Research Article

q,h-Opial-Type Inequalities via Hahn Operators

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In this paper, the well-known Hölder’s inequality is proved via Hahn differential and integral operators, which is a helping tool to establish some Opial-type inequalities via Hahn’s calculus. The weight functions involved in these Opial-type inequalities are positive and monotone. In search of applications, some new as well as some existing inequalities in the literature are obtained by applying suitable limits.

1. Introduction

In 1960, first time Opial’s inequality was founded by Opial [1]. He established the following important integral inequality.

\[ \int_{c}^{d} |v(t)||v'(t)| \, dt \leq \frac{(d - 4)}{4} \int_{c}^{d} |v'(t)|^2 \, dt, \tag{1} \]

where \( v \) is an absolutely continuous function on \([c, d] \) and \( v(c) = v(d) = 0 \), the constant \((d - 4)/4\) is the most suitable. Equality (1) holds, if and only if

\[ v(t) = \begin{cases} a(t - c), & c \leq v \leq \frac{d - c}{2} \\ a(d - t), & \frac{d - c}{2} \leq v \leq d, \end{cases} \tag{2} \]

where \( a \) is a constant.

Furthermore, the proof of Opial’s inequality is simplified by Olech [2], Beescak [3], Levison [4], Pederson [5], Mallows [6]. Levison [7] proved that if \( v \) is an absolutely continuous function on \((0, d)\) with \( v(0) = 0 \), in (1), then,

\[ \int_{0}^{d} |v(t)||v'(t)| \, dt \leq \frac{d}{4} \int_{0}^{d} |v'(t)|^2 \, dt. \tag{3} \]

For the extension of (1), Beescak [3] demonstrated that if \( u \) is an absolutely continuous function on \([c, x_1] \) with \( u(c) = 0 \), then,

\[ \int_{c}^{x_1} |u(t)||u'(t)| \, dt \leq \frac{1}{2} \int_{c}^{x_1} \frac{1}{p(t)} \, dt \int_{c}^{x_1} p(t)|u'(t)|^2 \, dt, \tag{4} \]

where \( p(t) \) is a continuous and positive function with \( \int_{c}^{x_1} (dt/p(t)) \, dt < \infty \), and if \( u(d) = 0 \), then,

\[ \int_{x_1}^{d} |u(t)||u'(t)| \, dt \leq \frac{1}{2} \int_{x_1}^{d} \frac{1}{p(t)} \, dt \int_{x_1}^{d} p(t)|u'(t)|^2 \, dt. \tag{5} \]

Keng [8] generalized the inequality (3) in the following form: If \( u \) is an absolutely continuous function with \( u(c) = 0 \), then,

\[ \int_{c}^{d} |u(t)||u'(t)| \, dt \leq \frac{(d - c)^{j}}{j + 1} \int_{c}^{d} |u'(t)|^{j+1} \, dt, \tag{6} \]

where \( j \) is a positive integer. Extensions of Beescak’s inequalities (4) and (5) are proved by Yang [9]. He assumed that if \( u \) is absolutely continuous on \([c, x_1] \) with \( u(c) = 0 \), then,
\[ 2 \int_{c}^{x} m(t)u(t)|u'(t)|dt \leq \frac{1}{r(t)} \int_{c}^{x} r(t)m(t)|u'(t)|^2 dt, \]

where \( r \) is continuous and positive function with \( \int_{c}^{x} (1/r(t))dt < \infty \) and \( m \) is positive, bounded and nonincreasing function on \([c, x]\). If \( u(d) = 0 \), then,

\[ 2 \int_{c}^{d} m(t)u(t)|u'(t)|dt \leq \frac{1}{x_{1}} \int_{c}^{d} r(t)m(t)|u'(t)|^2 dt, \]

where \( m \) is positive, bounded and nondecreasing function on \([x_{1}, d]\). Further he assumed that if \( u \) is an absolutely continuous function on \((c, d)\) with \( u(c) = u(d) = 0 \), then,

\[ \int_{c}^{d} m(t)|u(t)||u'(t)|^2 dt \leq \frac{1}{2} K \int_{c}^{d} r(t)m(t)|u'(t)|^2 dt, \]

where \( K \) is defined by \( K = \int_{c}^{x_{1}} (dt/r(t)) = \int_{x_{1}}^{d} (dt/r(t)) \) and \( m(t) \) is a positive bounded function on interval \([c, d]\). Further \( m(t) \) is nonincreasing on \([c, x_{1}] \) and \( m(t) \) is nondecreasing on \([x_{1}, d]\). Also, another extension of Opial's inequality is given by Yang [9] is generalization of (6). He proved that, if \( u(c) = u(d) = 0 \), then,

\[ (j + k) \int_{c}^{x_{1}} r^{k}(x)m^{k}(x)|u(x)|^j|u'(x)|^k dx \]

\[ \leq k \int_{c}^{x_{1}} r^{-k}(x)dx \int_{c}^{x_{1}} r^{j+k}(x)m^{k}(x)|u'(x)|^{j+k} dx, \]

where \( j \geq 0, k \geq 1 \) and \( r \) is positive with \( \int_{c}^{x_{1}} (dt/r^k(t)) < \infty \) and \( m \) is nonincreasing and positive on \([c, x_{1}] \). He also proved a generalization of inequality (8) and suppose that if \( u \) is an absolutely continuous function on \([x_{1}, d]\) and \( u(d) = 0 \), then,

\[ (j + k) \int_{x_{1}}^{d} r^{k}(x)m^{k}(x)|u(x)|^j|u'(x)|^k dx \]

