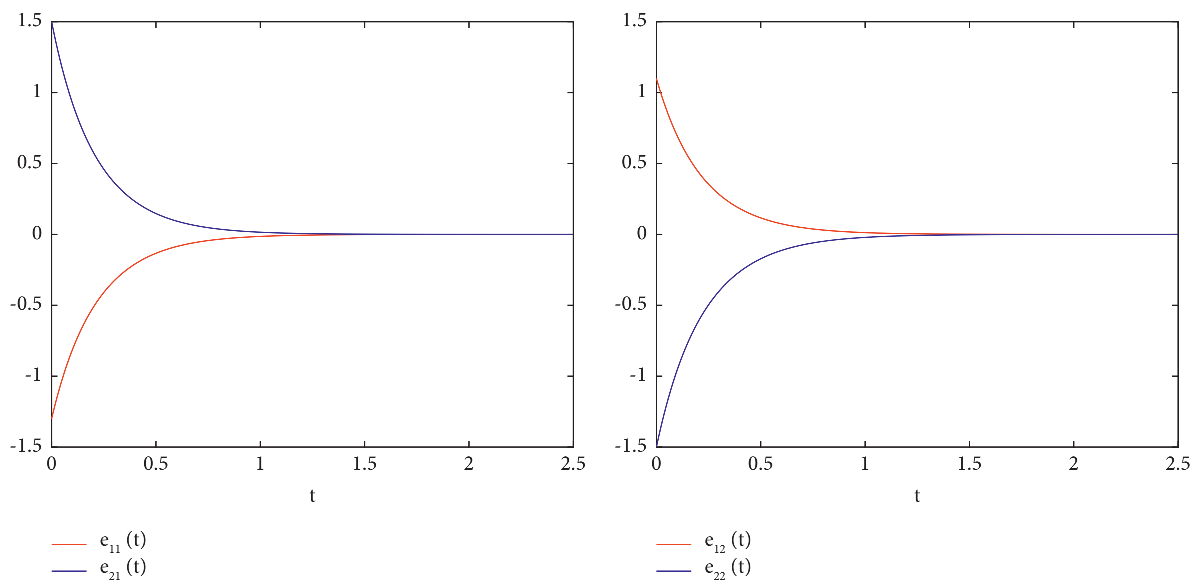

Figure 4: The evolvement behavior of error system (81) under a linear control law.

Clearly, all the requirements outlined in Theorem 2 are each fulfilled. In the light of Theorem 2 and Definition 1, we are able to deduce that error system (86) is exponentially stable, that is to say, system (84) can achieve exponential synchronization when $\rho = 1/20$. As indicated in Figure 5, the simulated findings closely correlate with the theory.

Figure 6 depicts that error system (86) is instable when $\rho = 1/2$. Moreover, in this situation, the parameters are not

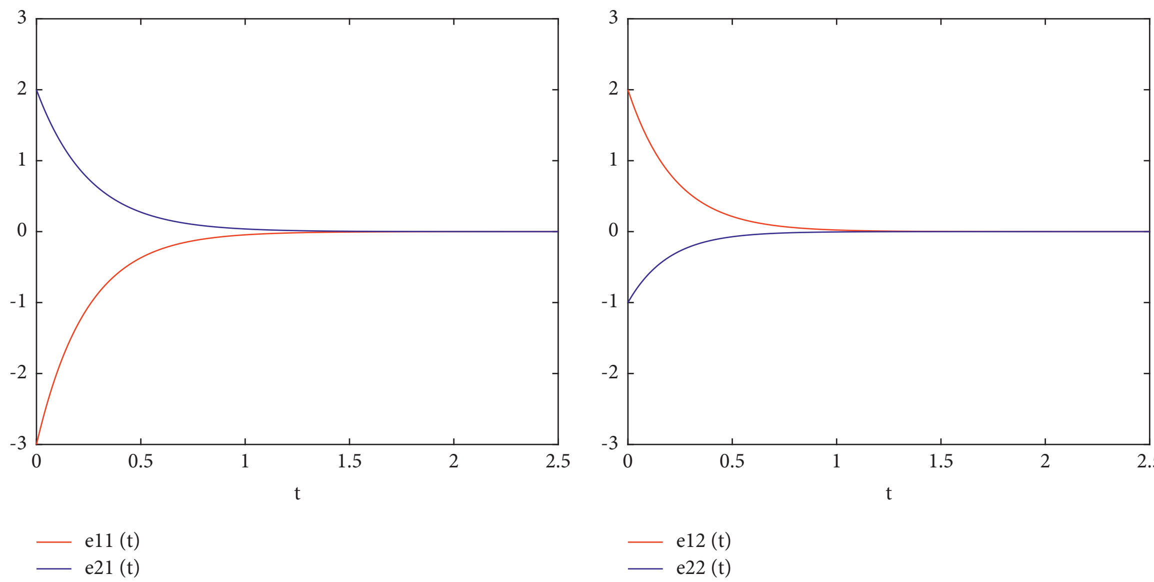

Figure 5: The evolvement behavior of error system (86) under a nonlinear control law.

