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Research Article

Optimal Asset Allocation for CRRA and CARA Insurers under the Vasicek Interest Rate Model

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This paper considers the reinsurance-investment problem with interest rate risks under constant relative risk aversion and constant absolute risk aversion preferences, respectively. Stochastic control theory and dynamic programming principle are applied to investigate the optimal proportional reinsurance-investment strategy for an insurer under the Vasicek stochastic interest rate model. Solving the corresponding Hamilton-Jacobi-Bellman equation via the Legendre transform approach, the optimal premium allocation strategies maximizing the expected utilities of terminal wealth are derived. In addition, several sensitivity analyses and numerical illustrations are given to analyze the impacts of different risk preferences and interest rate fluctuation on the optimal strategies. We find that the asset allocation and reinsurance ratio of the insurer are correlated with risk preference coefficient and interest rate fluctuation, and the insurance company may adjust the reinsurance-investment strategy to deal with interest rate risk.

1. Introduction

As a financial institution, insurance company plays an important role in the modern society, and its reinsurance and investment business is also the focus of the management because reinsurance and investment are effective at dispersing risks and making profits from surplus. Many literature studies have discussed the reinsurance and investment problem from different perspectives, and it is common to convert it into a problem of stochastic optimal control. In the last decades, stochastic control theory has been widely used in risk research. For instance, Browne [1] obtained the optimal investment strategy under the diffusion model through the Hamilton–Jacobi–Bellman (HJB) equation, creating a precedent of combining risk theory with stochastic control theory. Since then, there have been many papers in which the HJB equation was used to solve optimal control problems in insurance. According to the research content, different objective and constraint functions have been studied, such as minimizing ruin probability (Schmidli [2], David Promislow and Young [3], and Bai et al. [4]), maximizing adjustment coefficient (Hald and Schmidli [5] and Liang and Guo [6]), and maximizing expected utility (Irgens and Paulsen [7], Bai and Guo [8], Xu et al. [9], Cao and Wan [10], and Liang et al. [11]). In addition, mean-variance optimization also gets a lot of attention (Bi and Guo [12], Zeng and Li [13], and Wang et al. [14]).

In this paper, the objective of the insurer is to maximize the expected utility of terminal wealth in the finite horizon. We suppose that the insurer purchases a proportional reinsurance and is allowed to invest in the financial market. The problem is that the insurer intends to find the optimal strategy to balance the risk and profit. Considering the fact that interest rate is uncertain in the real-world environments, the optimal strategy under stochastic interest rate is more practical. There have been many studies on stochastic interest rate in dynamic portfolio problems; see Li and Wu [15], Noh and Kim [16], Chang [17], Wang and Li [18], and so on. For the reinsurance-investment problem, Liang et al. [19] used an Ornstein-Uhlenbeck process to describe the instantaneous rate of investment return under CRRA utility maximization, and inflation risks are further considered in Guan and Liang [20]. Li et al. [21] obtained the optimal time-consistent reinsurance-investment strategy under the mean-variance criterion. Compared with previous studies, the first contribution of this paper is that we consider the stochastic interest rate in the reinsurance-investment problem, and the stochastic interest rate model and surplus process are different from Guan and Liang [20]. Second, we investigate the optimal reinsurance-investment strategy under two different risk preferences, which may provide the insurer with a more suitable investment strategy. Stochastic dynamic programming is a classical method to solve optimal problems, but the nonlinear partial differential equation generated in it is not easy to solve. Therefore, on the basis of the stochastic control theory, we also use Legendre transformation to obtain the explicit expression of the optimal strategy. For more references on the Legendre transform technique, Jonsson and Sircar [22], Xiao et al. [23], Chang [24], and Hu et al. [25] can be seen. Finally, we analyze the effects of market parameters on the optimal trading strategies.

The rest of this paper is organized as follows. Section 2 formulates the reinsurance-investment problem with the Vasicek stochastic interest rate. Section 3 derives the explicit expressions of the optimal reinsurance-investment strategies under CRRA and CARA utilities. Section 4 provides several sensitive analyses of market parameters. Section 5 gives conclusions.

2. The Model

In this section, we formulate a continuous-time reinsurance-investment model where the insurers can trade in the financial market or the insurance market with no taxes or fees. The framework consists of four parts: the surplus process, the financial market, the wealth process, and the optimization criterion. Let $(\Omega, F, \{F_t\}_{0\le t\le T}, P)$ be a complete probability space with filtration $\{F_t, 0\le t\le T\}$, where $T>0$ is the time horizon and P is the probability. All stochastic processes in this paper are supposed to be well defined in this probability space.

2.1. Surplus Process. Typically, three types of models are used in the insurance market: the Cramer–Lundberg model, approximating diffusion model, and jump-diffusion model. We adopt the diffusion model to describe the surplus for the insurers. The claim process C is described as

$$\mathrm{d}C(t) = a\mathrm{d}t - b\mathrm{d}W_1(t), \tag{1}$$

where a and b are positive constants and $W_1(t)$ is a one-dimensional standard Brownian motion. According to the expected value premium principle, the pure premium rate of the insurer is $c=(1+\theta)a$ with safety loading $\theta>0$, and the reinsurance premium is paid at the constant rate $c_1=(1+\eta)a$ with safety loading $\eta>\theta>0$. Suppose that the insurer purchases the proportional reinsurance to transfer the underlying risk. For each $t\in[0,T]$, the value of risk exposure is denoted by $q(t)\in[0,+\infty)$ representing the retention level of reinsurance. When $q(t)\in(0,1]$, it corresponds to a proportional reinsurance cover. Let Y denote the total claim and $R\colon R^+\longrightarrow R^+$ denote the reinsurance function. Then, $R(Y)=(1-q(t))Y$, where $1-q(t)$ represents the proportion reinsured. The dynamics for the surplus process $R(t)$ associated with reinsurance strategy $q(t)$ is given by

$$\begin{aligned}\mathrm{d}R(t) &= c\mathrm{d}t - q(t)\mathrm{d}C(t) - (1-q(t))c_1\mathrm{d}t\\ &= [\theta-(1-q(t))\eta]a\mathrm{d}t + bq(t)\mathrm{d}W_1(t).\end{aligned} \tag{2}$$

2.2. Financial Market. In addition to the reinsurance, we assume that the insurer is allowed to invest its surplus in a financial market consisting of a risk-free asset (i.e., bond) and a risky asset (i.e., stock). The stochastic interest rate $r(t)$ follows the Vasicek model (see [26]).

$$\mathrm{d}r(t) = (\alpha-\lambda r(t))\mathrm{d}t + \beta\mathrm{d}W_0(t),\quad r(0)=r_0, \tag{3}$$

where the coefficients $\alpha,\lambda,$ and β are positive real constants and $W_0(t)$ is a standard Brownian motion which is independent of $W_1(t)$.

