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Research Article

Research on Partial Ordered Sets That Can Be Constructed as Effect Algebras

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In this paper, we study effect algebra-induced partial ordered sets. All possible cases of effect algebras generated by bounded partial ordered set M_I of height 2 are given. In addition, the structure of chain effect algebra is studied carefully and the corresponding results are obtained.

1. Introduction

In the past few decades, algebraic structure *models* used to describe objective things have emerged in large numbers, providing effective tools for our scientific research. In the field of quantum mechanics and quantum logic, there are effects algebra [1], mv-algebra and bck-algebra [2]. In particular, effect algebra greatly promotes the rapid development of quantum theory and quantum logic.

Effect algebra is an important concept introduced by Foulis and Bennett through algebraic abstraction when they studied quantum logic. Since 1994, the study of effect algebra has been favored by scholars.First, Foulis and Bennett gave a series of basic properties of effect algebra [1]. In 1996, Gudder proposed the concept of accurately measurable elements and pivot elements [3], and proved that the effect algebra in which all elements are pivot elements is an orthogonal modular lattice. In 1999, Riecanova removed the condition of existence of identity elements in the effect algebra and obtained the generalized effect algebra [4]. In 2002, Gudder and Greechie extracted some properties of sequence product operation in Hilbert space and proposed sequence effect algebra [5]. In 2019, Wu et al. proposed L-algebras in [6], which is a generalization of homogeneous effect algebra. The relationship between effect algebras and other algebraic structures has also been studied extensively.

Generally speaking, effects algebra is an algebraic structure with a binary partial operation and a unary operation, and contains elements 0,1. The elements of effects algebra are events that are not sharp or clear, such as fuzzy events, quantum effects. Therefore, we can think of effect algebra as fuzzy and ambiguous quantum logic [7]. Effects algebra, of course, is also an algebraic abstraction of the various physical models of quantum mechanics. At the same time, effect algebra is in the category of partial order structure, it is MV-algebra.

As we all know, effect algebra has been applied to quantum theory and quantum logic with great success. However, because effect algebra is a defect of partial algebra, its algebraic structure is not perfect, which brings inconvenience to the application research. The partial binary operation + of effect algebra are fully embodied in the partial order relation introduced. Therefore, it is a good idea to study effect algebra from the perspective of partial ordered sets.

To avoid the difficulty of partial algebras, starting from the concept of effect algebras, we use effect algebras to induce the unique partial order and make it a partial order set. We study effect algebras from this perspective. The types, quantities and structures of partial ordered sets are discussed in effect algebras. The results show that this method can give some interesting theorems describing the structure of effect algebras perfectly.

Below, we first give the basic definition and some properties used in the paper.

2. Preliminaries

First we introduce the concept of effects algebra and some properties that will be used.

Definition 1. [1] *An **effect algebra** (**EA** for short) is a system* $\mathcal{X} = (X, +, 0, 1)$ *where* $0, 1 \in X$ *and* $+$ *is a partial binary operation on* X *satisfying the conditions*:

(E1) for all $u, v \in X$, $u + v$ is defined $\Rightarrow v + u$ is defined, and $u + v = v + u$;

(E2) for all $u, v, w \in X$, $(u + v) + w$ is defined $\Rightarrow u + (v + w)$ is defined, and $(u + v) + w = u + (v + w)$;

(E3) for all $u \in X$ there is a unique $u' \in E$ such that $u + u' = 1$;

(E4) for all $u \in X$, if $1 + u$ is defined then $u = 0$.

The mapping $x \mapsto x'$ is a total unary operation on X We can define the so-called **induced order** $\leq_{\mathcal{X}}$ on X by stipulation

$$\leq_{\mathcal{X}} = \{(a, b) \in X \times X \mid \exists r \in X \; s.t. \; a + r = b\}. \tag{1}$$

Since $0 \leq_{\mathcal{X}} u \leq_{\mathcal{X}} 1$ is true for any $u \in X$, $(X, \leq_{\mathcal{X}})$ is a bounded poset, denoted by $P(\mathcal{X})$. When $\leq_{\mathcal{X}}$ is a lattice-order relation, we call the effect algebra $\mathcal{X}$ a **lattice-order effect algebra** (**LEA** for short). When $\leq_{\mathcal{X}}$ is a total-order relation, we call the effect algebra $\mathcal{X}$ a **chain effect algebra** (**CEA** for short).

Definition 2. Two effect algebras $\mathcal{X} = (X, +, ', 0_X, 1_X)$ *and* $\mathcal{Y} = (Y, \oplus, ^{\dagger}, 0_Y, 1_Y)$ *are **isomorphic** if there is a bijection* φ *from* X *to* Y *such that for every* u, v *in* X *the following four equations hold*:

(I1) $u + v$ is defined $\Leftrightarrow \varphi(u) \oplus \varphi(v)$ is defined and then $\varphi(u + v) = \varphi(u) \oplus \varphi(v)$;

(I2) $\varphi(0_X) = 0_Y$;

(I3) $\varphi(1_X) = 1_Y$;

(I4) $\varphi(u') = (\varphi(u))^{\dagger}$.

Such a φ *is called an **isomorphism**, denoted by* $\mathcal{X} \overset{\varphi}{\simeq} \mathcal{Y}$ *(*$\mathcal{X} \simeq \mathcal{Y}$ *for short).*

Definition 3. (see[8]). *Let* $(F, \leq)$ *be a poset, and* $f, g \in P$.

(1) *g **covers** f in F, denoted by* $f \prec g$, *iff* $<$ *g and* $\forall t \in P, f \leq t \leq g \Rightarrow f = t$ or $g = t$.

In a poset $(F, \leq)$ with a smallest element 0, the element covering the 0 is called an **atom**. Let $A(F) = \{a \in F | a$ is an atom $\}$.

(2) $f, g \in F$ are called ***comparable*** *iff* $\leq g$ *or* $g \leq f$. *Otherwise* f *and* g *are **incomparable**, which denoted by* $f \| g$.

The following are the basic properties of effect algebra given by Foulis and Bennett in 1994, which we will use.

Lemma 1 (see[1]). *Let* $\mathcal{X} = (X, +, 0, 1)$ *be an EA and* $g_1, g_2, k, l \in X$. *Then*

(1) $g_1 + g_2$ *is defined* $\Leftrightarrow g_2 \leq_{\mathcal{X}} g_1' \Leftrightarrow g_1 \leq_{\mathcal{X}} g_2'$;

(2) if $g_1 + g_2$ *is defined then* $k + l$ *is defined for all* $k \leq_{\mathcal{X}} g_1$ *and* $l \leq_{\mathcal{X}} g_2$;

(3) $g_1 \leq_{\mathcal{X}} g_2 \Leftrightarrow g_2' \leq_{\mathcal{X}} g_1'$;

(4) $k'' = k$;

(5) if $g_1 \leq_{\mathcal{X}} g_2$ *and* $g_2 + k$ *is defined then* $g_1 + k$ *is defined and* $g_1 + k \leq_{\mathcal{X}} g_2 + k$;

(6) $g_1 + g_2 = k \Leftrightarrow g_1' = g_2 + k' \Leftrightarrow g_1 = (g_2 + k')'$.

Theorem 1. *The necessary and sufficient condition for the EA* $\mathcal{X} = (X, +, ', 0_E, 1_E)$ *and* $\mathcal{Y} = (Y, \oplus, ^{\dagger}, 0_F, 1_F)$ *to be isomorphic is that there exists a bijective* φ *from* X *to* Y *such that* $\forall g, h \in X$ *the following condition (I1) is true.*

(I1) $g + h$ *is defined* $\Leftrightarrow \varphi(g) \oplus \varphi(h)$ *is defined and* $\varphi(g + h) = \varphi(g) \oplus \varphi(h)$,

Proof. ($\Leftarrow$): Since φ: $X \longrightarrow Y$ is a bijection, then $\exists j \in X$, such that $\varphi(j) = 1_Y$. Hence

$$\varphi(1_X) = \varphi(j + j') = \varphi(j) \oplus \varphi(j') = 1_Y \oplus \varphi(j'). \tag{2}$$

By (E3), we have $\varphi(j') = 0_Y$. Therefore, $\varphi(1_X) = 1_Y$, $j = 1_X$ and $j' = 0_X$, which shows that (I2) and (I3) holds.

For every $l \in X$, Since $l + l' = 1_X$, we have

$$\varphi(l) \oplus \varphi(l') = \varphi(l + l') = \varphi(1_X) = 1_Y. \tag{3}$$

Hence $\varphi(l') = (\varphi(l))^{\dagger}$, i.e. (I4) holds. Therefore, $\mathcal{X} \overset{\varphi}{\simeq} \mathcal{Y}$.

