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Research Article

Optimal Investment-Consumption Strategy of Household Based on CEV Model

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To maintain and increase household wealth, this study studies the optimal allocation ratio of household investment and consumption. When considering venture capital, it is assumed that the theoretical price of risky asset obeys the CEV model. Our goal was to maximize the expectation of household cumulative consumption and the discounted utility of terminal wealth and to solve the optimal consumption and investment ratio using the dynamic programming principle and HJB equation. Using logarithmic utility and isoelastic power utility function with residual utility, we get the analytical solution of the household investment-consumption ratio by means of guessing and variable transformation. Finally, the influence of general parameters on the optimal ratio in the market is analyzed by numerical simulation and diagram, which is consistent with the description of actual situation. This study not only enriches portfolio theory but also provides investors with investment strategies.

1. Introduction

With the growth of economy, the household income also increases. How to reasonably invest the family's remaining wealth and realize the maintenance and appreciation of family wealth is the core issue of our research. The systematic research on household investment began in the 1960s, starting from Samuelson [1] and Fama [2]. For a continuous time, in 1969, Merton [3] studied portfolio selection under uncertainty in the life cycle, thus opening the prelude to study the dynamic portfolio optimization under continuous time. In 1971, Merton [4] proposed the continuous-time model of the optimal consumption and portfolio rules. It opens a new milestone in the methodology of dynamic portfolio selection. However, the classical Merton model and its extension [5–7] usually assume that its risk assets obey the geometric Brownian motion (GBM) model. The volatility of the price of a risky asset described by the GBM model is constant and cannot reflect the actual market situation. The CEV model, first proposed by Cox and Ross, can make up for these shortcomings. It is a natural extension of the geometric Brownian motion model [8]. For a complete financial market, the CEV model considers the relationship between the price of risky assets and the volatility, which is more in line with the requirements of the financial market [9]. The price fluctuation of risk assets described by the CEV model is stochastic, which is more suitable for practical problems. Xiao et al. [10], for the first time, will be introduced to the CEV model to determine the payment type pension (defined contribution pension) portfolio optimization problem in the study, and the optimal strategy is given under the exponential utility function. In recent years, there are a lot of literature studies on pension problems based on the CEV model [11–14] and very few literature studies on investment optimization based on the CEV model. In 2013, Chang et al. [15] pointed out that the relevant literature on the issue of investment and consumption based on the CEV model has not been widely discussed. When the CEV model is introduced into the portfolio optimization problem, it is often difficult to obtain the explicit solution of these problems. By maximizing discounted expected utility of consumption and final value wealth, Chang et al. studied the CEV model of individual optimal investment strategy. Although Yuan et al. [16] studied the family optimal investment strategy under the CEV model, they think of it in terms of mean-variance. Later, Jia et al. [17] obtained the analytic solution of investment strategy using power function utility and exponential function utility. However, they used only an asymptotic analysis. For the portfolio optimization problem in continuous time, Pliska [18] gave the analytical solution of the optimal portfolio problem, Sundaresan and Zapatero [19] gave the optimal asset ratio, and Liu [20] derived the analytical solution of the portfolio problem in a random case. Stochastic optimal control theory is one of the effective methods to solve the problem of optimal portfolio in continuous time [21]. Summing up the above literature studies, there are not many research studies on household investment-consumption problem using the CEV model.

This research mainly focuses on the investment and consumption problems based on the CEV model. In household wealth investment, this study considers a risk-free asset and a risky asset. Assuming that the price of an asset at risk is subject to the CEV model, we aim at maximizing consumption and the expected utility of the final value of wealth with discount to acquire the optimal ratio. The innovations of this study are as follows: (1) considering both the CEV model and consumption factor in this section, the HJB equation of the corresponding value function is more difficult to solve than the equation in Gao [22]. (2) Inspired by Gao [22] and Liu [23], when a second-order nonlinear partial differential equation is transformed into a linear equation that is easy to handle, an important innovation is to assume that the structural expression of the solution of equation (25) is in the form of equation (26). It is also proved that equation (25) is equivalent to equation (29). (3) Using the method of variable transformation, the optimal consumption-consumption strategy under the exponential utility function and the equal elastic power utility function with giveaway utility function is solved. The analytic expressions of the optimal investment-consumption strategy under the two utility functions are obtained by the method of guessing.

2. Household Wealth Investment Model

We propose an investment model based on the investment and consumption decision problem under the CEV process. Assume that households have two continuously tradable financial assets in financial risk markets during an investment period [0, T]. Now, let us consider a specific risk asset based on the CEV model and a market composed of risk-free assets. At time t, the price of the risk-free asset satisfies the following random process:

$$dB_t = rB_t dt, \tag{1}$$

where r is the fixed interest rate.

At time t, let the price of the risky asset (stock) be $S(t)$, abbreviated as S_t, following the CEV model as follows:

$$\frac{dS_t}{S_t} = \mu dt + kS_t^{\beta} dW_t, \tag{2}$$

where μ represents the instantaneous return rate of the stock, which generally satisfies $\mu > \beta$. kS_t^{β} represents the instantaneous volatility, and generally, $\beta < 0$ represents the elasticity coefficient of constant variance. When $\beta = 0$, it is a general geometric Brownian motion process. $W_t (t \geq 0)$ is a one-dimensional standard Brownian motion, which is defined in the complete probability space (Ω, F, P) where P is the risk-neutral probability, and $F = \{F\}_t$ is a right continuous σ-algebra generated by BM W_t.

At the time of $t \in [0, T]$, let $V(t)$ denote the total additional disposable wealth of the household, and the initial household wealth is assumed to be $V_0 = V(0)$, and π_t and $1 - \pi_t$, respectively, represent the proportion of investment in risky assets and risk-free assets. The differential equation of household wealth investment can be expressed as follows:

$$\begin{aligned} dV_t &= \pi_t V_t\left(\mu\ dt + kS_t^{\beta} dW_t\right) + (1 - \pi_t)V_t r\ dt - C_t dt \\ &= [(\pi_t(\mu - r) + r)V_t - C_t]dt + \pi_t V_t k S_t^{\beta} dW_t. \end{aligned} \tag{3}$$

Let $\Gamma = \{(\pi_t, C_t)\colon 0 \leq t \leq T\}$ be the set of all feasible investment-consumption proportions. For any combination (π_t, C_t), the stochastic differential (3) has a special solution. Theoretically, investors want the ultimate wealth to have the greatest expected effect, namely, $\max_{u_t} E\{U(V_T)\}$, where $U(\cdot)$ is a concave and increasing utility function.

