Hindawi
Discrete Dynamics in Nature and Society
Volume 2022, Article ID 5172679, 12 pages
https://doi.org/10.1155/2022/5172679

Research Article

A Novel SIR Approach to Closeness Coefficient-Based MAGDM Problems Using Pythagorean Fuzzy Aczel–Alsina Aggregation Operators for Investment Policy

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Received 6 September 2022; Revised 15 November 2022; Accepted 18 November 2022; Published 13 December 2022

Academic Editor: Youssef N. Raffoul

In this study, a novel Pythagorean fuzzy aggregation operator is presented by combining the concepts of Aczel–Alsina ($\mathscr{A}\mathscr{A}$) T-norm and T-conorm operations for multiple attribute group decision-making (MAGDM) challenge for the superiority and inferiority ranking (SIR) approach. This approach has many advantages in solving real-life problems. In this study, the superiority and inferiority ranking method is illustrated and showed the effectiveness for decision makers by using multicriteria. The Aczel–Alsina aggregation operators on interval-valued IFSs, hesitant fuzzy sets (HFSs), Pythagorean fuzzy sets (PFSs), and T-spherical fuzzy sets (TSFSs) for multiple attribute decision-making (MADM) issues have been proposed in the literature. In addition, we propose a Pythagorean fuzzy Aczel–Alsina weighted average closeness coefficient ($\mathrm{PF}-\mathscr{A}\mathscr{A}-\mathrm{WA}-\mathscr{C}\mathscr{C}$) aggregation operator on the basis of the closeness coefficient for MAGDM challenges. To highlight the relevancy and authenticity of this approach and measure its validity, we conducted a comparative analysis with the method already in vogue.

1. Introduction

The superiority and inferiority ranking (SIR) approach for MAGDM is essential to decision-making (DM) challenges. It provides the most desirable and attractive option from a set of alternatives. The (SIR) approach was first introduced by Xu [1] in 2001. In this approach, alternatives are ranked by superiority and inferiority flows. The advantages of the SIR method are that it associates the properties of other MCDM problems such as PROMETHEE, SAW, and TOPSIS. Tam et al. [2] used this method for selecting concrete pumps in 2004. Tam and Tong [3] utilized this method in development projects with a grey aggregation approach in 2008. Liu [4] introduced the SIR approach for IFSs in 2010. Ma et al. [5] continued the SIR approach with HFSs and interval-valued HFSs in 2014. Peng and Yang [6] proposed this method for PFS and showed few results in 2015. Rouhani [7] employed the fuzzy SIR method in the IT field. Chen [8] introduced a PROMETHEE-based outranking approach for PFNs in 2018. Tavana et al. [9] introduced the IF-grey SIR approach in 2018. Selvaraj and Samayan [10] extended the SIR method for HPFSs for MCDM challenges in 2020.

1.1. Research Gap and Motivation of the Study. Atanassov [11] suggested the idea of IFS, a successful extension of Zadeh's fuzzy set theory [12] that deals with vagueness and uncertainties in the data. Every element in the IFS is represented by an ordered pair of the degree of membership and degree of nonmembership, whose sum ranges from zero to one. However, in some cases, the sum of membership and nonmembership degrees provided by the DM may be greater than one, but their square sum is less than or equal to one. Therefore, Yager [13–15] introduced PFS, which satisfies the condition that the sum of the square of its membership and nonmembership degrees is less than or equal to one. In addition, Yager [13–15] proposed different kinds of aggregation operators for DM problems, where ambiguity is found in the other basis of achievement.

Triangular norms play an essential role in decision-making problems. Menger [16] was the first to introduce triangular norms. Deschrijver et al. [17] discussed T-norm (TN) and T-conorm (TCN) for IFSs. There are many TN and TCN that are used for the aggregation of data. Those are Lukasiewicz TN and TCN [18], product TN, and sum TCN [19], Archimedian TN and TCN, [20], and drastic TN and TCN [21]. These norms have an essential role in the formation of aggregation operators. Aczel–Alsina ($\mathscr{A}\mathscr{A}$) [22] proposed a new idea of TN and TCN known as Aczel–Alsina ($\mathscr{A}\mathscr{A}$) TN and TCN. Many researchers used $\mathscr{A}\mathscr{A}$TN and TCN and concluded that these norms and conorms give better results due to their parameters. Senapati et al. [23–30] recognized the IF, IVIF, HF, PF, IVPF, and linguistic IF aggregation operators based on the $\mathscr{A}\mathscr{A}$-TN and $\mathscr{A}\mathscr{A}$-TCN by using the MADM challenges. Hussain et al. [31] used the $\mathscr{A}\mathscr{A}$ aggregation operators on TSFSs. Hussain et al. [32] considered $\mathscr{A}\mathscr{A}$ aggregation operators on PFSs using the MADM problems.

Many scholars worked on decision-making processes and also introduced different approaches. There are several approaches in the literature mentioned above. By inspiring this trend of scholars, we also invented a new approach for decision-making under multiple-attribute alternatives. We developed Pythagorean fuzzy Aczel–Alsina weighted average closeness coefficient ($\mathrm{PF}-\mathscr{A}\mathscr{A}-\mathrm{WA}-\mathscr{C}\mathscr{C}$) aggregation operators, applied them to the Pythagorean environment, and ranked the superiority and inferiority of the alternatives. Specially, we evaluate the group decision-making process of multicriteria known as MAGDM and for ranking the best and worst results of the alternatives. We used the superiority and inferiority ranking method under the Pythagorean data. Furthermore, we expressed the validity and effectiveness of the proposed approach and we solved a mathematical example for selecting the best stock of Internet stocks for a valuable investment. Finally, we compared the developed approach with existing studies and showed its graphical representations.