Let $S_0(t)$ denote the price process of the bond, which evolves according to

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\mathrm{d}t,\quad S_0(0)=s_0, \tag{4}$$

where $r(t)$ satisfies equation (3).

Let $S(t)$ denote the price process of the risky asset, which follows

$$\frac{\mathrm{d}S(t)}{S(t)} = (r(t)+u(t))\mathrm{d}t + \sigma\mathrm{d}W(t),\quad S(0)=s, \tag{5}$$

where $u(t)>0$ is a positive real-valued function, the constant $\sigma>0$ denotes the volatility rate of the risk asset, and $W(t)$ is another standard Brownian motion, which is independent with $W_1(t)$, and $W(t), W_0(t)$ satisfy $E[\mathrm{d}W_0(t)W(t)]=\rho\mathrm{d}t$, where $\rho\in[-1,1]$ is the correlation coefficient.

2.3. Wealth Process. Let $X(t)$ represent the wealth of the insurer at time t with initial value $X(0)=x_0$ and $\pi(t)$ be the amount of the wealth invested in the risky assets; then, the remainder $X(t)-\pi(t)$ is invested in the risk-free assets at time t. Since the insurer is allowed to buy reinsurance and invest in the financial market, the trading strategy is a pair of dynamic process which is denoted by $\Pi\colon=(q(t),\pi(t))$, where $q(t)$ represents the reinsurance strategy and $\pi(t)$ denotes the investment strategy. Adopting the reinsurance-investment strategy Π, the corresponding reserve $X(t)$ of the insurer is described by

$$\begin{aligned}\mathrm{d}X(t) &= \mathrm{d}R(t) + \pi(t)\frac{\mathrm{d}S(t)}{S(t)} + (X(t)-\pi(t))\frac{\mathrm{d}S_0(t)}{S_0(t)}\\ &= [r(t)X(t)+u(t)\pi(t)+\theta a-(1-q(t))\eta a]\mathrm{d}t\\ &\quad + \sigma\pi(t)\mathrm{d}W(t)\\ &\quad + bq(t)\mathrm{d}W_1(t).\end{aligned} \tag{6}$$

2.4. Optimization Criterion. We focus on maximizing the utility of the insurer's terminal wealth

$$\max_{\pi,q} E[U(X(T))], \tag{7}$$

where the utility function $U(\cdot)$ is typically increasing and concave with constraints (3) and (6). For an admissible strategy Π, the value function $H_\Pi(t,r,x)$ from state (r,x) at time t is defined by

$$H_\Pi(t,r,x) = E[U(X(T))|X(t) = x, r(t) = r], \tag{8}$$

and the objective function is

$$H(t,r,x) = \sup_{\pi,q} E[U(X(T))|X(t) = x, r(t) = r], \tag{9}$$

with boundary condition $H(T,r,x) = U(x)$. The insurer aims to find a pair of strategy $(q^*(t), \pi^*(t))$ such that $H^*_\Pi(t,r,x) = H(t,r,x)$, where $q^*(t)$ is called the optimal reinsurance strategy and $\pi^*(t)$ is called the optimal investment strategy.

3. Optimal Reinsurance-Investment Strategy

To solve optimal problem (7), we apply the dynamic programming approach described in Fleming and Soner [27]. Because of the value function $H(t,r,x)$, its partial derivatives $H_t, H_r, H_x, H_{rr}, H_{xx}$, and H_{xr} are continuous on $R^1_+ \times R^1$, and then $H(t,r,x)$ satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{aligned} &H_t + (\alpha - \lambda r(t))H_r + [r(t)x + (\theta - \eta)a]H_x + \frac{1}{2}\beta^2 H_{rr} \\ &+ \sup_{\pi(t)>0}\left\{u(t)\pi(t)H_x + \frac{1}{2}\sigma^2\pi^2(t)H_{xx} + \rho\beta\sigma\pi(t)H_{rx}\right\} \\ &+ \sup_{0\le q(t)\le 1}\left\{\eta a q(t)H_x + \frac{1}{2}b^2q^2(t)H_{xx}\right\} = 0, \end{aligned} \tag{10}$$

for $(t,x) \in [0,T] \times R$ with boundary condition $H(T,r,x) = U(x)$, where $H = H(t,r,x)$.

Differentiating equation (10) with respect to $q(t)$ and $\pi(t)$ and setting their derivatives equal to zero, we have

$$\begin{aligned} u(t)H_x + \sigma^2\pi(t)H_{xx} + \rho\beta\sigma H_{rx} &= 0, \\ \eta a H_x + b^2 q(t)H_{xx} &= 0. \end{aligned} \tag{11}$$

Using the first-order maximizing conditions for $(q(t), \pi(t))$ yields

$$\pi^*(t) = -\frac{u(t)}{\sigma^2}\frac{H_x}{H_{xx}} - \frac{\rho\beta}{\sigma}\frac{H_{rx}}{H_{xx}}, \tag{12}$$

$$q^*(t) = -\frac{\eta a}{b^2}\frac{H_x}{H_{xx}}. \tag{13}$$

Note that $q(t) > 0$. If $q(t) \le 1$, then $q^*(t)$ coincides with equation (13). If $q(t) > 1$, then we can let $q(t) = 1$ which means that the proportion of reinsurance is zero. We only consider the case $q(t) \le 1$.

Substituting equations (12) and (13) into the left side of equation (10), we obtain

$$\begin{aligned} &H_t + (\alpha - \lambda r(t))H_r + [r(t)x + (\theta - \eta)a]H_x + \frac{1}{2}\beta^2 H_{rr} \\ &- \frac{1}{2}\frac{\eta^2 a^2}{b^2}\frac{H_x^2}{H_{xx}} \\ &- \frac{\rho\beta u(t)}{\sigma}\frac{H_x H_{rx}}{H_{xx}} - \frac{1}{2}\beta^2\rho^2\frac{H_{rx}^2}{H_{xx}} - \frac{1}{2}\frac{u^2(t)}{\sigma^2}\frac{H_x^2}{H_{xx}} = 0. \end{aligned} \tag{14}$$

Now, the above stochastic control problem has been transformed into solving a partial differential equation for the value function $H(t,r,x)$. In the next step, we shall find the solution to equation (14) with boundary condition $H(T,r,x) = U(x)$.

