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Research Article

Basket Credit Default Swap Pricing with Two Defaultable Counterparties

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In this paper, we study the basket CDS pricing with two defaultable counterparties based on the reduced-form model. The default jump intensities of the reference firms and counterparties are all assumed to follow the mean-reverting constant elasticity of variance (CEV) processes. Taking the Vasicek process which is a special case of CEV process as an example, the approximate analytic solutions of the joint survival probability density, the probability densities of the first default and the first two defaults can be solved by using PDE method. In addition, we also extend the Vasciek process to the Vasciek process with cojumps and obtain the approximate closed-form solutions of the relevant default probability densities. Then with the expressions of the probability densities, we can get the formula of the basket CDS price with two defaultable counterparties. In the numerical analysis, we do sensitivity analysis and compare the basket CDS prices under our model with that with only one defaultable counterparty. The numerical results show that our model can be applied into practice.

1. Introduction

Credit derivatives have been widely used by market participants to manage and hedge credit risks. One of the most popular credit derivatives is the credit default swap (CDS). There are mainly two kinds of CDS contracts: single-name CDS and basket CDS. The difference between the two contracts is the number of the reference entities. Since the bankruptcy of the Worldcom Company and the Enron Corporation, people have paid more and more attention to the default probability of large companies. The subprime crisis has made people realize that the correlated default risk plays an important role in the pricing of a basket CDS. With the development of global economic integration, enterprises are more closely related, so the study of pricing the basket CDS has attracted more and more researchers. There are two common models to study the pricing of credit derivatives in the literature, namely the structural models introduced by Black and Scholes [1] and Merton [2], and the reduced-form intensity-based models pioneered by Jarrow and Turnbull [3].

In the structural model, the default of a firm is deemed to be triggered when the firm value falls below the liability level. Black and Cox [4] proposed the first passage model in which the default time was assumed to be the first time that the firm value broke down the constant barrier. Based on Merton [2] model and Black-Cox [4] model, Gökgöz et al. [5] studied the evaluation of a single-name CDS via the discounted cash flow method. Chen and He [6] proposed the multi-scale stochastic volatility (SV) model to price a CDS. Based on the structural model and introducing the concept of fuzziness, Wu et al. [7] proposed a new double exponential jump diffusion model with fuzziness for CDS pricing.

The reduced-form intensity-based model was introduced by Jarrow and Turnbull [3], in which the Poisson process was used to describe the exogenous default occurrence. Lando [8] proposed a Cox process to describe the default intensity and assumed that the risk-free interest rate satisfied the Vasicek model. Malherbe [9] applied a Poisson process to describe the default intensity. He assumed that the default intensities were constant between defaults, but could jump at the times of defaults. Herbertsson and Rootzén [10] used the matrix-analytic method to derive a closed-form expression for a basket CDS. Zheng and Jiang [11] used the total hazard construction method to derive an analytic formula for the joint distribution of default times.

The key of basket CDS pricing is to obtain the relevant default probability of multiple assets. There are mainly three methods in the literature to model the default risk of multiple assets: copula function, conditional independence model and contagion model. Using the copula approach, one can derive a joint distribution of default times by combining the marginal distributions of default times. Li [12] studied the default correlations among companies using CreditMetrics model with copula functions. Crépey and Jeanblanc [13] studied CDS pricing with counterparty risk under a Markov chain copula model. Harb and Louhichi [14] used the mixture copula to price the basket CDS with counterparty risk. In the conditional independence model, Finger [15] assumed that the common macro factor affected the default times of all the assets in the portfolio, while the default intensities were independent with each other. Based on the reduced-form default intensity model, Kijima and Muromachi [16] studied the pricing of basket CDS of the first-default type and obtained an analytic formula for the basket CDS price. Kijima and Muromachi [17] considered k^{th}-to-default basket CDS pricing, but they did not give the explicit solution. White [7] presented a new model for valuing a CDS contract which was affected by multiple credit risks. They showed that the default dependency had a significant impact on CDS pricing. Davis and Lo [18] firstly employed the contagion model to describe the default risk of the stock portfolio. Based on the contagion model, Jarrow and Yu [19] introduced the concept of the counterparty risk and illustrated the effect of the counterparty risk on CDS price. Hao and Wang [20] studied the CDS pricing with the contagious risk under the fractional Vasicek interest rate model. Dong and Wang [21] assumed the intensities of the default times were driven by macro-economy described by a Markov regime-switching model. Gu and Liu [22] established the attenuation model for the contagious risk and derived the pricing formula of CDS in the fractional dimension environment. Huang and Song [23] priced the basket CDS with counterparty risk under a multi-name contagion model.

In this paper, we study the basket CDS pricing with two defaultable counterparties based on the reduced-form model. When the reference asset is a basket of assets and if there are positive correlations among assets, the default probability may be high. In this case, the CDS sellers are willing to sign the CDS contract together to share the default risk. The investors wonder whether more counterparties can reduce the default risk. The main contributions of this paper are as follows. (1) To the best of our knowledge, there is no basket CDS contract with two defaultable counterparties traded in the market yet. However, this kind of contracts can be applied into practice when the time is ripe. At present, it is meaningful to carry out the theoretical research. (2) The default jump intensities of the reference entities and counterparties are all assumed to follow the mean-reverting constant elasticity of variance (CEV) processes. The CEV process is a general process that contains the Vasicek process, CIR process and geometric Brownian motion. Using PDE method, we obtain three PDEs for the joint survival probability density, the probability density of the first default and the probability density of the first two defaults among the reference entities and counterparties. (3) Taking the Vasicek process which is a special case of CEV process as an example, the approximate analytic solutions of the relevant default probability densities can be solved from PDEs. In addition, we also extend the Vasciek process to the Vasciek process with cojumps and obtain the approximate closed-form solutions of the relevant default probability densities. Then with the expressions of the probability densities, we can get the formula of the basket CDS price with two defaultable counterparties. In the numerical analysis, we find that the CDS buyer pay more for the basket CDS contract with two defautable counterparties and there will be almost no price difference if the number of reference assets is large enough. The numerical results show that our model can be applied into practice. It is worthy of note that our model can be extended to the jump model with stochastic volatility. For the introduction of this model, readers can refer to He and Lin [24, 25], He and Chen [26, 27]. Stochastic volatility model can describe the phenomenon of volatility clustering of default intensity, but the derivation of basket CDS price is quite difficult. When the volatilities of the default intensity of the reference assets and two counterparties are stochastic, the number of state variables will increase which makes the calculation more difficult. The solution for the PDEs satisfied by the relevant default probability densities will not necessarily have analytical solutions. If so, the CDS price can be solved by Monte Carlo simulation and other numerical methods.

The article is organized as follows. In Section 2, we assume the default jump intensities of the reference firms and counterparties follow the mean-reverting CEV processes. We obtain three PDEs for the joint survival probability density, the probability density of the first default and the probability density of the first two defaults. In Section 3, we determine the approximate closed-form solutions for relevant default probability densities under the Vasicek processes. In Section 4, we obtain the approximate closed-form solutions for relevant default probability densities under the Vasicek processes with cojumps. In Section 5, we derive the formula of the basket CDS price with two defaultable counterparties. In Section 6, we do sensitivity analysis under our model. We compare the price differences of the basket CDS with two defaultable counterparties and that with only one defaultable counterparty. Finally, we offer concluding remarks in Section 7.

2. Default Probability Density under Reduced-form Intensity Model

The traditional basket CDS contract is usually signed with only one credit protection seller. The disadvantage of this kind of contract is that when the credit protection seller defaults, the credit protection buyer is likely to lose the CDS fee or not be compensated. Therefore, we consider the basket CDS pricing with two credit protection sellers. If the reference assets in the basket do not default, the credit protection buyer will pay the CDS fee to the two credit protection sellers at the same time. If the default of the reference assets occurs, the sellers will compensate the buyer. In addition, in order to make our model more general and closer to the real market, we assume that two credit protection sellers may also default.

Let $T > 0$ be a finite time horizon and fix a probability space $(\Omega, \mathscr{F}, P)$, the probability measure P is the risk-neutral measure. The canonical filtration generated by the underlying stochastic structure is denoted by $\mathscr{F}_t$, which defines the information available at each time. The conditional probability measure given $\mathscr{F}_t$ is denoted by P_t and the associated conditional expectation operator is E_t. Let default time τ be a stopping time associated with the filtration $\mathscr{F}_t$. For sufficiently small $\Delta t \geq 0$, $\lambda(t)$ is an intensity process for τ if satisfied

$$P_t\{\tau \leq t + \Delta t | \tau > t\} = \lambda(t)\Delta t. \quad (1)$$

Suppose $\lambda(t)$ is a $(n+2)$-dimensional Markov process of *càdlàg* state variables drawn from space $V \subset \mathbb{R}^{n+2}$. We assume the reference asset is a basket of bonds $\{F_{B_i}(t), i = 1, 2, \ldots, n.\}$ which are issued by different companies. Each bond may default with the default intensity $\{\lambda_i(t), i = 1, 2, \ldots, n\}$. There are two sellers named F_C and F_D who will both compensate if one of the assets in the basket defaults. We assume the seller F_C has a stochastic default intensity of $\lambda_{n+1}(t)$ and seller F_D has a stochastic default intensity of $\lambda_{n+2}(t)$. All the default intensities $\{\lambda_i(t), i = 1, 2, \ldots, n, n+1, n+2\}$ are assumed to follow the mean-reverting constant elasticity of variance (CEV) processes

$$d\lambda_i(t) = a_i(b_i - \lambda_i(t))dt + \sigma_i(\lambda_i(t))^{\beta} dW_i(t), \quad (2)$$

where $a_i, b_i, \sigma_i, \beta$ are positive constants. Respectively, a_i is the mean-reverting rate. b_i represents the long-term level of jump intensity. σ_i is the volatility of the jump intensity. β can be interpreted as the elasticity. Each $W_i(t)$ is a standard Brownian motion. $dW_i(t)dW_j(t) = \rho_{ij}dt$ for $i \neq j$ and $\rho_{ij} = 1$ for $i = j$. The CEV process in (2) can include some existing processes as special cases. (1) If $\beta = 0$, process $\lambda_i(t)$ is the Vasicek process. (2) If $\beta = 1/2$, process $\lambda_i(t)$ is the CIR process. (3) If $\beta = 1$ and $b_i = 0$, process $\lambda_i(t)$ is the geometric Brownian motion.