1.2. Contributions of this Study. The rest of the study is structured as follows. In Section 2, basic concepts of IFS, PFS, TN, TCN, $\mathscr{A}\mathscr{A}-\mathrm{TN}$, and $\mathscr{A}\mathscr{A}-\mathrm{TCN}$ are briefly reviewed. Furthermore, in this section, we discuss $\mathscr{A}\mathscr{A}$ operations on Pythagorean fuzzy numbers (PFNs), and some Pythagorean fuzzy Aczel–Alsina (PF-$\mathscr{A}\mathscr{A}$) averaging aggregation operators are defined. In Section 3, we introduce a novel $\mathrm{PF}-\mathscr{A}\mathscr{A}-\mathrm{WA}-\mathscr{C}\mathscr{C}$ aggregation operator for solving MAGDM challenges for the SIR approach and with their properties. In Section 4, we developed the SIR approach to deal with MAGDM challenges for $\mathrm{PF}-\mathscr{A}\mathscr{A}-\mathrm{WA}-\mathscr{C}\mathscr{C}$ operator. In Section 5, an approach is depicted by a numerical example. In Section 6, a comparison of suggested results is done with the results already present. Finally, we concluded this study with Section 7.

2. Preliminaries

In this section, we summarize the requisite knowledge associated with IFSs and PFSs with their operations and operators utilizing Aczel–Alsina T-Norm ($\mathscr{A}\mathscr{A}$-TN). We also discuss more familiarized ideas, which are helpful in sequential analysis.

2.1. IFS and PFS. Atanassov [11] introduced IFS theory, which is the extension of Fuzzy set theory (FS) [12].

Definition 1 (see [11]). For the universe of discourse $\mathscr{X}$, an IFS "$\widehat{I}$" is defined as

$$\widehat{I} = \left\{ < x, \mu_{\widehat{I}}(x), \nu_{\widehat{I}}(x) > : x \in \mathscr{X} \right\}, \tag{1}$$

where $\mu_{\widehat{I}}(x) \in [0, 1]$ denotes the degree of membership and $\nu_{\widehat{I}}(x) \in [0, 1]$ denotes the degree of nonmembership of x in $\widehat{I}$, respectively, with the condition $0 \leq \mu_{\widehat{I}}(x) + \nu_{\widehat{I}}(x) \leq 1$, for all $x \in \mathscr{X}$.

Yager [13–15] proposed the idea of PFS as a generalization of IFS with the modified condition that the sum of squares of the degree of membership and degree of nonmembership is less than or equal to 1. In contrast to IFSs, PFSs include more space for the selection of grades.

Definition 2 (see [13–15]). For universe of discourse $\mathscr{X}$, a PFS "$\dot{P}$" is defined as

$$\dot{P} = \left\{ < x, \mu_{\dot{P}}(x), \nu_{\dot{P}}(x) > : x \in \mathscr{X} \right\}, \tag{2}$$

where $\mu_{\dot{P}} \in [0,1]$ denotes the degree of membership and $\nu_{\dot{P}} \in [0,1]$ denotes the degree of nonmembership of x in $\dot{P}$, respectively, with the condition $0 (\mu_{\dot{P}}(x))^2 + (\nu_{\dot{P}}(x))^2 \leq 1$. The degree of indeterminacy is$\pi_{\dot{P}}(x) = \sqrt{1 - (\mu_{\dot{P}}(x))^2 - (\nu_{\dot{P}}(x))^2}$.

For convenience, Zhang and Xu [33] denoted Pythagorean fuzzy number (PFN) $(\mu_{\dot{P}}(x), \nu_{\dot{P}}(x))$ by $(\mu_{\dot{P}}, \nu_{\dot{P}})$.

Definition 3 (see [33]). The distance between two Pythagorean fuzzy numbers $\dot{P}_1 = (\mu_{\dot{P}_1}, \nu_{\dot{P}_1})$and $\dot{P}_2 = (\mu_{\dot{P}_2}, \nu_{\dot{P}_2})$ is defined as

$$d\left(\dot{P}_1, \dot{P}_2\right) = \frac{1}{2}\left(\left|\left(\mu_{\dot{P}_1}\right)^2 - \left(\mu_{\dot{P}_2}\right)^2\right| + \left|\left(\nu_{\dot{P}_1}\right)^2 - \left(\nu_{\dot{P}_2}\right)^2\right| + \left|\left(\pi_{\dot{P}_1}\right)^2 - \left(\pi_{\dot{P}_2}\right)^2\right|\right), \tag{3}$$

Definition 4 (see [33]). The score function of PFN $\dot{P} = (\mu_{\dot{P}}, \nu_{\dot{P}})$ is defined as

$$Sc(\dot{P}) = (\mu_{\dot{P}})^2 - (\nu_{\dot{P}})^2, \tag{4}$$

where $Sc(\dot{P}) \in [-1, 1]$.

For Pythagorean fuzzy numbers $\dot{P}_1$ and $\dot{P}_2$,

(1) If $Sc\ (\dot{P}_1) < \dot{P}c\ (\dot{P}_2)$, then $\dot{P}_1 \prec \dot{P}_2$
(2) If $Sc\ (\dot{P}_1) > \dot{P}c\ (\dot{P}_2)$, then $\dot{P}_1 \succ \dot{P}_2$
(3) If $Sc\ (\dot{P}_1) = \dot{P}c\ (\dot{P}_2)$, then $\dot{P}_1 \sim \dot{P}_2$

Definition 5 (see [33, 34]). the accuracy function of Pythagorean fuzzy number $\dot{P} = (\mu_{\dot{P}}, \nu_{\dot{P}})$ is defined as

$$\widehat{A}c(\dot{P}) = (\mu_{\dot{P}})^2 + (\nu_{\dot{P}})^2, \tag{5}$$

where$'\,\widehat{A}c(\dot{P}) \in [0, 1]$.