Definition 1 (see [23]). Let $H\colon R \longrightarrow R$ be a convex function. For $z > 0$, define the Legendre transform

$$L(z) = \sup_{x>0}\{H(x) - zx\}. \tag{15}$$

The function $L(z)$ is called the Legendre dual of function $H(x)$.

Following the works of Xiao et al. [23], we define a Legendre transform

$$\widehat{H}(t,r,z) = \sup_{x>0}\{H(t,r,x) - zx\}, \tag{16}$$

$$g(t,r,z) = \inf_{x>0}\left\{x | H(t,r,x) \ge zx + \widehat{H}(t,r,z)\right\}, \tag{17}$$

where $z > 0$ denotes the dual variable to x. The function $\widehat{H}(t,r,z)$ is related to $g(t,r,z)$,

$$g(t,r,z) = -\widehat{H}_z(t,r,z). \tag{18}$$

Noting that $H(T,r,x) = U(x)$ at terminal time T, we have

$$\begin{aligned} \widehat{H}(T,r,z) &= \sup_{x>0}\{U(x) - zx\}, \\ g(T,r,z) &= \inf_{x>0}\left\{x | U(x) \ge zx + \widehat{H}(T,r,z)\right\}, \end{aligned} \tag{19}$$

from which we have

$$g(T,r,z) = (U')^{-1}(z). \tag{20}$$

Equation (20) implies that $g(T,r,z)$ is the inverse of marginal utility. From equation (16), we have $H_x(t,r,x) = z$, and

$$\begin{aligned} g(t,r,z) &= x, \\ \widehat{H}(t,r,z) &= H(t,r,g) - zg. \end{aligned} \tag{21}$$

Referring to Jonsson and Sircar [22], we have the following transformation rules:

$$\begin{aligned} H_t &= \widehat{H}_t, \\ H_r &= \widehat{H}_r, \\ H_{xx} &= -\frac{1}{\widehat{H}_{zz}}, \\ H_{rx} &= -\frac{\widehat{H}_{rz}}{\widehat{H}_{zz}}, \\ H_{rr} &= \widehat{H}_{rr} - \frac{\widehat{H}_{rz}^2}{\widehat{H}_{zz}}, \end{aligned} \tag{22}$$

where $\widehat{H} = \widehat{H}(t, r, z)$.

Letting $\rho^2 = 1$ and putting (22) into equation (14), we have

$$\begin{aligned} &\widehat{H}_t + (\alpha - \lambda r(t))\widehat{H}_r + [r(t)x + (\theta - \eta)a]z \\ &+ \frac{1}{2}\left[\frac{\eta^2 a^2}{b^2} + \frac{u^2(t)}{\sigma^2}\right] z^2 \widehat{H}_{zz} \\ &- \frac{\rho\beta u(t)}{\sigma} z\widehat{H}_{rz} + \frac{1}{2}\beta^2 \widehat{H}_{rr} = 0. \end{aligned} \tag{23}$$

Differentiating equation (23) with respect to z gives the following equation:

$$\begin{aligned} &g_t + (\alpha - \lambda r(t))g_r - r(t)g - r(t)zg_z - (\theta - \eta)a + K_1(t) \\ &\left(z^2 g_{zz} + 2zg_z\right) \\ &- \frac{\rho\beta u(t)}{\sigma}\left(g_r + zg_{rz}\right) + \frac{1}{2}\beta^2 g_{rr} = 0, \end{aligned} \tag{24}$$

where $K_1(t) = (1/2)[(\eta^2 a^2/b^2) + (u^2(t)/\sigma^2)]$, and the boundary condition $g(T, r, z) = (U')^{-1}(z)$.

Note that we have transformed the nonlinear partial differential equation (14) into a linear second-order partial differential equation (24). In the following sections, we provide the explicit solutions for equation (14) under CRRA and CARA utilities by the variable change method.

3.1. Power Utility. Assume that the insurer takes the power utility function (CRRA)

$$U(x) = \frac{x^p}{p}, \quad p < 1. \tag{25}$$

According to equation (20), we have

$$g(T, r, z) = z^{(1/p-1)}. \tag{26}$$

We conjecture a solution to equation (26) with the form

$$g(t, r, z) = f(t, r)z^{(1/p-1)} + h(t, r), \tag{27}$$

where $f(T, r)$ and $h(t, r)$ are suitable functions such that equation (27) is a solution of equation (24), and $f(T, r) = 1$ and $h(T, r) = 0$. The derivatives of $g(t, r, z)$ with respect to the variables t, r, and z are

$$\begin{aligned} g_t &= f_t(t, r)z^{(1/p-1)} + h_t(t, r), \\ g_r &= f_r(t, r)z^{(1/p-1)} + h_r(t, r), \\ g_z &= \frac{1}{p-1} f(t, r)z^{(2-p/p-1)}, \\ g_{rr} &= f_{rr}(t, r)z^{(1/p-1)} + h_{rr}(t, r), \\ g_{rz} &= \frac{1}{p-1} f_r(t, r)z^{(2-p/p-1)}, \\ g_{zz} &= \frac{2-p}{(p-1)^2} f(t, r)z^{(3-2p/p-1)}. \end{aligned} \tag{28}$$

Putting the above derivatives back into equation (24) leads to an equation of $f(t, r)$ and $h(t, r)$,

$$\begin{aligned} &z^{(1/p-1)}\left\{ f_t(t, r) + (\alpha - \lambda r(t)) f_r(t, r) - r(t) f(t, r) - \frac{\rho\beta u(t)}{\sigma}\left[f_r(t, r) + \frac{1}{p-1} f_r(t, r)\right] \right. \\ &+ \frac{1}{2}\beta^2 f_{rr}(t, r) - \frac{r(t)}{p-1} f(t, r) + K_1(t)\left[\frac{2-p}{(p-1)^2} f(t, r) + \frac{2}{p-1} f(t, r)\right] + h_t(t, r) \\ &+ (\alpha - \lambda r(t))h_r(t, r) + \frac{1}{2}\beta^2 h_{rr}(t, r) - \frac{\rho\beta u(t)}{\sigma} h_r(t, r) - r(t)h(t, r) + (\eta - \theta)a = 0. \end{aligned} \tag{29}$$