Let $\tau_1, \ldots \tau_n$ denote the default times of reference assets $F_{B_i}(i = 1, 2, \ldots n)$. τ_{n+1} and τ_{n+2} represent the default time of counterparty F_C and F_D respectively throughout this article. Given $\mathscr{F}_T$, the default times τ_j are conditionally independent. The initial time and expiration date are represented by t and T $(0 \leq t \leq T)$. Before we price the basket CDS, we need some conclusions about the default probability densities as shown in Theorem 2.1–2.3. Theorem 2.1. (Joint survival probability density under the mean-reverting CEV model) Denote $\lambda = \{\lambda_i, i = 1, 2, \ldots, n+2\}$, if all the reference assets $F_{B_i}(i = 1, 2, \ldots, n)$ and two counterparties do not default until time $s\,(t \leq s \leq T)$, the joint survival probability density $\widehat{P}(t, \lambda; s)$ satisfies the following PDE:

$$\begin{cases} \dfrac{\partial \widehat{P}}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n+2} \rho_{ij}\sigma_i\sigma_j\lambda_i^{\beta}\lambda_j^{\beta}\dfrac{\partial^2 \widehat{P}}{\partial\lambda_i\partial\lambda_j} + \displaystyle\sum_{i=1}^{n+2} a_i(b_i - \lambda_i)\dfrac{\partial \widehat{P}}{\partial \lambda_i} - \displaystyle\sum_{i=1}^{n+2}\lambda_i\widehat{P} = 0, \\ \widehat{P}(s, \lambda; s) = 1. \end{cases} \quad (3)$$

Proof. If no default events happen, the CDS buyer will pay the CDS fee continuously until the expiration date T. The conditional independence means the joint survival probability at time $s\,(t \leq s \leq T)$ can be given by

$$P_T\{\tau_1 > s, \ldots, \tau_{n+2} > s\} = \exp\left\{-\int_t^s \sum_{i=1}^{n+2} \lambda_i(u)du\right\}. \quad (4)$$

Because $\mathscr{F}_t \subset \mathscr{F}_T$, we have $E_t(1_{\{Event\}}) = E_t(E_T(1_{\{Event\}}))$. We denote the probability density $\widehat{P}(t, \lambda; s)$ as

$$\begin{aligned} \widehat{P}(t, \lambda; s) &= P_t\{\tau_1 > s, \ldots, \tau_{n+2} > s\} \\ &= E\left[\exp\left\{-\int_t^s \sum_{i=1}^{n+2}\lambda_i(u)du\right\}|\mathscr{F}_t\right] \\ &= E_t\left[\exp\left\{-\int_t^s \sum_{i=1}^{n+2}\lambda_i(u)du\right\}\right]. \end{aligned} \quad (5)$$

With Feynman-Kac theorem, we can get the PDE (3) that $\widehat{P}(t, \lambda; s)$ satisfies. Theorem 2.2 (The probability density of the first default for the i^{th} company under the mean-reverting CEV model). Among the reference assets $F_{B_i}(i = 1, 2, \ldots, n)$ issued by different companies and two counterparties, if the i^{th} company firstly defaults at time $\tau_i\,(s \leq \tau_i \leq s + ds)$, we denote this default probability density at time s to be $\widehat{q}_i(t, \lambda; s)$. Then the probability density $\widehat{q}_i(t, \lambda; s)$ satisfies the following PDE

$$\begin{cases} \dfrac{\partial \widehat{q}_i}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{j,k=1}^{n+2} \rho_{jk}\sigma_j\sigma_k\lambda_j^{\beta}\lambda_k^{\beta}\dfrac{\partial^2 \widehat{q}_i}{\partial\lambda_j\partial\lambda_k} + \displaystyle\sum_{j=1}^{n+2} a_j\left(b_j - \lambda_j\right)\dfrac{\partial \widehat{q}_i}{\partial \lambda_j} - \displaystyle\sum_{j=1}^{n+2}\lambda_j\widehat{q}_i = 0, \\ \widehat{q}_i(s, \lambda; s) = \lambda_i. \end{cases} \quad (6)$$

□

Proof. Among $n+2$ companies, the probability of the first default for the i^{th} company at time $\tau_i\,(s \le \tau_i \le s+ds)$ can be given by

$$\begin{aligned} &P_T\{\tau_1 > s, \ldots, \tau_{n+2} > s, \tau_i \le s + ds\} \\ &\quad = P_T\{\tau_1 > s, \ldots, \tau_{n+2} > s\}\lambda_i(s)ds \\ &\quad = \exp\left\{-\int_t^s \sum_{j=1}^{n+2} \lambda_j(u)\mathrm{d}u\right\}\lambda_i(s)ds. \end{aligned} \tag{7}$$

Because of $E_t(1_{\{\text{Event}\}}) = E_t(E_T(1_{\{\text{Event}\}}))$, the probability density of the first default for the i^{th} company at time s is

$$\hat{q}_i(t,\lambda;s) = E_t\left[\exp\left\{-\int_t^s \sum_{j=1}^{n+2} \lambda_j(u)\mathrm{d}u\right\}\lambda_i(s)\right]. \tag{8}$$

With Feynman-Kac theorem, we can get the PDE (6) that the probability density $\hat{q}_i(t,\lambda;s)$ satisfies. Theorem 2.3. (The probability density of the first two defaults for the i^{th} and j^{th} companies under the mean-reverting CEV model) Among the reference assets $F_{B_i}\,(i = 1, 2, \ldots, n)$ issued by different companies and two counterparties, if the i^{th} company is the first to default at time $\tau_i\,(t \le \tau_i \le s)$ and the j^{th} company is the second to default at time $\tau_j\,(s \le \tau_j \le s+ds)$ with the constrain that $\tau_i \le T, \tau_i \le \tau_j \le T$, denote $\bar{\lambda} = \{\lambda_j, j = 1, 2, \ldots, i-1, i+1, \ldots, n+2\}$ and then the probability density $\hat{q}_{ij}$ of the default event has a solution form as follows

$$\hat{q}_{ij}(t,\lambda;s) = \hat{Q}(t,\bar{\lambda};s) - \hat{q}_j(t,\lambda;s), \tag{9}$$

where $\hat{Q}(t,\bar{\lambda};s)$ satisfies the following PDE

$$\begin{cases} \dfrac{\partial \hat{Q}}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{k,l\ne i}^{n+2} \rho_{kl}\sigma_k\sigma_l\lambda_k^{\beta}\lambda_l^{\beta}\frac{\partial^2 \hat{Q}}{\partial\lambda_k\partial\lambda_l} + \sum_{k\ne i}^{n+2} a_k(b_k - \lambda_k)\frac{\partial \hat{Q}}{\partial \lambda_k} - \sum_{k\ne i}^{n+2}\lambda_k\hat{Q} = 0, \\ \hat{Q}(s,\bar{\lambda};s) = \lambda_j. \end{cases} \tag{10}$$

□

Proof. Among the $n+2$ defaultable companies (including the n reference assets and two counterparties), the i^{th} company is the first to default at time $\tau_i\,(u \le \tau_i \le u+du)$ and the j^{th} company is the second to default at time $\tau_j\,(s \le \tau_j \le s+ds)$ with the constrain that $\tau_i \le T, \tau_i \le \tau_j \le T$. The cumulative default process is denoted by

$$H_i(s) = \int_t^s \lambda_i(u)\mathrm{d}u, \quad s \ge t. \tag{11}$$

Then the probability of the default event is

$$\begin{aligned} &P_T\{\tau_i \le T, \tau_i \le \tau_j \le T, \tau_k > \tau_j, \text{ for all } k \ne i, j\} \\ &\quad = \int_t^T\int_u^T P_T\{u < \tau_i \le u + \mathrm{d}u,\ s < \tau_j \le s + \mathrm{d}s, \tau_k > s, \text{ for all } k \ne i, j\} \\ &\quad = \int_t^T\int_u^T e^{-H_i(u)}\lambda_i(u)e^{-H_j(s)}\lambda_j(s)\exp\left\{-\sum_{k\ne i,j} H_k(s)\right\}\mathrm{d}s\mathrm{d}u \\ &\quad = \int_t^T \exp\left\{-\sum_{k\ne i} H_k(s)\right\}\lambda_j(s)\int_t^s e^{-H_i(u)}\lambda_i(u)\mathrm{d}u\mathrm{d}s. \end{aligned} \tag{12}$$

Since

$$\int_t^s e^{-H_i(u)\lambda_i(u)\mathrm{d}u} = 1 - P_T\{\tau_i > s\}. \tag{13}$$

So

$$\begin{aligned} &P_T\{\tau_i \le T, \tau_i \le \tau_j \le T, \tau_k > \tau_j, \text{ for all } k \ne i, j\} \\ &\quad = \int_t^T \exp\left\{-\sum_{k\ne i} H_k(s)\right\}\lambda_j(s)\mathrm{d}s - \int_t^T \exp\left\{-\sum_{k=1}^{n+2} H_k(s)\right\}\lambda_j(s)\mathrm{d}s. \end{aligned} \tag{14}$$