\[ \leq k \int_{x_{1}}^{d} r^{-k}(x)dx \int_{x_{1}}^{d} r^{j+k}(x)m^{k}(x)|u'(x)|^{j+k} dx, \]

where \( k \geq 1 \) and \( j \geq 0 \) and \( r(t) \) is positive function with \( \int_{x_{1}}^{d} (dt/r^k(t)) < \infty \) and \( m(t) \) is nondecreasing and positive on \([x_{1}, d]\). Lee [10] has combined (11) and (12) with \( u(c) = u(d) = 0 \), to find extension of (9) in the following form:

\[ (j + k) \int_{c}^{x} r^{k}(x)m^{k}(x)|u(x)|^j|u'(x)|^k dx \]

\[ \leq kC \left( \int_{c}^{x} r^{j+k}(x)m^{k}(x)|u'(x)|^{j+k} dx \right), \]

where \( C = \left( \int_{c}^{x} r^{-k}(x)dx \right)^j \left( \int_{c}^{x} r^{j+k}(x)m^{k}(x)|u'(x)|^{j+k} dx \right)^j \). Further discussion on Opial-type inequalities may include the work: some Opial inequalities in q-calculus [11] by Mirković, q-Opial-type inequality by Alp, et al., in [12], refinements of Opial-type inequalities in two variables [13], Opial-type inequalities for conformable fractional integrals [14,15], dynamic Opial inequalities on time scales [16–19], Opial-type inequalities in (p, q)-calculus by Li et al., in [20], interval valued Opial-type inequalities by Zhao et al., in [21].

The history of quantum calculus is 300 years old. It is considered the most difficult subject to engage in mathematics by Bernoulli and Euler. The quantum has been derived from “quantus” a Latin word meaning “how much”, commonly quantum deals with the measurement to its smallest unit. Quantum calculus treats the sets of non-differentiable functions without using limits.

The h-calculus \((h > 0)\), deals with calculus of finite differences (Boole [22]). For the study of \( h - \text{calculus} \), readers are suggested to Milne Thomson [23]. Hahn’s calculus unifies h-calculus and q-calculus, initiated by Hahn [24]. It is utilized to construct families of orthogonal polynomials and to deal with some approximation problems [25].

In 2015, Saker et al. [18], Theorems 3.3 and 3.4 initiated the study of dynamic versions of (11) and (12) on time scales (a time scale is a closed subset of real line). In 2019, Fatma et al., [16] have also studied (11)–(13) with the help of time scales calculus. In this paper, we present \( q, h \)-analogues of Opial-type inequalities presented in [10,16,18] with the help of Hahn’s calculus.

The paper is arranged as follows: In Section 2, some basic concepts of Hahn’s calculus and useful lemmas are presented. In Section 3, Hölder inequality and some Opial-type inequalities for monotonic functions via Hahn integrals (also called Jackson Nörlund integrals) are established. Section 4 consists of the conclusion of the paper.

2. Some Essentials of Hahn’s Calculus

The generalization of both quantum calculus and h-calculus is another kind of quantum calculus called Hahn’s quantum calculus. Hahn’s difference operator is defined by Wolfgang Hahn in 1945 [24].

Definition 1. Let \( h > 0 \) and \( 0 < q < 1 \). Define \( h_{q} = (h/1 - q) \) and let \( I \) be any interval in \( R \) containing \( h_{q} \). Suppose \( u: I \rightarrow R \). The \( q, h \)-derivative of \( u \) is described by

\[ D_{q,h}u(t_{i}) = \frac{u(qt_{i} + h) - u(t_{i})}{(qt_{i} + h) - t_{i}}, \]

where \( t_{i} \neq h_{q} \), then, \( D_{q,h}u(t_{i}) = u'(h_{q}) \). \( D_{q,h} \) is the \( q, h \)-derivative of \( u \) at \( t_{i} \) and \( u \) is \( q, h \)-differentiable over \( I \), if \( D_{q,h}u(t_{i}) \) exists for all \( t_{i} \in I \). Generally, when \( q \rightarrow 1 \), we get \( h \)-derivative of \( u(\cdot) \) at \( t_{i} \).
\[ \Delta_h[u](t_1) = \frac{u(t_1 + h) - u(t_1)}{h}. \]  

(15)

When \( h = 0 \), we get q-derivative of \( u(\cdot) \) at \( t_1 \).

\[ D_q[u](t_1) = \frac{u(qt_1) - u(t_1)}{(q-1)t_1}. \]  

(16)

If \( t_1 \neq 0 \) and \( D_q[u](0) = u'(0) \), on condition that \( u'(0) \) exists. It is noted that, by making both restrictions \( q \rightarrow 1, h = 0 \) simultaneously we get usual derivative of \( u(\cdot) \) at \( t_1 \).

\[ \lim_{q \rightarrow 1} D_{q,0}[u](t_1) = u'(t_1). \]  

(17)

The arithmetical properties of Hahn-differential are simply concluded in the following theorem, which is given in [26].

**Theorem 1.** Assume \( u, v \) both are Hahn-differentiable (\( q,h \)-differentiable) at \( t_1 \in I \), then,

\[ D_{q,h}[u](t_1) = 0; \]  

(18)

where \( u \) is a constant.

\[ D_{q,h}[u + v](t_1) = D_{q,h}u(t_1) + D_{q,h}v(t_1); \]  

(19)

\[ D_{q,h}[uv](t_1) = D_{q,h}[u](t_1)v(t_1) + u(qt_1 + h)D_{q,h}[v](t_1); \]  

(20)

\[ u(qt_1 + h) = u(t_1) + (t_1(q - 1) + h)D_{q,h}[u](t_1); \]  

(21)

\[ D_{q,h}\left[ \frac{u}{v} \right](t_1) = \frac{D_{q,h}[u](t_1)v(t_1) - u(t_1)D_{q,h}[v](t_1)}{v(t_1)v(qt_1 + h)}; \]  

(22)

Provided by \( v(t_1)v(qt_1 + h) \neq 0 \).