To solve equation (27), we decompose it into the following two equations:

$$f_t(t,r) + \left[(\alpha - \lambda r(t)) - \frac{\rho\beta u(t)}{\sigma}\frac{p}{p-1} \right] f_r(t,r) + \left[\frac{pK_1(t)}{(p-1)^2} - \frac{pr(t)}{p-1} \right] f(t,r) + \frac{1}{2}\beta^2 f_{rr}(t,r) = 0, \tag{30}$$

with boundary condition $f(T,r) = 1$, and

$$h_t(t,r) + \frac{1}{2}\beta^2 h_{rr}(t,r) + \left[(\alpha - \lambda r(t)) - \frac{\rho\beta u(t)}{\sigma} \right] h_r(t,r) - r(t)h(t,r) + (\eta - \theta)a = 0, \tag{31}$$

with boundary condition $h(T,r) = 0$.

Lemma 1. *If a solution of equation (27) is in the form*

$$f(t,r) = A(t)e^{B(t)r}, \tag{32}$$

with the boundary conditions $A(T) = 1$ and $B(T) = 0$, then $A(t)$ and $B(t)$ are given by

$$A(t) = \exp\left\{ \int_t^T \left[\frac{1}{2}\beta^2 B^2(s) + \left(\alpha - \frac{\rho\beta u(s)}{\sigma}\frac{p}{p-1} \right) B(s) + \frac{pK_1(s)}{(p-1)^2} \right] \mathrm{d}s \right\}, \tag{33}$$

$$B(t) = \frac{p}{\lambda(p-1)}\left[e^{-\lambda(T-t)} - 1 \right]. \tag{34}$$

Proof. Plugging solution (32) into equation (30), we obtain

$$e^{rB(t)}\left\{ \begin{array}{c} A'(t) + rA(t)B'(t) + \frac{1}{2}\beta^2 A(t)B^2(t) + \left[\alpha - \frac{\rho\beta u(t)}{\sigma}\frac{p}{p-1} \right] A(t)B(t) \\ + \frac{pK_1(t)}{(p-1)^2} A(t) - \lambda r A(t)B(t) - \frac{p}{p-1} r A(t) \end{array} \right\} = 0, \tag{35}$$

where $A'(t)$ and $B'(t)$ denote the derivatives with respect to t. In order to eliminate the dependence on r, we decompose equation (35) into the following two equations:

$$rA(t)\left(B'(t) - \lambda B(t) - \frac{p}{p-1} \right) = 0, \tag{36}$$

$$\frac{A'(t)}{A(t)} + \frac{1}{2}\beta^2 B^2(t) + \left(\alpha - \frac{\rho\beta u(t)}{\sigma}\frac{p}{p-1} \right) B(t) + \frac{pK_1(t)}{(p-1)^2} = 0. \tag{37}$$

Solving the ordinary differential equation (36) with boundary condition $B(T) = 0$, we obtain equation (34). For equation (37) with $A(T) = 1$, the solution is given by equation (33). □

Lemma 2. *If a solution of equation (31) is of the structure*

$$h(t,r) = (\eta - \theta)a \int_t^T \widetilde{h}(s,r)\mathrm{d}s, \tag{38}$$

then $\widetilde{h}(t,r)$ *satisfies the following equation:*

$$\widetilde{h}_t(t,r)+\frac{1}{2}\beta^2\widetilde{h}_{rr}(t,r)+\left[(\alpha-\lambda r(t))-\frac{\rho\beta u(t)}{\sigma}\right]\widetilde{h}_r(t,r)-r(t)\widetilde{h}(t,r)=0, \tag{39}$$

with the boundary condition $\widetilde{h}(T,r)=1$.

Proof. We define the variational operator ∇ on $h(t,r)$ by

$$\nabla h(t,r)=-r(t)h(t,r)+\left[(\alpha-\lambda r(t))-\frac{\beta u(t)}{\sigma}\right]h_r(t,r)+\frac{1}{2}\beta^2 h_{rr}(t,r). \tag{40}$$

Then, equation (31) is rewritten in the form

$$\frac{\partial h(t,r)}{\partial t}+\nabla h(t,r)+(\eta-\theta)a=0,\quad h(T,r)=0. \tag{41}$$

Considering

$$h(t,r)=(\eta-\theta)a\int_t^T\widetilde{h}(s,r)\mathrm{d}s, \tag{42}$$

we derive

$$\frac{\partial h(t,r)}{\partial t}=(\eta-\theta)a\left[\int_t^T\frac{\partial\widetilde{h}(s,r)}{\partial s}\mathrm{d}s-\widetilde{h}(T,r)\right], \tag{43}$$

$$\nabla h(t,r)=(\eta-\theta)a\int_t^T\nabla\widetilde{h}(s,r)\mathrm{d}s. \tag{44}$$

Substituting equations (43) and (44) into (41), we get

$$(\eta-\theta)a\left[\int_t^T\left(\frac{\partial\widetilde{h}(s,r)}{\partial s}+\nabla\widetilde{h}(s,r)\right)\mathrm{d}s-\widetilde{h}(T,r)+1\right]=0. \tag{45}$$

Therefore, we obtain

$$\frac{\partial\widetilde{h}(s,r)}{\partial s}+\nabla\widetilde{h}(s,r)=0,\quad \widetilde{h}(T,r)=1, \tag{46}$$

which completes the proof. □

Lemma 3. *Assume that*

$$\widetilde{h}(t,r)=D(t)e^{E(t)r}, \tag{47}$$

is a solution of equation (39), with boundary conditions $D(T)=1$ *and* $E(T)=0$. *Then,* $D(t)$ *and* $E(t)$ *are given by*

$$D(t)=\exp\left\{\int_t^T\left[\frac{1}{2}\beta^2E^2(s)+\left(\alpha-\frac{\rho\beta u(s)}{\sigma}\right)E(s)\right]\mathrm{d}s\right\}, \tag{48}$$

$$E(t)=\frac{1}{\lambda}\left(e^{-\lambda(T-t)}-1\right). \tag{49}$$

Proof. Putting equation (47) into (39) yields

$$D(t)e^{E(t)r}\left[\frac{D'(t)}{D(t)}+rE'(t)+\left(\alpha-\lambda r-\frac{\rho\beta u(t)}{\sigma}\right)E(t)+\frac{1}{2}\beta^2E^2(t)-r\right]=0. \tag{50}$$

Eliminating the dependence on r, we decompose equation (50) into the following two equations:

$$r\left(E'(t)-\lambda E(t)-1\right)=0,\quad E(T)=0, \tag{51}$$

$$\frac{D'(t)}{D(t)}+\left(\alpha-\frac{\rho\beta u(t)}{\sigma}\right)E(t)+\frac{1}{2}\beta^2E^2(t)=0,\quad D(T)=1. \tag{52}$$

Using the same approach as that of solving equation (36), the solution to equation (51) with $E(T)=0$ is given by equation (49). For equation (52) with $D(T)=1$, we obtain equation (48).

