Research Article

On New Polynomial Sequences Constructed to Each Vertex in an n-Gon

Abdul Hamid Ganie, Engin Özkkan, Mine Uysal, and Afroza Akhter

1Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Abha Male 61421, Saudi Arabia
2Department of Mathematics, Faculty of Sciences and Arts, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, Turkey
3Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, Turkey
4Department of Mathematics, VIT Bhopal University, Sehore 466116, India

Correspondence should be addressed to Engin Özkkan; eozkan@erzincan.edu.tr and Mine Uysal; cmine.uysal@erzincan.edu.tr

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In this work, we bring to light the properties of newly formed polynomial sequences at each vertex of Pell polynomial sequences placed clockwise at each vertex in the n-gon. We compute the relation among the polynomials with such vertices. Moreover, in an n-gon, we generate a recurrence relation for a sequence giving the $m^{th}$ term formed at the $k^{th}$. Similar to the situations we are talking about, we applied to Pell-Lucas polynomials, Jacobsthal polynomials, and Jacobsthal-Lucas polynomials and have obtained new relations recurrence.

1. Introduction

It is an interesting tool for many researchers to know about number sequences, especially thrust is given to Fibonacci numbers. In this direction, many researchers work on it and study its wings, and the Lucas numbers became a vital role for them to study as can be seen in [1–8]. While framing these sequences, other sequences came into existence such as Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas by replacing the initial conditions [9–15].

As in [16–23], in engineering, science, and technology, the structure of such sequences with generalizations plays a vital significance art and architecture. In studying such discipline, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials have shown their significance as in [24–27].

Recently, the study of triangles with coordinates of vertices have been studied in [14, 19].

We have made a new study for polynomials of Pell, Pell-Lucas, Jacobsthal, and Jacobsthal by reference to [19] articles in this study.

The motivation of this work is to study the clockwise nature of the Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials to be located on the vertices of the polygons.

We have computed some interesting relations in n-gon concerning the vertex $B_k$ corresponding to $m^{th}$ term for $0 \leq k < n$. Also, a relation corresponding to a vertex among the numbers will be determined.

2. Materials and Methods

Definition 1. The polynomials of the given by linear recurrence relation

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x), \quad (1)$$
with \( P_0(x) = 0 \) and \( P_1(x) = 1 \) called as Pell polynomials. The negative Pell polynomials are defined by the recurrence relation,

\[
P_n(x) = (-1)^{n+1} P_n(x). \tag{2}
\]

The first few terms of this sequence are \( \{1, 2x, 4x^2 + 1, 8x^3 + 4x, 16x^4 + 12x^2 + 1, 32x^5 + 32x^3 + 6x, \ldots \} \). For Fibonacci sequence, the characteristic equation is \( r^2 = 2xr - 1 = 0 \). \( \tag{3} \)

The roots of this equation are \( \alpha(x) \) and \( \beta(x) \).

\[
\alpha(x) = x + \sqrt{x^2 + 1} \quad \text{and} \quad \beta(x) = x - \sqrt{x^2 + 1}. \tag{4}
\]

The Binet formula of this sequence is

\[
P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{5}\]

**Definition 2.** The linear recurrence relation for Pell-Lucas polynomials is

\[
Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x), \tag{6}
\]

with \( Q_0(x) = 2 \) and \( Q_1(x) = 2x \).

The first few terms of this sequence are \( \{2, 2x, 4x^2 + 2, 8x^3 + 6x, 16x^4 + 16x^2 + 2, 32x^5 + 40x^3 + 10x, \ldots \} \).

The negative Pell-Lucas polynomials are defined by the recurrence relation,

\[
Q_{n} = (-1)^{n+1} Q_{n}(x). \tag{7}
\]

The Binet formula of this sequence is

\[
Q_n(x) = \alpha^n(x) + \beta^n(x). \tag{8}
\]

**Definition 3.** The recurrence relation for Jacobsthal polynomials is

\[
J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x), \tag{9}
\]

with \( J_0(x) = 0 \) and \( J_1(x) = 1 \).