According to $E_t(1_{\{\text{Event}\}}) = E_t(E_T(1_{\{\text{Event}\}}))$, the probability density $\hat{q}_{ij}$ is

$$\hat{q}_{ij}(t,\lambda;s) = E_t\left[\exp\left\{-\int_t^s \sum_{k\neq i}^{n+2} \lambda_k(u)\mathrm{d}u\right\}\lambda_j(s)\right] - E_t\left[\exp\left\{-\int_t^s \sum_{k=1}^{n+2} \lambda_k(u)\mathrm{d}u\right\}\lambda_j(s)\right]. \tag{15}$$

Denote

$$\hat{Q}(t,\overline{\lambda};s) = E_t\left[\exp\left\{-\int_t^s \sum_{k\neq i}^{n+2} \lambda_k(u)\mathrm{d}u\}\lambda_j(s)\right]. \tag{16}$$

According to (15), we have

$$\hat{q}_{ij}(t,\lambda;s) = \hat{Q}(t,\overline{\lambda};s) - \hat{q}_j(t,\lambda;s). \tag{17}$$

With Feynman-Kac theorem, we can get the PDE (10) that $\hat{Q}(t,\overline{\lambda};s)$ satisfies. □

3. Basket CDS Pricing with the Vasicek Processes

For PDEs (3) (6) (10), there are no analytical solutions generally. The analytic solutions exist only when β is a special value. In this paper, we aim to take the Vasicek process (i e. $\beta = 0$) as an example to derive the analytic price for the basket CDS. How to get the general solution under CEV process, CIR process and geometric Brownian motion are the problems we need to solve in the future. Although the Vasicek process may take negative values which is economically unacceptable, the probability of such a pathology to arise is very small (please refer to Rogers [28], Hull and White [29]). Consequently, the inconvenience linked to the Vasicek process and its extensions can be neglected facing to the benefit they could bring. Theorem 3.1. (Joint survival probability density under the Vasicek model) Denote $\lambda = \{\lambda_i, i = 1,2,\ldots,n+2\}$, if all the reference assets $F_{B_i}(i = 1,2,\ldots,n)$ and two counterparties do not default until time $s(t\leq s\leq T)$, the joint survival probability density $P(t,\lambda;s)$ has a closed-form solution as follows

$$P(t,\lambda;s) = \exp\left\{A(t;s) - \sum_{i=1}^{n+2} B_i(t;s)\lambda_i(t)\right\}, \tag{18}$$

where

$$B_i(t;s) = \frac{1}{a_i}\left(1 - e^{-a_i(s-t)}\right), \tag{19}$$

$$A(t;s) = \int_t^s \left[\frac{1}{2}\sum_{i,j=1}^{n+2} \rho_{ij}\sigma_i\sigma_j B_i(u;s)B_j(u;s) - \sum_{i=1}^{n+2} a_i b_i B_i(u;s)\right]\mathrm{d}u. \tag{20}$$

Proof. According to Theorem 2.1, $P(t,\lambda;s)$ satisfies the following PDE if λ_i is a Vasicek process (i.e. $\beta = 0$)

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n+2} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 P}{\partial\lambda_i\partial\lambda_j} + \sum_{i=1}^{n+2} a_i(b_i - \lambda_i)\frac{\partial P}{\partial\lambda_i} - \sum_{i=1}^{n+2}\lambda_i P = 0, \\ P(s,\lambda;s) = 1. \end{cases} \tag{21}$$

According to ∅ ksendal(2003), $P(t,\lambda;s)$ has a solution with the following form

$$P(t,\lambda;s) = \exp\left\{A(t;s) - \sum_{i=1}^{n+2} B_i(t;s)\lambda_i(t)\right\}. \tag{22}$$

Substitute the above formula into Equation (21) to get two ODEs

$$\begin{cases} \dfrac{\partial A(t;s)}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n+2} \rho_{ij}\sigma_i\sigma_j B_i(t;s)B_j(t;s) - \sum_{i=1}^{n+2} a_i b_i B_i(t;s) = 0, A(s;s) = 0, \\ \dfrac{\partial B_i(t;s)}{\partial t} - a_i B_i(t;s) + 1 = 0,\ B(s;s) = 0. \end{cases} \tag{23}$$

Solve the above ODEs and obtain (19) and (20). Theorem 3.2. (The probability density of the first default for the i^{th} company under the Vasicek model) Among the reference assets $F_{B_i}(i = 1,2,\ldots,n)$ issued by different companies and two counterparties, if the i^{th} company firstly defaults at time $\tau_i(s\leq\tau_i\leq s+ds)$, we denote this default probability density at time s to be $q_i(t,\lambda;s)$. $q_i(t,\lambda;s)$ has a closed-form solution as follows

$$q_i(t,\lambda;s) = (C_i(t;s)\lambda_i + D_i(t;s))\exp\left\{A(t;s) - \sum_{k=1}^{n+2} B_k(t;s)\lambda_k(t)\right\}, \tag{24}$$

where

$$C_i(t;s) = e^{-a_i(s-t)}, \tag{25}$$

$$D_i(t;s) = b_i\left(1 - e^{-a_i(s-t)}\right) - \sum_{k=1}^{n+2} \frac{\rho_{ik}\sigma_i\sigma_k}{a_k}\left[\frac{1 - e^{-a_i(s-t)}}{a_i} - \frac{1 - e^{-(a_i+a_k)(s-t)}}{a_i + a_k}\right]. \tag{26}$$

$A(t;s)$ and $B_k(t;s)$ are expressed as in (19) and (20). Proof. According to Theorem 2.2, $q_i(t,\lambda;s)$ satisfies the following PDE if λ_i is a Vasicek process (i e. $\beta = 0$)

$$\begin{cases} \dfrac{\partial q_i}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{j,k=1}^{n+2} \rho_{jk}\sigma_j\sigma_k \frac{\partial^2 q_i}{\partial\lambda_j\partial\lambda_k} + \sum_{j=1}^{n+2} a_j\left(b_j - \lambda_j\right)\frac{\partial q_i}{\partial\lambda_j} - \sum_{j=1}^{n+2}\lambda_j q_i = 0, \\ q_i(s,\lambda;s) = \lambda_i. \end{cases} \tag{27}$$

According to ∅ ksendal(2003), the probability density has a solution with the following form

$$q_i(t,\lambda;s) = \left(C_i(t;s)\lambda_i + D_i(t;s)\right) \exp\left\{A(t;s) - \sum_{k=1}^{n+2} B_k(t;s)\lambda_k(t)\right\}. \tag{28}$$

$A(t;s)$ and $B_k(t;s)$ have been solved by (23). Substitute the above formula into (27) to get two ODEs as follows

$$\begin{cases} \dfrac{\partial D_i(t;s)}{\partial t} - \displaystyle\sum_{k=1}^{n+2} \rho_{ik}\sigma_i\sigma_k C_i(t;s)B_k(t;s) + a_i b_i C_i(t;s) = 0, D_i(s;s) = 0, \\ \dfrac{\partial C_i(t;s)}{\partial t} - a_i C_i(t;s) = 0, C_i(s;s) = 1. \end{cases} \tag{29}$$

Solve the above ODEs and obtain (25) and (26). Theorem 3.3. (The probability density of the first two defaults for the i^{th} and j^{th} companies) Among the reference assets $F_{B_i}(i = 1,2,\ldots,n)$ issued by different companies and two counterparties, if the i^{th} asset is the first to default at time $\tau_i\,(t \le \tau_i \le s)$ and the j^{th} asset is the second to default at time $\tau_j\,(s \le \tau_j \le s + ds)$ with the constrain that $\tau_i \le T, \tau_i \le \tau_j \le T$, denote $\overline{\lambda} = \left\{\lambda_j, j = 1,2,\ldots,i-1,i+1,\ldots n+2\right\}$ and then the probability density q_{ij} of the default event has a closed-form solution as follows

$$q_{ij}(t,\lambda;s) = Q(t,\overline{\lambda};s) - q_j(t,\lambda;s). \tag{30}$$

$Q(t,\overline{\lambda};s)$ has a form like

$$Q(t,\overline{\lambda};s) = \left(C_j(t;s)\lambda_j + \overline{D}_j(t;s)\right) \exp\left\{\overline{A}(t;s) - \sum_{k\ne i}^{n+2} B_k(t;s)\lambda_k(t)\right\}, \tag{31}$$

where

$$C_j(t;s) = e^{-a_j(s-t)}, \tag{32}$$

$$\overline{D}_j(t;s) = b_j\left(1 - e^{-a_j(s-t)}\right) - \sum_{k\ne i,k=1}^{n+2} \frac{\rho_{jk}\sigma_j\sigma_k}{a_k}\left[\frac{1 - e^{-a_j(s-t)}}{a_j} - \frac{1 - e^{-(a_j+a_k)(s-t)}}{a_j + a_k}\right], \tag{33}$$

$$B_k(t;s) = \frac{1}{a_k}\left(1 - e^{-a_k(s-t)}\right), \tag{34}$$

$$\overline{A}(t;s) = \int_t^s \left[\frac{1}{2}\sum_{j,k\neq i}^{n+2} \rho_{jk}\sigma_j\sigma_k B_j(u;s)B_k(u;s) - \sum_{k\neq i,k=1}^{n+2} a_k b_k B_k(u;s)\right]du. \tag{35}$$

□

Proof. According to Theorem 2.3, $Q(t,\overline{\lambda};s)$ satisfies the following PDE if λ_i is a Vasicek process (i.e. $\beta = 0$)

$$\begin{cases} \dfrac{\partial Q}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{k,l\neq i}^{n+2} \rho_{kl}\sigma_k\sigma_l \dfrac{\partial^2 Q}{\partial\lambda_k\partial\lambda_l} + \sum_{k\neq i}^{n+2} a_k(b_k-\lambda_k)\dfrac{\partial Q}{\partial\lambda_k} - \sum_{k\neq i}^{n+2}\lambda_k Q = 0, \\ Q(s,\overline{\lambda};s) = \lambda_j. \end{cases} \tag{36}$$