The target expected utility function is as follows:

$$\max_{(\pi_t, C_t)\in\Gamma} E\left[\int_0^T e^{-\delta t} U_1(C_t)\mathrm{d}t + e^{-\delta t} U_2(V_T)\right]. \tag{4}$$

The utility functions $U_1(\cdot)U_2(\cdot)$ are continuously differentiable and strictly concave utility functions on $(-\infty, +\infty)$, and δ is the discount rate. Since $U(\cdot)$ is strictly concave, there is a unique strategy (π_t, C_t) that makes (4) valid.

For a particular feedback function π, the value function can be defined as the Markov property.

Value function is as follows:

$$H(t, s, v) = \sup_{(\pi_t, C_t)\in\Gamma} E\left\{\int_0^T e^{-\delta t} U(C_t)\mathrm{d}t + e^{-\delta t} U_2(V_T) | S_t = s, V_t = v\right\}. \tag{5}$$

Boundary condition is as follows: $H(T,s,v) = e^{-\delta T}U(v)$.

The value function defined by equation (5) is the upper definite bound of the expected utility, so it reaches the optimal value under the stochastic constraint. Let us write $\pi_t = \pi$ and $C_t = c$; according to the dynamic programming principle and Itō′s lemma, we obtain the following HJB equation:

$$\begin{aligned} 0 &= H_t + (rv - c)H_v + \mu s H_s + \frac{1}{2}k^2 s^{2\beta+2}H_{ss} \\ &+ \sup_{(\pi_t, C_t)}\left\{\frac{1}{2}\pi^2 v^2 k^2 s^{2\beta}H_{vv} + \pi\left[(\mu - r)vH_v + vk^2 s^{2\beta+1}H_{vs}\right\}\right. \\ &+ e^{-\delta t}U_1(c). \end{aligned} \tag{6}$$

Boundary condition is as follows: $H(T,s,v) = U(v)$ where $H_t, H_v, H_s, H_{vv}, H_{ss}$, and H_{vs} represent the first-order and second-order partial differentials about time t, wealth v, and stock price s, respectively.

For the optimal strategy π^*, the first-order conditions lead to the following formula:

$$\pi_t^* = -\frac{(\mu - r)H_v + k^2 s^{2\beta+1}H_{vs}}{vk^2 s^{2\beta}H_{vv}}. \tag{7}$$

Similarly, for the optimal consumption strategy c^*, the first-order conditions lead to the following formula:

$$U_1'(c^*) = e^{\delta t}H_v. \tag{8}$$

Equation (7) is substituted into equation (6) of HJB to get the following formula:

$$\begin{aligned} &H_t + \mu s H_s + rvH_v + \frac{1}{2}k^2 s^{2\beta+2}H_{ss} \\ &- \frac{\left[(\mu - r)H_v + k^2 s^{2\beta+1}H_{vs}\right]^2}{2k^2 s^{2\beta}H_{vv}} - cH_v + e^{-\delta t}U_1(c) = 0. \end{aligned} \tag{9}$$

Boundary condition is as follows: $H(T,s,v) = U(v)$.

It can be seen that we turn the stochastic control problem into a problem of solving a nonlinear second-order partial differential equation. If H is solved from (9) and then substituted into equations (7) and (8), the optimal strategy π^* and optimal consumption strategy c^* can be obtained. In the following, we use the specific utility function. Here, we first select the logarithmic utility function and then the power utility function. Closed-form solutions of equation (9) are obtained by guessing solutions and variable separation methods.

3. Optimal Household Investment Strategy Based on Logarithmic Utility Function

The utility function is set as $U_1(x) = U_2(x) = \ln x$; suppose the solution of HJB equation (9) has the following form:

$$H(t,s,v) = m(t)e^{-\delta t}\ln v + n(t,s). \tag{10}$$

Boundary conditions are as follows: $m(T) = 1$ and $n(T,s) = 0$.

For simplicity, $m = m(t)$, $m_t = m'(t)$, $n = n(t,s)$, and $n_t = n'(t,s)$ are written. The first-order and second-order partial differentials of t, s, and v, respectively, are solved, and the solution results are as follows:

$$\begin{aligned} H_t &= m_t e^{-\delta t}\ln v - \delta m e^{-\delta t}\ln v + n_t; \\ H_v &= me^{-\delta t}\frac{1}{v}, \\ H_s &= n_s; \\ H_{vv} &= -me^{-\delta t}\frac{1}{v^2}; \\ H_{ss} &= n_{ss}; \\ H_{vs} &= 0. \end{aligned} \tag{11}$$

Because $U(c) = \ln c$, and $U_1'(c) = e^{\delta t}H_v$, with H_v in equation (10), we can get the following equation: $1/c = e^{\delta t}me^{-\delta t}(1/v)$. So, there is equation: $c = (v/m)$.

The expressions of equation (11) and c are substituted into equation (9) to get the following equation:

$$\begin{aligned} &n_t + \mu s n_s + \frac{1}{2}k^2 s^{2\beta+2}n_{ss} + \left[r + \frac{(\mu - r)^2}{2k^2 s^{2\beta}}\right]me^{-\delta t} \\ &- e^{-\delta t}(\ln m + 1) + e^{-\delta t}\ln v\,(m_t - \delta m + 1) = 0. \end{aligned} \tag{12}$$

We divide (12) into the following two equations:

$$\begin{aligned} &m_t - \delta m + 1 = 0; \\ &m(T) = 1, \\ &n_t + \mu s n_s + \frac{1}{2}k^2 s^{2\beta+2}n_{ss} + \left[r + \frac{(\mu - r)^2}{2k^2 s^{2\beta}}\right]me^{-\delta t} \\ &\quad - e^{-\delta t}(\ln m + 1) = 0, \\ &n(T,s) = 0. \end{aligned} \tag{13}$$

The first-order ordinary differential equation (12) is solved to get the following equation:

$$m(t) = m = \frac{1}{\delta} + \left(1 - \frac{1}{\delta}\right)e^{-\delta(T-t)}. \tag{14}$$

Suppose (13) has a solution of the following form:

$$n(t,s) = A(t) + B(t)y, \quad y = s^{-2\beta}. \tag{15}$$

Boundary conditions are as follows: $A(T) = 0$ and $B(T) = 0$.