(1) If $Sc\ (\dot{P}_1) < \dot{P}c\ (\dot{P}_2)$, then $\dot{P}_1 \prec \dot{P}_2$.
(2) If $Sc\ (\dot{P}_1) > Sc\ (\dot{P}_2)$, then $\dot{P}_1 \succ \dot{P}_2$.
(3) If $Sc\ (\dot{P}_1) = Sc\ (\dot{P}_2)$, then
  (a) If $\widehat{A}c\ (\dot{P}_1) > \widehat{A}c\ (\dot{P}_2)$, then $\dot{P}_1 \succ \dot{P}_2$
  (b) If $\widehat{A}c\ (\dot{P}_1) < \widehat{A}c\ (\dot{P}_2)$, then $\dot{P}_1 \prec \dot{P}_2$
  (c) If $\widehat{A}c\ (\dot{P}_1) = \widehat{A}c\ (\dot{P}_2)$, then $\dot{P}_1 \sim \dot{P}_2$

2.2. Triangular Norm, Triangular Co-Norm, and Aczel–Alsina Triangular Norm. Triangular norms (TNs) are the particular classes of functions that act as a tool for interpreting the conjunction of fuzzy logic and the intersection of fuzzy sets. Menger [16] was the first to introduce triangular norms for statistical metric spaces. They have many applications in decision-making and aggregation. We shall here examine some ideas necessary for this study's development.

Definition 6 (see [35] [36, 37]). A function $\check{T}$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a triangular norm (TN) if the following axioms are satisfied, $\forall \dot{\alpha}, \beta, \gamma \in [0, 1]$:

(1) Symmetry: $\check{T}\ (\dot{\alpha},\beta') = \check{T}\ (\beta',\dot{\alpha})$
(2) Associativity: $\check{T}\ (\dot{\alpha},\ \check{T}\ (\beta'\ ,\ \gamma))\ \dot{\alpha} = \check{T}\ (\check{T}\ (\dot{\alpha},\beta'),\ \gamma)$
(3) Monotonicity: $\check{T}\ (\dot{\alpha},\beta') \le \check{T}\ (\dot{\alpha},\gamma)$ if $\beta' \le \gamma$
(4) One Identity: $\check{T}\ (1,\ \dot{\alpha}) = \dot{\alpha}$

Examples of TNs are $\forall \dot{\alpha}, \beta, \gamma \in [0, 1]$:

(1) Product triangular norm: $\check{T}_{\mathrm{pro}}\ (\dot{\alpha},\ \beta') = \dot{\alpha}.\ \beta'$.
(2) Minimum triangular norm: $\check{T}_{\min}\ (\dot{\alpha}, \beta') = \min\ (\dot{\alpha}, \beta')$.
(3) Lukasiewicz triangular norm: $\check{T}_{\mathrm{luk}}\ (\dot{\alpha},\ \beta') = \max\ (\dot{\alpha} + \beta' - 1, 0)$.
(4) Drastic triangular norm:

$$\check{T}_{\mathrm{dra}}(\dot{\alpha}\beta') = \begin{cases} \dot{\alpha}, & if\,\beta' = 1, \\ \beta', & if\,\dot{\alpha} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Definition 7 (see [35] [36, 37]). A function $\check{S}$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is triangular conorm (TCN) if the following axioms are satisfied, $\forall \dot{\alpha}, \beta, \gamma \in [0, 1]$:

(1) Symmetry: $\check{S}\ (\dot{\alpha},\ \beta) = \check{S}\ (\beta',\dot{\alpha})$
(2) Associativity: $\check{S}\ (\dot{\alpha},\ \check{S}\ (\beta',\ )) = \check{S}\ (\check{S}\ (\dot{\alpha},\beta'),\ \gamma)$
(3) Monotonicity: $\check{S}\ (\dot{\alpha},\beta') \le \check{S}\ (\dot{\alpha},\gamma)$ if $\beta' \le \gamma$
(4) Zero Identity: $\check{S}\ (0,\ \dot{\alpha}) = \dot{\alpha}$

Examples of TCN*s* are $\forall \dot{\alpha}, \beta, \gamma \in [0, 1]$:

(1) Probabilistic sum triangular co-norm: $\check{S}_{\mathrm{PS}}\ (\dot{\alpha},\beta') = \dot{\alpha} + \beta' - \dot{\alpha}, \beta'$.
(2) Maximum triangular co-norm: $\check{S}_{\max}\ (\dot{\alpha},\beta') = \max\ (\dot{\alpha}\ ,\beta')$.
(3) Lukasiewicz triangular co-norm: $\check{S}_{\mathrm{luk}}\ (\dot{\alpha},\beta') = \min\ \{\dot{\alpha} + \beta', 1\}$.
(4) Drastic triangular co-norm:

$$\check{S}_{\mathrm{dra}}(\dot{\alpha},\beta') = \begin{cases} \dot{\alpha}, & if\,\beta' = 0, \\ \beta', & if\,\dot{\alpha} = 0, \\ 1, & \text{otherwise.} \end{cases} \tag{7}$$

Definition 8 (see [22]). Aczel–Alsina presented triangular norm in the circumstance of functional equations, known as Aczel–Alsina triangular norm ($\mathscr{A}\mathscr{A}$ − TN). For $\varphi \in [0, \infty]$, it stated as

$$\check{T}_A^{\varphi}(\dot{\alpha},\beta) = \begin{cases} \check{T}_{\mathrm{dra}}(\dot{\alpha},\beta'), & if\,\varphi = 0, \\ \min\,(\dot{\alpha},\beta'), & if\,\varphi = \infty, \\ e^{-\left(\left(-\log\dot{\alpha}\right)^{\varphi} + \left(-\log\beta'\right)^{\varphi}\right)^{1/\varphi}}, & \text{otherwise.} \end{cases} \tag{8}$$

