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Research Article

Phase Portraits and Bounded and Singular Traveling Wave Solution of Stochastic Nonlinear Biswas–Arshed Equation

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The main purpose of the current paper is to study the phase portraits and bounded and singular traveling wave solution of the stochastic nonlinear Biswas–Arshed equation by using the "three-step method" of Professor Li's method together with the phase orbit of planar dynamical system. Firstly, by employing the traveling wave transformation, the stochastic nonlinear Biswas–Arshed equation is simplified into deterministic nonlinear ordinary differential equation. Secondly, phase portraits of the stochastic nonlinear Biswas–Arshed equation are plotted by analyzing the planar dynamic system of the nonlinear ordinary differential equation. Finally, the bounded and singular traveling wave solutions of the stochastic nonlinear Biswas–Arshed equation are constructed.

1. Introduction

The stochastic differential equation (SDE) with multiplicative noise driven by Brownian motion was first proposed by Japanese mathematician Itô in 1951 [1]. Due to the wide application of stochastic differential equation in the fields of nonlinear optics, finance, communication, and control, the development of SDE has been very fast in recent years, and many experts and scholars have determined their research direction in this field [2–4]. The research directions of SDE mainly include the existence of solution, stability of solution, numerical solution, analytical solutions, martingale representation theory, variational inequality, stochastic control, and so on [5]. Compared with deterministic differential equations, in these research directions, physicists, financiers, and engineers are also trying to analyze the dynamic behavior of SDE and solve the analytical solutions of these equations.

In the past two years, many experts and scholars have adjusted their research in the analysis of traveling wave solutions of SDE, especially for the well-known stochastic partial differential equation (SPDE). Due to the complexity of randomness, the construction of analytical solutions of deterministic partial differential equations cannot be directly applied to the construction of analytical solutions of stochastic partial differential equations. By using some special transformations and traveling wave transformations, SPDE can be simplified into a deterministic ordinary differential equation. Therefore, the construction of analytical solution of SPDE has become a reality. Recently, many experts and scholars have applied some well-known methods to construct analytical solutions of SPDE. These methods mainly include the sine-cosine method [6], the extended (G'/G)-expansion method [7, 8], the generalized (G'/G)-expansion method [9, 10], the Jacobi-elliptic equation scheme [11], and so on [12].

The stochastic nonlinear Biswas–Arshed equation in the Itô calculus sense is a very important class of SPDE. The main purpose of this paper is to analyze the dynamic behavior and exact traveling wave solutions of the stochastic nonlinear Biswas–Arshed equation in the Itô calculus sense as follows (see [11]):

$$\begin{aligned} &iq_t + a_1 q_{xx} + a_2 q_{xt} + i\left(b_1 q_{xxx} + b_2 q_{xxt}\right) \\ &\quad + \sigma\left(q - ia_2 q_x + b_2 q_{xx}\right)\frac{\mathrm{d}W(t)}{\mathrm{d}t} \\ &\quad = i\left[\lambda\left(|q|^2 q\right)_x + \mu\left(|q|^2\right)_x q + \theta|q|^2 q_x\right], \end{aligned} \tag{1}$$

where $q = q(t,x)$ is the wave profile. $W(t)$ denotes the standard Wiener process. $\mathrm{d}W(t)/\mathrm{d}t$ stands for the white noise. The constant coefficients $a_1, a_2, \sigma, b_1, b_2, \lambda, \mu$, and θ represent different meanings in the nonlinear optics. (1) was first proposed by Elsayed et al. In [11], the optical soliton solution of (1) is discussed by using the unified Riccati equation method, the extended auxiliary equation method, and the Jacobi-elliptic equation method, respectively. Especially, when $\sigma = 0$, (1) becomes the well-known Biswas–Arshed equation.

The article is organized as follows. In Section 2, the phase portraits of the stochastic nonlinear Biswas–Arshed equation are plotted. In Section 3, the bounded wave solutions and single wave solutions of the stochastic nonlinear Biswas–Arshed equation are obtained, respectively. In Section 4, a conclusion is presented.

2. Phase Portraits of (1)

In order to simplify (1) into nonlinear ordinary differential equation, the following traveling wave transformation is considered:

$$\begin{aligned} q(t,x) &= Q(\xi)e^{i\left[\psi(t,x)+\sigma W(t)-\sigma^2 t\right]}, \psi(t,x) \\ &= -kx + wt, \xi = x - vt, \end{aligned} \tag{2}$$

where k, w, ξ, and v represent the frequency, the wave number, the wave variable, and the velocity, respectively.