($\Rightarrow$): By Definition 2, it is trivial. □

*Definition 4. By an **isomorphism** between two posets* $(S, \leq)$ *and* $(T, \leq)$, *is meant a one-one correspondence* ψ *between* S *and* T *such that*

$$s \leq t \Leftrightarrow \psi(s) \leq \psi(t). \tag{4}$$

Two posets are called **isomorphic** iff there exists an isomorphism between them, we write ψ: $S \overset{\psi}{\simeq}_{po} T$ *or just* $S \simeq T$; an isomorphism of a partly ordered set with itself is called an automorphism. A many-one corregiondence satisfying (4) is called **isotone** .

By the **converse** of a relation R is meant the relation R^c such that $xR^c y$ if and only if yRx.

*Definition 5. By the **dual** P^c of a poset* P *is meant that poset defined by the converse relation on the same elements.*

According to Definitions 2 and 4, we can obtain the following lemma. It gives the conclusion that the isomorphism of EA isomorphism can imply the order isomorphism.

Lemma 2. *Let* $\mathscr{X} = (X, +, ', 0_X, 1_X)$ *and* $\mathscr{Y} = (Y, \oplus, ^\dagger, 0_Y, 1_Y)$ *be EAs,* $\mathscr{X} \overset{\varphi}{\simeq} \mathscr{Y}$ *. Then* $(X, \leq_{\mathscr{X}}) \overset{\varphi}{\simeq}_{po} (Y, \leq_{\mathscr{Y}})$.

The condition (I1) can be further simplified by the following theorem.

Theorem 2. *Two effect algebras* $\mathscr{X} = (X, +, ', 0_X, 1_X)$ *and* $\mathscr{Y} = (Y, \oplus, ^\dagger, 0_Y, 1_Y)$ *are isomorphic iff* $\exists \varphi\colon X \longrightarrow Y$*, and for all* $w, v \in X$

> *(I1)* $'$ *if* $w + v$ *is defined then* $\varphi(w) \oplus \varphi(v)$ *is defined and* $\varphi(w + v) = \varphi(w) \oplus \varphi(v)$*,*
> *(li)* $(X, \leq_{\mathscr{X}}) \overset{\varphi}{\simeq}_{po} (Y, \leq_{\mathscr{Y}})$

Proof. ($\Leftarrow$): First of all, since $(X, \leq_{\mathscr{X}}) \overset{\varphi}{\simeq}_{po} (Y, \leq_{\mathscr{Y}})$, φ is bijective.

For all $u \in Y$, $0_X + \varphi^{-1}(u)$ is defined, then so does $\varphi(0_X) \oplus \varphi(\varphi^{-1}(u))$ and

$$\varphi(0_X) \oplus \varphi\left(\varphi^{-1}(u)\right) = \varphi\left(0_X + \varphi^{-1}(u)\right) = \varphi\left(\varphi^{-1}(u)\right) = u. \quad (5)$$

Thus $\varphi(0_X) = 0_Y$.

Since $\varphi\colon X \longrightarrow Y$ is a bijection, then $\exists h \in X$, such that $\varphi(h) = 1_Y$. Hence

$$\varphi(1_X) = \varphi(h + h') = \varphi(h) \oplus \varphi(h') = 1_Y \oplus \varphi(h'). \quad (6)$$

By (E3), we have $\varphi(h') = 0_Y$. Thus $h' = 0_X$ and $h = 1_Y$.Therefore, which shows that (I2) and (I3) holds.

For every $w \in X$, Since $w + w' = 1_X$, we have

$$\varphi(w) \oplus \varphi(w') = \varphi(w + w') = \varphi(1_X) = 1_Y. \quad (7)$$

Hence $\varphi(w') = (\varphi(w))^\dagger$, i.e. (I4) holds.

The following is the proof of condition (I1)$'$ and (li) implication condition (I1).

Let $k, l, r \in X$ and $\varphi(k) \oplus \varphi(l)$ is defined, $\varphi(k) \oplus \varphi(l) = \varphi(r)$ in $\mathscr{Y}$. Then

$$\varphi(k) \leq_{\mathscr{Y}} \varphi(r). \quad (8)$$

therefore

$$k \leq_{\mathscr{X}} r. \quad (9)$$

$\exists o \in X$, such that $k + o = r$. By (I1), we have $\varphi(k) \oplus \varphi(o) = \varphi(r)$. Then $\varphi(l) = \varphi(o)$ and $l = o$. i.e. (I1) holds.

Therefore, $\mathscr{X} \overset{f}{\simeq} \mathscr{Y}$.

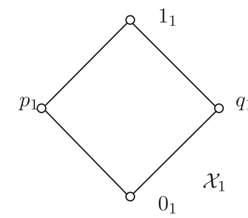
($\Rightarrow$):By Definition 2 and Lemma 2, it is trivial. □

Remark 1. The condition (I1)$'$ in Theorem 1 *does not imply the condition (I1), see* Example 1.

Example 1. Let$X_1 = \{0_1, 1_1, p_1, q_1,\}, X_2 = \{0_2, 1_2, p_2, q_2,\}$ *. It is easy to verify that* $\mathscr{X}_1 = (X_1, +, ', 0_1, 1_1)$ *and* $\mathscr{X}_2 = (X_2, \oplus, ^\dagger, 0_2, I_2)$ *are effect algebras, where*$+, '$ *,* $\oplus, ^\dagger$ *see the follows.*

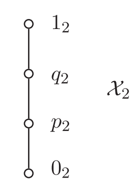
$+$	0_1	p_1	q_1	1_1
0	0_1	p_1	q_1	1_1
p_1	p_1	–	1_1	–
q_1	q_1	1_1	–	–
1_1	1_1	–	–	–

$0_1' = 1_1$, $p_1' = q_1$, $q_1' = p_1$, $1_1' = 0_1$,



$\oplus$	0_2	p_2	q_2	1_2
0_2	0_2	p_2	q_2	1_2
p_2	p_2	q_2	1_2	–
q_2	q_2	1_2	–	–
1_2	1_2	–	–	–

$0_2' = 1_2$, $p_2' = q_2$, $q_2' = p_2$, $1_2' = 0_2$,



Let $f\colon X_1 \longrightarrow X_2, 0_1 \mapsto 0_2, 1_1 \mapsto 1_2, p_1 \mapsto p_2, q_1 \mapsto q_2$. Then f is a bijection from X_1 to X_2, and satisfies condition (I1)$'$. But the effect algebras $\mathscr{X}_1$ and $\mathscr{X}_2$ are not isomorphic.

For each effect algebra $\mathscr{E}$, a unique partially ordered set $P(\mathscr{E})$ can be obtained under (1). Then, for a given bounded poset $(P, \leq)$, can we introduce partial binary operation $+$ and unary operation$'$such that $(P, +, ', 0, 1)$ is an EA and the induced order relation on P is exactly $\leq$?

The following counterexamples answers this question.

Example 2. Let $V_5 = \{0, 1, v_1, v_2, r\}, 0 < r < v_1 < 1, 0 < r < v_2 < 1$*, (see* Figure 1*).*

If $E(V_5) = (V_5, +, ', 0, 1)$ is an EA. Since $0 < r < v_1 < 1$, we have $r + r = v_1$. Similarly, since $0 < r < v_2 < 1$, we have $r + r = v_2$, hence $v_1 = v_2$. This is a contradiction. Thus, $(V_5, \leq)$ cannot be constructed as an EA.

Example 3. Let$V_4 = \{0, 1, v_1, v_2\}$*and partial binary operations* $+$*,* $\oplus$ *and unary*operations$'^{,\dagger}$ *are defined by*

$+$	0	v_1	v_2	1
0	0	v_1	v_2	1
v_1	v_1	1	-	-
v_2	v_2	-	1	-
1	1	-	-	-

$\oplus$	0	v_1	v_2	1
0	0	v_1	v_2	1
v_1	v_1	-	1	-
v_2	v_2	1	-	-
1	1	-	-	-

x	x'
0	1
v_1	v_1
v_2	v_2
1	0

x	$x^\dagger$
0	1
v_1	v_2
v_2	v_1
1	0

It is easy to prove that $\mathscr{E}_1(V_4) = (V_4, +, ', 0, 1)$ and $\mathscr{E}_2(V_4) = (V_4, \oplus, ^\dagger, 0, 1)$ are two completely different effect algebras, but they both induce partial ordered sets of V_4 (see Figure 2). In face, We have more general examples.

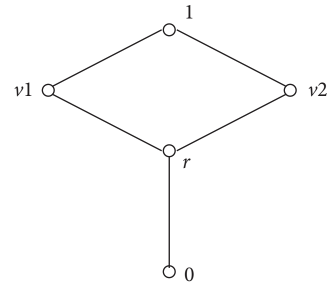


Figure 1: Lattice V_5.

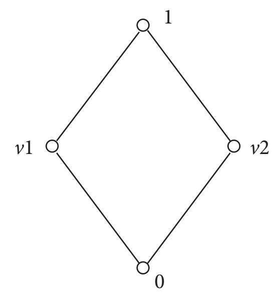


Figure 2: Lattice V_4.