**Theorem 2** (Chain Rule involving Hahn-differential operator) (see [27]).

Consider \( u : I \rightarrow R \) is Hahn-differentiable and continuous. Let \( v : R \rightarrow R \) is continuously differentiable. Then, there must exists \( c \) between \( t_1 \) and \( qt_1 + h \), such that,

\[ D_{q,h}(vu)(t_1) = v'(u(c))D_{q,h}u(t_1). \]  

(23)

The right inverse of Hahn-differential operator ([28], Chapter 6) is as follows:

**Definition 2.** Let \( I \) be any closed interval of real numbers, which contains \( y \). Suppose that \( g : I \rightarrow R \) and \( c,d \in I \), such that, \( c < d \). The Hahn – integral of \( g \) from \( c \) to \( d \) is defined by,

\[ \int_c^d g(t_1)d_{q,h}t_1 = \int_c^d g(t_1)d_{q,h}t_1 - \int_c^c g(t_1)d_{q,h}t_1 \]  

(24)

where

\[ \int_c^d g(t_1)d_{q,h}t_1 = (x(1 - q) - h)\sum_{k=0}^{\infty} q^k(xq^k + h[k]_q), \quad x \in I, \]  

(25)

\[ [k]_q = (1 - q^2 / 1 - q) \] and the series on the right hand side is convergent at \( x = c, x = d \).

The following properties of Hahn’s integral are given in ([28], Lemma 6.2.2).

**Lemma 1.** Assume \( u,v : I \rightarrow R \) are Hahn – integrable functions on interval \( I, I \in R \) and \( c,d,\lambda \in I \), where \( I \) is a constant. Then,

\[ \int_c^d u(t_1)d_{q,h}t_1 = 0; \]  

(26)

\[ \int_c^d \int_c^d \lambda(t_1)d_{q,h}t_1 = I \int_c^d \int_c^d u(t_1)d_{q,h}t_1; \]  

(27)

\[ \int_c^d u(t_1)d_{q,h}t_1 = - \int_d^c \int_c^d u(t_1)d_{q,h}t_1; \]  

(28)

\[ \int_c^d u(t_1)d_{q,h}t_1 = \lambda^c \int_c^d u(t_1)d_{q,h}t_1 + \int_c^d \lambda(t_1)d_{q,h}t_1 \]  

(29)

\[ \int_c^d (u(t_1) + v(t_1))d_{q,h}t_1 = \int_c^d u(t_1)d_{q,h}t_1 + \int_c^d v(t_1)d_{q,h}t_1. \]  

(30)

Next result can be found in ([28], Lemma 6.2.8).

**Lemma 2.** Assume \( u,v : I \rightarrow R \) are Hahn – integrable and \( y \in I \). Denote \([m]_q = (1 - q^m / 1 - q) \). If \([u(t)] \leq v(t) \forall t, c, d \in \{ yq^m + h[m]_q \}_{m=0}^{\infty} \) then,

\[ \int_c^d u(t)d_{q,h}t_1 \leq \int_c^d v(t)d_{q,h}t_1. \]  

(31)

In particular, (31) leads to the following inequality. If \( u : I \rightarrow R \) is Hahn – integrable, \( y \in I \), then \( \forall t, c, d \in \{ yq^m + h[m]_q \}_{m=0}^{\infty} \) one has that,

\[ \int_c^d |u(t)|d_{q,h}t_1 \leq \int_c^d |v(t)|d_{q,h}t_1. \]  

(32)

The following \( q,h \) integration by parts formula can be found in ([28], Lemma 6.2.8).

**Lemma 3.** If \( u,v : I \rightarrow R \) are continuous at \( h' \). Then,

\[ \int_c^d u(t_1)D_{q,h}v(t_1)d_{q,h}t_1 = u(t_1)v(t_1)\int_c^d \]  

\[ - \int_c^d D_{q,h}u(t_1)v(qt_1 + h)d_{q,h}t_1. \]  

(33)
3. Opial Inequalities for Monotone Functions

3.1. Hölder’s Inequality via Hahn’s Calculus. The Hölder inequality plays a fundamental role in the field of mathematics. Different variants can be found in [29,30]. All through this section, $j$ and $i$ are conjugate to each other’s, as $(1/j) + (1/i) = 1$. In order to extend Opial-type inequalities by using Hahn calculus, we first prove the Hölder’s inequality involving Hahn calculus.

**Theorem 3.** If $a, b \in I$ and $u, v : [a, b] \rightarrow R$ are continuous functions, then,

$$
\int_a^b |u(t)v(t)|d_{q,h}t \leq \left\{ \int_a^b |u(t)|^{i/j}d_{q,h}t \right\}^{(1/i)} \left\{ \int_a^b |v(t)|^{j/i}d_{q,h}t \right\}^{(1/j)},
$$

where $j > 1$ and $i = j/(j-1)$.