Plugging (2) into (1), the real part and the imaginary part of (1) are given, respectively, as

$$\text{Real part: } \left[a_1 + 3b_1 k - b_2\left(2kv + w - \sigma^2\right) - a_2 v\right]Q'' - k(\lambda+\theta)Q^3 - \left[a_1 k^2 + b_1 k^3 + \left(1 - a_2 k - b_2 k^2\right)\left(w - \sigma^2\right)\right]Q = 0, \tag{3}$$

and

$$\text{Imaginary part: } (b_1 - b_2 v)Q''' - (3\lambda + 2\mu + \theta)Q^2 Q, -\left[2a_1 k + 3b_1 k^2 - (a_2 - 2kb_2)\left(w - \sigma^2\right) + \left(1 - a_2 k - b_2 k^2\right)v\right]Q\prime = 0. \tag{4}$$

By integrating both sides of (4) at the same time, the following can be obtained:

$$(b_1 - b_2 v)Q'' - \frac{1}{3}(3\lambda + 2\mu + \theta)Q^3 - \left[2a_1 k + 3b_1 k^2 - (a_2 - 2kb_2)\left(w - \sigma^2\right) + \left(1 - a_2 k - b_2 k^2\right)v\right]Q = 0. \tag{5}$$

Since (3) and (5) are satisfied at the same time, we can get

$$\frac{a_1 + 3kb_1 - b_2\left(2kv + w - \sigma^2\right) - a_2 v}{b_1 - b_2 v} = \frac{3k(\lambda+\theta)}{3\lambda + 2\mu + \theta} = \frac{a_1 k^2 + b_1 k^3 + \left(1 - a_2 k - b_2 k^2\right)\left(w - \sigma^2\right)}{2a_1 k + 3b_1 k^2 - (a_2 - 2kb_2)\left(w - \sigma^2\right) + \left(1 - a_2 k - b_2 k^2\right)v}. \tag{6}$$

Next, combined with equation (6), equation (3) can be rewritten as

$$A\frac{\mathrm{d}^2 Q(\xi)}{\mathrm{d}\xi^2} - BQ^3(\xi) + CQ(\xi) = 0, \tag{7}$$

where $A = a_1 + 3b_1 k - b_2(2kv + w - \sigma^2) - a_2 v$, $B = a_1 k^2 + b_1 k^3 + (1 - a_2 k - b_2 k^2)(w - \sigma^2)$, $C = -k(\lambda + \theta)$, $w = (a_2 - 2b_2^2 k^3 - 2a_2 b_2 k^2)\sigma^2 + 2(a_1 b_2 + 2a_2 b_1)k^3 + 4b_1 b_2 k^4 + 2(a_1 a_2 - b_1)k^2 - a_1 k/a_2 - 2b_2^2 k^3 - 2a_2 b_2 k^2$, $v = (a_1 + 2b_1 k)(1 - 4b_2 k^2)/a_2 - 2b_2^2 k^3 - 2a_2 b_2 k^2$, and $\theta = \mu$.

Supposing $A \neq 0$ (otherwise, (7) becomes a linear system), system (7) is converted to two-dimensional plane system:

$$\begin{cases} \dfrac{\mathrm{d}Q(\xi)}{\mathrm{d}\xi} = y, \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{B}{A}Q^3 - \dfrac{C}{A}Q, \end{cases} \tag{8}$$

and its first integral is as follows:

$$H(Q,y) = \frac{1}{2}y^2 - \frac{B}{4A}Q^4 + \frac{C}{2A}Q^2 = h, \quad h \in \mathbb{R}. \tag{9}$$

Suppose that $F(Q) = B/AQ^3 - C/AQ$. When $BC > 0$, system (9) has three equilibrium points $P_1(0,0)$, $P_2(\sqrt{C/B}, 0)$, and $P_3(-\sqrt{C/B}, 0)$. Similarly, when $BC < 0$, system (9) has one equilibrium point $P_4(0,0)$. Assume that $\lambda_{1,2} = \pm\sqrt{F\prime(Q)}$, $h_0 = H(0,0) = 0$, and $h_1 = H(\pm\sqrt{C/B}, 0) = C^2/4AB$. When $F\prime(Q_i) > 0$, $F\prime(Q_i) = 0$, and $F\prime(Q_i) < 0$, we conclude that the equilibrium point $P_i(Q_i, 0)$ is saddle point, degraded saddle point, and center point, respectively. Thus, we can draw the phase diagram of the system (8) as shown in Figure 1.