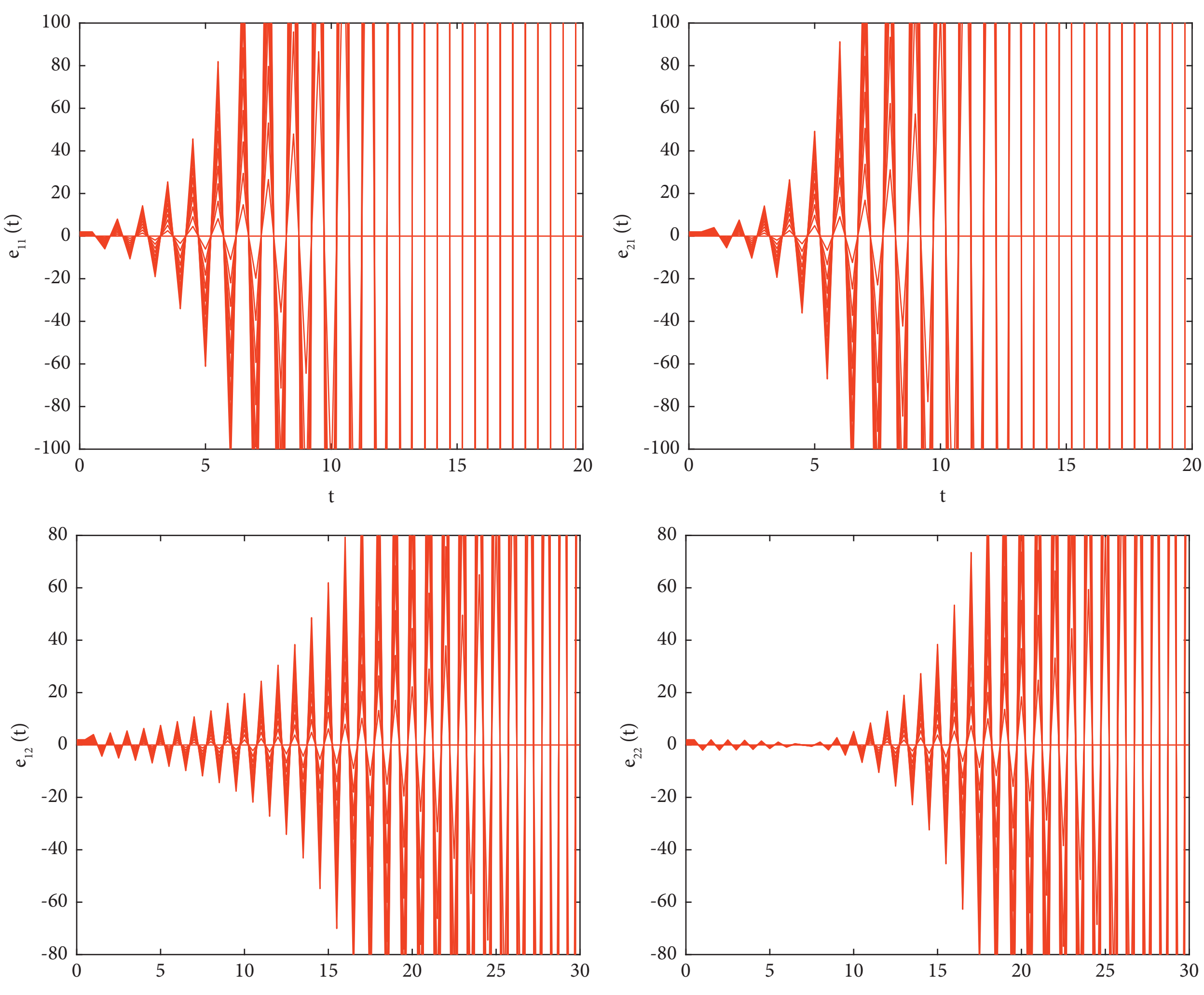

Figure 6: The instable behavior of error system (86) with $\rho = 1/2$ under a nonlinear control law.

appropriate for Theorem 2. Therefore, system (84) is not exponential synchronization.

5. Concluding Remarks

The robustness of the complex system to control laws evoked by deviating arguments is investigated. In this paper, two categories of control laws are provided and some corresponding sufficient criteria are put forward to prove the synchronization of CDNs with deviation argument. The findings show that a complex system containing a deviation function will keep exponential synchronization continuously as long as the interval length of the deviation function is lower than the derived upper limit. In view of the analysis and methodology discussed in this paper, more complex models will be considered for further topics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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