According to ∅ ksendal(2003), $Q(t,\overline{\lambda};s)$ has a solution with the following form

$$Q(t,\overline{\lambda};s) = \left(C_j(t;s)\lambda_j + \overline{D}_j(t;s)\right) \exp\left\{\overline{A}(t;s) - \sum_{k\neq i}^{n+2} B_k(t;s)\lambda_k(t)\right\}, \tag{37}$$

where $\overline{A}(t;s)$ and $B_k(t;s)$ satisfy the following ODEs which are similar to (23).

$$\begin{cases} \dfrac{\partial\overline{A}(t;s)}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{j,k\neq i}^{n+2} \rho_{jk}\sigma_j\sigma_k B_j(t;s)B_k(t;s) - \sum_{k\neq i}^{n+2} a_k b_k B_k(t;s) = 0,\ \overline{A}(s;s) = 0, \\ \dfrac{\partial B_k(t;s)}{\partial t} - a_k B_k(t;s) + 1 = 0,\ B(s;s) = 0. \end{cases} \tag{38}$$

Solve the above ODEs and obtain (34) and (35). And then, substitute the formulas (37) (34) (35) into equation (36) to get two ODEs

$$\begin{cases} \dfrac{\partial\overline{D}_j(t;s)}{\partial t} - \displaystyle\sum_{k\neq i}^{n+2} \rho_{jk}\sigma_j\sigma_k C_j(t;s)B_k(t;s) + a_j b_j C_j(t;s) = 0,\ \overline{D}_j(s;s) = 0, \\ \dfrac{\partial C_j(t;s)}{\partial t} - a_j C_j(t;s) = 0,\ C_j(s;s) = 1. \end{cases} \tag{39}$$

Solve the above ODEs and obtain (32) and (33). □

4. Basket CDS Pricing with Cojumps

In this section, we will extend the Vasicek processes to the jump processes. We assume there exist simultaneous jumps called cojumps among all the companies when an extreme event occurs. All the default intensities $\{\lambda_i(t), i = 1,2,\ldots,n,n+1,n+2\}$ are assumed to follow the Vasicek processes with cojumps

$$d\lambda_i(t) = a_i(b_i - \lambda_i(t))dt + \sigma_i dW_i(t) + \varepsilon_i dN(t), \tag{40}$$

where $N(t)$ is a Poisson counter with constant intensity λ_J. $\text{prob}(dN(t) = 1) = \lambda_J dt$ and $\text{prob}(dN(t) = 0) = 1 - \lambda_J dt$. ε_i is the percentage jump size (conditional on a jump occurring). Before we price the basket CDS, we will give some results as shown in Theorem 4.1–4.3 about the default probability densities under the jump models (40). Theorem 4.1. (Joint survival probability density) If all the reference assets $F_{B_i}(i = 1,2,\ldots,n)$ and two counterparties do not default until time $s\,(t \leq s \leq T)$, the joint survival probability density $P'(t,\lambda;s)$ will have a closed-form solution as follows:

$$P'(t,\lambda;s) = \exp\left\{A'(t;s) - \sum_{i=1}^{n+2} B_i(t;s)\lambda_i(t)\right\}, \tag{41}$$

where,

$$B_i(t;s)=\frac{1}{a_i}\left(1-e^{-a_i(s-t)}\right), \tag{42}$$

$$A'(t;s)=\int_t^s\left[\frac{1}{2}\sum_{i,j=1}^{n+2}\rho_{ij}\sigma_i\sigma_j B_i(u;s)B_j(u;s)-\sum_{i=1}^{n+2}\left(a_ib_i+\lambda_J\varepsilon_i\right)B_i(u;s)\right]\mathrm{d}u. \tag{43}$$

Proof. Similar with Theorem 3.1, according to Feynman-Kac theorem, the joint survival probability $P'(t,\lambda;s)$ at time $s\,(t\le s\le T)$ satisfies the following PDE

$$\begin{cases}\dfrac{\partial P'}{\partial t}+\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n+2}\rho_{ij}\sigma_i\sigma_j\dfrac{\partial^2P'}{\partial\lambda_i\partial\lambda_j}+\sum_{i=1}^{n+2}a_i(b_i-\lambda_i)\dfrac{\partial P'}{\partial\lambda_i}-\sum_{i=1}^{n+2}\lambda_iP'+\lambda_JE\left[P'(t,\lambda+\varepsilon;s)-P'(t,\lambda;s)\right]=0,\\ P'(s,\lambda;s)=1.\end{cases} \tag{44}$$

According to ∅ ksendal(2003), $P'(t,\lambda;s)$ has a solution with the following form

$$P'(t,\lambda;s)=\exp\left\{A'(t;s)-\sum_{i=1}^{n+2}B_i(t;s)\lambda_i(t)\right\}. \tag{45}$$

Substitute the above formula into (44) to obtain

$$\frac{\partial A'(t;s)}{\partial t}-\sum_{i=1}^{n+2}\frac{\partial B_i(t;s)}{\partial t}\lambda_i+\frac{1}{2}\sum_{i,j=1}^{n+2}\rho_{ij}\sigma_i\sigma_jB_i(t;s)B_j(t;s)-\sum_{i=1}^{n+2}a_i(b_i-\lambda_i)B_i(t;s)-\sum_{i=1}^{n+2}\lambda_i+\lambda_JE\left[e^{-\sum_{i=1}^{n+2}B_i(t;s)\varepsilon_i}-1\right]=0. \tag{46}$$

With the approximate formula

$$E\left[e^{-\sum_{i=1}^{n+2}B_i(t;s)\varepsilon_i}-1\right]=-\sum_{i=1}^{n+2}B_i(t;s)\varepsilon_i. \tag{47}$$

We substitute (47) into (46) to obtain two ODEs

$$\begin{cases}\dfrac{\partial A'(t;s)}{\partial t}+\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n+2}\rho_{ij}\sigma_i\sigma_jB_i(t;s)B_j(t;s)-\sum_{i=1}^{n+2}\left(a_ib_i+\lambda_J\varepsilon_i\right)B_i(t;s)=0,\ A'(s;s)=0,\\ \dfrac{\partial B_i(t;s)}{\partial t}-a_iB_i(t;s)+1=0, B(s;s)=0.\end{cases} \tag{48}$$

Solve the above ODEs and obtain (42) and (43). Theorem 4.2. (The probability density of the first default for the i^{th} company) Among the reference assets $F_{B_i}(i=1,2,\ldots,n)$ issued by different companies and two counterparties, if the i^{th} asset firstly defaults at time $\tau_i\,(s\le\tau_i\le s+ds)$, the default probability density $q_i'(t,\lambda;s)$ at time s has a closed-form solution as follows

$$q_i'(t,\lambda;s)=\left(C_i(t;s)\lambda_i+D_i'(t;s)\right)\exp\left\{A'(t;s)-\sum_{k=1}^{n+2}B_k(t;s)\lambda_k(t)\right\}, \tag{49}$$

where

$$C_i(t;s)=e^{-a_i(s-t)}. \tag{50}$$

And

$$D_i'(t;s)=\left(b_i+\frac{\lambda_J\varepsilon_i}{a_i}\right)\left(1-e^{-a_i(s-t)}\right)-\sum_{k=1}^{n+2}\frac{\rho_{ik}\sigma_i\sigma_k+\lambda_J\varepsilon_i\varepsilon_k}{a_k}\left[\frac{1-e^{-a_i(s-t)}}{a_i}-\frac{1-e^{-(a_i+a_k)(s-t)}}{a_i+a_k}\right]. \tag{51}$$

$A'(t;s)$ and $B_k(t;s)$ are expressed as in (42) and (43). □

Proof. Similar with Theorem 3.2, according to Feynman-Kac theorem, the probability density $q_i'(t,\lambda;s)$ satisfies the following PDE

$$\begin{cases} \dfrac{\partial q_i'}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{j,k=1}^{n+2} \rho_{jk}\sigma_j\sigma_k \frac{\partial^2 q_i'}{\partial\lambda_j\partial\lambda_k} + \sum_{j=1}^{n+2} a_j\left(b_j-\lambda_j\right)\frac{\partial q_i'}{\partial\lambda_j} - \sum_{j=1}^{n+2}\lambda_j q_i' + \lambda_J E\left[q_i'(t,\lambda+\varepsilon;s) - q_i'(t,\lambda;s)\right] = 0, \\ q_i'(s,\lambda;s) = \lambda_i. \end{cases} \quad (52)$$

According to ∅ ksendal(2003), $q_i'(t,\lambda;s)$ has a solution with the following form

$$q_i'(t,\lambda;s) = \left(C_i(t;s)\lambda_i + D_i'(t;s)\right) \exp\left\{A'(t;s) - \sum_{k=1}^{n+2} B_k(t;s)\lambda_k(t)\right\}. \quad (53)$$

Substitute the above formula into (52), we have

$$\begin{aligned} &\frac{\partial C_i(t;s)}{\partial t}\lambda_i + \frac{\partial D_i'(t;s)}{\partial t} + \left(C_i(t;s)\lambda_i + D_i'(t;s)\right)\left\{\frac{\partial A'(t;s)}{\partial t} - \sum_{i=1}^{n+2}\frac{\partial B_i(t;s)}{\partial t}\lambda_i + \frac{1}{2}\sum_{i,j=1}^{n+2}\rho_{ij}\sigma_i\sigma_j B_i(t;s)B_j(t;s)\right. \\ &\left. - \sum_{i=1}^{n+2} a_i(b_i-\lambda_i)B_i(t;s) - \sum_{i=1}^{n+2}\lambda_i + \lambda_J E\left[e^{-\sum_{i=1}^{n+2}B_i(t;s)\varepsilon_i} - 1\right]\right\} + a_i(b_i-\lambda_i)C_i(t;s) - \sum_{k=1}^{n+2}\rho_{ik}\sigma_i\sigma_k C_i(t;s)B_k(t;s) \\ &+ \lambda_J\varepsilon_i C_i(t;s)E\left[e^{-\sum_{i=1}^{n+2}B_i(t;s)\varepsilon_i}\right] = 0. \end{aligned} \quad (54)$$