For $\varphi \in [0, \infty]$, $\mathscr{A}\mathscr{A}$ − TCN is stated as

$$\check{S}_A^{\varphi}(\dot{\alpha},\beta') = \begin{cases} \check{S}_{\mathrm{dra}}(\dot{\alpha},\beta'), & if\,\varphi = 0, \\ \max\,(\dot{\alpha},\beta'), & if\,\varphi = \infty\ , \\ 1 - e^{-\left(\left(-\log\,(1-\dot{\alpha})\right)^{\varphi} + \left(-\log\,(1-\beta')\right)^{\varphi}\right)^{1/\varphi}}, & \text{otherwise.} \end{cases} \tag{9}$$

Definition 9 (see [32]). Let $\dot{P}$ and $\dot{R}$ be two PFSs. Then, the generalizations of intersection and union are defined as

$$\begin{aligned} \dot{P} \cap_{\check{T},\check{S}} \dot{R} &= \left\{ < x, \check{T}\left\{\mu_{\dot{P}}(x), \mu_{\dot{R}}(x)\right\}, \check{S}\left\{\nu_{\dot{P}}(x), \nu_{\dot{R}}(x)\right\} >: x \in \mathscr{X} \right\}, \\ \dot{P} \cup_{\check{T},\check{S}} \dot{R} &= \left\{ < x, \check{S}\left\{\mu_{\dot{P}}(x), \mu_{\dot{R}}(x)\right\}, \check{T}\left\{\nu_{\dot{P}}(x), \nu_{\dot{R}}(x)\right\} >: x \in \mathscr{X} \right\}, \end{aligned} \tag{10}$$

where $\check{T}$ is for TN and $\check{S}$ is for TCN.

2.3. Aczel–Alsina Operations on Pythagorean Fuzzy Numbers. We suppose TN $\check{T}$ and TCN $\check{S}$ represent $\mathscr{A}\mathscr{A}$ product $\check{T}_A$ and $\mathscr{A}\mathscr{A}$ sum $\check{S}_{\dot{A}}$, respectively. Then, the generalization of intersection and union over two PFSs "$\dot{P}$" and "$\dot{R}$" turn out to be the $\mathscr{A}\mathscr{A}$ product (represented by $\dot{P} \otimes \dot{R}$) and $\mathscr{A}\mathscr{A}$ sum (represented by $\dot{P} \oplus \dot{R}$) over two PFSs "$\dot{P}$" and "$\dot{R}$," respectively, [32]:

$$\begin{aligned}\dot{P}\otimes\dot{\mathrm{R}} &= \left\{< x, \check{\mathrm{T}}_{\mathrm{A}}\left\{\mu_{\dot{P}}(x), \mu_{\dot{\mathrm{R}}}(x)\right\}, \check{\mathrm{S}}_{\mathrm{A}}\left\{\nu_{\dot{P}}(x), \nu_{\dot{\mathrm{R}}}(x)\right\} >: x \in \mathscr{X}\right\},\\ \dot{P}\oplus\dot{\mathrm{R}} &= \left\{< x, \check{\mathrm{S}}_{\mathrm{A}}\left\{\mu_{\dot{P}}(x), \mu_{\dot{\mathrm{R}}}(x)\right\}, \check{\mathrm{T}}_{\mathrm{A}}\left\{\nu_{\dot{P}}(x), \nu_{\dot{\mathrm{R}}}(x)\right\} >: x \in \mathscr{X}\right\}.\end{aligned} \tag{11}$$

Definition 10 (see [32]). Let $\dot{P} = (\mu_{\dot{P}}, \nu_{\dot{P}})$, $\dot{P}_1 = (\mu_{\dot{P}_1}, \nu_{\dot{P}_1})$, and $\dot{P}_2 = (\mu_{\dot{P}_2}, \nu_{\dot{P}_2})$ be three Pythagorean fuzzy numbers, with $\beth \geq 1$ and $\Omega > 0$. Then, operations of PFNs with Definition 8 is defined as

$$\begin{aligned}\dot{P}_1\oplus\dot{P}_2 &= \left(\sqrt{1 - e^{-\left(\left(-\log\left(1-\left(\mu_{\dot{P}_1}\right)^2\right)\right)^{\beth}+\left(-\log\left(1-\left(\mu_{\dot{P}_2}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}}, e^{-\left(\left(-\log\left(\nu_{\dot{P}_1}\right)\right)^{\beth}+\left(-\log\left(\nu_{\dot{P}_2}\right)\right)^{\beth}\right)^{1/\beth}}\right),\\ \dot{P}_1\oplus\dot{P}_2 &= \left(e^{-\left(\left(-\log\left(\mu_{\dot{P}_1}\right)\right)^{\beth}+\left(-\log\left(\mu_{\dot{P}_2}\right)\right)^{\beth}\right)^{1/\beth}}, \sqrt{1 - e^{-\left(\left(-\log\left(1-\left(\nu_{\dot{P}_1}\right)^2\right)\right)^{\beth}+\left(-\log\left(1-\left(\nu_{\dot{P}_2}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}}\right),\\ \Omega\dot{P} &= \left(\sqrt{1 - e^{-\left(\Omega\left(-\log\left(1-\left(\mu_{\dot{P}}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}}, e^{-\left(\Omega\left(-\log\left(\nu_{\dot{P}}\right)\right)^{\beth}\right)^{1/\beth}}\right),\\ \dot{P}^{\Omega} &= \left(e^{-\left(\Omega\left(-\log\left(\mu_{\dot{P}}\right)\right)^{\beth}\right)^{1/\beth}}, \sqrt{1 - e^{-\left(\Omega\left(-\log\left(1-\left(\nu_{\dot{P}}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}}\right).\end{aligned} \tag{12}$$

2.4. Pythagorean Fuzzy Aczel–Alsina (PF-$\mathscr{A}\mathscr{A}$) Average Aggregation Operators (AOs). Hussain et al. [32] introduced some PF-$\mathscr{A}\mathscr{A}$ average AOs under PFNs. Additionally, some related properties are also discussed.