For simplicity, $A = A(t)$, $A_t = A'(t)$, $B = B(t)$, and $B_t = B'(t)$ are written. The first and second partial derivatives of t, s, respectively, are solved for (15), and the results are as follows:

$$n_t = A_t + B_t y,$$
$$n_s = -2\beta B s^{-2\beta-1}, \tag{16}$$
$$n_{ss} = 2\beta(2\beta+1)Bs^{-2\beta-2}.$$

The above equation is substituted into (13) to get the following equation:

$$y\left[B_t - 2\beta\mu B + \frac{(\mu-r)^2}{2k^2}e^{-\delta t}m\right] + A_t + 2\beta(2\beta+1)k^2B + re^{-\delta t} - e^{-\delta t}(\ln m + 1) = 0. \tag{17}$$

(17) is decomposed into two equations as follows:

$$B_t - 2\beta\mu B + \frac{(\mu-r)^2}{2k^2}e^{-\delta t}m = 0;$$
$$B(T) = 0, \tag{18}$$
$$A_t + 2\beta(2\beta+1)k^2B + re^{-\delta t} - e^{-\delta t}(\ln m + 1) = 0,$$
$$(T) = 0.$$

(17) is solved to get the following equation:

$$B(t) = \frac{m(\mu-r)^2}{2k^2(\delta+2\mu\beta)}e^{-\delta T}\left[e^{\delta(T-t)} - e^{-2\mu\beta(T-t)}\right]. \tag{19}$$

$B(t)$ is substituted into (18), and it is solved to get the following equation:

$$A(t) = \frac{r - \ln m - 1}{2\beta(2\beta+1)k^2B(t) - \delta}\left[e^{[2\beta(2\beta+1)k^2B(t)-\delta](T=t)} - e^{-\delta t}\right]. \tag{20}$$

The following conclusion is obtained from the above solution process.

Theorem 1. *If the utility function $U_1(x) = U_2(x) = \ln x$, then the solution of HJB equation (9) has the following form:*

$$H(t,s,v) = \frac{1}{\delta}\left[1 - e^{-\delta(T-t)}\right]e^{-\delta t}\ln v + A(t) + B(t)s^{-2\beta}, \tag{21}$$

where $A(t) = r - \ln(m - 1/2\beta(2\beta+1)k^2B(t) - \delta)$ $[e^{[2\beta(2\beta+1)k^2B(t)-\delta](T=t)} - e^{-\delta t}]$, *and*

$$B(t) = \frac{m(\mu-r)^2}{2k^2(\delta+2\mu\beta)}e^{-\delta T}\left[e^{\delta(T-t)} - e^{-2\mu\beta(T-t)}\right]. \tag{22}$$

Theorem 2. *Under the logarithmic utility function, the optimal investment ratio and optimal consumption ratio in the stocks of risky assets are listed as follows:*

$$\pi_t^* = M(\sigma_t),$$
$$C_t^* = \frac{v}{m} = \frac{v}{m(t)}. \tag{23}$$

$M(\sigma_t) = (\mu-r)/\sigma_t^2$, $\sigma_t = kS_t^\beta$, and $m(t) = (1/\delta) + (1 - (1/\delta))e^{-\delta(T-t)}$ are written.

Proof. From equation (10), we can get the following:

$$\frac{H_v}{H_{vv}} = \frac{me^{-\delta t}(1/v)}{-me^{-\delta t}(1/v^2)} = -v, \quad \frac{H_{vs}}{H_{vv}} = 0. \tag{24}$$

Substituting the above equation into equations (7) and (8), we can get the optimal investment ratio π_t^* and the optimal consumption ratio C_t^*.

$$\pi_t^* = -\frac{\mu-r}{vk^2s^{2\beta}}\frac{H_v}{H_{vv}} - \frac{s}{v}\frac{H_{vs}}{H_{vv}} = \frac{\mu-r}{k^2s^{2\beta}}$$
$$C_t^* = \frac{v}{m} = \frac{v}{m(t)}. \tag{25}$$

□

4. Optimal Household Investment Strategy Based on Equal Elastic Power Utility Function

For the objective function (4), where $U_2(V_T)$ has retention utility, isoelastic power utility function is defined as follows:

$$U(\mathrm{v}) = \phi^\gamma\frac{v^{1-\gamma}}{1-\gamma},$$
$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \text{where } \gamma > 0 \text{ also } U(V_T) = \phi^\gamma\frac{V_T^{1-\gamma}}{1-\gamma}, \tag{26}$$

where $\phi = \phi(v)$ is a weighting adjustment factor for the bequest value function. Its coefficient is the utility function of constant relative risk aversion (CRRA). Now, it is assumed that the solution form of the objective function $H(t,s,v)$ is as follows:

$$H(t,s,v) = e^{-\delta t}g(t,s)\frac{v^{1-\gamma}}{1-\gamma}. \tag{27}$$

Boundary condition is as follows: $g(T,s) = \phi^\gamma$.

For simplicity, $g(t,s) = g$ is written. The first-order and second-order partial differentials of $t, s,$ and v are solved, respectively. The results are as follows:

$$
\begin{aligned}
H_t &= e^{-\delta t} g_t \frac{v^{1-\gamma}}{1-\gamma} - \delta e^{-\delta t} g \frac{v^{1-\gamma}}{1-\gamma};\\
H_v &= g e^{-\delta t} v^{-\gamma};\\
H_s &= e^{-\delta t} g_s \frac{v^{1-\gamma}}{1-\gamma};\\
H_{vv} &= -\gamma e^{-\delta t} g v^{-\gamma-1};\\
H_{ss} &= e^{-\delta t} g_{ss} \frac{v^{1-\gamma}}{1-\gamma};\\
H_{sv} &= e^{-\delta t} g_s v^{-\gamma}.
\end{aligned} \tag{28}
$$

Because $U(c) = c^{1-\gamma}/(1-\gamma)$ and $U_1'(c) = c^{-\gamma}$, with the equation (8) $U_1'(c) = e^{\delta t} H_v$ and the equation in H_v, we have the equation $e^{\delta t} H_v = e^{\delta t} e^{-\delta t} g v^{-\gamma} = c^{-\gamma}$. So, $c = g^{-(1/\gamma)} v$.