With (42) and (43), we can obtain two ODEs

$$\begin{cases} \dfrac{\partial D_i'(t;s)}{\partial t} - \displaystyle\sum_{k=1}^{n+2}\rho_{ik}\sigma_i\sigma_k C_i(t;s)B_k(t;s) - \lambda_J\varepsilon_i C_i(t;s)\sum_{k=1}^{n+2}\varepsilon_k B_k(t;s) + \left(a_i b_i + \lambda_J\varepsilon_i\right)C_i(t;s) = 0, D_i'(s;s) = 0; \\ \dfrac{\partial C_i(t;s)}{\partial t} - a_i C_i(t;s) = 0, C_i(s;s) = 1. \end{cases} \quad (55)$$

Solve the above ODEs and obtain (50) and (51). Theorem 4.3 (the probability density of the first two defaults for the i^{th} and j^{th} companies). Among the reference assets $F_{B_i}(i=1,2,\ldots,n)$ issued by different companies and two counterparties, if the i^{th} asset is the first to default at time $\tau_i\,(t\le\tau_i\le s)$ and the j^{th} asset is the second to default at time $\tau_j\,(s\le\tau_j\le s+ds)$ with the constrain that $\tau_i\le T, \tau_i\le\tau_j\le T$, then the probability density q_{ij}' of the default event has a closed-form solution

$$q_{ij}'(t,\lambda;s) = Q'(t,\overline{\lambda};s) - q_j'(t,\lambda;s). \quad (56)$$

$Q'(t,\overline{\lambda};s)$ has a form like

$$Q'(t,\overline{\lambda};s) = \left(C_j(t;s)\lambda_j + \overline{D}_j'(t;s)\right) \exp\left\{\overline{A}'(t;s) - \sum_{k\ne i}^{n+2} B_k(t;s)\lambda_k(t)\right\}, \quad (57)$$

where

$$C_j(t;s) = e^{-a_j(s-t)}, \tag{58}$$

$$\overline{D}_j'(t;s) = \left(b_j + \frac{\lambda_J \varepsilon_j}{a_j}\right)\left(1 - e^{-a_j(s-t)}\right) - \sum_{k\neq i,k=1}^{n+2} \frac{\rho_{jk}\sigma_j\sigma_k + \lambda_J\varepsilon_j\varepsilon_k}{a_k}\left[\frac{1-e^{-a_j(s-t)}}{a_j} - \frac{1-e^{-\left(a_j+a_k\right)(s-t)}}{a_j+a_k}\right], \tag{59}$$

$$B_k(t;s) = \frac{1}{a_k}\left(1 - e^{-a_k(s-t)}\right), \tag{60}$$

$$\overline{A'}(t;s) = \int_t^s \left[\frac{1}{2}\sum_{j,k\neq i}^{n+2} \rho_{jk}\sigma_j\sigma_k B_j(u;s)B_k(u;s) - \sum_{k\neq i,k=1}^{n+2}\left(a_k b_k + \lambda_J\varepsilon_k\right)B_k(u;s)\right]\mathrm{d}u. \tag{61}$$

Proof. Similar with Theorem 3.3, the probability density $q_{ij}'(t,\lambda;s)$ has a form as follows

$$q_{ij}'(t,\lambda;s) = Q'(t,\overline{\lambda};s) - q_j'(t,\lambda;s), \tag{62}$$

where

$$Q'(t,\overline{\lambda};s) = E_t\left[\exp\left\{-\int_t^s \sum_{k\neq i}^{n+2}\lambda_k(u)\mathrm{d}u\right\}\lambda_j(s)\right]. \tag{63}$$

Denote $\overline{\varepsilon} = \left\{\varepsilon_j, j = 1,2,\ldots,i-1,i+1,\ldots,n+2\right\}$ and with Feynman-Kac theorem, $Q'(t,\overline{\lambda};s)$ satisfies the following PDE

$$\begin{cases} \dfrac{\partial Q'}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{k,l\neq i}^{n+2}\rho_{kl}\sigma_k\sigma_l\dfrac{\partial^2 Q'}{\partial\lambda_k\partial\lambda_l} + \sum_{k\neq i}^{n+2} a_k\left(b_k - \lambda_k\right)\dfrac{\partial Q'}{\partial\lambda_k} - \sum_{k\neq i}^{n+2}\lambda_k Q' + \lambda_J E\left[Q'(t,\overline{\lambda}+\overline{\varepsilon};s) - Q'(t,\overline{\lambda};s)\right] = 0, \\ Q'(s,\overline{\lambda};s) = \lambda_j. \end{cases} \tag{64}$$

According to ∅ ksendal(2003), $Q'(t,\overline{\lambda};s)$ has a solution with the following form

$$Q'(t,\overline{\lambda};s) = \left(C_j(t;s)\lambda_j + \overline{D}_j'(t;s)\right) \exp\left\{\overline{A}'(t;s) - \sum_{k\neq i}^{n+2} B_k(t;s)\lambda_k(t)\right\}. \tag{65}$$

By solving similar equations like those in (48), we have $\overline{A}'(t;s)$ and $B_k(t;s)$ expressed in (60) and (61). And then, substitute the formulas (65) (60) (61) into equation (64) to get two ODEs

$$\begin{cases} \dfrac{\partial\overline{D}_j'(t;s)}{\partial t} - \displaystyle\sum_{k\neq i}^{n+2}\rho_{jk}\sigma_j\sigma_k C_j(t;s)B_k(t;s) - \lambda_J\varepsilon_j C_j(t;s)\sum_{k\neq i}^{n+2}\varepsilon_k B_k(t;s) + \left(a_j b_j + \lambda_J\varepsilon_j\right)C_j(t;s) = 0, \overline{D}_j'(s;s) = 0; \\ \dfrac{\partial C_j(t;s)}{\partial t} - a_j C_j(t;s) = 0, C_j(s;s) = 1. \end{cases} \tag{66}$$

Solve the above ODEs and obtain (58) and (59). □

5. Basket CDS Price

In this section, we will discuss the pricing of the basket CDS with two defaultable counterparties under the Vasicek model and the jump model in (40) respectively. Firstly, we consider the basket CDS price under the Vasicek model using Theorems 3.1–3.3 as an example. We assume the credit protection buyer F_A holds a basket of n reference assets $F_{B_i}(i = 1,2,\ldots,n)$. At initial time t, F_A buys a basket CDS contact with the credit protection sellers F_C and F_D. The maturity of the basket CDS is T. During time t to T, the CDS buyer F_A will pay the CDS fee continuously to the CDS sellers F_C and F_D until T if no defaults happen. Once one of the assets $F_{B_i}(i = 1,2,\ldots n)$ defaults first, the CDS sellers F_C and F_D will both compensate to the CDS buyer F_A. However, if the CDS seller F_C or F_D defaults first, the CDS buyer F_A will stop paying premiums to F_C or F_D and not receive any compensations from F_C or F_D. Note that when one counterparty defaults, the CDS buyer will still pay the premium to the other counterparty that does not default. The default events include three types of defaults, namely, the default of reference asset F_{B_i}, the default of counterparty F_C and the default of counterparty F_D. That is, we need to consider the default order of F_{B_i}, F_C and F_D.

Now we analysis all the possible default events once the basket CDS contract becomes effective from time t :

Situation 1. For any $i\,(i = 1, 2, \ldots n)$, reference asset F_{B_i} is the first to default at time τ_i.

Situation 2. Counterparty F_C defaults firstly at time τ_{n+1},

(1) For any $i\,(i = 1, 2, \ldots n)$, reference asset F_{B_i} is the second to default at time τ_i

(2) Counterparty F_D is the second to default at time τ_{n+2}

(3) All the reference assets $F_{B_i}\,(i = 1, 2 \ldots n)$ and counterparty F_D do not default

Situation 3. Counterparty F_D defaults firstly at time τ_{n+2},

(1) For any $i\,(i = 1, 2, \ldots n)$, reference asset F_{B_i} is the second to default at time τ_i

(2) Counterparty F_C is the second to default at time τ_{n+1}

(3) All the reference assets $F_{B_i}\,(i = 1, 2 \ldots n)$ and counterparty F_C do not default

Situation 4. No defaults happen until the maturity T.

Next we will discuss how to compute the basket CDS price that the credit protection buyer F_A pay to counterparty F_C. We assume the CDS costs are continuously paid by F_A and denote the cost rate to be W_1. Firstly, we need to analyze the present value at time t of the basket CDS costs received by counterparty F_C under different situations.

Situation 1. According to Theorem 3.2, the default probability density under this situation is $q_i(t, \lambda; s)$.

The present value of the CDS costs received by counterparty F_C is $W_1 \int_t^T e^{-r(s-t)} q_i(t, \lambda; s) ds$.

Situation 2. Correspondingly, according to the notation stated in Theorem 3.3,

(1) For any $i\,(i = 1, 2, \ldots n)$, the probability density of the default event is $q_{n+1,i}(t, \lambda; s)$

(2) The probability density of the default event is $q_{n+1,n+2}(t, \lambda; s)$

(3) The probability density of the default event is $q_{n+1}(t, \lambda; s) - \sum_{i=1}^{n} q_{n+1,i}(t, \lambda; s) - q_{n+1,n+2}(t, \lambda; s)$

The present value of the total CDS costs received by counterparty F_C is $W_1 \int_t^T e^{-r(s-t)} q_{n+1}(t, \lambda; s) ds$.