Note that

\[
J_{n}(x) = -(-1)^{n}(2x)^{-n} J_{n}(x). \tag{10}
\]

The first few terms of this sequence are \( \{0, 1, 1 + 2x, 1 + 4x, 1 + 6x + 4x^2, 1 + 8x + 12x^2, \ldots \} \).

The characteristic equation for the Jacobsthal polynomials sequence is

\[
k^2 - k - 2x = 0. \tag{11}
\]

The roots of this equation are \( \gamma(x) \) and \( \delta(x) \),

\[
\gamma(x) = \frac{1 + \sqrt{1 + 8x}}{2} \quad \text{and} \quad \delta(x) = \frac{1 - \sqrt{1 + 8x}}{2}. \tag{12}
\]

The Binet formula of this sequence is

\[
J_{n}(x) = \gamma^n(x) - \delta^n(x)\gamma(x) - \delta(x). \tag{13}
\]

**Definition 4.** The recurrence relation for Jacobsthal-Lucas polynomials is

\[
j_n(x) = j_{n-1}(x) + 2xj_{n-2}(x), \tag{14}\]

with \( j_0(x) = 2 \) and \( j_1(x) = 1 \).

Note that

\[
j_{n}(x) = (-1)^{n}(2x)^{-n} j_{n}(x). \tag{15}\]

The first few terms of this sequence are \( \{2, 1 + 4x, 1 + 8x + 8x^2, 1 + 10x + 20x^2, \ldots \} \).

The Binet formula of this sequence is

\[
j_{n}(x) = \gamma^n(x) + \delta^n(x). \tag{16}\]

**3. Results and Discussion**

**3.1. New Recurrence Relations for the Pell and Pell-Lucas Numbers.** From here onwards, we shall consider a continuously clockwise sense approach to the polygon of the Pell polynomials and Pell-Lucas polynomials. We will continue to n-gon starting this placement from a point. For such a polygon, let us call each vertex point as \( B_0, B_1, B_2, \ldots \) for \( k = 0, 1, 2, \ldots \). Here, we aim at finding a relationship that locates any term of new \( B_k \) sequences to every vertex of the polygon. We establish such a relationship in Theorem 3.1.1 to Pell polynomials and in Theorem 3.1.3 to Pell-Lucas polynomials.

Let the vertex number be represented by \( k \) and \( m \) represents the order of any term of the new sequence occurring at any vertex in any n-gon, with \( 0 \leq k < n \) and \( m \geq 1 \).

Taking \( n = 1 \), the Pell polynomials in succession over a dot are given, as depicted in Figure 1.

For \( n = 1 \) yields \( k = 0 \) and giving a recurrence relation,

\[
P_{(m-1)n+k} = 2xP_{m-2} + P_{m-3}(x), \tag{17}\]

with \( P_0(x) = 0 \) and \( P_1(x) = 1 \) of the form of Pell polynomials.

For \( n = 2 \), choose the Pell polynomials in succession at the endpoints of a line, as depicted in Figure 2. So, it determines a recurrence relation of the form

\[
P_{(m-1)n+k} = \left(4x^2 + 2\right)P_{m-22} + P_{m-32}(x) \tag{18}\]

For \( n = 3 \), the vertices of the triangle will be labeled by Pell polynomials clockwise, as given in Figure 3. So, it determines a recurrence relation of the form

\[
P_{(m-1)n+k} = \left(8x^3 + 6x\right)P_{m-32} + P_{m-33}(x) \tag{19}\]

Now, for \( n = 4 \), the Pell numbers are taken clockwise on the vertices of a tetragon, as given in Figure 4.
So, it determines a recurrence relation of the form

\[ P_{(m-1)n+k}(x) = P_{(m-1)3+k}(x) = (16x^4 + 16x^2 + 2)P_{(m-2)3+k}(x) - P_{(m-3)3+k}(x). \]  

Similarly, for an n-gon, we take clockwise the Pell numbers on its vertices, as given in Figure 5.