Example 4. Let $E_0 = [0,1] \subseteq \mathbb{R}$ ($\mathbb{R}$ *is a set of real numbers*) *and define* $\oplus_n$ *and* $'_n$ *as follows*:

$$x \oplus_n y = \begin{cases} (x^n + y^n)^{1/n} & x^n + y^n \leq 1 \\ - & x^n + y^n > 1 \end{cases} \quad x'^{n} = (1 - x^n)^{1/n}.\ (n \in Z^+). \tag{10}$$

Then $\mathscr{E}_n = (E_0, \oplus_n, '^{n}, 0, 1)$ is a LEA with the induced order $\leq_n$. For all $u_1, u_2 \in [0,1]$, since

$$u_1 \leq_n u_2 \Leftrightarrow \exists h \in [0,1]\ s.t.\ u_1 \oplus_n d = u_2. \tag{11}$$

Thus, if $(u_1, u_2) \in \leq_n$, then $u_2 = u_1 \oplus_n d = \sqrt[n]{u_1^n + d^n} \geqslant u_1$, i.e. $(u_1, u_2) \in \leqslant$, where $\leqslant$ is usual orders on $\mathbb{R}$. Hence $\leq_n \subseteq \leqslant$.

Conversely, let $(s,t) \in \leqslant$, i.e. $0 \leqslant s \leqslant t \leqslant 1$, then we have $0 \leqslant s^n \leqslant t^n \leqslant 1$ and $\sqrt[n]{t^n - s^n} \in [0,1]$, $s \oplus_n (\sqrt[n]{t^n - s^n}) = t$, hence $s \leq_n t$, i.e. $\leq \subseteq \leq_n$. Thus $\leq_n = \leqslant (n = 1, 2, \cdots)$.

Theorem 3. *Let* $(X, \leq)$ *be a bounded poset with* $|X| = n$ ($n \in \mathbb{Z}^+$). *If* $n \leq 4$, *then poset* $(X, \leq)$ *can be used to construct an EA.*

Proof

(1) When $n = 1$, the statement is clearly true.

(2) If $n = 2$, then $X = \{0, 1\}$, $0 + 0 = 0, 0 + 1 = 1 + 0 = 1$, $0' = 1, 1' = 0$, and $\mathscr{X} = (X, +, 0, 1)$ is an EA.

(3) Since $(X, \leq)$ is a bounded poset, when $n = 3$, we have $X = \{0, a, 1\}$, and $0 < a < 1$, i.e. poset $(X, \leq)$ is a 3-element chain. Define $+$ and$'$ as follows:

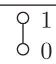
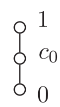
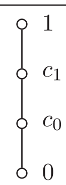
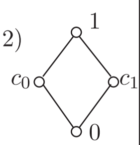
$+$	0	a	1
0	0	a	1
a	a	1	$-$
1	1	$-$	$-$

x	x'
0	1
a	a
1	0

therefor, $\mathscr{X} = (X, +, 0, 1)$ is an EA.

(4) For a bounded poset $(X, \leq)$, there are two cases when $n = 4$, one is a chain of four elements and the other is V_4 (see Example 1). So by Theorem 2 and Example 1, we get $(X, +)$, which can be converted into EA. The proof is complete. □

Remark 2. These results can be summarized in the following table.

n	P	$(P, \leq)$	$+$ and $'$	Uniqueness
1	$\{0\}$	○	$0+0=0,\ 0'=0$	Yes
2	$\{0,1\}$		$0+0=0,\ 0'=1$ $0+1=1,\ 1'=0$	Yes
3	$\{0, c_0, 1\}$		$\begin{array}{c\|ccc} + & 0 & c_0 & 1 \\ \hline 0 & 0 & c_0 & 1 \\ c_0 & c_0 & 1 & - \\ 1 & 1 & - & - \end{array}$ $0'=1,\ c_0'=c_0,\ 1'=0,$	Yes
4	$\{0, c_0,\ c_1, 1\}$	1)  2) 	1) $\begin{array}{c\|cccc} + & 0 & c_0 & c_1 & 1 \\ \hline 0 & 0 & c_0 & c_1 & 1 \\ c_0 & c_0 & c_1 & 1 & - \\ c_1 & c_1 & 1 & - & - \\ 1 & 1 & - & - & - \end{array}$ $0'=1,\ c_0'=c_1,\ c_1'=c_0,\ 1'=0,$ 2) see Example 3	1) Yes 2) No

We use $\mathscr{E}(n)$ to denote n-element effect algebras $(n = 1, 2, 3)$.

Remark 3. Example 2 *shows lattice V_5 with the least number of elements in non-effect algebra.*

Lemma 3. *Let $\mathscr{K} = (K, +, 0, 1)$ be an EA. Then $'$: $K \simeq K^c$, that is. $(K, \leq_{\mathscr{K}})$ is automorphic.*

Proof. for all $k_1, k_2 \in K$, if $k_1 \leq k_2$, then $k_2' \geq k_1'$ by Lemma 1 (3). And if $k_2' \geq k_1'$, then

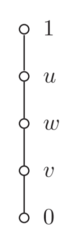
$$k_1 = k_1^{''} = (k_1') \leq (k_2'), = k_2^{''} = k_2. \quad (12)$$

by Lemma 1 (4). Thus $'$: $K \simeq K^c$. □

Theorem 4. *In the isomorphism sense, there are only four types of effect algebra for five elements, which are:*

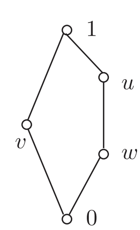
(1) $C_5 = \{0, u, v, w, 1\}$. Define $+$ and $'$ as follows:

$$\begin{array}{c|ccccc} + & 0 & v & w & u & 1 \\ \hline 0 & 0 & v & w & u & 1 \\ v & v & w & u & 1 & - \\ w & w & u & 1 & - & - \\ u & u & 1 & - & - & - \\ 1 & 1 & - & - & - & - \end{array} \qquad \begin{array}{l} 0' = 1, \\ u' = v, \\ v' = u, \\ w' = w, \\ 1' = 0, \end{array}$$


$(C_5, \leq)$.

(2) $N_5 = \{0, u, v, w, 1\}$. Define $+$ and $'$ as follows:

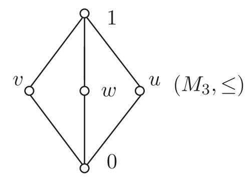
$$\begin{array}{c|ccccc} + & 0 & v & w & u & 1 \\ \hline 0 & 0 & v & w & u & 1 \\ v & v & 1 & - & - & - \\ w & w & - & - & 1 & - \\ u & u & - & 1 & - & - \\ 1 & 1 & - & - & - & - \end{array} \qquad \begin{array}{l} 0' = 1, \\ u' = w, \\ v' = v, \\ w' = u, \\ 1' = 0, \end{array}$$


$(N_5, \leq)$.

(3) $M_3 = \{0, u, v, w, 1\}$. Define + and ′ as follows:

(4) $M_3 = \{0, u, v, w, 1\}$. Define + and ′ as follows:

+	0	v	w	u	1	
0	0	v	w	u	1	$0' = 1$,
v	v	–	1	–	–	$u' = u$,
w	w	1	–	–	–	$v' = w$,
u	u	–	–	1	–	$w' = v$,
1	1	–	–	–	–	$1' = 0$,



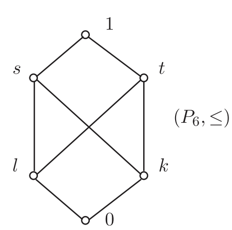
Proof. Since there are only three kinds of automorphic bounded five-element partial ordered sets: C_5, N_5, and M_3, the theorem holds. □

Remark 4. The five-element effect algebras whose induced poset isM_3 are not unique. There are altogether four of them. We'll look at this in the next section.

Corollary 1. *Let $\mathscr{L} = (L, +, ', 0, 1)$ be an EA with $|E| = n$ ($n \in \mathbb{Z}^+$). If $n < 6$, then $\mathscr{L} = (L, +, ', 0, 1)$ is a LEA.*

Example 5. [6] *The$\mathscr{P}_6 = (P_6, +, ', 0, 1)$is an EA, where$+, '$see the follows.*

+	0	l	k	s	t	1	
0	0	l	k	s	t	1	$0' = 1$,
l	l	t	s	1	–	–	$l' = s$,
k	k	s	t	–	1	–	$k' = t$,
s	s	1	–	–	–	–	$s' = l$,
t	t	–	1	–	–	–	$t' = k$,
1	1	–	–	–	–	–	$1' = 0$,



Let $(F, \leq)$ be a poset, $r, s \in F$ and $r < s$. Here are the definitions of the intervals:

$$[r, s] = \{d \in F | r \leq d \leq s\}, (r, s) = \{d \in F | r < d < s\}. \tag{13}$$

Note that P_6 is not a lattice, next, we will consider lattice-ordered effect algebras.