Situation 3. Correspondingly,

(1) For any $i\,(i = 1, 2, \ldots n)$, the probability density of the default event is $q_{n+2,i}(t, \lambda; s)$

(2) The probability density of the default event is $q_{n+2,n+1}(t, \lambda; s)$

(3) The probability density of the default event is $q_{n+2}(t, \lambda; s) - \sum_{i=1}^{n} q_{n+2,i}(t, \lambda; s) - q_{n+2,n+1}(t, \lambda; s)$

The present value of the total CDS costs received by counterparty F_C is $W_1 \int_t^T e^{-r(s-t)} q_{n+2}(t, \lambda; s) ds$.

Situation 4. According to Theorem 3.1, the joint survival probability density is $P(t, \lambda; s)$.

The present value of the CDS costs received by counterparty F_C is $W_1 \int_t^T e^{-r(s-t)} P(t, \lambda; s) ds$.

Then we will analyze the present values of compensations paid by counterparty F_C under different situations. Denote R to be the recovery rate and L_i to be the face value of the reference assets $F_{B_i}\,(i = 1, 2 \ldots n)$. We assume that the counterparty F_C and counterparty F_D prefer to share default risk. Under 1, counterparty F_C will compensate $(1/2)L_i(1 - R) \int_t^T e^{-r(s-t)} q_i(t, \lambda; s) ds$ to the basket CDS buyer F_A. Under 2, counterparty F_C is the first to default, so there will be no compensations paid by counterparty F_C. Under 3, only when the counterparty F_D defaults firstly and F_{B_i} defaults secondly, counterparty F_C will compensate. The present values of the compensations paid by counterparty F_C is $(1/2)L_i(1 - R) \int_t^T e^{-r(s-t)} q_{n+2,i}(t, \lambda; s) ds$. Otherwise, there will be no compensations paid by counterparty F_C. Under Situation 4, there will be no compensations paid by counterparty F_C if no defaults happen from time t to T.

Finally, according to the no-arbitrage pricing principal, the present value of the total CDS costs received by counterparty F_C should be equal to the present value of the total compensations paid by counterparty F_C. Thus we have

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^{n} \int_t^T L_i (1 - R_i) e^{-r(s-t)} q_i(t, \lambda; s) ds + \frac{1}{2} \sum_{i=1}^{n} \int_t^T L_i (1 - R_i) e^{-r(s-t)} q_{n+2,i}(t, \lambda; s) \\ &\quad = W_1 \sum_{i=1}^{n+2} \int_t^T e^{-r(s-t)} q_i(t, \lambda; s) ds + W_1 \int_t^T e^{-r(s-t)} P(t, \lambda; s) ds ds. \end{aligned} \tag{67}$$

So the CDS price W_1 the buyer F_A paid to counterparty F_C is

$$W_1 = \frac{1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + 1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_{n+2,i}(t,\lambda;s)\mathrm{d}s}{\sum_{i=1}^{n+2}\int_t^T e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + \int_t^T e^{-r(s-t)}P(t,\lambda;s)\mathrm{d}s}. \quad (68)$$

Similarly, the CDS price W_2 paid to counterparty F_D should satisfy the following equation

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + \frac{1}{2}\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_{n+1,i}(t,\lambda;s)\mathrm{d}s\\ &\quad = W_2\sum_{i=1}^{n+2}\int_t^T e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + W_2\int_t^T e^{-r(s-t)}P(t,\lambda;s)\mathrm{d}s.\end{aligned} \quad (69)$$

So

$$W_2 = \frac{1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + 1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_{n+1,i}(t,\lambda;s)\mathrm{d}s}{\sum_{i=1}^{n+2}\int_t^T e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + \int_t^T e^{-r(s-t)}P(t,\lambda;s)\mathrm{d}s}. \quad (70)$$

When there exist cojumps, using the similar method and the results in Theorems 4.1–4.3, the CDS price the buyer F_A paid to counterparty F_C and F_D can be given respectively:

$$W_1' = \frac{1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_i'(t,\lambda;s)\mathrm{d}s + 1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_{n+2,i}'(t,\lambda;s)\mathrm{d}s}{\sum_{i=1}^{n+2}\int_t^T e^{-r(s-t)}q_i'(t,\lambda;s)\mathrm{d}s + \int_t^T e^{-r(s-t)}P'(t,\lambda;s)\mathrm{d}s}, \quad (71)$$

$$W_2' = \frac{1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_i'(t,\lambda;s)\mathrm{d}s + 1/2\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_{n+1,i}'(t,\lambda;s)\mathrm{d}s}{\sum_{i=1}^{n+2}\int_t^T e^{-r(s-t)}q_i'(t,\lambda;s)\mathrm{d}s + \int_t^T e^{-r(s-t)}P'(t,\lambda;s)\mathrm{d}s}. \quad (72)$$

With the closed-form solution, the sensitivity of CDS price to the initial default intensity can be measured by partial derivative. Denote $f(t,\lambda;s)$, $g(t,\lambda;s)$ as follows,

$$f(t,\lambda;s) = \frac{1}{2}\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + \frac{1}{2}\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}q_{n+2,i}(t,\lambda;s)\mathrm{d}s, \quad (73)$$

$$g(t,\lambda;s) = \sum_{i=1}^{n+2}\int_t^T e^{-r(s-t)}q_i(t,\lambda;s)\mathrm{d}s + \int_t^T e^{-r(s-t)}P(t,\lambda;s)\mathrm{d}s. \quad (74)$$

And we can have analytical partial derivatives

$$\frac{\partial W_1}{\partial \lambda_j} = \frac{\left(\partial f(t,\lambda;s)/\partial \lambda_j\right) g(t,\lambda;s) - f(t,\lambda;s)\left(\partial g(t,\lambda;s)/\partial \lambda_j\right)}{g(t,\lambda;s)^2}, \quad (j = 1,2,\ldots,n+2), \tag{75}$$

where

$$\begin{aligned}\frac{\partial f(t,\lambda;s)}{\partial \lambda_j} &= \frac{1}{2}\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}\frac{\partial q_i(t,\lambda;s)}{\partial \lambda_j}ds \\ &+ \frac{1}{2}\sum_{i=1}^{n}\int_t^T L_i(1-R_i)e^{-r(s-t)}\frac{\partial q_{n+2,i}(t,\lambda;s)}{\partial \lambda_j}ds.\end{aligned} \tag{76}$$

And

$$\begin{aligned}\frac{\partial g(t,\lambda;s)}{\partial \lambda_j} &= \sum_{i=1}^{n+2}\int_t^T e^{-r(s-t)}\frac{\partial q_i(t,\lambda;s)}{\partial \lambda_j}ds \\ &+ \int_t^T e^{-r(s-t)}\frac{\partial P(t,\lambda;s)}{\partial \lambda_j}ds.\end{aligned} \tag{77}$$

Other derivatives $(\partial W_2/\partial \lambda_j)$, $(\partial W_1'/\partial \lambda_j)$, $(\partial W_2'/\partial \lambda_j)$ can be obtained in a similar way.

6. Numerical Analysis

In this section, we will do some numerical analysis to show the impacts of main parameters on CDS prices. In order to verify the correctness of our formulas for CDS prices, we do Monte Carlo simulations. Due to the similar structures of CDS prices, we compute W_1' under the model in Section 4 using formula (71) as an example. In order to obtain W_1', we need to know the relevant probability densities $P'(t,\lambda;s), q_i'(t,\lambda;s), q_{n+2}'(t,\lambda;s)$. Without the approximate closed-form solutions for $P'(t,\lambda;s), q_i'(t,\lambda;s), q_{n+2}'(t,\lambda;s)$, they can also be numerically solved, which can serve as the benchmark solution to the original problem. Thus we use the solution for CDS price calculated by Monte Carlo simulation as the benchmark solution. For $i = 1,2,\ldots,n,n+1,n+2$, we use the following formulas in a discrete time form to simulate the paths of the default intensities for reference assets and counterparties.

$$\begin{aligned}\lambda_i(t+\Delta t) &= \lambda_i(t) + a_i(b_i - \lambda_i(t))\Delta t \\ &+ \sigma_i \xi_i \sqrt{\Delta t} \\ &+ \sum_{j=1}^{N(t)} \varepsilon_i^j,\end{aligned} \tag{78}$$

where ξ_i follows the multivariate standard normal distribution with an $(n+2)\times(n+2)$ correlation matrix. $N(t)$ is a Poisson processes with constant intensities λ_J. We assume the jump sizes ε_i^j are all constants for simplicity. By simulating the default intensities, we can calculate the relevant probability densities, so as to obtain the price of CDS. When the jump intensity λ_J is equal to zero, the simulation results can be used to calculate the CDS price of formula (68) under the continuous model in Section 3. In the procedure of the Monte Carlo simulations, we set the number of time step to be 100, and the number of simulations to be 100000. In Table 1, we show the relative percentage differences are all less than 1%. Therefore, formula (71) is effective to compute CDS prices. Formula (68) can be also proved to be effective using the same method (the results here are omitted).

Our paper mainly considers two kinds of models, that is, the Vasicek model and the Vasicek model with cojumps (hereinafter referred to as VCJ model). Next, we will use the numerical solution to do some numerical analysis under the two kinds of models. From Figures 1–4, the curve "-" represents the CDS price under the Vasieck model and the curve "- -" represents the CDS price under the VCJ model. Some of the parameters values we used in the following numerical analysis refer to Wang and Liang [30], Leung and Kwok [31]. The basic parameters values are $T = 5, r = 0.05$ and $a_i = 0.5, b_i = 0.1, \sigma_i = 0.05$ and $L_i = 1$ for $i = 1,2,\ldots,n,n+1,n+2$.