Note that

$$\frac{H_x}{H_{xx}}=zg_z(t,r,z)=\frac{1}{p-1}(x-h(t,r)), \tag{53}$$

$$\frac{H_{rx}}{H_{xx}}=-g_r(t,r,z)=-B(t)(x-h(t,r))-h_r(t,r). \tag{54}$$

Substituting equations (53) and (54) into trading strategies (12) and (13), we get the following theorem. □

Theorem 1. *Let* $\rho^2=1$, *and assume that the utility is given by a power utility function (25) for the optimal investment-*

reinsurance problem (7). There exists a solution $g(t,r,z)$ to the dual Hamilton–Jacobi–Bellman equation (24) with boundary condition $g(T,r,z)=z^{(1/p-1)}$. The corresponding optimal investment $\pi_p^*(t)$ and proportional strategy $q_p^*(t)$ of problem (7) are given by

$$\begin{aligned}\pi_p^*(t)&=-\frac{u(t)}{\sigma^2}\frac{H_x}{H_{xx}}-\frac{\rho\beta}{\sigma}\frac{H_{rx}}{H_{xx}}\\&=\frac{u(t)}{\sigma^2}\frac{1}{1-p}(x-h(t,r))+\frac{\rho\beta}{\sigma}\left[B(t)(x-h(t,r))+h_r(t,r)\right],\\q_p^*(t)&=-\frac{\eta a}{b^2}\frac{H_x}{H_{xx}}=\frac{\eta a}{b^2}\frac{1}{1-p}(x-h(t,r)),\end{aligned}\tag{55}$$

where $h(t,r)=(\eta-\theta)a\int_t^T\tilde{h}(s,r)ds$ is given in Lemmas 2 and 3.

3.2. Exponential Utility. Assume that the insurer takes an exponential utility function (CARA)

$$U(x)=-\frac{1}{\gamma}e^{-\gamma x},\quad \gamma>0,\tag{56}$$

where γ represents the absolute risk aversion coefficient. The exponential utility function (56) plays a prominent role in insurance mathematics and actuarial practice.

According to terminal condition (20), we have

$$g(T,r,z)=-\frac{\ln z}{\gamma}.\tag{57}$$

We conjecture a solution to equation (24) with the form

$$g(t,r,z)=-\frac{1}{\gamma}k(t,r)[\ln z+v(t,r)]+w(t,r),\tag{58}$$

with boundary conditions given by $k(T,r)=1, v(T,r)=0$, and $w(T,r)=0$.

A direct calculation yields the partial derivatives

$$\begin{aligned}g_t(t,r,z)&=-\frac{k_t(t,r)}{\gamma}[\ln z+v(t,r)]-\frac{k(t,r)}{\gamma}v_t(t,r)+w_t(t,r),\\g_r(t,r,z)&=-\frac{k_r(t,r)}{\gamma}[\ln z+v(t,r)]-\frac{k(t,r)}{\gamma}v_r(t,r)+w_r(t,r),\\g_z(t,r,z)&=-\frac{k(t,r)}{\gamma}\frac{1}{z},\\g_{zz}(t,r,z)&=\frac{k(t,r)}{\gamma}\frac{1}{z^2},\\g_{rz}(t,r,z)&=-\frac{k_r(t,r)}{\gamma}\frac{1}{z},\\g_{rr}(t,r,z)&=-\frac{k_{rr}(t,r)}{\gamma}[\ln z+v(t,r)]-\frac{2k_r(t,r)v_r(t,r)+k(t,r)v_{rr}(t,r)}{\gamma}+w_{rr}(t,r).\end{aligned}\tag{59}$$

Introducing the above derivatives back into equation (24), we derive that

$$
\begin{aligned}
&-\frac{\ln z}{\gamma}\left[k_t(t,r)+(\alpha-\lambda r(t))k_r(t,r)-r(t)k(t,r)+\frac{1}{2}\beta^2 k_{rr}(t,r)-\frac{\rho\beta u(t)}{\sigma}k_r(t,r)\right]\\
&+w_t(t,r)+(\alpha-\lambda r(t))w_r(t,r)-r(t)w(t,r)+\frac{1}{2}\beta^2 w_{rr}(t,r)-\frac{\rho\beta u(t)}{\sigma}w_r(t,r)+(\eta-\theta)a\\
&-\frac{k(t,r)}{\gamma}\\
&\left[\begin{array}{c}
\dfrac{v(t,r)k_t(t,r)}{k}+(\alpha-\lambda r(t))\dfrac{v(t,r)k_r(t,r)}{k}+(\alpha-\lambda r(t))v_r(t,r)-r(t)v(t,r)\\
+\dfrac{1}{2}\beta^2\dfrac{v(t,r)k_{rr}(t,r)}{k}+\beta^2\dfrac{v_r(t,r)k_r(t,r)}{k}-\dfrac{\rho\beta u(t)}{\sigma}\\
\left[\dfrac{v(t,r)k_r(t,r)}{k}+v_r(t,r)+\dfrac{k_r(t,r)}{k}\right]+v_t(t,r)-r(t)+K_1(t)+\dfrac{1}{2}\beta^2 v_{rr}(t,r)
\end{array}\right]=0.
\end{aligned}
\tag{60}
$$