Proof. For nonnegative $\alpha, \beta$ real numbers, the basic inequality holds.

$$
\alpha^{(1/j)} \beta^{(1/i)} \leq \frac{\alpha}{j} + \frac{\beta}{i},
$$

(35)

Now consider,

$$
\left\{ \int_a^b |u(t)|^{i/j}d_{q,h}t \right\}^{(1/i)} \left\{ \int_a^b |v(t)|^{j/i}d_{q,h}t \right\}^{(1/j)} \neq 0.
$$

(36)

In (35), choose,

$$
\alpha(t) = \int_a^b |u(t)|^{i/j}d_{q,h}t, \beta(t) = \int_a^b |v(t)|^{j/i}d_{q,h}t.
$$

(37)

Use (35) and integrating the resultant inequality from $a$ to $b$ to obtain

$$
\int_a^b \alpha^{(1/j)} \beta^{(1/i)}(t)d_{q,h}t = \int_a^b \frac{|u(t)|^{i/j}}{\int_a^b |u(t)|^{i/j}d_{q,h}t} \frac{|v(t)|^{j/i}}{\int_a^b |v(t)|^{j/i}d_{q,h}t}d_{q,h}t
$$

$$
\leq \int_a^b \left( \frac{1}{j} \int_a^b |u(t)|^{i/j}d_{q,h}t + \frac{1}{i} \int_a^b |v(t)|^{j/i}d_{q,h}t \right) d_{q,h}t
$$

$$
= \int_a^b \frac{|u(t)|^{i/j}}{\int_a^b |u(t)|^{i/j}d_{q,h}t} \frac{|v(t)|^{j/i}}{\int_a^b |v(t)|^{j/i}d_{q,h}t} d_{q,h}t = 1.
$$

(38)

Hence, we get (34). The proof is complete.

Next, consider the notations $\tilde{y}_m = yq^m + h[m]$, $\tilde{y} = (y + nh)$ and $\tilde{y} = yq^m$, where $y \in \{a, x_1, b\}$.

**Remark 1.** In form of sums (34) can be written as

$$
(b(1-q)-h) \sum_{m=0}^{\infty} q^m \left| u(\tilde{b}_m)v(\tilde{b}_m) \right| - (a(1-q)-h) \sum_{m=0}^{\infty} q^m \left| u(\tilde{a}_m)v(\tilde{a}_m) \right|
$$

$$
\leq \left\{ (b(1-q)-h) \sum_{m=0}^{\infty} q^m \left| u(\tilde{b}_m) \right|^{(1/i)} - (a(1-q)-h) \sum_{m=0}^{\infty} q^m \left| u(\tilde{a}_m) \right|^{(1/i)} \right\}^{(1/j)}
$$

$$
\times \left\{ (b(1-q)-h) \sum_{m=0}^{\infty} q^m \left| v(\tilde{b}_m) \right|^{(1/j)} - (a(1-q)-h) \sum_{m=0}^{\infty} q^m \left| v(\tilde{a}_m) \right|^{(1/j)} \right\}^{(1/i)}.
$$

(39)

When $h = 0$ in (39), it recaptures the Hölder’s inequality in q-calculus [31].
\[
\sum_{m=0}^{\infty} q^m ((b|u(\hat{b})\nu(\hat{b})) - (a|u(\hat{a})\nu(\hat{a})))
\leq \left\{ \sum_{m=0}^{\infty} q^m (b|u(\hat{b})|a - a|u(\hat{a})|a) \right\}^{(l/j)}
\leq \left\{ \sum_{m=0}^{\infty} q^m (b|v(\hat{b})|a - a|v(\hat{a})|a) \right\}^{(l/j)}.
\]

By using \( q \to 1 \) in (39), we get Hölder’s inequality in \( h \)-discrete calculus, which is ([32], Theorem 3.1).

\[
\sum_{m=0}^{\infty} |u(\hat{a})\nu(\hat{a})| - |u(\hat{b})\nu(\hat{b})| \leq \left\{ \sum_{m=0}^{\infty} |u(\hat{a})|^l - |u(\hat{b})|^l \right\}^{(l/j)}
\leq \left\{ \sum_{m=0}^{\infty} |v(\hat{a})|^l - |v(\hat{b})|^l \right\}^{(l/j)}.
\]  

(41)

When \( h = 0, q \to 1 \) in (34), we obtain Hölder’s inequality in classical calculus, which can be found in [33].

3.2. Opial-Type Inequalities on \([a, x_1]\)

**Theorem 4.** Let \( I \) be any interval, \( y \in I \) and \( a, x_1 \in \{ yq^m + h[m] \}_{m=0}^{\infty} \). Assume \( I(\cdot), s(\cdot) \) are positive continuous functions on \([a, x_1]\). \( s(\cdot) \) is non-increasing function on \([a, x_1]\) and \( \int_{a}^{x_1} l^{-k}(z) d_{q,h} \nu < \infty \). Furthermore, suppose \( u \) is a continuous function on \([a, x_1]\) with \( u(a) = 0 \). Then, \( \forall j \geq 1, k \geq 1, \) and \( \forall t, z \in \{ yq^m + h[m] \}_{m=0}^{\infty} \), we have,

\[
(j + k) \int_{a}^{x_1} l^{-k}(z)s^j(z)u(z)|D_{q,h}u(z)|^k d_{q,h} \nu
\leq k \left( \int_{a}^{x_1} l^{-k}(z)s^j(z) \right) \left( \int_{a}^{x_1} l^{-k}(z)s^j(z)D_{q,h}u(z) |D_{q,h}u(z)|^k d_{q,h} \nu \right).
\]  

(42)

**Proof.** Let us consider the following integral

\[
T(z) = \int_{a}^{z} l^{-k}(t)s^j(t) |D_{q,h}u(t)|^k d_{q,h} \nu.
\]  

(43)

Since \( T(a) = 0 \), one has that,

\[
D_{q,h}T(z) = l^{-k}(z)s^j(t) |D_{q,h}u(t)|^k.
\]  

(44)