Definition 11. Let $\dot{P}_i = (\mu_{\dot{P}}, \nu_{\dot{P}_i})$, $(i = 1, 2, \ldots, n)$, be an accumulation of PFNs. Let the weight vector $\text{ɰ} = (\text{ɰ}_1, \text{ɰ}_2, \ldots, \text{ɰ}_n)^T$ of $\dot{P}_i$, , $(i = 1, 2, \ldots, n)$ with $\text{ɰ}_i > 0$, $\text{ɰ}_i \in [0, 1]$,and $\sum_{i=1}^{n} \text{ɰ}_i = 1$. Then, PF-$\mathscr{A}\mathscr{A}$-WA operator is a function, (PF-$\mathscr{A}\mathscr{A}$-WA): $(\dot{P})^n \longrightarrow \dot{P}$, defined as

$$PF - \mathscr{A}\mathscr{A} - \mathscr{W}\mathscr{A}\left(\dot{P}_1, \dot{P}_2 \ldots, \dot{P}_n\right) = \oplus_{i=1}^{n}\left(\text{ɰ}_i\dot{P}_i\right) = \text{ɰ}_1\dot{P}_1\oplus\text{ɰ}_2\dot{P}_2\oplus, \ldots, \oplus\text{ɰ}_n\dot{P}_n. \tag{13}$$

Theorem 1. *Let $\dot{P}_i = (\mu_{\dot{P}_i}, \nu_{\dot{P}_i})$, $(i = 1, 2, \ldots, n)$, be an accumulation of PFNs. Let the weight vector $\text{ɰ} = (\text{ɰ}_1, \text{ɰ}, \ldots, \text{ɰ}_n)^T$ of $\dot{P}_i$, $(i = 1, 2, \ldots, n)$ with $\text{ɰ}_i > 0$, $\text{ɰ}_i \in [0, 1]$, and $\sum_{i=1}^{n} \text{ɰ}_i = 1$. Then, aggregated values of PFNs by PF-$\mathscr{A}\mathscr{A}$-WA operator is defined as*

$$PF - \mathscr{A}\mathscr{A} - \mathscr{W}\mathscr{A}\left(\dot{P}_1, \dot{P}_2, \ldots, \dot{P}_n\right) = \left(\sqrt{1 - \mathrm{e}^{-\left(\sum_{i=1}^{n}\text{ɰ}_i\left(-\log\left(1-\left(\mu_{\dot{P}_i}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}}, \mathrm{e}^{-\left(\sum_{i=1}^{n}\text{ɰ}_i\left(-\log\left(\nu_{\dot{P}_i}\right)\right)^{\beth}\right)^{1/\beth}}\right). \tag{14}$$

3. Proposed $PF - \mathscr{A}\mathscr{A} - WA - \mathscr{C}\mathscr{C}$ Operator for Solving MAGDM Problems

Now, we introduce PF – $\mathscr{A}\mathscr{A}$ – WA – $\mathscr{C}\mathscr{C}$ operators with their properties under PFSs to deal with MAGDM challenges for decision makers.

Definition 12. suppose $\dot{P}_i = (\mu_{\dot{P}_i}, \nu_{\dot{P}_i})$, $(i = 1, 2, \ldots, n)$ is an accumulation of PFNs. Let ξ_k $(k = 1, 2, \ldots, l)$ be the individual measure degree via weight $\text{ɰ} = \text{ɰ}_1, \text{ɰ}_2, \ldots, \text{ɰ}_l$ of experts. Let $\mathrm{S}_k = (\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_l)^T$ be the normalized vector of individual measure degrees. Then, PF – $\mathscr{A}\mathscr{A}$ – WA – $\mathscr{C}\mathscr{C}$ operator is a function: (PF – $\mathscr{A}\mathscr{A}$ – WA – $\mathscr{C}\mathscr{C}$): $(\dot{P})^n \longrightarrow \dot{P}$ defined as

$$PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n\right)=\oplus_{i,k=1}^{n}\left(S_k\dot{P}_i\right)=S_1\dot{P}_1\oplus S_2\dot{P}_2\oplus,\ldots,\oplus S_n\dot{P}_n. \tag{15}$$

Theorem 2. *suppose $\dot{P}_i=(\mu_{\dot{P}_i},\nu_{\dot{P}_i})$, $(i=1,2,\ldots,n)$, is an accumulation of PFNs. Let ξ_k $(k=1,2,\ldots,l)$ be the individual measure degree of experts via weight $\text{ɰ}=\text{ɰ}_1,\text{ɰ}_2,\ldots,\text{ɰ}_l$. Let $S_k=(S_1,S_2,\ldots,S_l)^T$ be the normalized vector of individual measure degrees. Then, aggregated values of PFNs by $PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}$ operator are defined as*

$$PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_i^{(1)},\dot{P}_i^{(2)},\ldots,\dot{P}_i^{(l)}\right)=\left(\sqrt{1-e^{-\left(\sum_{k=1}^{l}S_k\left(-\log\left(1-\left(\mu_i^{(k)}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}},e^{-\left(\sum_{k=1}^{l}S_k\left(-\log\left(\nu_i^{(k)}\right)\right)^{\beth}\right)^{1/\beth}}\right). \tag{16}$$