For example, in a new sequence to determine the second term of it occurring at the vertex \( B_1 \) in a 3-gon.

\[ P_{(m-1)n+k}(x) = P_{(m-1)3+k}(x) = (8x^3 + 6x)P_{(m-2)3+k}(x) + P_{(m-3)3+k}(x), \]

so that

\[ P_{(3-1)3+1} = (8x^3 + 6x)P_{(3-2)3+1}(x) + P_{(3-3)3+1}(x), \]
\[ P_1(x) = (8x^3 + 6x)P_0(x) + P_1(x), \]
\[ 64x^6 + 80x^4 + 24x^2 + 1 = (8x^3 + 6x)(8x^3 + 4x) + 1 \]
\[ = 64x^6 + 32x^4 + 48x^4 + 24x^2 + 1. \]  

(22)

In Figure 5, we define a new sequence created at the vertex \( B_k \):
\[ B_k = \{x_1, x_2, \ldots \} = \{P_k, P_{n+k}, P_{2n+k}, P_{3n+k}, \ldots \}. \]

As \( P_k = P_{0+n+k} \) is the first term to this sequence, the second term is \( P_{n+k}(x) = P_{1n+k}(x) \) and the third term is \( P_{2n+k}(x) \), and if it continues like this, it is observed that the \( m \)th term will be \( P_{(m-1)n+k}(x) \) of the new sequence formed in an n-gon.

In addition, the coefficient of \( P_{(m-2)n+k}(x) \) is Pell-Lucas polynomials \( Q_m(x) \) and \((-1)^n\) is the coefficient of \( P_{(m-3)n+k}(x) \).

So, we have the following result.

**Theorem 1.** For \( 0 \leq k < n \), the relation
\[ P_{(m-1)n+k}(x) = Q_m(x)P_{(m-2)n+k}(x) - (-1)^nP_{(m-3)n+k}(x) \]

is satisfied, giving \( m \)th term of the sequence with respect to vertex \( B_k \).

**Proof.** Using Binet formulas of Pell and Pell-Lucas polynomials, we see
Because of $\alpha(x)\beta(x) = -1$, we have

$$Q_n(x)P_{(m-2)n+k}(x) - (-1)^nP_{(m-3)n+k}(x)$$

$$= (\alpha^n(x) + \beta^n(x))\left(\frac{\alpha^{(m-2)n+k}(x) - \beta^{(m-2)n+k}(x)}{\alpha(x) - \beta(x)}\right) - (-1)^n\left(\frac{\alpha^{(m-3)n+k}(x) - \beta^{(m-3)n+k}(x)}{\alpha(x) - \beta(x)}\right)$$

$$= \frac{\alpha^{(m-1)n+k}(x) - \beta^{(m-1)n+k}(x)}{\alpha(x) - \beta(x)} + \frac{\alpha^n(x)\beta^{(m-2)n+k}(x)}{\alpha(x) - \beta(x)} + \frac{\beta^n(x)\alpha^{(m-2)n+k}(x)}{\alpha(x) - \beta(x)}$$

$$- \frac{(-1)^n, \alpha^{(m-3)n+k}(x)}{\alpha(x) - \beta(x)} + \frac{(-1)^n, \beta^{(m-3)n+k}(x)}{\alpha(x) - \beta(x)}$$

$$= \frac{\alpha^{(m-1)n+k}(x) - \beta^{(m-1)n+k}(x)}{\alpha(x) - \beta(x)} + \frac{\alpha^{(m-3)n+k}(x)\beta^n(x)\alpha^n(x) - (-1)^n}{\alpha(x) - \beta(x)}$$

$$- \beta^{(m-3)n+k}(x)\left(\frac{\alpha^n(x)\beta^n(x) - (-1)^n}{\alpha(x) - \beta(x)}\right).$$

This is equation (24).

Because of $\alpha(x)\beta(x) = -1$, we have

$$= \frac{\alpha^{(m-3)n+k}(x)\beta^n(x) - (-1)^n}{\alpha(x) - \beta(x)}$$

$$= \frac{\alpha^{(m-1)n+k} - \beta^{(m-1)n+k}}{\alpha - \beta}$$

$$= P_{(m-1)n+k}(x).$$

This is equation (25).
Let us indicate the coefficients corresponding to this sequence in Table 1.