3. Homo-Ordered Effect Algebras

Next we will study the poset induced by the effect algebra with the same property, and first give the definition of the same order effect algebra.

*Definition 6. Two effect algebras $\mathscr{X} = (X, +, 0_X, 1_X)$ and $\mathscr{Y} = (Y, +, 0_Y, 1_Y)$ are called**Homo-ordered** if the posets $(X, \leq_X)$ and $(Y, \leq_Y)$ are isomorphic, denoted by $\mathscr{X} \overset{po}{\cong} \mathscr{Y}$.*

In Example 3, $\mathscr{E}_1(V_4) \overset{po}{\cong} \mathscr{E}_2(V_4)$ and in Example 4, $\mathscr{E}_n \overset{po}{\cong} \mathscr{E}_1, n = 1, 2, \cdots$ holds. Below, we have more general results.

Theorem 5. *Let $\mathscr{Z} = (Z, +, 0_Z, 1_Z)$ be an EA, $(K, \leq)$ be a poset. If$P(\mathscr{Z}) \overset{h}{\simeq} (K, \leq)$, then $\mathscr{Z} \overset{po}{\cong} \mathscr{K}$, where $\mathscr{K} = (K, \oplus, ^\dagger, 0_K, 1_K)$, $\forall l_1, l_2 \in K$:*

$$l_1 \oplus l_2 = \begin{cases} h\left(h^{-1}(l_1) + h^{-1}(l_2)\right) & h^{-1}(l_1) + h^{-1}(l_2) \text{is defined}, \\ - & \text{otherwise}, \end{cases} l_1^\dagger = h\left(h^{-1}(l_1)\right). \tag{14}$$

Proof. First, we prove that $\mathscr{K} = (K, \oplus, ^\dagger, 0_K, 1_K)$ is an EA.

For all $l_1, l_2 \in K$, if $l_1 \oplus l_2$ is defined, then $h^{-1}(l_1) + h^{-1}(l_2)$ is defined, and $h^{-1}(l_1) + h^{-1}(l_2) = h^{-1}(l_2) + h^{-1}(l_1)$ is defined, hence $l_2 \oplus l_1$ is defined and

$$\begin{aligned} l_2 \oplus l_1 &= h\left(h^{-1}(l_2) + h^{-1}(l_1)\right) \\ &= h\left(h^{-1}(l_1) + h^{-1}(l_2)\right) = l_1 \oplus l_2. \end{aligned} \tag{15}$$

i.e. (E1) holds. Similarly, we can prove that (E2) holds as well.

$$\begin{aligned} l_1 \oplus l_1^{\dagger} &= l_1 \oplus \left(h\left(\left(h^{-1}(l_1)\right)'\right)\right) \\ &= h\left(h^{-1}(l_1) + h^{-1}\left(h\left(\left(h^{-1}(l_1)\right)'\right)\right)\right), \\ &= h\left(h^{-1}(l_1) + \left(\left(h^{-1}(l_1)\right)'\right)\right) \\ &= h(1_Z) = 1_K. \end{aligned} \tag{16}$$

Then (E3) holds.

Let $1_K \oplus l$ is defined $(l \in K)$, then $h^{-1}(1_K) + h^{-1}(l) = 1_Z + h^{-1}(l)$ is defined. Thus, $h^{-1}(l) = 0_Z$,

$$l = h\left(h^{-1}(l)\right) = h(0_Z) = 0_K. \tag{17}$$

i.e. (E4) holds. Hence $\mathscr{K} = (K, \oplus, \dagger, 0_K, 1_K)$ is an EA.

Next, we show that $P(\mathscr{K}) = (K, \leq)$. i.e. $\leq_{\mathscr{K}} = \leq$.

Let $k_1, k_2 \in K$ and $k_1 \leq k_2$. Then we have $h^{-1}(k_1) \leq h^{-1}(k_2)$ and $\exists y \in Z$, $h^{-1}(k_1) + y = h^{-1}(k_2)$.i.e.

$$h^{-1}(k_1) + h^{-1}(h(y)) = h^{-1}(k_2). \tag{18}$$

Therefore $k_1 \oplus h(y) = h(h^{-1}(k_1) + h^{-1}(h(y))) = h(h^{-1}(k_2)) = k_2$, i.e. $k_1 \leq_{\mathscr{K}} k_2$.

Since $k_1 \leq_{\mathscr{K}} k_2$, then $\exists r \in K$, $k_1 \oplus r = k_2$. therefore we have $h^{-1}(k_1) + h^{-1}(r) = h^{-1}(k_2)$. Hence $h^{-1}(k_1) \leq_{\mathscr{Z}} h^{-1}(k_2)$. Since $P(\mathscr{Z}) \overset{h}{\simeq} (K, \leq)$, we have $k_1 \leq k_2$. Thus, we conclude that $P(\mathscr{K}) = (K, \leq)$ holds as well. Therefore, $P(\mathscr{Z})$ and $P(\mathscr{K})$ are isomorphic.

Thus, $\mathscr{Z} \overset{po}{\cong} \mathscr{K}$, the proof is complete. □

Remark 5

(1) This theorem gives a way to construct a new EA from the poset of an EA.

(2) *This method is not sufficient, see* Example 3, $\mathscr{E}_1(V_4) \overset{po}{\cong} \mathscr{E}_2(V_4)$*holds, but*+*and* $\oplus$ *do not satisfy the relationship of* Theorem 5.

Definition 7. Let $\mathscr{K}_1 = (K_1, +_{K_1}, '^{K_1}, 0, 1)$ *and* $\mathscr{K}_2 = (K_2, +_{K_2}, '^{K_2}, 0, 1)$ *are effect algebras and* $K_1 \cap K_2 = \{0, 1\}$, $M = K_1 \cup K_2$. *If we put*

$$r + t := \begin{cases} r +_{K_1} t, & r +_{K_1} t \text{ is defined } r, t \in K_1, \\ r +_{K_2} t, & r +_{K_2} t \text{ is defined } r, t \in K_2, r' \\ - & \text{otherwise,} \end{cases}$$
$$:= \begin{cases} r'^{K_1}, & r \in K_1, \\ r'^{K_2}, & r \in K_2, \end{cases} \tag{19}$$

for all $r, t \in M$ *then* $\mathscr{M} = (M, +, ', 0, 1)$ *is EA, we call* $\mathscr{M}$ *a* ***union effect algebra*** *of* $\mathscr{K}_1$ *and* $\mathscr{K}_2$, *denoted by* $\mathscr{M} = \mathscr{K}_1 \sqcup \mathscr{K}_2$ *(see* Figure 3.

In Example 3, if we put $G = \{0, a, 1\}, H = \{0, b, 1\}$, then $\mathscr{E}_1(V_4) = \mathscr{G} \sqcup \mathscr{H}$.

Definition 8. Let $\mathscr{K} = (K, +_K, '^K, 0_K, 1_K)$ *and* $\mathscr{L} = (L, +_L, '^L, 0_L, 1_L)$ *are EA. If we put*

$$(k_1, h_1) + (k_2, h_2) := \begin{cases} (k_1 +_K k_2, h_1 +_L h_2), & k_1 +_K k_2 \text{ and } h_1 +_L h_2 \text{ are defined,} \\ -, & \text{otherwise,} \end{cases}$$
$$(k_1, h_1)' := \left(k_1'^K, h_1'^L\right). \tag{20}$$

for all $(k_1, h_1), (k_2, h_2) \in K \times L$, obviously $\mathscr{K} \otimes \mathscr{L} = (K \times L, +, ', (0_K, 0_L), (1_K, 1_L))$ *is EA, we call* $\mathscr{K} \otimes \mathscr{L}$ a **direct product effect algebra** of $\mathscr{K}$ and $\mathscr{L}$.

In Example 3, if we put $G = H = \{0, 1\}$, $\mathscr{G} = (G, +, ', 0, 1)$, $\mathscr{H} = (H, +, ', 0, 1)$ and $0 + 0 = 0, 0 + 1 = 1 + 0 = 1$, $0' = 1, 1' = 0$, then $\mathscr{E}_2(V_4) = \mathscr{G} \otimes \mathscr{H} \cong \mathscr{E}(2) \otimes \mathscr{E}(2)$.

If all sub-chains in a poset $P = (P, \leq)$ contain at most $m + 1$ element $(m \in N)$, then we say that the **height** of the poset $P = (P, \leq)$ is m, denoted by $h(P) = m$.

Lemma 4. *Let* $H = (H, \leq)$ *be a bounded poset with* $h(H) = 2$, *then* $H = M_I$, *where* $M_I = (M_I, \leq_M)$, $M_I = \{0, 1\} \cup I, I \neq \varnothing, 0 \leq_M a \leq_M 1$, *for all* $a \in I$ *(see Figure 4).*

Proof. The proof can be obtained directly from the boundedness and height of the poset $(H, \leq)$. □

Theorem 6. *Let* $H = (H, \leq)$*be a bounded poset with* $h(H) = 2$, *then there is an EA* $\mathscr{X} = (H, +, ', 0, 1)$ *such that* $P(\mathscr{X}) = (H, \leq)$.