Proof. By induction, for $n=2$,

$$\begin{aligned}
&PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2\right)=\oplus_{i,k=1}^{2}\left(S_k\dot{P}_i\right)=S_1\dot{P}_1\oplus S_2\dot{P}_2\\
&=\left(\sqrt{1-e^{-\left(S_1\left(-\log\left(1-\left(\mu_{\dot{P}_1}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}},e^{-\left(S_1\left(-\log\left(\nu_{\dot{P}_1}\right)\right)^{\beth}\right)^{1/\beth}}\right)\oplus\left(\sqrt{1-e^{-\left(S_2\left(-\log\left(1-\left(\mu_{\dot{P}_2}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}},e^{-\left(S_2\left(-\log\left(\nu_{\dot{P}_2}\right)\right)^{\beth}\right)^{1/\beth}}\right)\\
&=\left(\sqrt{1-e^{-\left(S_1\left(-\log\left(1-\left(\mu_{\dot{P}_1}\right)^2\right)\right)^{\beth}+S_2\left(-\log\left(1-\left(\mu_{\dot{P}_2}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}},e^{-\left(S_1\left(-\log\left(\nu_{\dot{P}_1}\right)\right)^{\beth}+S_2\left(-\log\left(\nu_{\dot{P}_2}\right)\right)^{\beth}\right)^{1/\beth}}\right)\\
&=\left(\sqrt{1-e^{-\left(\sum_{i,k=1}^{2}S_k\left(-\log\left(1-\left(\mu_{\dot{P}_i}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}},e^{-\left(\sum_{i,k=1}^{2}S_k\left(-\log\left(\nu_{\dot{P}_i}\right)\right)^{\beth}\right)^{1/\beth}}\right),
\end{aligned} \tag{17}$$

which is true for $n=2$.

We suppose it is true for n, that is,

$$PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n\right)=\oplus_{i,k=1}^{n}\left(S_k\dot{P}_i\right)=S_1\dot{P}_1\oplus S_2\dot{P}_2\oplus,\ldots,\oplus S_n\dot{P}_n. \tag{18}$$

We have to show that it is true for $n+1$; that is,

$$
\begin{aligned}
&PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_{n+1}\right)=\oplus_{i,k=1}^{n+1}\left(S_k\dot{P}_i\right)=S_1\dot{P}_1\oplus S_2\dot{P}_2\oplus,\ldots,\oplus\zeta_{nS1}\dot{P}_{n+1}\\
&=\oplus_{i,k=1}^{n}\left(\zeta_k\dot{P}_i\right)\oplus\zeta_{n+1}\dot{P}_{n+1}\\
&=\left(\sqrt{1-e^{-\left(\sum_{i,k=1}^{n}S_k\left(-\log\left(1-\left(\mu_{\dot{P}_i}\right)^2\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}},e^{-\left(\sum_{i,k=1}^{n}S_k\left(-\log\left(\nu_{\dot{P}_i}\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}\right)\\
&\oplus\left(\sqrt{1-e^{-\left(\zeta_{n+1}\left(-\log\left(1-\left(\mu_{\dot{P}_{n+1}}\right)^2\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}},e^{-\left(\zeta_{n+1}\left(-\log\left(\nu_{\dot{P}_{n+1}}\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}\right)=\left(\sqrt{1-e^{-\left(\sum_{i,k=1}^{n+1}\zeta_k\left(-\log\left(1-\left(\mu_{\dot{P}_i}\right)^2\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}},e^{-\left(\sum_{i,k=1}^{n+1}\zeta_k\left(-\log\left(\nu_{\dot{P}_i}\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}\right),
\end{aligned}
\tag{19}
$$

which is true for $n+1$. Thus, it is true for all values of n. □

Theorem 3. *suppose $\dot{P}_i=(\mu_{\dot{P}_i},\nu_{\dot{P}_i})\,(i=1,2,\ldots,n)$ is an accumulation of PFNs. Let the weight vector be $\text{ɰ}=(\text{ɰ}_1,V_2,\ldots,V_n)^T$ of $\dot{P}_i\,(i=1,2,\ldots,n)$ with $\text{ɰ}_i>0$, $\text{ɰ}_i\in[0,1]$, and $\sum_{i=1}^{n}\text{ɰ}_i=1$. Then, the following properties are satisfied.*

(P1) Idempotency: for equal PFNs $\dot{P}_i=(\mu_{\dot{P}_i},\nu_{\dot{P}_i})\,(i=1,2,\ldots,n)$, that is, $\dot{P}_i=\dot{P}\forall i$, we have

$$PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n\right)=\dot{P}. \tag{20}$$

(P2) Boundedness: let $\dot{P}_i=(\mu_{\dot{P}_i},\nu_{\dot{P}_i})\,(i=1,2,\ldots,n)$, be an accumulation of PFNs, with $\dot{P}^-=\min(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n)$ and $\dot{P}^+=\max(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n)$. Then,

$$\dot{P}^-\le PF-\mathscr{A}\mathscr{A}-WA-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n\right)\le\dot{P}^+. \tag{21}$$

(P3) Monotonicity: we suppose $\dot{P}_i=(\mu_{\dot{P}_i},\nu_{\dot{P}_i})$ and $\dot{P}_i=(\mu_{\dot{P}_i},\nu_{\dot{P}_i})\,(i=1,2,\ldots,n)$ are any two accumulations of PFNs, and if $\dot{P}_i\le\dot{P}_i$ and $\forall\,(i=1,2,\ldots,n)$, with $\dot{P}^-=\min(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n)$ and $\dot{P}^+=\max(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n)$, then

$$PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n\right)\le PF-\mathscr{A}\mathscr{A}-\mathscr{W}\mathscr{A}-\mathscr{C}\mathscr{C}\left(\dot{P}_1,\dot{P}_2,\ldots,\dot{P}_n\right). \tag{22}$$