Table 1 clearly indicates that the first components are Pell-Lucas polynomials, \([2x, 4x^2 + 1, 8x^3 + 6x, 16x^4 + 2, 32x^5 + 40x^6 + 10x, 64x^7 + 96x^8 + 36x^2 + 2, 128x^9 + 224x^{10} + 112x^3 + 14x, \ldots] \).
Table 1: The coefficients.

<table>
<thead>
<tr>
<th>Dot</th>
<th>Segment</th>
<th>Triangle</th>
<th>Tetragon</th>
<th>Pentagon</th>
<th>Hexagon</th>
<th>Heptagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2x,</td>
<td>(4x^2 + 1,</td>
<td>(8x^3 + 6x,</td>
<td>(16x^4 + 16x^3 + 2,</td>
<td>(32x^5 + 40x^4 + 10x,</td>
<td>(64x^6 + 96x^5 + 36x^4 + 2,</td>
<td>(128x^7 + 224x^6 + 112x^5 + 14x)</td>
</tr>
<tr>
<td>1)</td>
<td>1)</td>
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</tbody>
</table>

\[
P_{(s+1)n+k} = P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-3(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 1 \\ v \end{pmatrix} Q_{n}^{2r-2v} (x) \right) + P_{k}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-5(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 2 \\ v \end{pmatrix} Q_{n}^{2r-2v-2} (x) \right) - (-1)^n P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-7(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 3 \\ v \end{pmatrix} Q_{n}^{2r-3} (x) \right) + (-1)^{n+1} P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-9(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 4 \\ v \end{pmatrix} Q_{n}^{2r-4} (x) \right) - (-1)^{n+1} P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-11(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 5 \\ v \end{pmatrix} Q_{n}^{2r-5} (x) \right)
\]

(30)

When \( s = 2r \), we have

\[
P_{(s+1)n+k} = P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-3(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 1 \\ v \end{pmatrix} Q_{n}^{2r-2v} (x) \right) + P_{k}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-5(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 2 \\ v \end{pmatrix} Q_{n}^{2r-2v-2} (x) \right) - (-1)^n P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-7(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 3 \\ v \end{pmatrix} Q_{n}^{2r-3} (x) \right) + (-1)^{n+1} P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-9(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 4 \\ v \end{pmatrix} Q_{n}^{2r-4} (x) \right) - (-1)^{n+1} P_{nst}(x) \left( \sum_{r=0}^{v \left[ \frac{2m-11(-1)^{r}}{2} \right]} (-1)^{(n+1)r} \begin{pmatrix} 2r - v - 5 \\ v \end{pmatrix} Q_{n}^{2r-5} (x) \right)
\]
\[ + P_k \left( (-1)^{s+1} C_n^{2s-1} (x) + (-1)^{2(s+1)} \binom{2r - 2}{0} Q_n^{2r-3} (x) + \ldots + (-1)^{(s+1)r} \binom{r}{r-1} Q_n (x) \right) \]

\[ = P_{n+k} \left( \sum_{v=0}^{r-1} (-1)^{(s+1)v} \binom{2r - v}{v} Q_n^{2r-2v} (x) \right) \]

\[ + P_k \left( \sum_{v=0}^{r-1} (-1)^{(s+1)(v+1)} \binom{2r - v - 1}{v} Q_n^{2r-2v-1} (x) \right) \]

\[ = P_{n+k} \left( \sum_{v=0}^{2s-1 - (-1)^{s+1}} (-1)^{(s+1)v} \binom{s - v}{v} Q_n^{2r-2v} (x) \right) \]

\[ + P_k \left( \sum_{v=0}^{2s - 3 - (-1)^{s+1}} (-1)^{(s+1)(v+1)} \binom{s - v - 1}{v} Q_n^{2r-2v-1} (x) \right) \]  

\[ = P_{n+k} \left( \sum_{v=0}^{2s-1 - (-1)^{s+1}} (-1)^{(s+1)v} \binom{s - v}{v} Q_n^{2r-2v} (x) \right) \]

\[ + P_k \left( \sum_{v=0}^{2s - 3 - (-1)^{s+1}} (-1)^{(s+1)(v+1)} \binom{s - v - 1}{v} Q_n^{2r-2v-1} (x) \right) \]  