Figure 1 shows the impact of number of assets in the basket on CDS price under the Vasicek model and VCJ model. We set $\rho_{ij} = 0.3\,(i \neq j), R = 0.4, \lambda_i(t) = 0.1$ for all $i = 1,2,\ldots,n+2$. Generally the larger the number of assets in the basket is, the default event is more likely to occur. If the default probability increases, the CDS buyer should pay more for the basket CDS contract against lager default risk. Compared with the Vasicek model, the CDS price under the VCJ model is higher. This is because cojumps make the risk of default higher.

Figure 2 shows the impact of the initial default intensities on CDS price under the Vasicek model and VCJ model. We set $\rho_{ij} = 0.3\,(i \neq j), n = 10, R = 0.4$. Higher initial default intensities of the companies lead to higher default risks, it is seen from Figure 2 that the basket CDS prices increase with the initial default intensities. The basket CDS price under the VCJ model is higher than that in the Vasicek model. We can also find that the curves in Figure 2 become flatter with larger initial default intensity. Intuitively, if the initial default intensity is too large, the CDS buyers will be less willing to pay more for the CDS contract. Therefore, with the increase of the initial jump intensity, the basket CDS prices under the VCJ model and Vasicek model tend to be consistent.

Figure 3 shows the impact of the recovery rate on CDS price under the Vasicek model and VCJ model. In Figure 3, we set $\rho_{ij} = 0.3\,(i \neq j), n = 10, \lambda_i(t) = 0.1$ for all $i = 1,2,\ldots,n+2$. Recovery rate R reflects the extent of loss. The greater the recovery rate is, the smaller the loss will be. As seen from Figure 3, the basket CDS price decreases when the recovery rate increases. Although the basket CDS price under the VCJ model is higher than that under the Vasicek model, the CDS prices under the two models become close with the increase of recovery rate.

Figure 4 shows the impact of the correlation coefficient among the reference assets and counterparties on CDS price under the Vasicek model and VCJ model. We set

Table 1: Compare the CDS prices derived from formula (71) and that from Monte Carlo method. Parameters values are $L = 1, n = 2, R = 0.4, r = 0.05, T = 5, a_i = 0.5, b_i = 0.1, \sigma_i = 0.05, \rho_{ij} = 0.3\,(i \neq j), \lambda_i(t) = 0.1, \varepsilon_i = 0.01$.

λ_J	Derived from formula (71)	Monte Carlo	% Difference
0.00	0.049747969035060	0.049759355891914	0.0229
0.01	0.049790128812098	0.049804011930421	0.0279
0.02	0.049832250853993	0.049848113228926	0.0318
0.03	0.049874335198439	0.049893900136012	0.0392
0.04	0.049916381883091	0.049934619943680	0.0365
0.05	0.049958390945565	0.049946025125867	0.0247
0.06	0.050000362423441	0.049982809103013	0.0351
0.07	0.050042296354261	0.050028570979433	0.0274
0.08	0.050084192775529	0.050049815970814	0.0686
0.09	0.050126051724711	0.050109782978705	0.0325
0.10	0.050167873239235	0.050191298437472	0.0467

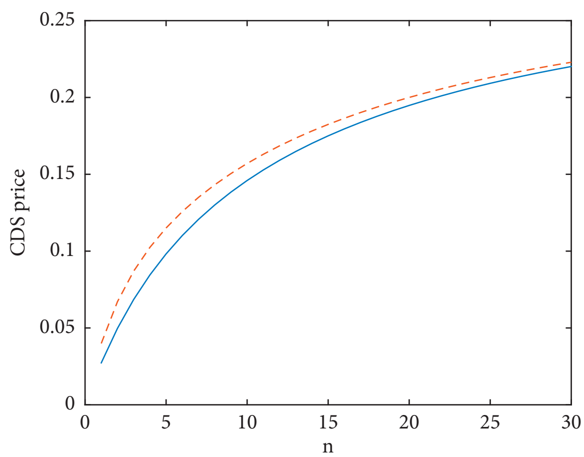

Figure 1: The CDS prices under different number of assets.

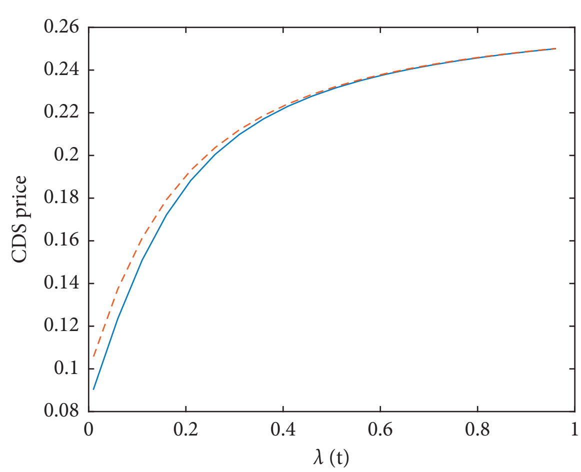

Figure 2: The CDS prices under different initial jump intensities.

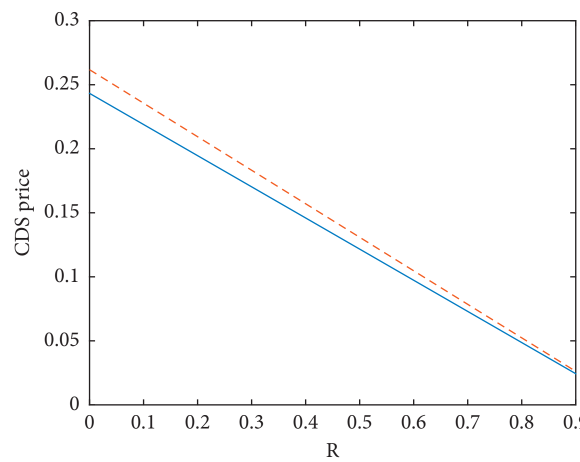

Figure 3: The CDS prices under different recovery rates.

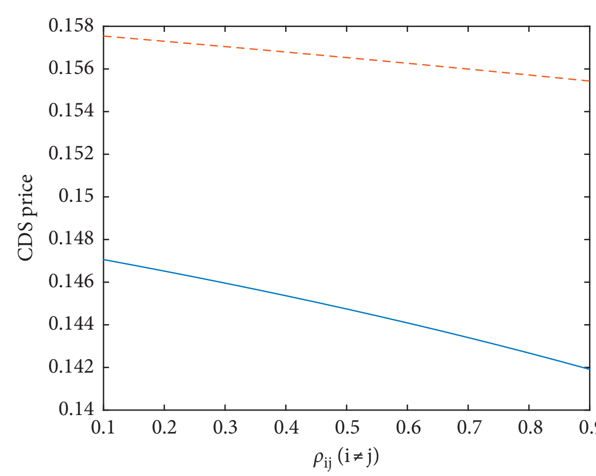

Figure 4: The CDS prices under different correlations.

$R = 0.4, n = 10, \lambda_i(t) = 0.1$ for all $i = 1, 2, \ldots, n+2$. As seen from Figure 4, the basket CDS price with cojumps is higher and the CDS price under both models decreases when the correlation coefficient increases. Larger correlation coefficient means that if one of the companies defaults, the default probabilities of the counterparties will be quite larger. If both of the two counterparties default, the basket CDS buyer will lose a lot and get no compensations, thus the CDS contract will become worthless.

Figure 5 and Figure 6 study the sensitivity of CDS price to initial default intensity under the Vasicek model and VCJ model respectively. For example, the partial derivative $(\partial W_1/\partial \lambda_j)$ means the change speed of W_1 with λ_j when other parameters are fixed. In Figure 5, the curve "-"

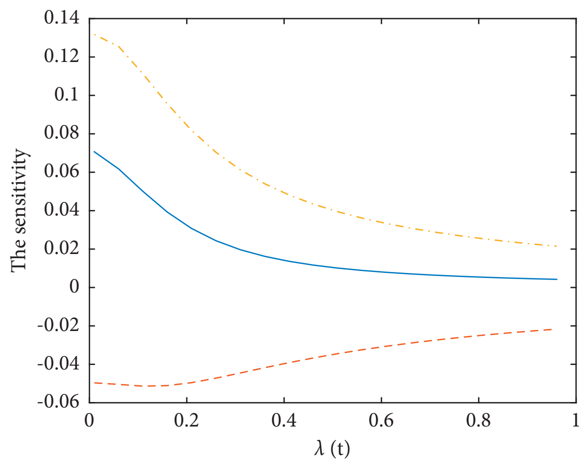

Figure 5: The sensitivity of jump intensity on CDS prices under the Vasicek model.