The expressions of (28) and c are substituted into (9) and arranged as follows:

$$
\left\{ g_t - \delta g + \mu s g_s + r(1-\gamma) g + \frac{1}{2} k^2 s^{2\beta+2} g_{ss} + \frac{(1-\gamma)(\mu - r)^2}{2\gamma k^2 s^{2\beta}} g + \frac{k^2 s^{2\beta}(1-\gamma)}{2\gamma} \frac{g_s^2}{g} + \frac{1}{\gamma}(\mu - r)(1-\gamma) s g_s + \gamma g^{(\gamma-1)/\gamma} \right\} \frac{v^{1-\gamma}}{1-\gamma} e^{-\delta t} = 0. \tag{29}
$$

The dependence on v is removed, and you get the following equation:

$$
\begin{aligned}
& g_t - \delta g + \mu s g_s + r(1-\gamma) g + \frac{1}{2} k^2 s^{2\beta+2} g_{ss}\\
& + \frac{(1-\gamma)(\mu - r)^2}{2\gamma k^2 s^{2\beta}} g + \frac{k^2 s^{2\beta}(1-\gamma)}{2\gamma} \frac{g_s^2}{g}\\
& + \frac{1}{\gamma}(\mu - r)(1-\gamma) s g_s + \gamma g^{(\gamma-1)/\gamma} = 0.
\end{aligned} \tag{30}
$$

Based on the inspiration of the method in Gao [21], we will use the following power transformation and variable transformation methods.

Let $g(t,s) = h(t,y)$, and $y = s^{-2\beta}$.

For simplicity, $g(t,s) = g$ is written, and the first and second partial derivatives of t and s are solved, respectively. The results are as follows:

$$
\begin{aligned}
g_t &= h_t,\\
g_s &= -2\beta s^{-2\beta-1} h_y,\\
g_{ss} &= 2\beta(2\beta+1) s^{-2\beta-2} h_y + 4\beta^2 s^{-4\beta-2} h_{yy}.
\end{aligned} \tag{31}
$$

Substituting the expression in equation (31) into (30), the following equation is obtained:

$$
\begin{aligned}
& h_t + [r(1-\gamma) - \delta] h - 2\beta\mu y h_y + \beta(2\beta+1) k^2 h_y + \frac{(\mu - r)^2}{2k^2} \frac{1-\gamma}{\gamma} y h\\
& + 2\beta^2 k^2 y h_{yy} + 2\beta^2 k^2 \frac{1-\gamma}{\gamma} y \frac{h_y^2}{h} - 2\beta(\mu - r) \frac{1-\gamma}{\gamma} y h_y + \gamma h^{1-(1/\gamma)} = 0.
\end{aligned} \tag{32}
$$

Boundary condition is as follows: $h(T,y) = \phi^\gamma$.

Let $h(t,y) = [f(t,y)]^\gamma$, using the method of variable transformation, and then, $h(T,y) = [f(T,y)]^\gamma = \phi^\gamma$. So, $f(T,y) = \phi$. For simplicity, $f(t,s) = f$ is written. The first and second partial derivatives of t and y are solved, respectively. The results are as follows:

$$
\begin{aligned}
h_t &= \gamma f^{\gamma-1} f_t,\\
h_y &= \gamma f^{\gamma-1} f_y,\\
h_{yy} &= \gamma(\gamma-1) f^{\gamma-2} f_y^2 + \gamma f^{\gamma-1} f_{yy}.
\end{aligned} \tag{33}
$$

Substituting the expression in equation (33) into equation (32), we get the following equation:

$$\gamma f^{\gamma-1}\begin{bmatrix} f_t+\dfrac{r(1-\gamma)-\delta}{\gamma}f-2\beta\mu y f_y+\beta(2\beta+1)k^2 f_y+2\beta^2k^2 y f_{yy} \\ +\dfrac{(\mu-r)^2}{2k^2}\dfrac{1-\gamma}{\gamma^2}yf-2\beta(\mu-r)\dfrac{1-\gamma}{\gamma}yf_y+1 \end{bmatrix}=0, \tag{34}$$

which simplifies to the following partial differential equation:

$$\begin{aligned} &f_t+\frac{1}{\gamma}[r(1-\gamma)-\delta]f-2\beta\mu y f_y+\beta(2\beta+1)k^2 f_y+2\beta^2k^2 y f_{yy} \\ &+\frac{(\mu-r)^2}{2k^2}\frac{1-\gamma}{\gamma^2}yf-2\beta(\mu-r)\frac{1-\gamma}{\gamma}yf_y+1=0. \end{aligned} \tag{35}$$

Boundary condition is as follows: $f(T,y)=\phi$.

Notice that equation (35) has changed into a linear second-order partial differential equation, which is still difficult to deal. Based on the inspiration of the method in Liu [22], we solve the solution of (35).

Theorem 3. *Assume that $f(t,y)=\int_t^T\overline{f}(u,y)du+\phi\overline{f}(u,y)$ is a solution to (25), and it can be proved that $\overline{f}(u,y)$ satisfies the following equation:*

$$\begin{aligned} &\overline{f}_t+\left[\frac{1}{\gamma}(r(1-\gamma)-\delta)+\frac{(\mu-r)^2}{2k^2}\frac{1-\gamma}{\gamma^2}y\right]\overline{f} \\ &-\left[2\beta\mu+2\beta(\mu-r)\frac{1-\gamma}{\gamma}\right]y\overline{f}+\beta(2\beta+1)k^2\overline{f}_y \\ &+2\beta^2k^2y\overline{f}_{yy}=0. \end{aligned} \tag{36}$$

Boundary condition is as follows: $\overline{f}(t,y)=1$.