3. Traveling Wave Solutions of (1)

By using the "three-step method" of Professor Li's method together with the phase orbit of system (8) [13–17], the traveling wave solution of (1) can be constructed.

3.1. The Bounded Traveling Wave Solutions of (1)

(1) $AC > 0$, $AB > 0$

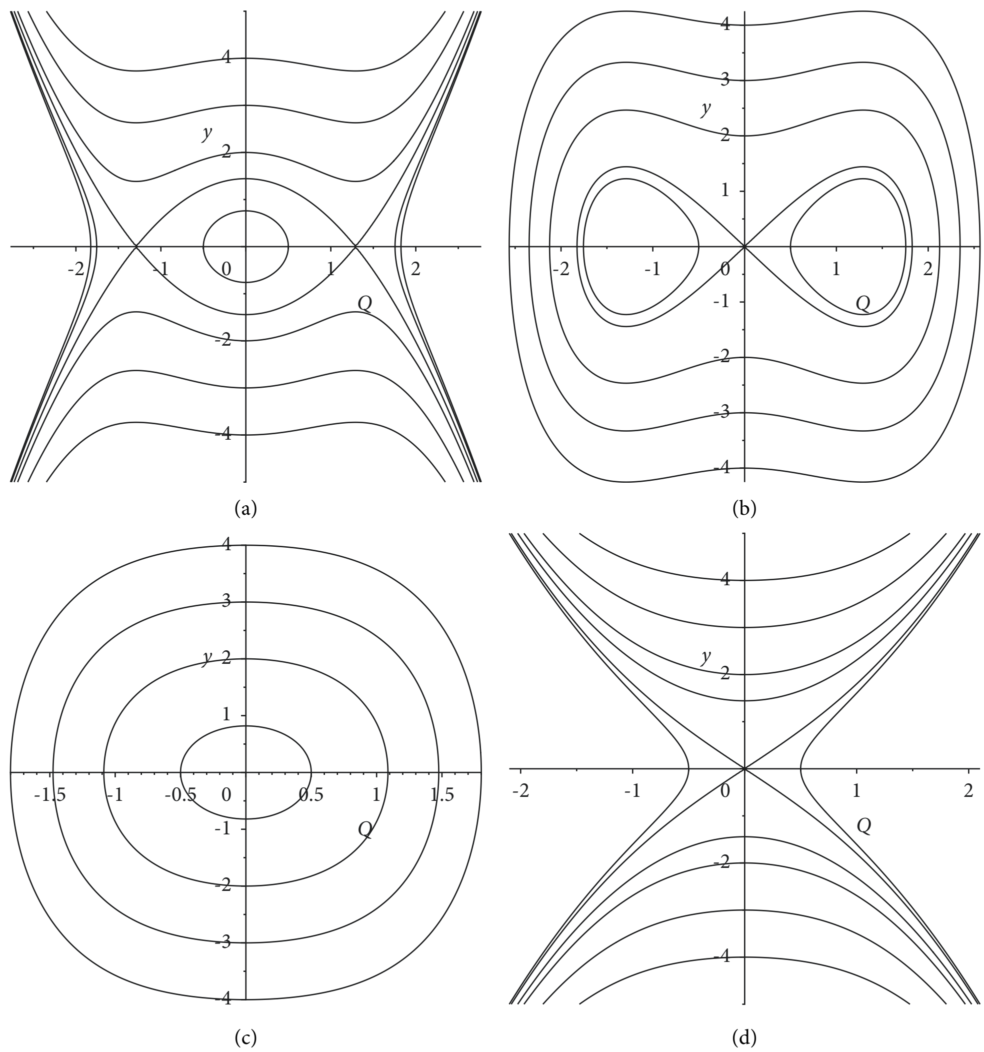

Figure 1: The phase portraits of system (8). (a) $A=2$, $B=1$, $C=2$. (b) $A=-2$, $B=1$, $C=2$. (c) $A=2$, $B=-1$, $C=2$. (d) $A=2$, $B=1$, $C=-2$.

(i) When $0<h<h_1$, equation (9) can be rewritten as

$$y^2=\frac{B}{2A}Q^4-\frac{C}{A}Q^2+2h=\frac{B}{2A}\left(Q^2-\phi_1^2\right)\left(Q^2-\phi_2^2\right), \tag{10}$$

where $\phi_1=\sqrt{C+\sqrt{C^2-4Ah}/B}$ and $\phi_2=\sqrt{C-\sqrt{C^2-4Ah}/B}$.