Proof. Let $0, 1$ be the smallest and largest element of a bounded poset $(H, \leq)$, that is: $0 \leq x \leq 1$, for any $x \in H$.

Since $h(H) = 2$, $H = I \cup \{0, 1\}, I = \{x, y, \cdots\}$ by Lemma 4 (see Figure 4). Obviously, $\mathscr{X} = (H, +, ', 0, 1)$ is an EA and $P(\mathscr{X}) = (H, \leq)$, where $+, '$ see the follows.

$$\begin{aligned} &0 + j = j + 0 := j, j + j := 1; j' = j, \text{for all } j \in I, \\ &0 + 1 = 1 + 0 = 1, 0 + 0 = 0; 0' = 1, 1' = 0. \end{aligned} \tag{21}$$

□

Theorem 7. *Let* $\mathscr{X} = (X, +, ', 0, 1)$*be an EA and* $I = X/\{0, 1\}$. *Then the following are equivalent:*

(1) $P(\mathscr{X}) = M_I$;

(2) For all $u, v \in I$, *if* $u + v$ *is defined then* $u + v = 1$.

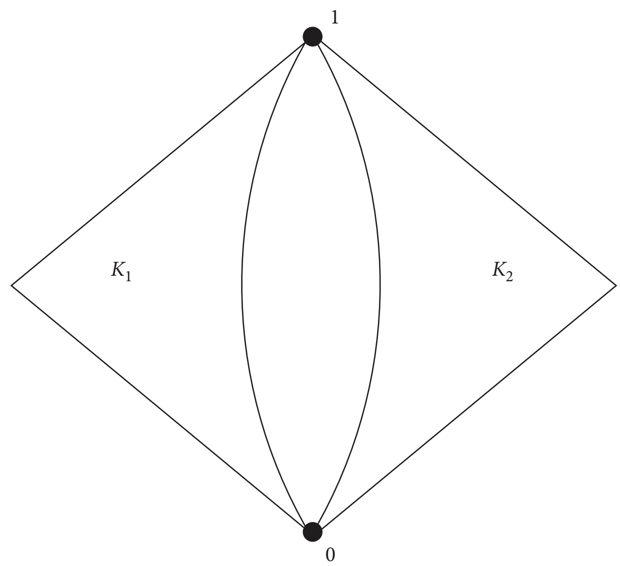


FIGURE 3: Union effect algebra $\mathscr{K}_1 \sqcup \mathscr{K}_2$.

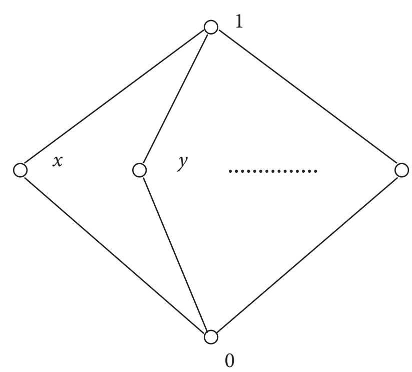


FIGURE 4: Bounded poset M_I.

Proof

(1) $\Rightarrow$ (2). For all $u, v \in I$, if $u + v$ is defined, then $u, v \leq u + v$ and $u \vee v \leq u + v$. In M_I, $u \vee v = 1$ for all $u, v \in I$. Thus we have $u + v = 1$.

(2)$\Rightarrow$ (1). For all $u, v \in I$, when $u \neq v$, we have $u < u \vee v$ or $v < u \vee v$. Then $\exists s, t \in I$, such that

$$u \vee v = u + s, \text{or}\, u \vee v = v + t. \tag{22}$$

and $u \vee v = 1$ by (2).

Since $X \simeq X^c$ by Lemma 3, $u \wedge v = 0$ for all $u, v \in I$. Hence $P(\mathscr{X}) = M_I$, the proof is complete. □

Theorem 8. *Let $\mathscr{Z} = (Z, +, ', 0, 1)$ be an EA with $P(\mathscr{Z}) = M_I$, $I = Z/\{0, 1\}$. Then*

$$\mathscr{Z} = \sqcup\{\mathscr{Z}_a | a \in I\}, \tag{23}$$

where $\mathscr{Z}_a = (Z_a, +, ', 0, 1)$, $Z_a = \{0, a, a', 1\}, a \in I$.

Proof. For all $a \in I$, $\mathscr{Z}_a = (\{0, a, a', 1\}, +, ', 0, 1)$ and

$$Z_a = \{0, a, a', 1\} = \begin{cases} \{0, a, 1\}, & a = a'; \\ \{0, a, a', 1\}, & a \neq a'. \end{cases} \tag{24}$$

It is easy to verify that $\mathscr{Z}_a$ is an EA and

$$\mathscr{Z}_r \cap \mathscr{Z}_s = \begin{cases} \{0, 1\}, & r \neq s\, \text{and}\, r \neq s'; \\ \mathscr{Z}_r = \mathscr{Z}_s, & r = s\, \text{or}\, r = s'. \end{cases} \tag{25}$$

for all $r, s \in I$. Thus, $\sqcup\{\mathscr{Z}_a | a \in I\}$ is defined and $\cup_{a \in I} Z_a = Z$, therefore, $u + v$ is defined iff $v \in \mathscr{Z}_u$ by Theorem 7. Hence $\mathscr{Z} = \sqcup\{\mathscr{Z}_a | a \in I\}$, the proof is complete. □

Corollary 2. *Let $|I| = n \in \mathbb{Z}^+$. In the isomorphism sense, there are altogether $[n/2] + 1$ different homo-ordered effect algebras with M_I as the induced partial ordered set.*

Remark 6

(1) In Theorem 8, when $a \neq a'$, $\mathscr{E}_a \simeq \mathscr{E}(2) \otimes \mathscr{E}(2)$. Therefore, the effect algebra $\mathscr{E}$ with $P(\mathscr{E}) = M_I$ is

obtained by some 2-element effect algebras and 3-element effect algebras through $\otimes$ and $\sqcup$ operations.

(2) We find out the structure of the EA of height 2 of its partial ordered set.

The structure of the EA of height 3 of its partial ordered set. Here are some examples.

(1) $+,'$ of $\mathscr{C}_1 = \mathscr{E}(2) \otimes \mathscr{E}(2) \otimes \mathscr{E}(2)$:

$+$	0	s_1	r_1	k_1	s_2	r_2	k_2	1	
0	0	s_1	r_1	k_1	s_2	r_2	k_2	1	$0' = 1,$
s_1	s_1	$-$	s_2	r_2	$-$	$-$	1	$-$	$s_1' = k_2,$
r_1	r_1	s_2	$-$	k_2	$-$	1	$-$	$-$	$r_1' = r_2,$
k_1	k_1	r_2	k_2	$-$	1	$-$	$-$	$-$	$k_1' = s_2,$
s_2	s_2	$-$	$-$	1	$-$	$-$	$-$	$-$	$s_2' = k_1,$
r_2	r_2	$-$	1	$-$	$-$	$-$	$-$	$-$	$r_2' = r_1,$
k_2	k_2	1	$-$	$-$	$-$	$-$	$-$	$-$	$k_2' = s_1,$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$1' = 0,$

(2) $+,'$ of $\mathscr{C}_2 = (\mathscr{E}(2) \sqcup \mathscr{E}(2)) \otimes \mathscr{E}(2)$:

$+$	0	s_1	r_1	k_1	s_2	r_2	k_2	1	
0	0	s_1	r_1	k_1	s_2	r_2	k_2	1	$0' = 1,$
s_1	s_1	s_2	$-$	r_2	$-$	1	$-$	$-$	$s_1' = r_2,$
r_1	r_1	$-$	s_2	k_2	$-$	$-$	1	$-$	$r_1' = k_2,$
k_1	k_1	r_2	k_2	$-$	1	$-$	$-$	$-$	$k_1' = s_2,$
s_2	s_2	$-$	$-$	1	$-$	$-$	$-$	$-$	$s_2' = k_1,$
r_2	r_2	1	$-$	$-$	$-$	$-$	$-$	$-$	$r_2' = s_1,$
k_2	k_2	$-$	1	$-$	$-$	$-$	$-$	$-$	$k_2' = r_1,$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$1' = 0,$

Example 6. $\mathscr{P}_6$, $\mathscr{C}_1 = \mathscr{E}(2) \otimes \mathscr{E}(2) \otimes \mathscr{E}(2)$ *and* $\mathscr{C}_2 = (\mathscr{E}(2) \sqcup \mathscr{E}(2)) \otimes \mathscr{E}(2)$ *are effect algebras whose partial ordered sets have height 3. But the poset* $P(\mathscr{P}_6)$ *of* $\mathscr{P}_6$ *is not a lattice, and* $\mathscr{C}_1 \overset{po}{\cong} \mathscr{C}_2$, $P(\mathscr{C}_1) = P(\mathscr{C}_2)$ *is a cube* $C_2 \times C_2 \times C_2$ *(see* Figure 5(a)*).*

Here is another example of an EA $\mathscr{X}$ whose poset $P(\mathscr{X})$ is not a lattice.