Equation (60) is split into the following equations:

$$k_t(t,r)-r(t)k(t,r)+\left[(\alpha-\lambda r(t))-\frac{\rho\beta u(t)}{\sigma}\right]k_r(t,r)+\frac{1}{2}\beta^2 k_{rr}(t,r)=0, \tag{61}$$

with boundary condition $k(T,r)=1$,

$$w_t(t,r)-r(t)w(t,r)+\left[(\alpha-\lambda r(t))-\frac{\rho\beta u(t)}{\sigma}\right]w_r(t,r)+\frac{1}{2}\beta^2 w_{rr}(t,r)+(\eta-\theta)a=0, \tag{62}$$

with boundary condition $w(T,r)=0$, and

$$
\begin{aligned}
&\frac{v(t,r)k_t(t,r)}{k(t,r)}+v_t(t,r)-r(t)v(t,r)+\left[(\alpha-\lambda r(t))-\frac{\rho\beta u(t)}{\sigma}\right]\frac{v(t,r)k_r(t,r)}{k(t,r)}\\
&\quad+\left[(\alpha-\lambda r(t))-\frac{\rho\beta u(t)}{\sigma}\right]v_r(t,r)+\frac{1}{2}\beta^2\frac{v(t,r)k_{rr}(t,r)}{k(t,r)}+\beta^2\frac{v_r(t,r)k_r(t,r)}{k(t,r)}\\
&\quad-r(t)+\frac{1}{2}\beta^2 v_{rr}(t,r)-\frac{\rho\beta u(t)}{\sigma}\frac{k_r(t,r)}{k(t,r)}+K_1(t)=0,
\end{aligned}
\tag{63}
$$

with boundary condition $v(T,r)=0$.

Lemma 4. *Assume that a solution of equation (61) is in the form*

$$k(t,r)=e^{\widetilde{A}(t)+\widetilde{B}(t)r}, \tag{64}$$

with the boundary conditions $\widetilde{A}(T)=0$ *and* $\widetilde{B}(T)=0$. *Then,* $\widetilde{A}(t)$ *and* $\widetilde{B}(t)$ *are given by*

$$\widetilde{A}(t)=\int_t^T\left[\left(\alpha-\frac{\rho\beta u(s)}{\sigma}\right)\widetilde{B}(s)+\frac{\beta^2}{2}\widetilde{B}^2(s)\right]ds, \tag{65}$$

$$\widetilde{B}(t)=\frac{1}{\lambda}\left(e^{-\lambda(T-t)}-1\right). \tag{66}$$

Proof. Putting solution (64) into equation (61) yields

$$k(t,r)\left[\widetilde{A}'(t)+r\widetilde{B}'(t)-r+\left(\alpha-\lambda r-\frac{\rho\beta u(t)}{\sigma}\right)\widetilde{B}(t)+\frac{\beta^2}{2}\widetilde{B}^2(t)\right]=0. \quad (67)$$

We separate equation (67) into two equations:

$$\widetilde{A}'(t)+\left(\alpha-\frac{\rho\beta u(t)}{\sigma}\right)\widetilde{B}(t)+\frac{\beta^2}{2}\widetilde{B}^2(t)=0, \quad (68)$$

$$r\left(\widetilde{B}'(t)-\lambda\widetilde{B}(t)-1\right)=0. \quad (69)$$

Solving equation (69) with $\widetilde{B}(T)=0$, we obtain equation (66). For equation (68) with $\widetilde{A}(T)=0$, we have equation (67). □

Lemma 5. *Assume that a solution of equation (62) takes the structure*

$$w(t,r)=(\eta-\theta)a\int_t^T\widetilde{h}(s,r)\mathrm{d}s. \quad (70)$$

Then, $\widetilde{h}(t,r)$ satisfies equation (39) in Lemma 2.

Proof. Observe that equation (62) has the same solution with equation (31), i.e., $w(t,r)=h(t,r)$. The proof is the same as that of Lemmas 2 and 3; we omit its proof. □

Lemma 6. *Assume that a solution of equation (63) is in the form*

$$v(t,r)=\widetilde{D}(t)+\widetilde{E}(t)r, \quad (71)$$

with the boundary conditions $\widetilde{D}(T)=0$ and $\widetilde{E}(T)=0$. Then, $\widetilde{D}(t)$ and $\widetilde{E}(t)$ are given by

$$\widetilde{D}(t)=\int_t^T\left[\left(\alpha-\frac{\rho\beta u(s)}{\sigma}+\beta^2\widetilde{B}(s)\right)\widetilde{E}(s)+K_1(s)-\frac{\rho\beta u(s)}{\sigma}\widetilde{B}(s)\right]\mathrm{d}s, \quad (72)$$

$$\widetilde{E}(t)=\frac{1}{\lambda}\left(e^{-\lambda(T-t)}-1\right). \quad (73)$$

Proof. From equation (72), we have $k_r(t,r)=\widetilde{B}(t)k(t,r)$.

Introducing equation (60) and $k_r(t,r)$, we simplify equation (61) in the form

$$v_t(t,r)-r(t)+\left[(\alpha-\lambda r(t))-\frac{\rho\beta u(t)}{\sigma}+\beta^2\widetilde{B}(t)\right]v_r(t,r)+\frac{1}{2}\beta^2 v_{rr}(t,r)-\frac{\rho\beta u(t)}{\sigma}\widetilde{B}(t)+K_1(t)=0. \quad (74)$$

Substituting solution (71) into equation (74) yields

$$\widetilde{D}'(t)+\left(\alpha-\frac{\rho\beta u(t)}{\sigma}+\beta^2\widetilde{B}(t)\right)\widetilde{E}(t)+r\left(\widetilde{E}'(t)-\lambda\widetilde{E}(t)-1\right)+K_1(t)-\frac{\rho\beta u(t)}{\sigma}\widetilde{B}(t)=0. \quad (75)$$

Splitting equation (75) into two equations, we have

$$\widetilde{D}'(t)+\left[\alpha-\frac{\rho\beta u(t)}{\sigma}+\beta^2\widetilde{B}(t)\right]\widetilde{E}(t)+K_1(t)-\frac{\rho\beta u(t)}{\sigma}\widetilde{B}(t)=0, \quad (76)$$

$$r\left[\widetilde{E}'(t)-\lambda\widetilde{E}(t)-1\right]=0. \quad (77)$$

Taking into account the boundary condition $\widetilde{E}(T)=0$, the solution to equation (77) is given by equation (64). Solving equation (76) with $\widetilde{D}(T)=0$, we obtain equation (63).