Substituting (10) into $\mathrm{d}Q(\xi)/d\xi=y$ and integrating it, we can get its integral expression as

$$\int_Q^{\phi_2}\frac{\mathrm{d}s}{\sqrt{\left(s^2-\phi_1^2\right)\left(s^2-\phi_2^2\right)}}=\sqrt{\frac{B}{2A}}|\xi|. \tag{11}$$

From (2) and (11), we obtain

$$\phi_1(t,x)=\pm\frac{\sqrt{\phi_2^2-\phi_1^2\left(\mathbf{sn}\left(\sqrt{2}/2\sqrt{B/A}\phi_2(x-vt),\phi_1/\phi_2\right)\right)^2}\,\mathbf{cn}\left(\sqrt{2}/2\sqrt{B/A}\,\phi_2(x-vt),\phi_1/\phi_2\right)\phi_1\phi_2}{\phi_1^2\left(\mathbf{sn}\left(\sqrt{2}/2\sqrt{B/A}\phi_2(x-vt),\phi_1/\phi_2\right)\right)^2-\phi_2^2}\mathbf{e}^{i\left[-kt+wt+\sigma W(t)-\sigma^2t\right]}. \tag{12}$$

(ii) When $h = h_1$, equation (9) can be rewritten as

$$y^2 = \frac{B}{2A}\left(Q^4 - \frac{2C}{B}Q^2 + \frac{4Ah_1}{B}\right) = \frac{B}{2A}\left(Q - \sqrt{\frac{C}{B}}\right)^2\left(Q + \sqrt{\frac{C}{B}}\right)^2. \quad (13)$$

Substituting (13) into $dQ(\xi)/d\xi = y$ and integrating it, we get

$$\int_Q^{\sqrt{C/B}} \frac{d\phi}{(\sqrt{C/B} - \phi)(\phi + \sqrt{C/B})} = \sqrt{\frac{B}{2A}}|\xi|. \quad (14)$$

From (2) and (14), we obtain

$$q_4(t,x) = \pm \frac{\sqrt{C/B}\left(-1 + e^{\sqrt{2C/A}\,(x-vt)}\right)}{1 + e^{\sqrt{2C/A}\,(x-vt)}} e^{i[-kt+wt+\sigma W(t)-\sigma^2 t]}. \quad (15)$$

(2) $AC \geq 0$, $AB < 0$

When $0 < h < +\infty$, (9) can be rewritten as

$$y^2 = -\frac{B}{2A}\left(-Q^4 + \frac{2C}{B}Q^2 - \frac{4Ah_1}{B}\right) = -\frac{B}{2A}\left(Q^2 + \varphi_2^2\right)\left(\varphi_1^2 - Q^2\right), \quad (16)$$

where $\varphi_1^2 = |C - \sqrt{C^2 - 4ABh}/B|$ and $\varphi_2^2 = |-C + \sqrt{C^2 - 4ABh}/B|$.

Substituting (16) into $dQ(\xi)/d\xi = y$ and integrating it, we have

$$\int_Q^{\varphi_1} \frac{d\phi}{\sqrt{\left(\phi^2 + \varphi_2^2\right)\left(\varphi_1^2 - \phi^2\right)}} = \sqrt{\frac{-B}{2A}}|\xi|. \quad (17)$$

From (2) and (17), we have

$$q_3(t,x) = \pm \pm \varphi_1 \mathbf{cn}\left(\sqrt{\frac{-B\left(\varphi_1^2 + \varphi_2^2\right)}{2A}}\,(x - vt), \frac{\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}}\right)\mathbf{e}^{i[-kt+wt+\sigma W(t)-\sigma^2 t]}. \quad (18)$$

(3) $AC < 0$, $AB < 0$

(i) When $h = 0$, we can obtain

$$Q_4(\xi) = \pm \frac{8Ce^{-\sqrt{-C/A}\xi}}{8C + Be^{-2\sqrt{-C/A}\xi}}. \quad (19)$$

That is,

$$q_4(t,x) = \pm \frac{8Ce^{-\sqrt{-C/A}\,(x-vt)}}{8C + Be^{-2\sqrt{-C/A}\,(x-vt)}} e^{i[-kt+wt+\sigma W(t)-\sigma^2 t]}. \quad (20)$$