Example 7. It is easy to verify that $\mathscr{P}_8 = (P_8, +, ', 0, 1)$ *is an EA, where* $+,'$ *see the follows.*

$+$	0	s_1	r_1	k_1	s_2	r_2	k_2	1	
0	0	s_1	r_1	k_1	s_2	r_2	k_2	1	$0' = 1,$
s_1	s_1	$-$	s_2	r_2	$-$	$-$	1	$-$	$s_1' = r_2,$
r_1	r_1	s_2	r_2	k_2	$-$	1	$-$	$-$	$r_1' = k_2,$
k_1	k_1	r_2	k_2	$-$	1	$-$	$-$	$-$	$k_1' = s_2,$
s_2	s_2	$-$	$-$	1	$-$	$-$	$-$	$-$	$s_2' = k_1,$
r_2	r_2	$-$	1	$-$	$-$	$-$	$-$	$-$	$r_2' = s_1,$
k_2	k_2	1	$-$	$-$	$-$	$-$	$-$	$-$	$k_2' = r_1,$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$1' = 0,$

$(P_8, \leq)$ is not a lattice (see Figure 5(b)).

Example 8. The poset in Figure 6*is not an induced poset of any lattice effect algebra* $(n \geq 4)$*. At the same time, we notice that at* $n \neq 2, Z_n$ *is all lattice, and we call* Z_n *crown lattice.*

Figure 7 below shows the crown lattice Z_1, Z_2, Z_3, and Z_4 with $n = 1, 2, 3$, and 4.

4. Chain Effect Algebra (CEA)

In the previous section we obtained the complete structure of a class of effect algebras.They are constructed from 2-element and 3-element effect algebra by $\otimes$ and $\sqcup$ operations. Since both 2-element and 3-element effect algebras are chain effect algebras, we will discuss chain effect algebras in this section.

Lemma 5. *Let* $\mathscr{X} = (X, +, ', 0, 1)$*be an EA and* $u, v, p, q \in X$*. Then*

(1) if $u + p = u + q$ *then* $p = q$*;*

(2) if $u + p = p$ *then* $u = 0$*;*

(3) $u \lessdot v$ *iff there exists a atom* $p \in X$ *such that* $u + p = v$*;*

(4) if $u \lessdot v$ *then* $v' \lessdot u'$*.*

Proof

(1) Let $w = u + p = u + q$, then $p' = u + w', q' = u + w'$ by Lemma 1 (4). Thus $p' = q'$, that is $p = q$.

(2) Since $u + p = p = 0 + p$, Hence $p = 0$ by (1).

(3) If $u \lessdot v$, then $\exists d \in X$, $u + d = v$. Let $y \in X, 0 \leq y < d$, since $u + d = v$, we have $u + y$ is defined and $u \leq u + y < u + d = b$ by (1), then $u = u + y$ and $y = 0$ by (2).

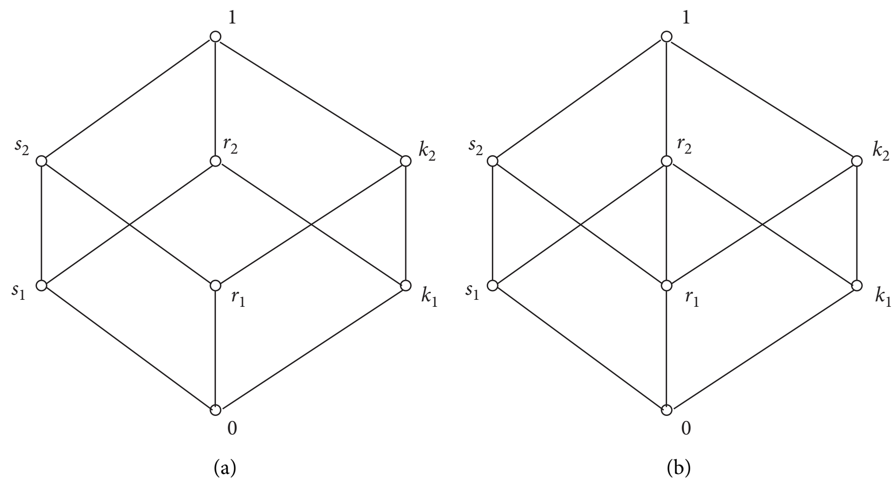


Figure 5: Lattice $C_2 \times C_2 \times C_2$ and poset P_8. (a) $C_2 \times C_2 \times C_2$ (b) P_8.

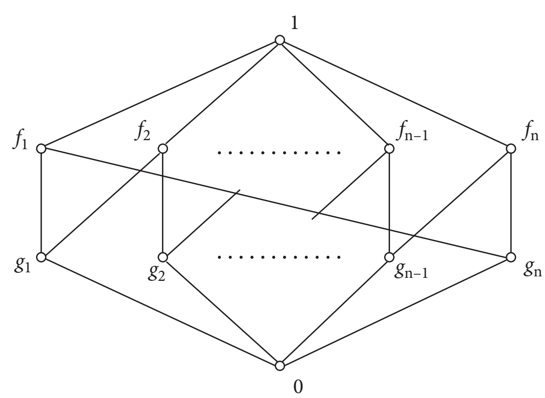


Figure 6: The crown lattice Z_n.

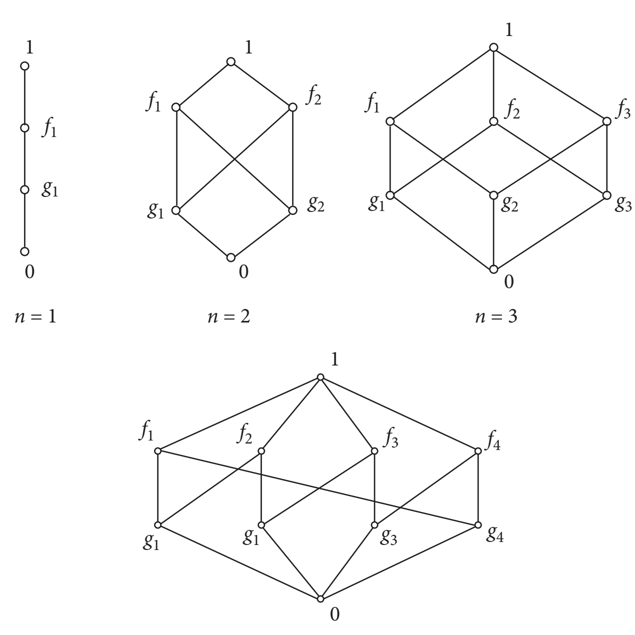


Figure 7: The crown lattices Z_1, Z_2, Z_3, Z_4.

Hence d is atom of $(X, \leq)$. Conversely, let d be an atom of X, and $u + d = v$, then $u \leq v$. If there $\exists e \in X$ $u \leq e < v$. that is $e = u + y < v = u + d$ for some $y \in X$. Then $0 \leq y < d$, Since d is an atom, we have $y = 0$, i.e. $u = e$. Hence $u \prec v$.

(4) Since $u \prec v$, then $\exists w \in X$, $u + w = v$ and w is atom of X by (3). We have $u' + w = v'$ by Lemma 1 (4), Then $v' \prec u'$, the proof is complete. □

Theorem 9. *If* $(L, \leq)$ *is an* n *-element chain* $\{0 = l_1 < l_2 < \cdots < l_{n-1} < l_n = 1\}$, *there is and only one effect algebra constructed by poset* $(P, \leq)$, *and its*+*and*$\prime$*operations are as follows:*

Proof. Obviously, the + and $\prime$ operations given in the theorem satisfy the condition (E1) – (E4), that is, $\mathscr{L} = (L, +, ', 0, 1)$ is an EA. The order relation induced on $\mathscr{L}$ is $\leq$.

The following shows that the effect algebra constructed by $(L, \leq)$ is unique.