Note that

$$\frac{H_x}{H_{xx}} = zg_z(t,r,z) = -\frac{1}{\gamma}k(t,r), \tag{78}$$

$$\frac{H_{rx}}{H_{xx}} = -g_r(t,r,z) = -\widetilde{B}(t)(x-w(t,r)) + \frac{1}{\gamma}\widetilde{E}(t)k(t,r) - w_r(t,r). \tag{79}$$

Substituting equations (78) and (79) into trading strategies (14) and (15), we get the following theorem. □

Theorem 2. *Let $\rho^2=1$, and assume that the utility is given by an exponential utility function (56) for the optimal investment-reinsurance problem (7). There exists a solution $g(t,r,z)$ to the dual Hamilton–Jacobi–Bellman equation (24) with boundary condition $g(t,r,z)-(\ln z/\gamma)$. The corresponding optimal investment $\pi_e^*(t)$ and proportional strategy $q_e^*(t)$ of problem (7) are given by*

$$\pi_e^*(t) = -\frac{u(t)}{\sigma^2}\frac{H_x}{H_{xx}} - \frac{\rho\beta}{\sigma}\frac{H_{rx}}{H_{xx}} = \frac{u(t)}{\sigma^2}\frac{1}{\gamma}k(t,r) + \frac{\rho\beta}{\sigma}\left[\widetilde{B}(t)(x-w(t,r)) - \frac{1}{\gamma}\widetilde{E}(t)k(t,r) + w_r(t,r)\right], \tag{80}$$

$$q_e^*(t) = -\frac{\eta a}{b^2}\frac{H_x}{H_{xx}} = \frac{\eta a}{b^2}\frac{1}{\gamma}k(t,r), \tag{81}$$

where $k(t,r)$ and $w(t,r)$ are given in Lemmas 4 and 5.

4. Sensitivity Analyses and Numerical Illustrations

In this section, we analyze the effects of market parameters on the optimal reinsurance-investment strategy, especially the parameters of interest rate and CARA and CRRA utilities, and provide several numerical simulations to illustrate our results. Throughout the numerical analyses, unless otherwise stated, the basic parameters are given by $\rho=1$, $a=0.6$, $b=0.4$, $\theta=0.2$, $\eta=0.25$, $\alpha=0.3,\lambda=1$, $\sigma=0.8, t=0, T=5$, and $x=1$.

4.1. Sensitivity Analyses on the Optimal Investment Strategy. From equation (3), we know that the parameter β represents the volatility of short interest rate. It means that the bigger the value of β is, the bigger the volatility resulted from interest rate is. The effect of β on the optimal investment strategy under the CRRA utility is shown in Figure 1, from which we see that $\pi_p^*(t)$ decreases with the parameter β in the case of $\rho=1$. Due to the positive correlation between interest rate and stock price dynamics, the volatility of stock price will become larger. It implies that the underlying risk becomes larger when the risk of interest rate becomes larger.

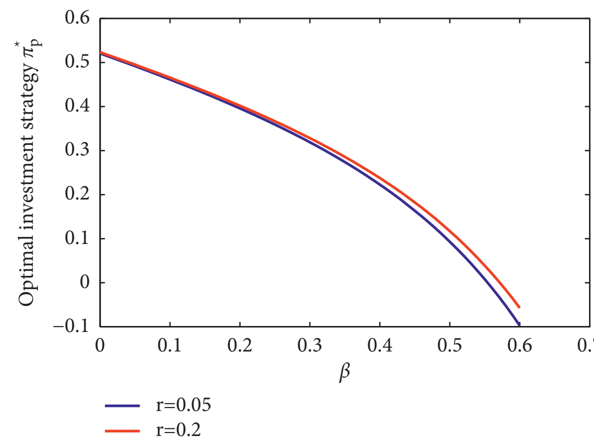

Figure 1: The effects of β on $\pi_p^*(t)$.

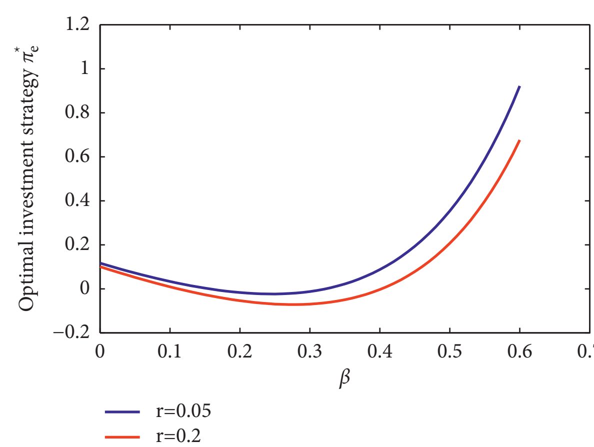

Figure 2: The effects of β on $\pi_e^*(t)$.

Therefore, in order to avoid risks, the investor will decrease the investment in the stocks.

Figure 2 illustrates that the optimal investment strategy under the CARA utility first declines slightly and then increases with β. When interest rate fluctuations are small, the insurer will not change their holdings of risk assets too much. However, the insurer will increase risky investment while interest rate fluctuations become larger.

For the CRRA utility, the absolute risk aversion coefficient $A_{\text{CRRA}} = -(U''(x)/U'(x)) = 1-p/x$, which implies that the risk aversion level of the investor decreases as p increases. Therefore, as it is illustrated in Figure 3, π_p^* increases as p becomes larger, and the insurer is willing to invest more money in the financial markets.

For the CARA utility, we obtain the absolute risk aversion coefficient $A_{\text{CARA}} = -(U''(x)/U'(x)) = \gamma$. Thus, π_e^* decreases with γ, which is shown in Figure 4. The larger γ is, the more risk averse the insurer will be and then will reduce the investment in risky assets.

4.2. Sensitivity Analyses on the Optimal Reinsurance Strategy. For the CRRA utility, the volatility of interest rate β has a negative effect on the optimal reinsurance strategy π_p^* (see Figure 5). The larger β is, the less the insurer's retention is.

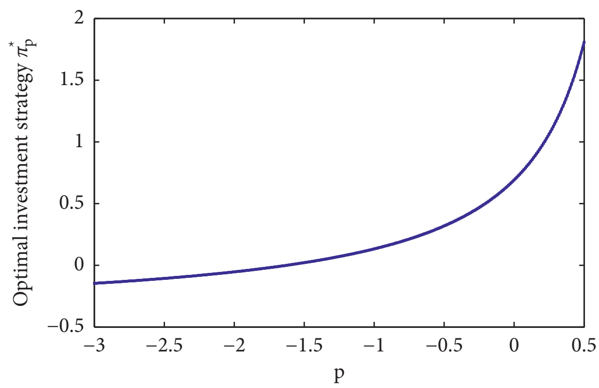

FIGURE 3: The effects of p on $\pi_p^*(t)$.