The equality has been shown for \( s = 2r \). Now, when \( s = 2r - 1 \), we get

\[ = P_{n+k} \left( \sum_{v=0}^{2s-1 - (-1)^{s+1}} (-1)^{(s+1)v} \binom{s - v}{v} Q_n^{2r-2v} (x) \right) \]

\[ + P_k \left( \sum_{v=0}^{2s - 3 - (-1)^{s+1}} (-1)^{(s+1)(v+1)} \binom{s - v - 1}{v} Q_n^{2r-2v-1} (x) \right) \]  

\[ = P_{n+k} \left( \sum_{v=0}^{2s-1 - (-1)^{s+1}} (-1)^{(s+1)v} \binom{s - v}{v} Q_n^{2r-2v} (x) \right) \]

\[ + P_k \left( \sum_{v=0}^{2s - 3 - (-1)^{s+1}} (-1)^{(s+1)(v+1)} \binom{s - v - 1}{v} Q_n^{2r-2v-1} (x) \right) \]
\[
P_{n,k}(x) \left( \sum_{v=0}^{r-1} (-1)^{v+1} \left( \begin{array}{c} 2r - v - 1 \\ v \\ \end{array} \right) Q_{n}^{2r-2v-1}(x) \right) + \sum_{v=0}^{r-1} (-1)^{v+1} \left( \begin{array}{c} 2r - v - 2 \\ v \\ \end{array} \right) Q_{n}^{2r-2v-2}(x)
\]

Thus, the proof is obtained.

Let us give example as an application. For \( n = 3, k = 2, \) and \( m = 2, \) we have

\[
P_8(x) = P_3(x) + P_5(x),
\]

\[
P_8(x) = P_3(x)Q_3(x) + P_5(x) + P_7(x) + 1,
\]

\[
128x^7 + 192x^5 + 80x^3 + 8x = (16x^4 + 12x^2 + 1)(8x^3 + 6x) + 2x
\]

\[
= 128x^7 + 96x^5 + 96x^3 + 72x^3 + 8x^3 + 6x + 2x
\]

\[
= 128x^7 + 192x^5 + 80x^3 + 8x.
\]

Now, let us place the Pell-Lucas polynomials like the Pell polynomials continuously clockwise at the vertices of a regular polygon.

For this new sequence, the \( m \)-th term with respect to the vertex \( B_k \) of an \( n \)-gon is \( Q_{(m-1)n+k}(x) \).

So, we have the following theorem about this situation.

\[
Q_{(m-2)n+k}(x)Q_n(x) - (-1)^n Q_{(m-3)n+k}(x) \\
= (a^{(m-2)n+k}(x) + \beta^{(m-2)n+k}(x)) (a^n(x) + \beta^n(x)) - (-1)^n (a^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)) \\
= a^{(m-1)n+k}(x) + \beta^{(m-1)n+k}(x) (a^n(x) + \beta^n(x)) - (-1)^n (a^{(m-2)n+k}(x) + \beta^{(m-2)n+k}(x)) \\
- (-1)^n (a^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)) \\
= a^{(m-1)n+k} + \beta^{(m-1)n+k} (a^n(x) + \beta^n(x)) - (-1)^n (a^{(m-3)n+k}(x) + \beta^{(m-3)n+k}(x)) \\
= (a^n(x) + \beta^n(x)) (1 - a^{(m-2)n+k}(x) - \beta^{(m-2)n+k}(x)).
\]

Because of \( a\beta = -1, \) we have

\[
\text{Theorem 3. If on the vertices of an n-gon, the Pell-Lucas polynomials are placed clockwise, then for } 0 \leq k < n, \text{ the } m \text{th term of sequence with respect to the vertex } B_k \text{ is }

Q_{(m-1)n+k}(x) = Q_{(m-2)n+k}(x)Q_n(x) - (-1)^n Q_{(m-3)n+k}(x).
\]