Let $(L, \oplus, ^{\dagger}, 0, 1)$ be an EA and the induced poset is the $(L, \leq)$. Since $l_1 = 0$, we have

$$l_1 \oplus l_r = l_r, r = 1, 2, \ldots, n. \tag{26}$$

Since $l_r \prec l_{r+1}$, then $\exists x \in L$, $l_r \oplus x = l_{r+1}$ $(r = 1, 2, \cdot n - 1)$, then $x = l_2$ by Lemma 5 (1). Hence

$$l_2 \oplus l_r = l_{r+1} \ (r = 1, 2, \ldots, n-1), \text{ and } l_2 \oplus l_n \text{ is not defined.} \tag{27}$$

Therefore, $l_{r+2} = l_2 \oplus l_{r+1} = l_2 \oplus (l_2 \oplus l_r) = (l_2 \oplus l_2) \oplus l_r = l_3 \oplus l_r$, that is

$$l_3 \oplus l_r = \begin{cases} l_{r+2} & (r = 1, \ldots, n-2), \\ - & (r = n-1, n). \end{cases} \tag{28}$$

Similarly, we can show that

$$\begin{aligned} l_4 \oplus l_r &= \begin{cases} l_{r+3} & (r = 1, \ldots, n-3), \\ - & (r = n-2, n-1, n), \end{cases} \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ l_i \oplus l_r &= \begin{cases} l_{r+(i-1)} & (r = 1, \ldots, n+1-i), \\ - & (r = n+2-i, \ldots, n-1, n), \end{cases} \\ i &= 1, 2, 3, \ldots, n-1. \end{aligned} \tag{29}$$

Considering the above mentioned, we can get: $\oplus = +$. And, according to the above equation, $l_k \oplus l_{n-k+1} = 1$, hence $l_k^{\dagger} = l_{n-k+1}$, $(k = 1, 2, \ldots, n)$, i.e. $\dagger = '$.

Thus, the effect algebra constructed by poset $(L, \leq)$ is unique, the proof is complete. □

Definition 9 (see[9]). *Let*$(P, \leq)$*be a partial ordered set.*

(1) $(P, \leq)$*has the* ***ascending chain condition (ACC)*** *if it has no infinite strictly ascending sequences, that is, for any ascending sequence*

$$a_1 \leq a_2 \leq a_3 \leq \cdots. \tag{30}$$

$\exists m \in N$, $a_{m+r} = a_m$ for all $r \geq 0$.

(2) $(P, \leq)$*has the* ***descending chain condition (DCC)*** *if it has no infinite strictly descending sequences, that is, for any descending sequence*

$$a_1 \geq a_2 \geq a_3 \geq \cdots. \tag{31}$$

$\exists m \in N$, $a_{m+r} = a_m$ for all $r \geq 0$.

(3) An effect algebra $\mathscr{X} = (X, +, ', 0, 1)$has the ***ACC (DCC)*** *if*$(X, \leq)$has the ***ACC (DCC)***.

where $\leq$ is induced order of X.

Definition 10 (see[9]). *A poset*$(P, \leq)$*is said to have a* ***maximal condition*** *if each non-empty subset of* $(P, \leq)$*contains a maximal element. Dually, the poset*$(P, \leq)$*can be defined to have* ***minimal conditions****.*

Lemma 6 (see[9]). *Let* $(X, \leq)$ *be a poset, then*

(1) The sufficient and necessary condition for $(X, \leq)$ *to satisfy ACC is that* $(X, \leq)$ *has the maximum condition.*

(2) The sufficient and necessary condition for $(X, \leq)$ *to satisfy DCC is that* $(X, \leq)$ *has the minimal condition.*

Theorem 10. *Let* $\mathscr{X} = (X, +, 0, 1)$ *be an EA, then* $\mathscr{X}$*has the ACC iff it has the DCC.*

Proof. If $\mathscr{X}$ has the ACC and let $\{p_1, p_2, p_3, \cdots\} \subseteq X$ be descending sequence, i.e.

$$p_1 \geq p_2 \geq p_3 \geq \cdots. \tag{32}$$

Then $p_1' \leq p_2' \leq p_3' \leq \cdots$, therefor, $\exists\, m \in N$, $a_{m+k}' = a_m'$ for all $k \geq 0$ by ACC. Thus $a_{n+k} = a_n$ for all $k \geq 0$, and $\mathscr{X} = (X, +, ', 0, 1)$ has the DCC, i.e. ACC $\Rightarrow$ DCC and vice versa. The proof is complete.

Using the above two theorems, we get the following result. □

Theorem 11. *A chain effect algebra* $\mathscr{C} = (C, +, ', 0, 1)$ *must is one of the following:*

(1) Cis a finite set$\{0 = p_1, p_2, \ldots, p_{n-1}, p_n = 1\}$*and*

$$0 = p_1 < p_2 < \cdots < p_{n-1} < p_n = 1. \tag{33}$$

(2) Chave an infinite strictly ascending chain

$$1 > q_1 > q_2 > q_3 > \cdots, \tag{34}$$

and an infinite strictly descending chain

$$0 < q_1' < q_2' < q_3' < \cdots. \tag{35}$$

Proof. If C is a finite set. Obviously, (1) is true.

Now let's assume that C is an infinite set, and let's prove that C can only be (2). In face, C fails to have the DCC and ACC (if not, C has the ACC, then C has the DCC by Theorem 10, hence C is a finite set. This is a contradiction.). Hence $(C, \le)$ have an infinite strictly ascending chain

$$1 > q_1 > q_2 > q_3 > \cdots. \tag{36}$$

Obviously,

$$0 < q_1' < q_2' < q_3' < \cdots. \tag{37}$$

is an infinite strictly descending chain in $(C, \le)$. The proof is complete.

Here is the simplest example of an infinite chain effect algebra. □

Example 9. Let $C_0 = \{0 = a_0, a_1, \ldots, a_n, \ldots, b_n, b_{n-1}, \ldots, b_1, b_0 = 1\}$ *, and define* $+$*and* $\prime$ *as follows*:

$$a_s + a_t = a_{s+t}, b_s + b_t = -, a_s + b_t = \begin{cases} b_{t-s} & (s \le t), \\ - & (s > t), \end{cases} (\forall s, t = 0, 1, 2, \cdots), \quad a_s' = b_s, b_s' = a_s \, (\forall s = 0, 1, 2, \cdots). \tag{38}$$

Then $\mathscr{C}_0 = (C_0, +, ', 0, 1)$ is an infinite CEA. And

$$0 = a_0 < a_1 < \cdots < a_n < \cdots < b_n < b_{n-1} < \cdots < b_1 < b_0 = 1. \tag{39}$$

Theorem 12. *Let* $\mathscr{X} = (X, +, 0, 1)$ *be an EA,* $\le$ *its induced order,* $t, u, w \in X$ *. If* $t < u$ *and* $u + w$ *is defined then* $([t, u], \le) \simeq ([t + w, u + w], \le)$.

Proof. Since $u + w$ is defined, $t < u$, we have $z + w$ is defined and $z + w \in [t + w, u + w] \, (\forall w \in [t, u])$ by Lemma 1 (2) and (5).

Let $f: ([t, u], \le) \longrightarrow ([t + w, u + w], \le), z \mapsto z + w$, $(\forall z \in [t, u])$.

For every $l, k \in [t, u]$

$$l \le k \Leftrightarrow l + w \le k + w \Leftrightarrow f(l) \le f(k). \tag{40}$$

Thus, $([t, u], \le) \simeq ([t + w, u + w], \le)$. □

Corollary 3. *Let* $\mathscr{L} = (L, +, 0, 1)$ *be a CEA,* $p \in C$. *If* $np = \overbrace{p + p + \cdots + p}^{n}$ *is defined then we have:*

$$([0, p], \le) \simeq ([p, 2p], \le) \simeq ([2p, 3p], \le) \simeq \cdots \simeq ([(n-1)p, np], \le). \tag{41}$$

Theorem 13. *Let* $\mathscr{X} = (X, +, 0, 1)$ *be an EA with* $(X, \le)$ *has no atoms. If* $l < k, l, k \in X$ *then* $\exists w \in X$ *such that* $l < w < k$.

Proof. Consider $l, k \in X, l < k$. So $\exists s \in X, s \ne 0$ such that $l + s = k$ by Definition 1 (E1). Since $(X, \le)$ has no atoms, we have: $\exists y \in X$ such that $0 < y < s$, therefore $l < l + y < l + s = k$, the result holds. □

Corollary 4. *Let* $\mathscr{M} = (M, +, 0, 1)$ *be a finite EA. If* $(M, \le)$ *has a atom* $p \in M$*, such that* $\forall t \in M/\{0\}$, $p \le t$*, then* $(M, \le)$ *is a chain.*

Proof. For the sequence $d, 2d, 3d, \cdots$ in $(M, \le)$, since M is finite, we have: $\exists k \in N$, $kd \in M$ but $(k+1)d$ is undefined.

Since $k\,d \le 1$, we have $k\,d + m = 1$ for some $m \in M$. If $m \ne 0$, then $d \le b$ and $(k+1)d = kd + d$ is defined by Lemma 1 (2). This is a contradiction, hence $m = 0$. Thus $kd = 1$. Now let's drove that M is equal to $\{0, d, 2d, \ldots, (k-1)d, 1\}$.