Proof. The following differential operator is defined:

$$\begin{aligned} \nabla f(t,y)&=2\beta^2k^2y\overline{f}_{yy}+\beta(2\beta+1)k^2\overline{f}_y \\ &+\left[\frac{1}{\gamma}(r(1-\gamma)-\delta)+\frac{(\mu-r)^2}{2k^2}\frac{1-\gamma}{\gamma^2}y\right]\overline{f} \\ &-\left[2\beta\mu+2\beta(\mu-r)\frac{1-\gamma}{\gamma}\right]y\overline{f}_y. \end{aligned} \tag{37}$$

Equation (35) can then be rewritten in the form as follows:

$$\begin{aligned} &\frac{\partial f(t,y)}{\partial t}+\nabla f(t,y)+1=0,\quad f(T,y)=\phi \\ &\frac{\partial f(t,y)}{\partial t}=-\overline{f}(t,y)+\phi\frac{\partial\overline{f}(t,y)}{\partial t} \\ &=\left[\int_t^T\frac{\partial\overline{f}(t,y)}{\partial u}du-\overline{f}(T,y)\right]+\phi\frac{\partial\overline{f}(t,y)}{\partial t}. \end{aligned} \tag{38}$$

The operator of equation (26) is obtained as follows:

$$\nabla f(t,y)=\int_t^T\nabla\overline{f}(u,y)\mathrm{d}u+\phi\nabla\overline{f}(t,y). \tag{39}$$

(30) and (31) are substituted into equation (29) to obtain the following equation:

$$\left[\int_t^T\left(\frac{\partial\overline{f}(t,y)}{\partial u}+\nabla\overline{f}(u,y)\right)du\right]+\phi\left[\frac{\partial\overline{f}(t,y)}{\partial t}+\nabla\overline{f}(u,y)\right]$$
$$-\overline{f}(T,y)+1=0. \tag{40}$$

Since $\overline{f}(T,y)=1$, the above equation can be written in the following form:

$$\frac{\partial\overline{f}(t,y)}{\partial t}+\nabla\overline{f}(u,y)=0,\quad \overline{f}(T,y)=1. \tag{41}$$

Theorem 3 is proved. For equation (27), the form of the guessing solution is as follows:

$$\overline{f}(t,y)=e^{A(t)+B(t)y},\quad A(T)=0,\ B(T)=0. \tag{42}$$

(42) is substituted into (36), and the following equation is obtained:

$$y\left[B_t+2\beta^2k^2B^2-\left(2\beta\mu+2\beta(\mu-r)\frac{1-\gamma}{\gamma}\right)B+\frac{(\mu-r)^2(1-\gamma)}{2k^2\gamma^2}\right]$$
$$+A_t+\beta(2\beta+1)k^2B+\frac{1}{\gamma}(r(1-\gamma)-\delta)=0. \tag{43}$$

To eliminate the dependence on y, (43) is decomposed into two equations as follows:

$$\begin{aligned}&B_t+2\beta^2k^2B^2-\left(2\beta\mu+2\beta(\mu-r)\frac{1-\gamma}{\gamma}\right)B+\frac{(\mu-r)^2(1-\gamma)}{2k^2\gamma^2}=0,\quad B(T)=0,\\&A_t+\beta(2\beta+1)k^2B+\frac{1}{\gamma}(r(1-\gamma)-\delta)=0,\quad A(T)=0.\end{aligned} \tag{44}$$

The method of solving equations (35) and (36) is the same as that of Gao [22]. First, (35) is solved. The process is as follows.

$a=2\beta^2$, $b=2\beta[\mu+(\mu-r)1-\gamma/\gamma]$, and $c=-((\mu-r)^2(1-\gamma)/2\gamma^2)$ are set. Then, equation (35) can be changed into the following form:

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t}=ak^2B^2(t)+bB(t)+\frac{c}{k^2},\quad B(T)=0. \tag{45}$$

Both sides of (45) are integrated with respect to t to get the following equation:

$$\frac{1}{ak^2(m_1-m_2)}\int\frac{1}{B(t)-m_1}-\frac{1}{B(t)-m_2}\mathrm{d}B(t)=t+c_1, \tag{46}$$

where c_1 is a constant and m_1 and m_2 are the solutions of the quadratic equation.

$$ak^2m^2+bm+\frac{c}{k^2}=0, \tag{47}$$

where $m_{1,2}=[\gamma\mu+(\mu-r)(1-\gamma)]\pm(\sqrt{[\gamma\mu+(\mu-r)(1-\gamma)]^2-(\mu-r)^2(1-\gamma)}/2\beta\gamma k^2)$.

Considering the boundary condition $B(T)=0$, the solution of (45) is as follows:

$$B(t)=\frac{m_1-m_1e^{ak^2(m_1-m_2)(t-T)}}{1-m_1/m_2e^{ak^2(m_1-m_2)(t-T)}}. \tag{48}$$

$\lambda_{1,2}=[\gamma\mu+(\mu-r)(1-\gamma)]\pm(\sqrt{[\gamma\mu+(\mu-r)(1-\gamma)]^2-(\mu-r)^2(1-\gamma)}/2\beta\gamma)$ is defined.

Then, (48) can be rewritten, so the solution of equation (35) is listed as follows:

$$B(t)=k^{-2}I(t), \tag{49}$$

where $I(t)=\lambda_1\lambda_2(1-e^{2\beta^2(\lambda_1-\lambda_2)(T-t)})/(\lambda_2-\lambda_1e^{2\beta^2(\lambda_1-\lambda_2)(T-t)})$.

(44) is solved, which can be obtained directly from (44).

$$dA(t)=-\left[\beta(2\beta+1)I(t)+\frac{r(1-\gamma)-\delta}{\gamma}\right]dt. \tag{50}$$

Both sides of $I(t)$ are differentiated to get the following equation:

$$\int I(t)\mathrm{d}t=\lambda_1t+\frac{1}{2\beta^2}\ln\left(\lambda_2-\lambda_1e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}\right)+c_2. \tag{51}$$

On differentiating both sides of (51), the solution of (44) can be obtained from (42) and the boundary condition $A(T)=0$ as follows:

$$\begin{aligned}A(t)=&\left[\lambda_1\beta(2\beta+1)k^2+\frac{1}{\gamma}(r(1-\gamma)-\delta)\right](T-t)\\&+\frac{2\beta+1}{2\beta}\ln\left(\frac{\lambda_2-\lambda_1}{\lambda_2-\lambda_1e^{2\beta^2(\lambda_1-\lambda_2)(T-t)}}\right).\end{aligned} \tag{52}$$

□

Theorem 4. *If the utility function is $u(v)=\phi^\gamma(v^{1-\gamma}/1-\gamma)$, $v(c)=(c^{1-\gamma}/1-\gamma)$, $\gamma>0$, $\gamma\neq1$, then the optimal investment and consumption strategy of problem (5) are listed as follows:*

$$\begin{aligned}\pi_t^*&=\frac{\mu-r}{\gamma k^2s^{2\beta}}-\frac{2\beta}{s^{2\beta}}\frac{f_y}{f},\\c_t^*&=\frac{v}{f},\end{aligned} \tag{53}$$