Proof. To prove the result, by employing Binet formula for Pell-Lucas polynomials, we see
\[ B_k \text{ vertex lastly at the vertices of an } n\text{-gon.} \]

By using the Binet formulas of Jacobsthal and Jacobsthal-Lucas polynomials, we have the following:

\[ B_k \text{ vertex lastly at the vertices of the triangle, and the clockwise sense of the Jacobsthal polynomials on the Jacobsthal-Lucas polynomials.} \]

3.2. Recurrences Notion Corresponding to Jacobsthal and Jacobsthal-Lucas Polynomials. In this direction, we involve the clockwise sense of the Jacobsthal polynomials on the point, at extreme points, at vertices of the triangle, and lastly at the vertices of an \( n \)-gon.

Since for this new sequence, the \( m \)th term computed at vertex \( B_k \) is \( I_{(m-1)\alpha} (x) \) for an \( n \)-gon. Also, the coefficient of \( I_{(m-2)\alpha} (x) \) is Jacobsthal-Lucas polynomials \( j_n (x) \) and the coefficient of \( I_{(m-3)\alpha} (x) \) is \( (-2x)^n \).

In this case, we have the following result.

\[ Q_{mn+k} (x) = Q_{rn+k} (x) \left( \sum_{v=0}^{2m-3(-1)^{m+1}} (-1)^{v+1} \binom{m-v-1}{v} Q_n^{m-2v-1} (x) \right) + Q_r (x) \left( \sum_{v=0}^{2m-5(-1)^{m+1}} (-1)^{v+1} \binom{m-v-2}{v} Q_n^{m-2v-2} (x) \right), \]

between the numbers with respect to the vertex \( B_k \).

\[ Q_n (x) = Q_5 (x) \left( \sum_{v=0}^{0} (-1)^{(5)v} \binom{1-v}{v} Q_4^{1-2v} (x) \right) + Q_4 (x) \left( \sum_{v=0}^{0} (-1)^{(5)v+1} \binom{-v}{v} Q_4^{-2v} (x) \right) \]

\[ Q_n (x) = Q_5 (x) Q_4 (x) - Q_4 (x) \\
= 512x^9 + 1152x^7 + 864x^5 + 240x^3 + 18x \\
= (32x^5 + 40x^3 + 10x)(16x^4 + 16x^2 + 2) - 2x \\
= 512x^9 + 512x^7 + 64x^5 + 640x^3 + 640x^5 + 80x^3 + 160x^5 + 160x^3 + 20x - 2x \\
= 512x^9 + 1152x^7 + 864x^5 + 240x^3 + 18x. \]

\[ \square \]

**Theorem 4.** For an \( n \)-gon, if Pell-Lucas polynomials are kept on the vertices clockwise sense. Then, for \( 0 \leq k < n \), there exists a relation of the form

\[ \square \]

**Proof.** The result can be proved in a similar fashion as Theorem 3.1.2.

For example, when \( n = 4, k = 1 \) and \( m = 2 \), we have

\[ \square \]

**Theorem 5.** On an \( n \)-gon, suppose Jacobsthal numbers are kept on its vertices clockwise. Then, for \( 0 \leq k < n \), we have

\[ I_{(m-1)\alpha} (x) = j_n (x) I_{(m-2)\alpha} (x) - (-2x)^n I_{(m-3)\alpha} (x), \]

and it determines the \( m \)th term of the sequence with respect to vertex \( B_k \).

**Proof.** By using the Binet formulas of Jacobsthal and Jacobsthal-Lucas polynomials, we have the following:
\[ j_n(x)J_{(m-2)\nu+n+k}(x) - (-2x)^n J_{(m-3)\nu+n+k}(x) \]

\[ = (y^n(x) + \delta^n(x)) \left( y^{(m-2)\nu+n+k}(x) - \delta^{(m-2)\nu+n+k}(x) \right) \frac{y(x) - \delta(x)}{y(x) - \delta(x)} \]

\[ - (-2x)^n \left( y^{(m-3)\nu+n+k}(x) - \delta^{(m-3)\nu+n+k}(x) \right) \frac{y(x) - \delta(x)}{y(x) - \delta(x)} \]