Assume that $c \in M$ and $c \notin \{0, d, 2d, \ldots, (k-1)d, 1\}$. Since $d < c$ and $0 \prec d \prec 2d \prec \cdots \prec (k-1)d \prec 1$, then $\exists t \, (1 \le t < k)$, $td < c$, but $(t+1)d < c$. Hence

$$c = t\,d + y \, (\exists\, y \in M). \tag{42}$$

Obvious, $y \ne 0$, and $y \ge d$, thus $c = td + y \ge td + d$. This is a contradiction. Hence

$$M = \{0, d, 2\,d, \ldots, (k-1)d, 1\}, |M| = k + 1, \tag{43}$$

and $(M, \le)$ is a chain.

The following example shows that Corollary 4 fails when L is an infinite EA. □

Example 10. Let $K = \{0, 1, kp, (kp)' | k = 1, 2, \cdots\} \cup \{a_t | t = 0, \pm 1, \cdots\} \cup \{b_t | t = 0, \pm 1, \cdots\}$, *and*

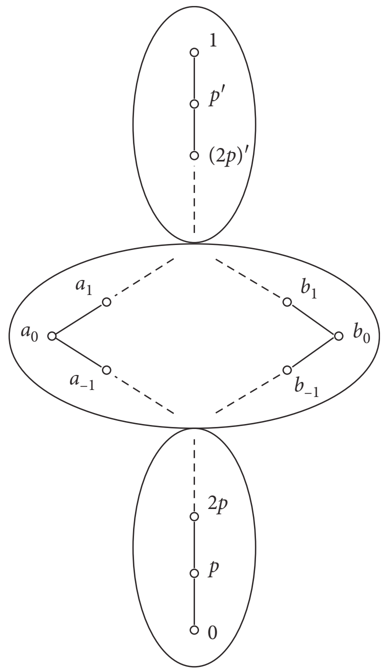


Figure 8: Poset $(K, \leq)$.

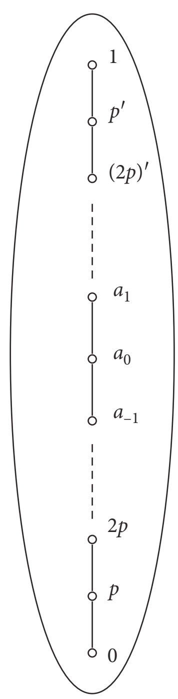


Figure 9: Chain $(K, \leq)$.

$$
\begin{aligned}
sp+tp=(s+t)p, sp+(tp)' &= \begin{cases} ((t-s)p)' & s<t, \\ 1 & s=t, (s,t\in Z^{+}), \\ - & s>t, \end{cases} \\
np+a_i = a_{n+i}, a_i+a_j &= \begin{cases} 1 & i+j=0, \\ ((-i-j)p)' & i+j<0, (i,j\in Z), \\ - & i+j>0, \end{cases} \\
np+b_i = b_{n+i}, b_i+b_j &= \begin{cases} 1 & i+j=0, \\ ((-i-j)p)' & i+j<0, \\ - & i+j>0, \end{cases} (i,j\in Z).
\end{aligned} \tag{44}
$$

then $\mathscr{K}=(K,+,0,1)$ is an EA, but $(K,\leq)$ is not chain (See Figure 8). In face, $(K,\leq)$ is not even a lattice ($\{a_0,b_0\}$ has no least upper bound in $(k,\leq)$).

Example 11. Let

$$
\begin{aligned}
K_0 = &\{0,p,2p,\ldots,np,\ldots,(np)',\ldots,(2p)',p',1\} \\
&\cup\{a_0,a_1,a_{-1},\cdots\},
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
&sp+tp=(s+t)p, sp+(tp)' \\
&\quad = \begin{cases} ((t-s)p)' & s<t, \\ 1 & s=t, (s,t\in Z^{+}), \\ - & s>t, \end{cases} \\
&np+a_i = a_{n+i}, a_i+a_j \\
&\quad = \begin{cases} 1 & i+j=0, \\ ((-i-j)p)' & i+j<0, (i,j\in Z). \\ - & i+j>0, \end{cases}
\end{aligned} \tag{46}
$$

then $\mathscr{K}_0=(K_0,+,0,1)$ is an EA, and $(K_0,\leq)$ is a chain (See Figure 9).

We naturally ask the question: in Corollary 4, if $\mathscr{L}=(L,+,0,1)$ is a LEA must $(L,\leq)$ be a chain?

The following theorem answers this question.

Theorem 14. *Let $\mathscr{L}=(L,+,0,1)$ be a LEA. If $(L,\leq)$ has a atom $p\in L$, such that $\forall l\in L/\{0\}$, $p\leq l$, then $(L,\leq)$ is a chain.*

Proof. If $\exists m\in N$ such that $mp\in L$ but $(m+1)p$ is undefined. Then by Corollary 4, we know that the theorem is true. The theorem will be proved in the case where np is defined ($\forall n\in N$).

Obvious, for all $k\in N$, $0<kp<1$. According to the proof of Corollary 4, similarly, we can get

$$
\begin{aligned}
\{x\in L\mid x\leq np\} &= \{0,p,2p,\ldots,(n-1)p,np\}, \\
0&<p<2p<\cdots<np,
\end{aligned} \tag{47}
$$

and its dual

$$
\begin{aligned}
\{x\in L\mid x\geq (np')\} &= \{1,p',(2p)',\ldots,(np)'\}, \\
(np)'&<\cdots<(2p)'<p'<1.
\end{aligned} \tag{48}
$$

Since $p<p'$, we have $np<(np)'$ $(n\in N)$.

Next, we will prove that L is a chain. Since

$$
\{0,p,2p,\ldots,np,\ldots,(np)',\ldots,(2p)',p',1\}\subseteq C. \tag{49}
$$

Assume that $u,v\in L$ are incomparable. Then

$$
w,v\notin\{0,p,2p,\ldots,np,\ldots,(np)',\ldots,(2p)',p',1\}, \tag{50}
$$

and, $\forall s,t\in N$, $sp<w,v<(tp)'$. Since $(L,\leq)$ is a lattice, we have $w\wedge v\in L$, let $d=w\wedge v$. Since $d<w,v$, we have $\exists x,y\in L, x,y\geq p$ such that $d+x=w, d+y=v$. Then

$$
d=w\wedge v=(d+x)\wedge(d+y)=d+(x\wedge y), \tag{51}
$$

hence $x\wedge y=0$, This is a contradiction, thus $(L,\leq)$ is a chain. □

Definition 11. Let Lbe a lattice, $j,m\in L$,

(1) j *is* *join*-irreducible *if* $j=u\vee v\Rightarrow j=u$ or $j=v$, *$(u,v\in L)$.*

(2) m is *meet-irreducible* *if* $m=s\wedge t\Rightarrow m=s$ or $m=t$, *$(s,t\in L)$.*

$J(L)=\{j\in L\mid j$ is join-irreducible$\}$ and $M(L)=\{m\in L\mid m$ is meet-irreducible$\}$.

Theorem 15. *Let $\mathscr{C}=(C,+,0,1)$ be a LEA. Then the following conditions are equivalent:*

(1) $(C,\leq)$is a chain.

(2) $1$is join-irreducible element of$(C,\leq)$.

(3) $0$is meet-irreducible element of$(C,\leq)$.

Proof

(1) ⇒ (2): Let $s\vee t=1, s,t\in C$. Since $(C,\leq)$ is a chain, we have: x and y are comparable. Hence $s\vee t=s$ or $s\vee t=t$, that is. $1=s$ or $1=t$. Thus, 1 is join-irreducible.

(2) ⇒ (3): By Lemma 2.

(3) ⇒ (1): Let $\mathscr{C} = (C, +, 0, 1)$ be a LEA. Suppose that $(C, \leq)$ is not a chain. then $\exists p, q \in C$ have $p \| q$ holds.

Since $(C, \leq)$ is a lattice, $p \wedge q \in C$, let $d = p \wedge q$. Since $p \| q$, we have: $d < p, q$, then $\exists x, y \in C, x, y > 0$ such that $d + x = p, d + y = q$. Hence

$$d = p \wedge q = (d + x) \wedge (d + y) = d + (x \wedge y), \qquad (52)$$

that is. $x \wedge y = 0$, That contradicts the fact that 0 is meet-irreducible element. Thus $(C, \leq)$ is a chain. □

Corollary 5. *Let $\mathscr{L} = (L, +, 0, 1)$ be a LEA. If 0 is meet-irreducible element of $(L, \leq)$, then $L = J(L) = M(L)$.*

5. Conclusion

The main content of this paper is to study the properties and structures of LEAs from the perspective of partial ordered sets. We study the characterization of original effect algebras by partial ordered sets induced by EAs. The structure and number of effect algebras generated by M_I bounded partially ordered sets of height 2 are solved.

We study the chain effect algebra and give some necessary and sufficient conditions for determining the LEA as a CEA. It is proved that a finite EA is a CEA if and only if it has only one atom, and some counterexamples are given.

Data Availability

All data from this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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