(ii) When $h_1 < h < 0$, a family of periodic solutions of (1) can be obtained:

$$q_5(\xi) = \pm \frac{\varphi_1\varphi_2}{\sqrt{\varphi_1^2\left(\mathbf{cn}\left(\sqrt{-B/2A}\,\varphi_1(x - vt), \sqrt{\varphi_1^2 - \varphi_2^2}/\varphi_1\right)\right)^2 + \sqrt{\varphi_1^2\left(\mathbf{sn}\left(\sqrt{-B/2A}\,\varphi_1(x - vt), \sqrt{\varphi_1^2 - \varphi_2^2}/\varphi_1\right)\right)^2}}} e^{i[-kt+wt+\sigma W(t)-\sigma^2 t]}, \quad (21)$$

where $\varphi_1 = \sqrt{C + \sqrt{C^2 - 4ABh}/B}$ and $\varphi_2 = \sqrt{C - \sqrt{C^2 - 4ABh}/B}$.

(iii) When $0 < h < +\infty$, we obtain

$$Q_6(\xi) = \pm \varphi_1 \mathbf{cn}\left(\frac{1}{2}\sqrt{\frac{-2B}{A}}\sqrt{\varphi_1^2 + \varphi_2^2}\,\xi, \frac{\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}}\right), \quad (22)$$

where $\varphi_1^2 = (C + \sqrt{C^2 - 4ABh}/|B|)$ and $\varphi_2^2 = -(C - \sqrt{C^2 - 4ABh}/|B|)$. That is,

$$q_6(t,x) = \pm \varphi_1 \mathbf{cn}\left(\sqrt{\frac{-B\left(\varphi_1^2 + \varphi_2^2\right)}{2A}}\,(x - vt), \frac{\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}}\right)\mathbf{e}^{i[-kt+wt+\sigma W(t)-\sigma^2 t]}. \quad (23)$$

3.2. The Singular Traveling Wave Solution of (1) When $AC > 0$, $AB > 0$

(i) When $h = 0$, (9) can be recollected as

$$y^2 = \frac{B}{2A}\left(Q^4 - \frac{2C}{B}Q^2\right) = \frac{B}{2A}Q^2\left(Q + \sqrt{\frac{2C}{B}}\right)\left(Q - \sqrt{\frac{2C}{B}}\right). \tag{24}$$

Substituting (24) into $(dQ(\xi)/d\xi) = y$ and integrating it, we obtain

$$\int_Q^{+\infty} \frac{d\phi}{\phi\sqrt{(\phi + \sqrt{2C/B})(\phi - \sqrt{2C/B})}} = \sqrt{\frac{B}{2A}}|\xi|. \tag{25}$$

From (2) and (25), we can obtain

$$q_7(t,x) = \sqrt{\frac{2C}{B}}\, \frac{1}{\mathbf{sin}(\sqrt{C/A}\,(x - vt))}\,\mathbf{e}^{i[-kt + wt + \sigma W(t) - \sigma^2 t]}. \tag{26}$$

(ii) If $h = h_1$, we have

$$\int_Q^{+\infty} \frac{d\phi}{(\phi + \sqrt{C/B})(\phi - \sqrt{C/B})} = \sqrt{\frac{B}{2A}}|\xi|. \tag{27}$$

From (27), the exponential function solutions of (8) can be obtained:

$$Q_8(\xi) = \pm\sqrt{C/B}\,\frac{\mathbf{e}^{\sqrt{2C/A}\xi} + 1}{\mathbf{e}^{\sqrt{2C/A}\xi} - 1}. \tag{28}$$

That is,

$$q_8(t,x) = \pm\sqrt{\frac{C}{B}}\,\frac{\mathbf{e}^{\sqrt{2C/A}(x-vt)} + 1}{\mathbf{e}^{\sqrt{2C/A}(x-vt)} - 1}\,\mathbf{e}^{i[-kt + wt + \sigma W(t) - \sigma^2 t]}. \tag{29}$$

4. Conclusion

In the paper, we have obtained the bounded and singular traveling wave solution of the stochastic nonlinear Biswas–Arshed equation by using theory of planar dynamical system. Using Maple software, we draw the phase portraits of equation (8). Through the orbital analysis of the phase portrait, we obtain a series of traveling wave solutions, which include not only hyperbolic function solutions, exponential function solutions, rational function solutions, and trigonometric function solutions but also more general Jacobian elliptic function solutions. As far as we know, the solutions we obtained are new and intentional. In future research, our research work will focus on the bounded and singular traveling wave solution of higher-order partial differential equation, and the order of the planar system of system (8) considered will be higher.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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