\[ = \frac{y^{(m-1)\nu+n+k}(x) - \delta^{(m-1)\nu+n+k}(x)}{y(x) - \delta(x)} + \frac{\delta^n(x)y^{(m-2)\nu+n+k}(x) - y^n(x)\delta^{(m-2)\nu+n+k}(x)}{y(x) - \delta(x)} \]

\[ - (-2x)^n \left( y^{(m-3)\nu+n+k}(x) - \delta^{(m-3)\nu+n+k}(x) \right) \frac{y(x) - \delta(x)}{y(x) - \delta(x)} \]

\[ = \frac{y^{(m-1)\nu+n+k}(x) - \delta^{(m-1)\nu+n+k}(x)}{y(x) - \delta(x)} + \frac{\delta^n(x)y^{(m-3)\nu+n+k}(x) - (-2x)^n}{y(x) - \delta(x)} \]

Since \( y(x)\delta(x) = -2x, \ y^n(x)\delta^n(x) - (-2x)^n = 0 \), thus

\[ j_n(x)J_{(m-2)\nu+n+k}(x) - (-2x)^n J_{(m-3)\nu+n+k}(x) = J_{(m-4)\nu+n+k}(x). \]

(41)

In Table 2, we write the coefficients concerning the above sequences.

So, clearly in Table 2, the Jacobsthal-Lucas polynomials \( \{1, 1 + 4x, 1 + 6x, 1 + 8x + 8x^2, 1 + 10x + 20x^2, 1 + 12x + 36x^2 + 16x^3, 1 + 14x + 56x^3 + 56x^4\} \) are first components.

The second components in Table 2 are \( \{2x, -(2x)^2, (2x)^3, -(2x)^4, 2^3, -(2x)^6, (2x)^7, \ldots\} \). \( \{1, -1, 1, -1, 1, -1, \ldots\} \) is enumerate as sequences A084633 in the On-Line Encyclopedia of Integer Sequences [25].

\( \square \)

**Theorem 6.** On an \( n \)-gon, let the Pell-Lucas polynomials be kept on its vertices clockwise. Then, for \( 0 \leq k < n \), there exists a relation corresponding to numbers with respect to the vertex \( B_k \) as

\[ \sum_{t=0}^{4} (-1)^{m-1} (2x)^m \left( \begin{array}{c} m-t-1 \\ t \end{array} \right) j_{m-2t-1}(x) \]

\[ \sum_{t=0}^{4} (-1)^{m-1} (2x)^m(t+1) \left( \begin{array}{c} m-t-2 \\ t \end{array} \right) j_{m-2t-2}(x) \]

(42)

**Proof.** As in Theorem 2, the proof can be performed.

Now, for an \( n \)-gon, taking the Jacobsthal-Lucas polynomials on the vertices clockwise and in this \( n \)-gon, the \( m \)th term formed to this new sequence at the vertex \( B_k \) is \( j_{(m-2)\nu+n+k}(x) \).

We have the following result in this direction. \( \square \)

**Theorem 7.** On an \( n \)-gon, if the Jacobsthal-Lucas polynomials are kept on their vertices clockwise. Then, for \( 0 \leq k < n \), the \( m \)th term with respect to the vertex \( B_k \) is

\[ j_{(m-2)\nu+n+k}(x) = j_n(x)j_{(m-2)\nu+n+k}(x) - (-2x)^n j_{(m-3)\nu+n+k}(x). \]

(43)
Table 2: The coefficients.