4. Application of the SIR Approach for PFSs with Novel Aggregation Operators in MAGDM

In this section, we use the $PF-\mathscr{A}\mathscr{A}-WA-\mathscr{C}\mathscr{C}$ operator to consider the MAGDM issues for PF data. Let $x=\{x_1,x_2,\ldots x_m\}$ be a family of alternatives and $c=\{c_1,c_2,\ldots c_n\}$ the collection of attributes. Let $e=\{e_1,e_2,\ldots e_l\}$ be a set of experts with $\text{ɰ}=(\text{ɰ}_1,\text{ɰ}_2,\ldots,\text{ɰ}_l)^T$ as weight vector. Let $\dot{P}(k)=(\dot{P}_{ij}^{(k)})_{m\times n}$, $(i=1,2,\ldots,m)$, $(j=1,2,\ldots,n)$, and $(k=1,2,\ldots,l)$, be the PF decision matrix (DM) given by the decision maker. $\mathscr{P}_{ij}^{(k)}$ represents the attribute value such that the alternative x_i satisfies the attribute c_j suggested by expert e_k. $\omega=(\text{ɰ}_j^{(k)})_{l\times n}$ is the attribute weight DM, where $\text{ɰ}_j^{(k)}$ denotes the weightage of attribute c_j suggested by expert e_k.

Step 1. find individual measure degree $\xi_k\,(k=1,2,\ldots,l)$ through weights of experts. The relative closeness coefficient is obtained as

$$\xi_k = \frac{d\left(\text{ɰ}_k, \text{ɰ}^-\right)}{d\left(\text{ɰ}_k, \text{ɰ}^-\right) + d\left(\text{ɰ}_k, \text{ɰ}^+\right)}. \tag{23}$$

$\text{ɰ}^- = \left(\min\left\{\mu_{\dot{P}_i}\right\}, \max\left\{\nu_{\dot{P}_i}\right\}\right)$ and $\dot{P}^+ = \left(\max\left\{\mu_{\dot{P}_i}\right\}, \min\left\{\nu_{\dot{P}_i}\right\}\right)$ with $0 \le \text{ɰ}_k \le 1$.

If $\xi_k \longrightarrow 1$, then $\text{ɰ}_k \longrightarrow \text{ɰ}^+$; if $\xi_k \longrightarrow 0$,then $\text{ɰ}_k \longrightarrow \text{ɰ}^-$.

Step 2. normalize $\xi_k\,(k = 1, 2, \ldots, l)$ for sum as a unit:

$$\mathrm{S}_k = \frac{\xi_k}{\sum_{k=1}^{l} \xi_k}, \tag{24}$$

which is the normalized vector $\mathrm{S}_k = \left(\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_l\right)^T$ as individual measure degrees.

Step 3. use $\mathrm{PF} - \mathscr{A}\mathscr{A} - \mathrm{WA} - \mathscr{C}\mathscr{C}$ operator to aggregate group viewpoints.

(a) Individual attributes' weight integration:

$$\overline{\text{ɰ}}_j = PF - \mathscr{A}\mathscr{A} - \mathscr{W}\mathscr{A} - \mathscr{C}\mathscr{C}\left(\text{ɰ}_j^{(1)}, \text{ɰ}_j^{(2)}, \ldots, \text{ɰ}_j^{(l)}\right) = \left(\sqrt{1 - e^{-\left(\sum_{k=1}^{l} S_k\left(-\log\left(1-\left(\mu_j^{(k)}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}}, e^{-\left(\sum_{k=1}^{l} S_k\left(-\log\left(\nu_j^{(k)}\right)\right)^{\beth}\right)^{1/\beth}}\right), \tag{25}$$

(b) Individual decision matrix integration:

$$\overline{\dot{P}}_{ij} = PF - \mathscr{A}\mathscr{A} - \mathscr{W}\mathscr{A} - \mathscr{C}\mathscr{C}\left(\dot{P}_{ij}^{(1)}, \dot{P}_{ij}^{(2)}, \ldots, \dot{P}_{ij}^{(l)}\right) = \left(\sqrt{1 - \mathrm{e}^{-\left(\sum_{k=1}^{l} S_k\left(-\log\left(1-\left(\mu_{ij}^{(k)}\right)^2\right)\right)^{\beth}\right)^{1/\beth}}}, e^{-\left(\sum_{k=1}^{l} S_k\left(-\log\left(\nu_{ij}^{(k)}\right)\right)^{\beth}\right)^{1/\beth}}\right). \tag{26}$$

The attribute weight vector $\overline{\text{ɰ}} = \left(\overline{\text{ɰ}}_1, \overline{\text{ɰ}}_2, \ldots, \overline{V}_n\right)$ and group integrated decision matrix $(\overline{\dot{P}}_{ij})_{m\times n}$ are obtained.

Step 4. determine PF superiority matrix (SM) and inferiority matrix (IM).