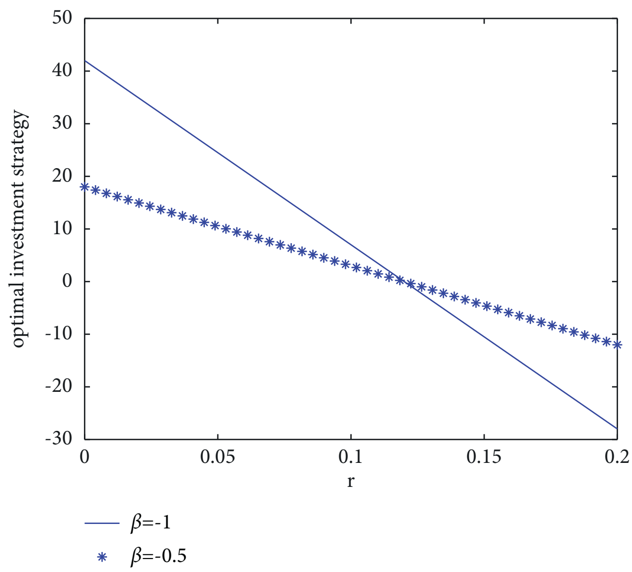

Figure 1: Effect of r on optimal investment ratio π_t^*.

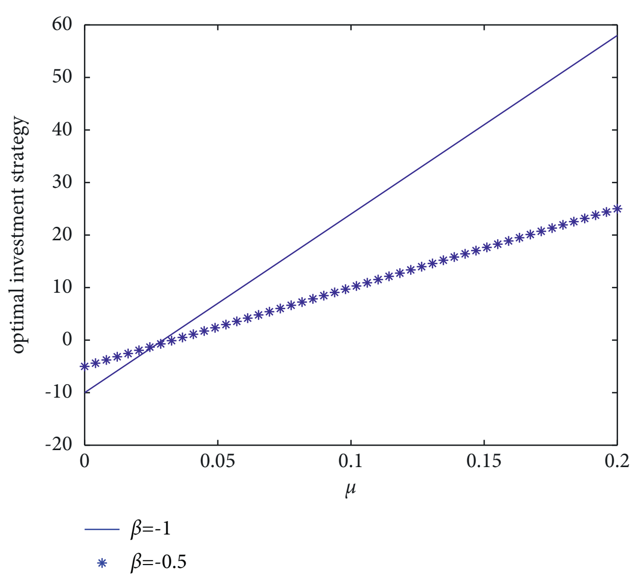

Figure 2: Influence of μ on optimal investment ratio π_t^*.

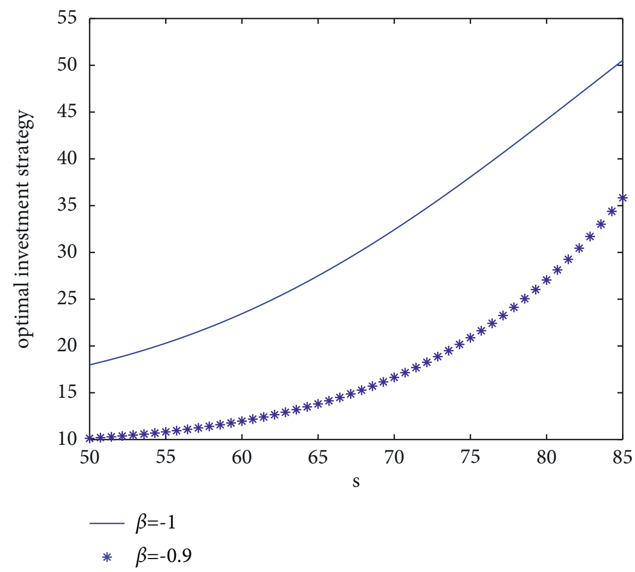

Figure 3: Influence of S on optimal investment ratio π_t^*.

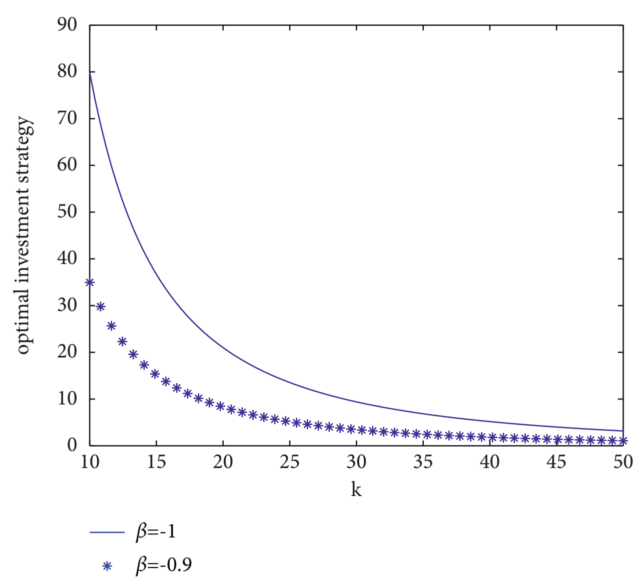

Figure 4: Influence of k on optimal investment ratio π_t^*.

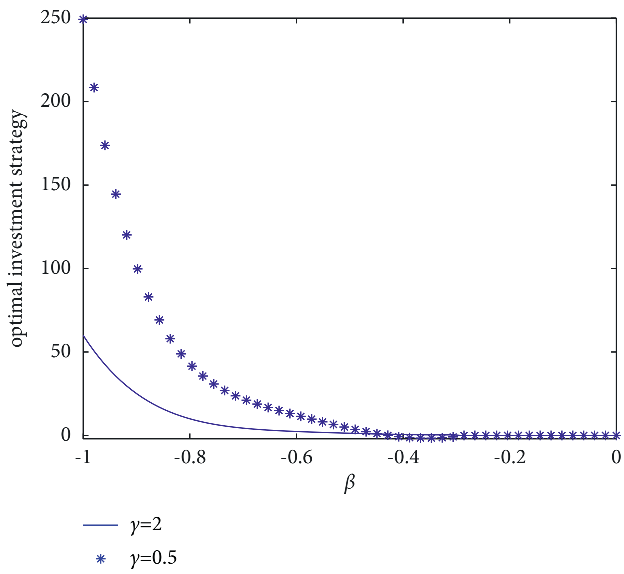

Figure 5: Influence of β on optimal investment ratio π_t^*.