<table>
<thead>
<tr>
<th>Dot</th>
<th>Segment</th>
<th>Triangle</th>
<th>Tetragon</th>
<th>Pentagon</th>
<th>Hexagon</th>
<th>Heptagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2x)</td>
<td>(1 + 4x, −4x²)</td>
<td>(1 + 6x, 8x³)</td>
<td>(1 + 8x + 8x⁵, −16x⁴)</td>
<td>(1 + 10x + 20x⁶, 32x⁸)</td>
<td>(1 + 12x + 36x⁹ + 16x¹¹, −64x¹⁴)</td>
<td>(1 + 14x + 56x¹² + 56x¹⁴, 128x¹⁷)</td>
</tr>
</tbody>
</table>
Proof. To establish this result, Binet formula for Jacobsthal-Lucas numbers is employed and gives

\[
j_n(x) j_{(m-2)n+k} (x) - (-2x)^n j_{(m-3)n+k} (x) \\
= (y^m (x) + \delta^n (x))(y^{(m-2)n+k} (x) + \delta^{(m-2)n+k} (x)) \\
- (-2x)^n (y^{(m-3)n+k} (x) + y^{(m-3)n+k} (x)) \\
= y^{(m-1)n+k} (x) + \delta^{(m-1)n+k} (x) + y^n (x) \delta^{(m-2)n+k} (x) + \delta^n (x) y^{(m-2)n+k} (x) \\
- (-2x)^n \delta^{(m-3)n+k} (x) \\
= y^{(m-1)n+k} (x) + \delta^{(m-1)n+k} (x) + y^{(m-3)n+k} (x) (y^n (x) \delta^n (x) - (-2x)^n) \\
+ \delta^{(m-3)n+k} (x) (y^n (x) \delta^n (x) - (-2x)^n).
\]

(44)

Since \( y(x) \delta (x) = -2x \), \( y^n (x) \delta^n (x) - (-2x)^n = 0 \), thus

\[
j_n(x) j_{(m-2)n+k} (x) - (-2x)^n j_{(m-3)n+k} (x) = j_{(m-1)n+k} (x).
\]

(45)

Theorem 8. On an \( n \)-gon, let the Jacobsthal-Lucas numbers be kept on its vertices clockwise. Then, for \( 0 \leq k < n \), the numbers with respect to the vertex \( A_k \) are

\[
j_{mnk} (x) = j_{nk} (x) \left( \frac{2m - 3 - (-1)^m}{4} \sum_{\nu=0}^{\nu=n} (-1)^{(n+1)(\nu)} (2x)^{\nu} \left( \begin{array}{c} m - \nu - 1 \\ \nu \end{array} \right) j_n^{m-2\nu-1} (x) \right) \\
+ j_k (x) \left( \frac{2m - 5 - (-1)^{m-1}}{4} \sum_{\nu=0}^{\nu=n} (-1)^{(n+1)(\nu+1)} (2x)^{\nu+1} \left( \begin{array}{c} m - \nu - 2 \\ \nu \end{array} \right) j_n^{m-2\nu-2} (x) \right).
\]

(46)

Proof. The result can be proved in a similar fashion as Theorem 2.

For example, when \( n = 2, k = 1 \) and \( m = 2 \), we get

\[
j_5(x) = j_3(x) \left( \sum_{\nu=0}^{\nu=2} (-1)^{3\nu} (2x)^{2\nu} \left( \begin{array}{c} 1 - \nu \\ \nu \end{array} \right) j_2^{1-2\nu} (x) \right) \\
+ j_1(x) \left( \sum_{\nu=0}^{\nu=2} (-1)^{(3\nu+1)} (2x)^{2\nu+1} \left( \begin{array}{c} -\nu \\ \nu \end{array} \right) j_2^{-2\nu} (x) \right)
\]

\[
j_5(x) = j_3(x) j_2(x) - j_1(x) (4x^2)
\]

\[
20x^2 + 10x + 1 = (1 + 6x)(1 + 4x) - 4x^2
\]

\[
= 1 + 4x + 6x + 24x^2 - 4x^2
\]

\[
= 20x^2 + 10x + 1.
\]
4. Conclusions

In this paper, we have described the clockwise sense of Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas polynomials on the vertices of polygons. We have computed the recurrence relations for $P_{n+k}(x)$ numbers by sorting the Pell polynomials around the polygons. Also, we have determined new correlations between the terms with respect to each vertex and generated new findings by employing these results. The work can be further extended to $k$-Jacobsthal and Jacobsthal-Lucas numbers and $k$–Fibonacci and Lucas numbers. [20–27]

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References