Equation (32) implies,

\[
|u(z)| = |u(z) - u(a)|
\leq \left( \int_{a}^{z} D_{q,h}u(t) |D_{q,h}u(t)|^k d_{q,h} \nu \right) \left( \int_{a}^{z} l^{-k}(t) D_{q,h}u(t) |D_{q,h}u(t)|^k d_{q,h} \nu \right)^{(1/k)}.
\]  

(45)

Use of Hölder inequality (34) with indices \( k \) and \( (k/(k - 1)) \), provides,

\[
|u(z)| \leq \left( \int_{a}^{z} l^{-k}(t) D_{q,h}u(t) |D_{q,h}u(t)|^k d_{q,h} \nu \right)^{(1/k)}.
\]  

(46)

Since \( s(\cdot) \) is nonincreasing and positive on \([a, x_1]\), one has that,

\[
s^{j(k/(j+k))}(z) |u(z)|^l \leq s^{j(k/(j+k))}(z) \left( \int_{a}^{z} l^{-k}(t) D_{q,h}u(t) |D_{q,h}u(t)|^k d_{q,h} \nu \right)^{(j/k)} \left( \int_{a}^{z} l^{-k}(t) D_{q,h}u(t) |D_{q,h}u(t)|^k d_{q,h} \nu \right)^{(1/k)}.
\]  

(47)

\[
\left( \int_{a}^{z} l^{-k}(t) D_{q,h}u(t) |D_{q,h}u(t)|^k d_{q,h} \nu \right)^{(j/k)}.
\]  

(48)
Multiply (48) by $p^{k(k-1)}(z)s^{(k^2/j+k)}(z)|D_{q,h}u(z)|^k \geq 0$, to get
\begin{equation}
p^{k(k-1)}(z)s^{(k^2/j+k)}(z)|D_{q,h}u(z)|^k \leq p^{k(k-1)}(z)\left(\int_a^x l^{-k}(t)d_q,t\right)^{(k-1)/k}\frac{T^{(j/k)}(z)s^{(k^2/j+k)}(z)|D_{q,h}u(z)|^k}{T^{(j/k)}(z)|D_{q,h}u(z)|^k}. \tag{49}
\end{equation}

Multiply (49) by $(j + k)$, integrate it from $a$ to $x$, and use monotonicity of $(\int_a^x l^{-k}(t)d_q,t)$ to obtain
\begin{equation}
(j + k)\int_a^x p^{k(k-1)}(z)s^{k}(z)|u(z)|^k|D_{q,h}u(z)|^k d_q,z \leq \frac{(j + k)}{k}\frac{\int_a^x l^{-k}(t)d_q,t}{\left(\int_a^x l^{-k}(t)d_q,t\right)^{(j-1)/k}}\frac{T^{(j/k)}(z)s^{(k^2/j+k)}(z)|D_{q,h}u(z)|^k}{T^{(j/k)}(z)|D_{q,h}u(z)|^k}. \tag{50}
\end{equation}

Now from Chain rule (23), we have
\begin{equation}
D_{q,h}\left(T^{(j/k)}(z)\right)(z) = \frac{(j + k)}{k}T^{(j/k)}(z)D_{q,h}T(z), c \in [z, qz + h]. \tag{51}
\end{equation}
Since $D_{q,h}T(z) \geq 0$ and $z \leq c$, one gets
\begin{equation}
D_{q,h}\left(T^{(j/k)}(z)\right)(z) = \frac{(j + k)}{k}T^{(j/k)}(z)D_{q,h}T(z). \tag{52}
\end{equation}

From (50) and (52), it is noted that $T(a) = 0$ and
\begin{equation}
(j + k)\int_a^x p^{k(k-1)}(z)s^{k}(z)|u(z)|^k|D_{q,h}u(z)|^k d_q,z \leq \frac{k}{k}\left(\int_a^x l^{-k}(z)d_q,z\right)^{(j(k-1))/k}\left(\int_a^x D_{q,h}\left(T^{(j/k)}(z)\right)d_q,z\right). \tag{53}
\end{equation}

(by using (43)),
\begin{equation}
\leq k\left(\int_a^x l^{-k}(z)d_q,z\right)^{(j(k-1))/k}T^{(j/k)}(z_1) \tag{54}
\end{equation}

Denote,
\begin{equation}
L(x_1) = \left(\int_a^x l^{k(k-1)}(z)s^{(k^2/j+k)}|D_{q,h}u(z)|^k d_q,z\right)^{(j(k-1))/k}. \tag{55}
\end{equation}

By applying Hölder inequality (34) with indices $(j + k)/j$ and $(j + k)/k$ on $L$, one gets
\begin{equation}
L = \left(\int_a^x l^{(-k(1+j/k))}(z)s^{(k^2/j+k)}|D_{q,h}u(z)|^k d_q,z\right)^{(j(k-1))/k} = \left(\int_a^x \left(\int_a^x l^{(-k(1+j/k))}(z)s^{(k^2/j+k)}|D_{q,h}u(z)|^k d_q,z\right)^{(j(k-1))/k} d_q,z\right)^{(j(k-1))/k} = \left(\int_a^x l^{-k}(z)d_q,z\right)^{(j(k-1))/k} \left(\int_a^x l^{k(k-1)}(z)s^{(k^2/j+k)}|D_{q,h}u(z)|^k d_q,z\right)^{(j(k-1))/k}. \tag{56}
\end{equation}

By combining (54) and (56), we get
\begin{equation}
(j + k)\int_a^x p^{k(k-1)}(z)s^{(k^2/j+k)}|u(z)|^k|D_{q,h}u(z)|^k d_q,z \leq \left(\int_a^x l^{-k}(z)d_q,z\right)^{(j(k-1))/k}\left(\int_a^x l^{k(k-1)}(z)s^k|D_{q,h}u(z)|^k d_q,z\right). \tag{57}
\end{equation}