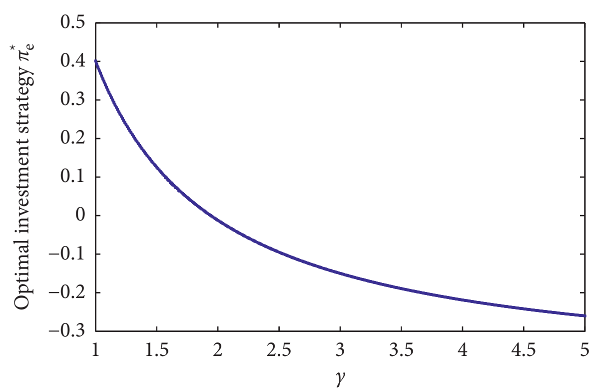

FIGURE 4: The effects of γ on $\pi_e^*(t)$.

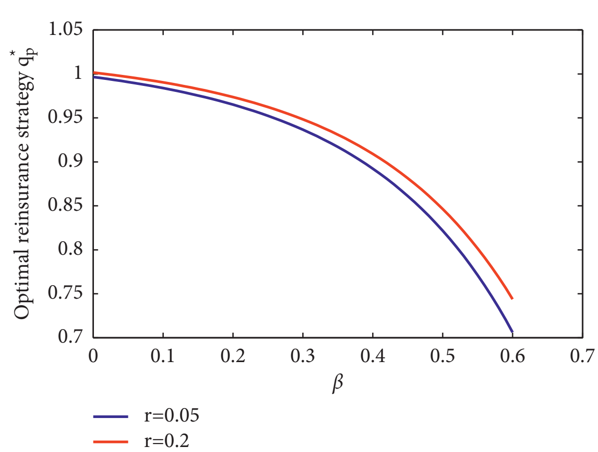

FIGURE 5: The effects of β on $q_p^*(t)$.

However, for the CARA utility, the effect of β on the optimal reinsurance strategy π_e^* is positive (see Figure 6). Accordingly, the insurer's reinsurance strategy is influenced by the interest rate risk and the utility function.

From equation (55), we derive that

$$\frac{\partial q_p^*}{\partial p} = \frac{\eta a}{b^2} \frac{1}{(1-p)^2} (x - h(t,r)) > 0, \quad (82)$$

which implies that $q_p^*(t)$ increases as p increases as it is shown in Figure 7. The larger p is, the smaller the absolute risk aversion coefficient is, and the insurer would like to take risks on their own and reduce the proportion of reinsurance.

From equation (81), we derive that

$$\frac{\partial q_e^*}{\partial \gamma} = -\frac{\eta a}{\gamma^2 b^2} e^{\widetilde{A}(t)+\widetilde{B}(t)r} < 0, \quad (83)$$

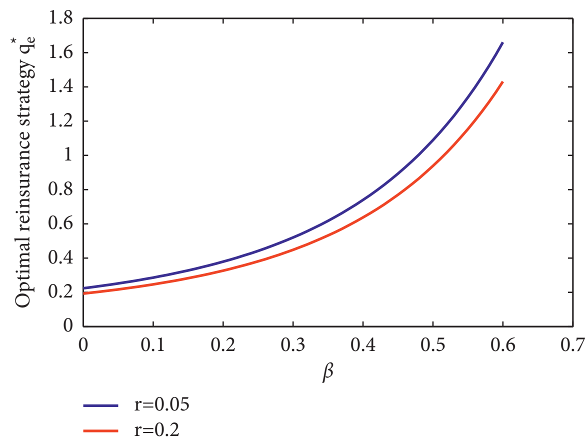

Figure 6: The effects of β on $q_e^*(t)$.

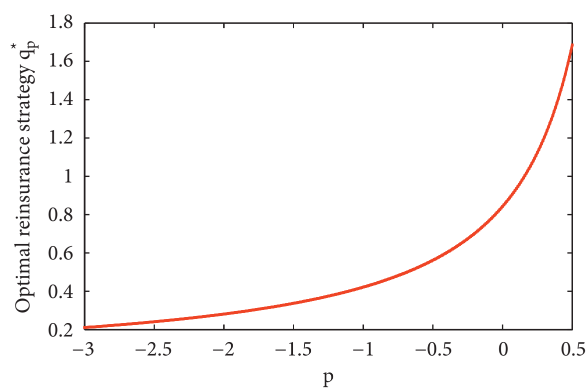

Figure 7: The effects of p on $q_p^*(t)$.

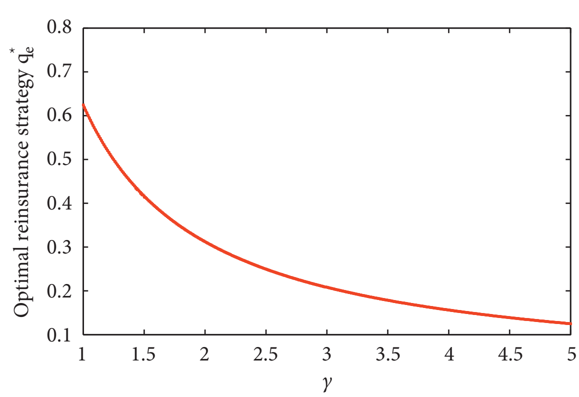

Figure 8: The effects of γ on $q_e^*(t)$.

which implies that $q_e^*(t)$ decreases with the risk aversion coefficient γ for the CARA utility as it is shown in Figure 8. For larger γ, the insurer is more risk averse and expects to reduce retention and transfer risks.

5. Conclusion

This paper investigates the investment-reinsurance problem with stochastic interest rate, in which interest rate is assumed to follow the Vasicek model and be correlated with stock price. The optimal reinsurance-investment strategies for CRRA and CARA utilities are derived by applying the stochastic dynamic programming and Legendre transformation. Through several sensitive analyses of the market parameters, we find that the optimal reinsurance strategy is not only affected by the parameters of reinsurance but also related to the risk preference coefficient and interest rate fluctuation, and the optimal investment strategy is influenced by both financial market and insurance market. For further research, considering the chaos dynamics in the financial market (Vaidyanathan et al. [28] and Sukono et al. [29]), it might be an interesting attempt to study the investment strategy combining the fractional-order financial risk chaotic system.

Data Availability

No data were involved in this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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