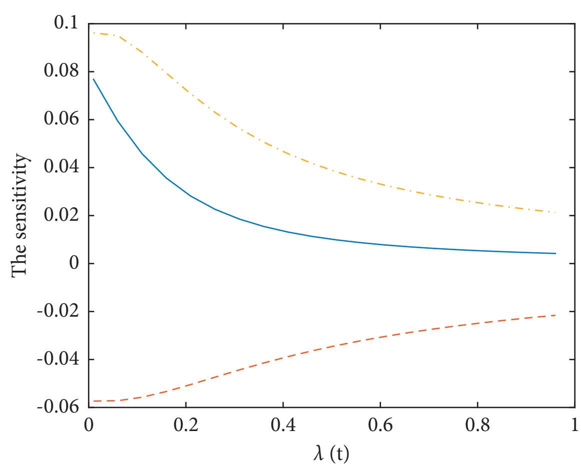

Figure 6: The sensitivity of jump intensity on CDS prices under the Vasicek model with cojumps.

represents the partial derivative of W_1 about $\lambda_j\,(j = 1, 2, \ldots, n)$. The curve "- -" represents the partial derivative of W_1 about λ_{n+1}. The curve "-." represents the partial derivative of W_1 about λ_{n+2}. W_1 is the premium paid by the credit protection buyer to the counterparty F_C. When the initial default intensity of the reference asset and the counterparty F_D become larger, the higher compensation the CDS buyer can obtain from the counterparty F_C, so the partial derivative $(\partial W_1/\partial \lambda_j)\,(j = 1, 2, \ldots, n)$ and $(\partial W_1/\partial \lambda_{n+2})$ are always positive. On the contrary, when the initial default intensity of counterparty F_C increases, the lower compensation that CDS buyer can obtain from the counterparty F_C, so the partial derivative $(\partial W_1/\partial \lambda_{n+1})$ is always negative. It is seen from Figure 5 that the sensitivity of CDS price to the initial default intensity of counterparty F_D is greater than that of the reference asset. This may be because the default of counterparty F_D will increase the fear of credit protection buyer about the default of counterparty F_C, so the CDS price is more sensitive to the initial default intensity of counterparty F_D. The absolute value of the sensitivity of CDS price to initial default intensity is decreasing and tends to be stable with the increase of initial default intensity. The reason is that when the default risk is larger, the willingness of CDS buyer to pay excess premium for the default risk is decreasing. In Figure 6, the curve "-" represents the partial derivative of W_1' about $\lambda_j\,(j = 1, 2, \ldots, n)$. The curve "- -" represents the partial derivative of W_1' about λ_{n+1}. The curve "-." represents the partial derivative of W_1' about λ_{n+2}. The sensitivity of CDS price to initial default intensity under the VCJ model is similar to that in Figure 5. Compared with the Vasicek model, it is worth noting that the sensitivity of CDS price to the initial default intensity of counterparty F_D under the VCJ model decreases slightly. The default risk of counterparty F_D may lead to a higher default risk of counterparty F_C under the VCJ model. Therefore, this concern will reduce the willingness of CDS buyer to pay premiums to counterparty F_C.

From the above sensitivity analysis, we can see that the existence of cojump does have impacts on CDS prices. And then, we investigate the impacts of cojumps on the basket CDS prices. We show the basket CDS prices under different λ_J in Figure 7 and 8. Without loss of generality, we set ε_i to be constants for all $i \in \{1, 2, \ldots, n+2\}$. From Figure 7, we see the basket CDS price increases when λ_J increases if ε_i are positive constants. Positive constants ε_i mean the default intensities jump upward, which can be caused by bad news. A larger λ_J means the bad news is more likely to happen, thus the default probability and the basket CDS price will both increase. On the contrary, negative constants ε_i mean the default intensities jump downward, which can be caused by good news. A larger λ_J means the good news is more likely to happen, thus the default probability and the basket CDS price will decrease.

Figure 9 shows the basket CDS prices under different ε_i when λ_J is fixed. From the previous analysis in Figures 7 and 8, we can find that the basket CDS prices increase when ε_i increase. It can be seen that the curve in Figure 9 becomes flatter with larger ε_i. Intuitively, larger ε_i means larger default intensity, thus the CDS buyer will be less willing to pay more for the basket CDS contract.

It is generally assumed that there is only one defaultable counterparty in the traditional basket CDS pricing model. We investigate the basket CDS price differences with two defaultable counterparties and with only one defaultable counterparty as shown in Figure 10. The curve "-" represents the result under the Vasieck model and the curve "- -" represents the result under the VCJ model. Take the Vasieck model as an example, when there are two defaultable counterparties, we assume the CDS buyer F_A spends a total cost of $W'\,(W' = W_1 + W_2)$ on the basket CDS contract. If there is only one defaultable counterparty under the Vasieck model, the CDS price is assumed to be W. While all the other model assumptions remain unchanged, we compute the price difference $W' - W$ under different number of assets in the basket. In Figure 10, we set $\rho_{ij} = 0.3\,(i \neq j)$, $R = 0.4, \lambda_i(t) = 0.1$ for all $i = 1, 2, \ldots, n+2$. The basket CDS buyer will need to pay more for the basket CDS contract when there are two defautable counterparties. Especially when the number of reference assets in the basket is small

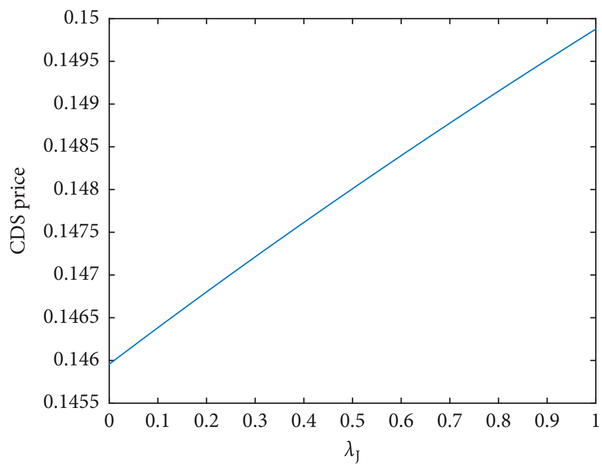

Figure 7: The CDS prices under different λ_J if $\varepsilon_i = 0.01$.

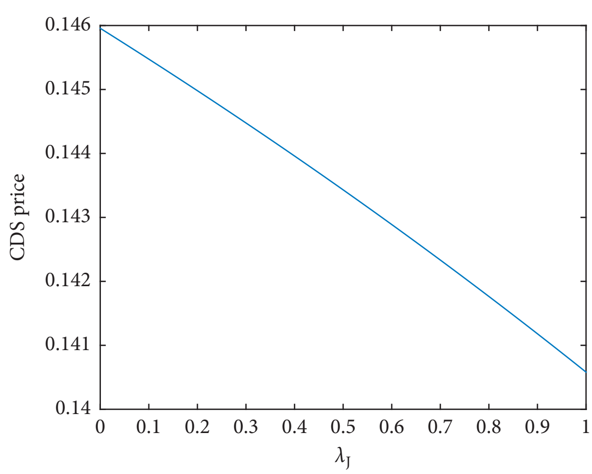

Figure 8: The CDS prices under different λ_J if $\varepsilon_i = -0.01$.

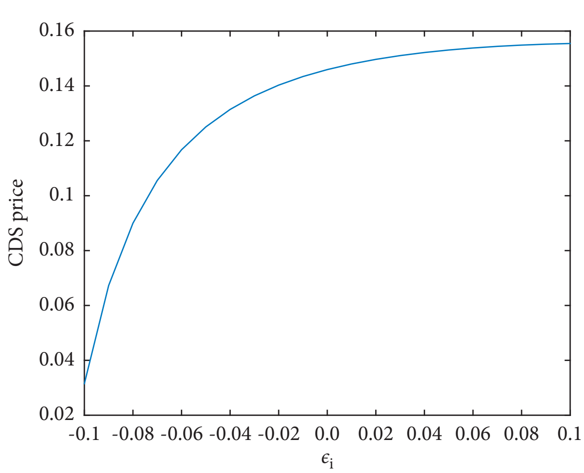

Figure 9: The CDS prices under different λ_J if $\varepsilon_i = 0.01$.

(for example, less than 5), the price difference increases with the increase of the number of reference assets. This may be because the default risk is overestimated when considering two counterparties if the number of reference assets in the basket is small. However, when the number of assets in the reference assets is larger (for example, lager than 5), the probability that at least one asset in the basket defaults will increase. Higher default risk makes the CDS buyer do not want to pay more for the basket CDS contract. Thus no matter how many counterparties there are, there will be almost no price difference if the number of reference assets is large enough. For the VCJ model, we assume the CDS buyer F_A spends a total cost of W'' ($W'' = W_1' + W_2'$) on the basket CDS contract. The price difference of premium is $W'' - W$. Compared with the Vasieck model, there are similar conclusions for the VCJ model. Because the default risk under the VCJ model is higher than that under the Vasieck model, the price difference under the VCJ model is larger. From the above analysis, we find that the basket CDS pricing with two defaultable counterparties has the advantage to explain some empirically phenomenons.

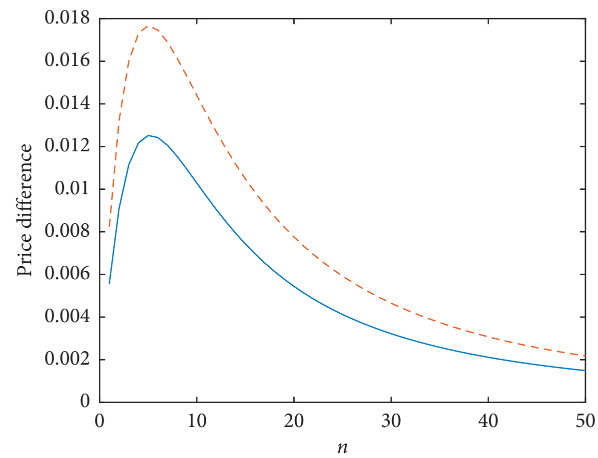

Figure 10: The price differences of basket CDS with two defaultable counterparties and that with only one defaultable counterparty.

7. Conclusion

In this paper, we mainly study the pricing of the basket CDS with two defaultable counterparties based on the Vasicek processes and the Vasicek processes with cojumps. Using the PDE method, we obtain the approximate closed-form formula of the basket CDS price. In the numerical analysis, we analyze the impacts of main parameters such as the number of reference assets, initial jump intensity, recovery rate and correlations among assets, jump size and jump intensity of cojumps on the basket CDS prices. Comparing the basket CDS prices with two defaultable counterparties and that with only one defaultable counterparty, we find that the price difference will be obvious if the number of assets in the basket is not too large. However, there will be almost no price difference if the number of assets is large enough. The numerical results help us better understand the basket CDS pricing with multiple defaultable counterparties. Moreover, we investigate the basket CDS pricing under the reduced-form model with cojumps, which help us better understand the impact of jump risk on the basket CDS prices.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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