Hence, (42) is obtained. \hfill \Box

**Corollary 1**

Case 1. When $a \to 1$, then (42) reduces to the following Opial inequality in $h$ - discrete calculus.
\[
\sum_{m=0}^{\infty} \left( I^{k(k-1)}(\bar{a}) s^k(\bar{a}) u^j(\bar{a}) |\Delta_{\bar{a}} u(\bar{a})|^k - I^{(k-1)}(\bar{x}_1) s^k(\bar{x}_1) u^j(\bar{x}_1) |\Delta_{\bar{x}_1} u(\bar{x}_1)|^k \right)
\]
\[
\leq L_1 \sum_{m=0}^{\infty} \left( I^{(j+k-1)}(\bar{a}) s^k(\bar{a}) |\Delta_{\bar{a}} u(\bar{a})|^{j+k} - I^{(j+k-1)}(\bar{x}_1) s^k(\bar{x}_1) |\Delta_{\bar{x}_1} u(\bar{x}_1)|^{j+k} \right),
\]
where \( L_1 = (kh^j/(j+k)) \left( \sum_{m=0}^{\infty} (I^{-k}(\bar{a}) - I^{-k}(\bar{x}_1))^j \right). \)

Case 2. When \( h = 0 \) in (42), it is converted to the following Opial inequality in \( q \)-calculus.

\[
\sum_{m=0}^{\infty} q^k \left( x_1 \left( I^{k(k-1)}(\bar{x}_1) s^k(\bar{x}_1) u^j(\bar{x}_1) |\Delta_{\bar{x}_1} u(\bar{x}_1)|^k \right) - a \left( I^{(k-1)}(\bar{a}) s^k(\bar{a}) u^j(\bar{a}) |\Delta_{\bar{a}} u(\bar{a})|^k \right) \right)
\]
\[
\leq L_2 \sum_{m=0}^{\infty} q^k \left( x_1 I^{(j+k-1)}(\bar{x}_1) s^k(\bar{x}_1) |\Delta_{\bar{x}_1} u(\bar{x}_1)|^{j+k} - a \left( I^{(j+k-1)}(\bar{a}) s^k(\bar{a}) |\Delta_{\bar{a}} u(\bar{a})|^{j+k} \right) \right),
\]
where \( L_2 = (k(1-q)^j/(j+k)) \left( \sum_{m=0}^{\infty} q^k (x_1 I^{-k}(\bar{x}_1) - a I^{-k}(\bar{x}_1))^j \right). \)

Case 3. When \( q \rightarrow 1 \) and \( h = 0 \) in (42). We get (11), which is ([10], Theorem 1.1), ([16], Corollary 3.2).

\[
(j+k) \int_a^{x_1} I^{k(k-1)}(z) s^k(z) |u(z)|^j |u'(z)|^k dz \leq \left( \int_a^{x_1} I^{-k}(z) dz \right) \left( \int_a^{x_1} I^{j(k-1)}(z) s^k(z) |u'(z)|^{j+k} dz \right).
\]

Case 4. When \( q \rightarrow 1 \), \( h = 0 \) and \( j = k = 1 \). We obtain (7) given by Yang in ([19], Theorem 3), ([16], Remark 3.3).

\[
2 \int_a^{x_1} s(z) |u(z)||u'(z)| dz \leq \left( \int_a^{x_1} I^{-1}(z) dz \right) \left( \int_a^{x_1} I(z) s(z) |u'(z)|^2 dz \right).
\]

Remark 2. If \( h = 1 \) in (26), then, it recaptures Opial-type discrete inequality given in ([16], Corollary 1).

3.3. Opial-Type Inequalities on \([x_1, b] \)

**Theorem 5.** Let \( I \) be any interval, \( y \in I \) and \( x_1, b \in \{ yq^m + h[m]_q \}_{m=0}^{\infty} \). Assume \( I(\cdot), s(\cdot) \) are positive continuous functions on \([x_1, b]\), \( s(\cdot) \) is nondecreasing on \([x_1, b]\), and \( \int_{z}^{b} I^{-k}(z) dz < \infty \). Furthermore, assume \( u(\cdot) \) is a continuous function on \([x_1, b]\) with \( u(b) = 0 \), then \( \forall j \geq 1, k \geq 1, z, t \in \{ yq^m + h[m]_q \}_{m=0}^{\infty} \) we have,

\[
(j+k) \int_{z}^{t} I^{k(k-1)}(z) s^k(z) |u(qz + h)||D_{q,h} u(z)\|^k dz \leq L \left( \int_{z}^{t} I^{-k}(z) dz \right) \left( \int_{z}^{t} I^{j(k-1)}(z) s^k(z) |D_{q,h} u(z)|^{j+k} dz \right).
\]

**Proof.** Let us consider the following integral:

\[
T(z) := - \int_{z}^{b} I^{k(k-1)}(t) s^{(k/(j+k))}(t) |D_{q,h} u(t)|^k dz.
\]

By using \( T(h) = 0 \), we get

\[
D_{q,h} T(z) = I^{k(k-1)}(z) s^{(k/(j+k))}(z) |D_{q,h} u(z)|^k.
\]

Use of (32) gives

\[
|u(qz + h)| = |u(qz + h)| = |u(b) - u(qz + h)| \leq \int_{qz+h}^{b} |D_{q,h} u(t)| dz \leq \int_{qz+h}^{b} |D_{q,h} u(t)| dz.
\]