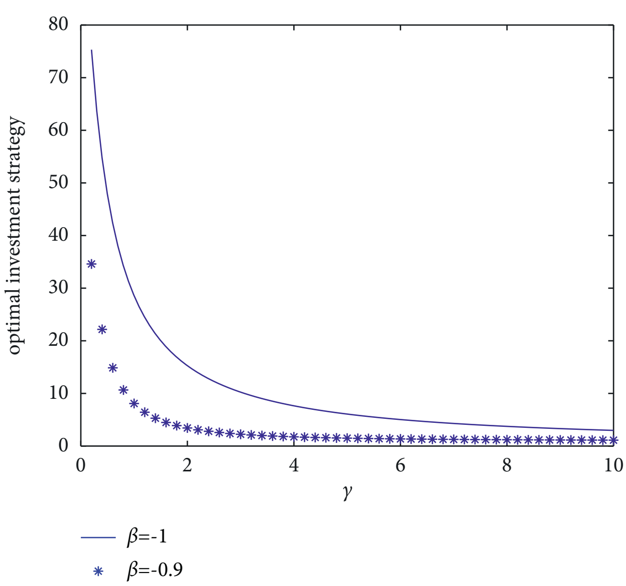

Figure 6: Influence of γ on optimal investment ratio π_t^*.

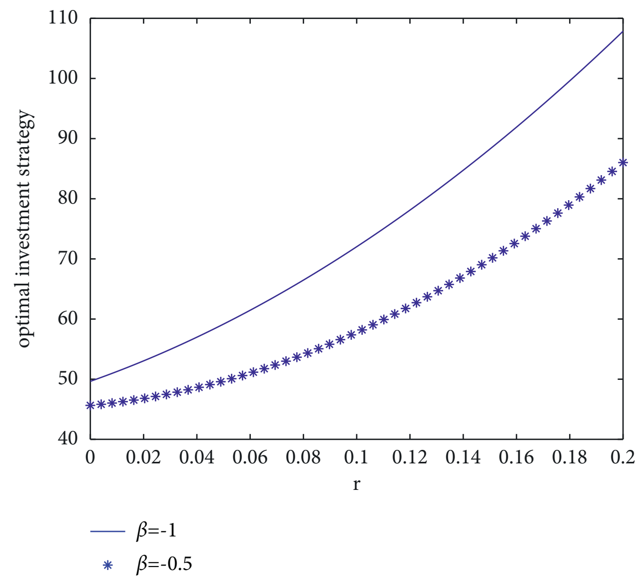

Figure 7: Influence of r on optimal consumption ratio C_t^*.

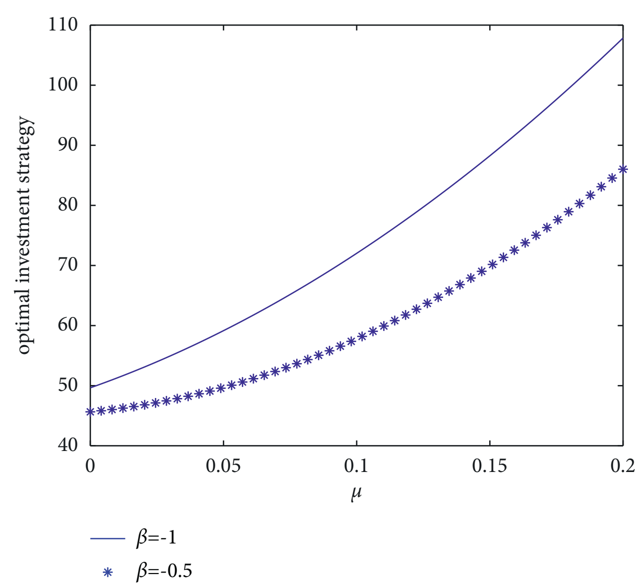

Figure 8: Influence of μ on optimal consumption ratio C_t^*.

where $f = f(t, y) = \int_t^T e^{A(u)+B(u)y} du + \phi e^{A(t)+B(t)y}$, *and* $y = s^{-2\beta}$, *$A(t)$ and $B(t)$ are given by equations (49) and (52).*

Proof. Following equations can be obtained from equation (7):

$$\begin{aligned}\pi_t^* &= -\frac{\mu - r}{v k^2 s^{2\beta}} \frac{H_v}{H_{vv}} - \frac{s}{v} \frac{H_{vs}}{H_{vv}} \\ &= \frac{\mu - r}{\gamma k^2 s^{2\beta}} - \frac{s g_s}{\gamma g} \text{ (According to Equation (20))} \\ &= \frac{\mu - r}{\gamma k^2 s^{2\beta}} - \frac{2\beta s^{-2\beta} h_y}{\gamma h} \text{ (According to Equation (22))} \\ &= \frac{\mu - r}{\gamma k^2 s^{2\beta}} - \frac{2\beta}{\gamma s^{2\beta}} \frac{\gamma f^{\gamma-1} f_y}{f^\gamma} \text{ (According to Equation (24))} \\ &= \frac{\mu - r}{\gamma k^2 s^{2\beta}} - \frac{2\beta}{s^{2\beta}} \frac{f_y}{f}.\end{aligned} \tag{54}$$

Because $U(c) = c^{1-\gamma}/(1-\gamma)$ and $U_1'(c) = c^{-\gamma}$, according to (8): $U_1'(c) = e^{\delta t} H_v$ and the equation in H_v, we have the equation$e^{\delta t} H_v = e^{\delta t} e^{-\delta t} g v^{-\gamma} = c^{-\gamma}$. Therefore, $c = g^{-1/\gamma} v$. Then, by $g = h(t, y)$ and $h(t, y) = [f(t, y)]^{1/\gamma}$, the optimal ratio is obtained as follows:

$$c_t^* = \frac{v}{f}. \tag{55}$$

So far, Theorem 4 is proved.

Special cases are discussed as follows.

(1) When $\gamma \longrightarrow 1$, the optimal investment and consumption problem (5) of the optimal strategy are as follows:

$$\pi_t^* = \frac{\mu - r}{k^2 s^{2\beta}}, \tag{56}$$

$$c_t^* = \frac{v}{f} = \frac{v}{\phi e^{-\delta(T-t)} + \left(1/\delta\left(1 - e^{-\delta(T-t)}\right)\right)}. \tag{57}$$

It shows that when $\phi = 1$, $U_1(x) = U_2(x)$ is power utility function, and $\gamma \longrightarrow 1$ power utility function degenerates into an exponential utility function. We know from (56) and (57) that the result is exactly the optimal ratio under the exponential utility function.