Use of (34) with indices \( k \) and \( (k/(k-1)) \) implies
\[ |u(qz+h)| \leq \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{1/k} \]

By taking power \( j \) on both sides, one gets

\[ |u(qz+h)|^j \leq \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{(j(1-k)/k)} \]

\[ \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{(j/k)} \] (67)

\[ s^{(k^j)(j+k)}(z)|u(qz+h)|^j \leq s^{(k^j)(j+k)}(z) \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{(j(1-k)/k)} \leq \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{(j/k)} \] (68)

Multiply both sides of (69) by \( l^{k(1-k)}(z) s^{(k^j)(j+k)}(z) |D_{q,t} u(z)|^k \leq 0 \), to find,

\[ l^{k(1-k)}(z) s^{(k^j)(j+k)}(z)|u(qz+h)|^j \leq l^{k(1-k)}(z) \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{(j(1-k)/k)} \] (69)

(63) implies,

\[ \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{(j(1-k)/k)} \] (70)

(70) can be written in the following subsequent form,

\[ l^{k(1-k)}(z) s^{(k^j)(j+k)}(z)|u(qz+h)|^j \left| D_{q,t} u(z) \right|^k \leq l^{k(1-k)}(z) \left( \int_{qz+h}^{b} \Gamma^{(1-k)}(t) \left| D_{q,t} u(t) \right|^k \right)^{(j(1-k)/k)} \] (71)
Now, from Chain rule (23), we have

\[ (-D_{q,h}(-T^{((j+k)/k)})(z)) = \frac{(j+k)}{k} (-T)^{(j/k)}(c)D_{q,h}T(z), c \in [z, qz + h]. \]

(73)

Since \( D_{q,h}T(z) \geq 0 \) and \( c \leq qz + h \), one gets

\[ D_{q,h}(-(-T^{((j+k)/k)})(z)) = \frac{(j+k)}{k} (-T)^{(j/k)}(c)D_{q,h}T(z) \geq \frac{(j+k)}{k} ((-T)(qz + h))D_{q,h}T(z). \]

(74)

It is noted that \( T(b) = 0 \). (72) and (74) yields,

\[ (j+k) \int_{x_i}^{b} \tilde{l}^{(k-1)}(z)\tilde{s}(z)u(qz + h)\left|D_{q,h}u(z)\right|^k d_{q,h}z \]

\[ \leq (j+k) \left( \int_{x_i}^{b} \tilde{l}^{(j/k)}(z)D_{q,h}T(z)\tilde{s}(z)\left|D_q u(z)\right|^k d_{q,h}z \right) \]

\[ \leq k \left( \int_{x_i}^{b} \tilde{l}^{(j/k)}(z)D_{q,h}T(z)\tilde{s}(z)\left|D_{q,h}^2 u(z)\right|^k d_{q,h}z \right). \]

(75)

\[
\begin{align*}
\left( \int_{x_i}^{b} \tilde{l}^{(k-1)}(z)\tilde{s}(z)\left|D_q u(z)\right|^k d_{q,h}z \right) \left|D_{q,h}u(z)\right|^k d_{q,h}z \\
\leq \left( \int_{x_i}^{b} \tilde{l}^{(j/k)}(z)\tilde{s}(z)\left|D_{q,h}^2 u(z)\right|^k d_{q,h}z \right) \left|D_{q,h}u(z)\right|^k d_{q,h}z \\
\leq \left( \int_{x_i}^{b} \tilde{l}^{(j/k)}(z)\tilde{s}(z)\left|D_{q,h}^2 u(z)\right|^k d_{q,h}z \right) \left( \int_{x_i}^{b} \tilde{l}^{(1-1/(j+k))}(z)\tilde{s}(z)\left|D_{q,h}^2 u(z)\right|^k d_{q,h}z \right) \\
= \left( \int_{x_i}^{b} \tilde{l}^{(j/k)}(z)\tilde{s}(z)\left|D_{q,h}^2 u(z)\right|^k d_{q,h}z \right) \left( \int_{x_i}^{b} \tilde{l}^{(1-1/(j+k))}(z)\tilde{s}(z)\left|D_{q,h}^2 u(z)\right|^k d_{q,h}z \right). \end{align*}
\]

(77)

By combining (76) and (77), we get
\[
(j + k) \int_{x_i}^{b} \int_{x_i}^{b} (s(x))u(z)|u'(z)|dz \leq k \left( \int_{x_i}^{b} l^{-1}(z)dz \right) \left( \int_{x_i}^{b} l^{(j+1-k)}(z)s(z)|u'(z)|^{j+k}dz \right).
\]

3.4. Opial-Type Inequalities on \((a,b)\)

**Theorem 6.** Suppose \(I\) is any interval, \(y \in I\) and \(a, b \in \{y q^m + h[m]_{m=0}^{\infty}\} \cup \emptyset\). Assume \(f(\cdot), s(\cdot)\) are positive continuous functions on \([a,b]\). \(s(\cdot)\) is nonincreasing on \([a,x_1]\) and \(s(\cdot)\) is nondecreasing on \([x_1,b]\)

\[
2 \int_{x_i}^{b} s(z)|u(z)||u'(z)|dz \leq \left( \int_{x_i}^{b} l^{-1}(z)dz \right) \left( \int_{x_i}^{b} l^{(j+1)}(z)s(z)|u'(z)| dz \right).
\]

Hence, (62) is proved.

**Corollary 2.** Case I: When \(q \to 1\), then, (62) is converted to the following Opial's inequality in \(h\) discrete-calculus.