(2) If $\beta \longrightarrow 0$, the CEV model degenerates to GBM model.

Equations (35) and (36) are simplified as follows:

$$B_t + \frac{(\mu - r)^2 (1 - \gamma)}{2k^2 \gamma^2} = 0, \quad B(T) = 0, \tag{58}$$

$$A_t + \frac{1}{\gamma} (r(1 - \gamma) - \delta) = 0, \quad A(T) = 0. \tag{59}$$

(58) and (59), respectively, are solved to get the following results:

$$\begin{aligned}A(t) &= \frac{r(1-\gamma) - \delta}{\gamma} (T - t), \\ B(t) &= \frac{(\mu - r)^2 (1 - \gamma)}{2\gamma^2 k^2} (T - t).\end{aligned} \tag{60}$$

Assume $D = ((\mu - r)^2 (1 - \gamma)/2\gamma^2 k^2) + r(1 - \gamma) - \delta/\gamma$. Then,

$$f(t, y) = f(t) = \frac{1}{D}\left(e^{D(T-t)} - 1\right) + \phi e^{D(T-t)}. \tag{61}$$

It can be known from (61) that $f_y/f = 0$, and then, (44) and (45) degenerate into $\pi_t^* = \mu - r/\gamma k^2$ and $c_t^* = v/f$, where f is given by equation (50).

It shows that if the CEV model degenerates into the GBM model, the optimal investment ratio is only affected by the risk aversion coefficient.

(3) When $\phi = 0$, it means there is no retention utility. Assuming that the general form of the residual function is $B(V(T), T) = G(T)\ (V(T))^{1-\gamma}/\ (1 - \gamma)$, in this study $G(t) = \phi^\gamma$.So, when there is no bequest, there is $B(V(T), T) = 0 \Leftrightarrow \phi = 0$.

When $\phi = 1$, $B(V(t), t) = (V(t))^{1-\gamma}/(1 - \gamma)$; that is, it is the ordinary power utility function. In this case, (44) and (45) are the optimal investment-consumption ratio under the general power utility function. □

5. The Numerical Simulation Analysis

Suppose there are two forms of assets in the financial market. One is bank deposits or bonds, and the other is stocks. Referring to the estimation of Hong Kong stock option market by Yuen et al. [24], the parameter values are set as follows: $r = 0.03$, $\mu = 0.12$, $k = 16.16$, $\beta = -1$, and $S_0 = 67$. With reference to Chang et al. [15], $\delta = 0.1$, $\gamma = 3$ ($\gamma = 5$), and $V_0 = 100$ are set. Without loss of generality, let $\phi = 1$, $t \in [0, T]$, and $T = 5$.

(1) From Figure 1, we can see the influence of risk-free interest rate r on the optimal ratio π_t^*. As can be seen from Figure 1, the optimal ratio π_t^* is negatively correlated with the risk-free interest rate r; that is, when the deposit interest rate rises, investors are more willing to deposit or buy bonds and reduce the proportion of buying stocks. This phenomenon echoes the intuition.

(2) From Figure 2, we can see the effect of the expected instantaneous rate μ of return of a risky asset on the optimal ratio π_t^*. The optimal investment strategy π_t^* is positively correlated with the expected instantaneous rate μ. That is, when stocks have a high yield, people are always willing to put more money into stocks in order to get more yield.

(3) From Figure 3, we can see the influence of stock price S on the optimal investment ratio π_t^*. As can be seen from Figure 3, π_t^* is positively correlated with the price S of risky assets. Under the CEV model, when the stock price increases, the instantaneous volatility of the stock price ks_t^{β} becomes smaller, and the investment risk accordingly decreases. Household investors are more likely to invest in stocks with high returns and relatively low risk in order to earn more wealth.

(4) From Figure 4, we can see the influence of k on the optimal investment ratio π_t^*. As can be seen from Figure 4, the optimal investment strategy π_t^* is negatively correlated with k. Under the CEV model, the higher the value of k, the higher the instantaneous volatility ks_t^{β} of the stock price will be, and the higher the investment risk will be. As a result, investors will reduce their exposure to stocks.

(5) From Figure 5, we can see the influence of β on the optimal ratio π_t^*. As can be seen from Figure 5, the optimal investment ratio π_t^* is negatively correlated with the constant elasticity coefficient β. Under the CEV model, generally $\beta < 0$, when the constant elastic coefficient β increases slightly, the instantaneous volatility ks_t^{β} of the stock price becomes larger and larger, which leads to higher investment risks. We can also draw the conclusion that the amount of assets that household investors invest in stocks under the CEV model is more than that under the GBM model.

(6) From Figure 6, we can see the influence of the risk aversion coefficient γ on π_t^*. Figure 6 also shows that the optimal ratio π_t^* is negatively correlated with the risk aversion coefficient γ. In fact, under the power utility function, the greater the risk aversion coefficient γ is, the more risk aversion household investors will have, so they are unwilling to invest more in stocks with high risks. This situation is consistent with the reality.

(7) Figure 7 shows the effect of risk-free interest rate r on C_t^*. As can be seen from Figure 7, optimal consumption C_t^* is positively correlated with risk-free interest rate r. In other words, when the deposit interest rate rises, investors are more willing to deposit or buy bonds and less to buy stocks, but the overall expectation of wealth increases, so investors are willing to spend more.

(7) Figure 8 shows the effect of the expected instantaneous rate μ on C_t^*. Intuitively, there is a positive correlation between optimal consumption and μ. That is, when the return rate of stocks is high, the share of wealth invested in stocks will increase, so will the overall expected return, and people will increase consumption. This phenomenon is verified by Figure 8.

As for the factors that affect the consumption strategy, we draw a conclusion that all the factors that can cause the increase in the total expected income can cause the increase in consumption.

6. Summary

The research of this study makes rational investment of household wealth to achieve the maximization of investment-consumption utility. We assume that there are two types of assets in the household financial market, namely, risk-free assets and risky assets, in which it is assumed that the price of risky assets obeys the CEV model. Firstly, we obtain the HJB equation by dynamic programming method. Secondly, we select specific utility functions, such as logarithmic utility function and power utility function, and apply variable transformation method to obtain the display solution, that is, optimal ratio. Finally, the effects of market general parameters on optimal ratio are analyzed by numerical simulation and graphical results, and the results are consistent with the actual situation. In further research, we will consider more factors, such as transaction costs and borrowing.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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