**Case II.** When \(h = 0\) in (62), it is converted to the following Opial inequality in \(q\)-calculus.

\[
\sum_{m=0}^{\infty} q^{k} \left( b^{(k-1)}(b)u^{j}(\bar{b})[\Delta_{h}u(\bar{b})]^{k} - b^{(k-1)}(\bar{b})u^{j}(\bar{b})[\Delta_{h}u(\bar{b})]^{k} \right) \leq L_{3} \sum_{m=0}^{\infty} q^{j} \left( b^{(j+k-1)}(b)u^{j} \Delta_{h}u(\bar{b})^{j+k} - b^{(j+k-1)}(\bar{b})u^{j} \Delta_{h}u(\bar{b})^{j+k} \right).
\]

Moreover, if \(h = 1\) in (62), it recaptures ([16], Corollary 3.8).

\[
\sum_{m=0}^{\infty} q^{k} \left( b^{(k-1)}(\bar{b})u^{j}(\bar{b})u^{j}(\bar{b})[\Delta_{h}u(\bar{b})]^{k} \right) \leq L_{4} \sum_{m=0}^{\infty} q^{j} \left( b^{(j+k-1)}(\bar{b})u^{j} \Delta_{h}u(\bar{b})^{j+k} - b^{(j+k-1)}(\bar{b})u^{j} \Delta_{h}u(\bar{b})^{j+k} \right).
\]

3.4. Opial-Type Inequalities on \((a,b)\)

**Theorem 6.** Suppose \(I\) is any interval, \(y \in I\) and \(a, b \in \{y q^m + h[m]_{m=0}^{\infty}\} \cup \emptyset\). Assume \(f(\cdot), s(\cdot)\) are positive continuous functions on \([a,b]\). \(s(\cdot)\) is nonincreasing on \([a,x_1]\) and \(s(\cdot)\) is nondecreasing on \([x_1,b]\), therefore by using (42) and (62) in (84), one gets
We get (13) ([10], Theorem 2), ([16], Corollary 3.10).

Moreover, if

\[ (j + k) \int_a^b p^{(k-1)}(z) s^k(z) u(z) dz \leq (j + k) \int_a^b p^{(k-1)}(z) s^k(z) u(z) dz + (j + k) \int_{x_i}^b p^{(k-1)}(z) s^k(z) u(qz + h) dz \]

\[ \leq k \left( \int_a^{x_i} I^k(z) dz \right)^j \left( \int_a^{x_i} p^{(j+k-1)}(z) s^k(z) D_q u(z) dz \right)^j d_q z \]

\[ \leq k \left( \int_a^{x_i} I^k(z) dz \right)^j \left( \int_a^{x_i} p^{(j+k-1)}(z) s^k(z) D_q u(z) dz \right)^j d_q z \]

which is the required inequality (83).

Corollary 3. Case I: When \( q \rightarrow 1 \), then, the inequality (83) becomes the Opial’s inequality in \( h - \) calculus.

\[ \sum_{m=0}^\infty q^k \left( b^{p^{(k-1)}(\bar{b})} s^k(\bar{b}) | \Delta_q u(\bar{a}) |^k - a^{p^{(k-1)}(\bar{a})} s^k(\bar{a}) | \Delta_q u(\bar{a}) |^k \right) \]

\[ \leq C_1 \sum_{m=0}^\infty q^k \left( b^{p^{(j+k-1)}(\bar{b})} s^k(\bar{b}) | \Delta_q u(\bar{b}) |^{j+k} - a^{p^{(j+k-1)}(\bar{a})} s^k(\bar{a}) | \Delta_q u(\bar{a}) |^{j+k} \right), \]

where \( C_1 = \max\{L_1, L_3\} \).

Moreover, if \( h = 1 \) in (86), it recaptures ([16], Corollary 3.11).

\[ \sum_{m=0}^\infty g^k \left( b^{p^{(k-1)}(\bar{b})} s^k(\bar{b}) | \Delta_q u(\bar{b}) |^k - a^{p^{(k-1)}(\bar{a})} s^k(\bar{a}) | \Delta_q u(\bar{a}) |^k \right) \]

\[ \leq C_2 \sum_{m=0}^\infty g^k \left( b^{p^{(j+k-1)}(\bar{b})} s^k(\bar{b}) | \Delta_q u(\bar{b}) |^{j+k} - a^{p^{(j+k-1)}(\bar{a})} s^k(\bar{a}) | \Delta_q u(\bar{a}) |^{j+k} \right), \]

where \( C_2 = \max\{L_2, L_4\} \).

Case III. When \( q \rightarrow 1 \) and \( h = 0 \) in the inequality (83). We get (13) ([10], Theorem 2), ([16], Corollary 3.10).

\[ (j + k) \int_a^b p^{(k-1)}(z) s^k(z) u(z) | u'(z) |^k dz \leq k C \]

\[ \left( \int_a^b p^{(j+k-1)}(z) s^k(z) u'(z) | u'(z) |^{j+k} dz \right). \]

\[ 2 \int_a^b s(z) | u(z) | u'(z) | dz \leq C \left( \int_a^b l(z) s(z) | u'(z) |^2 dz \right). \]
4. Conclusions

In this article, Hölder’s inequality is proved with the help of Hahn calculus whose special cases contain Hölder in q-calculus [31], in h-discrete calculus [32], and in usual calculus [33]. Furthermore, the obtained Hölder’s inequality is utilized to find some Opial-type inequalities. Limiting cases of newly proved Opial-type inequalities coincide with some Opial-type inequalities of [9,10,16]. Some extensions of Opial-type inequalities in q-calculus (q > 1) and in h-discrete calculus (h > 0) as well [28].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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