(a) Performance function:

$$S_{ij} = S\left(\overline{\dot{P}}_{ij}\right) = \frac{d\left(\overline{\dot{P}}_{ij}, \overline{\dot{P}}^-\right)}{d\left(\overline{\dot{P}}_{ij}, \overline{\dot{\mathrm{P}}}^-\right) + d\left(\overline{\dot{P}}_{ij}, \overline{\dot{P}}^+\right)},$$
$$\overline{\dot{P}}^- = \left\{c_j, \left(\min\left\{\mu_{\dot{P}_{ij}}\right\}, \max\left\{\nu_{\dot{P}_{ij}}\right\}\right)\right\}, \tag{27}$$
$$\overline{\dot{P}}^+ = \left\{c_j, \left(\max\left\{\mu_{\dot{P}}\right\}, \min\left\{\nu_{\dot{P}_{ij}}\right\}\right)\right\},$$

with $0 \le S_{ij} \le 1$; if $S_{ij} \longrightarrow 1$, then $\overline{\dot{P}}_{ij} \longrightarrow \overline{\dot{P}}^+$; if $S \longrightarrow 0$, then $\dot{P}_{ij} \longrightarrow \overline{\dot{P}}^-$.

(b) Preference intensity $PI_j\,(x_i, x_t)$: w define, $PI_j\,(x_i, x_t)$ $(t, i = 1, 2, \ldots, m, t \ne i;\ j = 1, 2, \ldots, n)$ as preference intensity of alternative x_i with alternative x_t to the parallel attribute c_j; that is,

$$PI_j\left(x_i, x_t\right) = \Psi_j\left(S_{ij} - S_{tj}\right) = \Psi_j\,(d), \tag{28}$$

here $\Psi_j\,(d)$ is generalized threshold functions or may defined themselves by experts.

(c) determine superiority index (S-I): $S = \left(S_{ij}\right)_{m\times n}$:

$$S_{ij} = \sum_{t=1}^{n} PI_j\left(x_i, x_t\right) = \sum_{t=1}^{n} \Psi_j\left(S_{ij} - S\right). \tag{29}$$

(d) determine inferiority index (I-I): $\mathrm{I} = \left(I_{ij}\right)_{m\times n}$

$$I_{ij} = \sum_{t=1}^{n} PI_j\left(x_t, x_i\right) = \sum_{t=1}^{n} \Psi_j\left(S_{tj} - S_{ij}\right). \tag{30}$$

Step 5 determine superiority flow:

$$\Psi^{>}(x_i) = PF - \mathscr{AA} - \mathscr{WA} - \mathscr{CC}\left(\overline{\text{ш}}_1, \overline{\text{ш}}_2, \ldots, \overline{\text{ш}}_n\right) = \left(\sqrt{1 - e^{-\sum_{j=1}^{n} S_{ij}\left(\sum_{k=1}^{l} S_k\left(-\log\left(1-\left(\mu_j^{(k)}\right)^2\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}}, e^{-\sum_{j=1}^{n} S_{ij}\left(\sum_{k=1}^{l} S_k\left(-\log\left(\nu_j^{(k)}\right)\right)^{\sqsupset}\right)^{1/\sqsupset}} \right). \tag{31}$$

Inferiority flow:

$$\Psi^{<}(x_i) = PF - \mathscr{AA} - \mathscr{WA} - \mathscr{CC}\left(\overline{\text{ш}}_1, \overline{\text{ш}}_2, \ldots, \overline{\text{ш}}_n\right) = \left(\sqrt{1 - e^{-\sum_{j=1}^{n} I_{ij}\left(\sum_{k=1}^{l} S_k\left(-\log\left(1-\left(\mu_j^{(k)}\right)^2\right)\right)^{\sqsupset}\right)^{1/\sqsupset}}}, e^{-\sum_{j=1}^{n} I_{ij}\left(\sum_{k=1}^{l} S_k\left(-\log\left(\nu_j^{(k)}\right)\right)^{\sqsupset}\right)^{1/\sqsupset}} \right). \tag{32}$$

We calculate the score function of $\Psi^{>}(x_i)$ and of $\Psi^{<}(x_i)$. This gives S-flow and I-flow of alternatives x_i as$x_i\left(\Psi^{>}(x_i), \Psi^{<}(x_i)\right)$. For greater $\Psi^{>}(x_i)$ and smaller $\Psi^{<}(x_i)$, alternative x_i is better.

Step 6. Superiority/ inferiority ranking.

(a) Superiority ranking rules.

(1) If $\Psi^{>}(x_i) > \Psi^{>}(x_t)$ and $\Psi^{<}(x_i) < \Psi^{<}(x_t)$, then $x_i \succ x_t$
(2) If $\Psi^{>}(\mathrm{x_i}) > \Psi^{>}(\mathrm{x_t})$ and $\Psi^{<}(\mathrm{x_i}) = \Psi^{<}(\mathrm{x_t})$, then $\mathrm{x_i} \succ \mathrm{x_t}$
(3) If $\Psi^{>}(\mathrm{x_i}) = \Psi^{>}(\mathrm{x_t})$ and $\Psi^{<}(\mathrm{x_i}) < \Psi^{<}(\mathrm{x_t})$, then $\mathrm{x_i} \succ \mathrm{x_t}$

(b) Inferiority ranking rules:

(1) If $\Psi^{>}(x_i) < \Psi^{>}(x_t)$ and $\Psi^{<}(x_i) > \Psi^{<}(x_t)$, then $x_i \prec x_t$
(2) If $\Psi^{>}(x_i) < \Psi^{>}(x_t)$ and $\Psi^{<}(x_i) = \Psi^{<}(x_t)$, then $x_i \prec x_t$
(3) If $\Psi^{>}(x_i) = \Psi^{>}(x_t)$ and $\Psi^{<}(x_i) > \Psi^{<}(x_t)$, then $x_i \prec x_t$

Step 7. combine the rules of superiority/ inferiority ranking for the best alternative (x_i).

5. Numerical Example

An investment company is interested in investing in Internet stocks. So, the company employs three brands of experts: market maker (e_1), dealer (e_2), and finder (e_3). They select four stocks: x_1 is GAD, x_2 is FUT, x_3 is NET, and x_4 is PUM, for three attributes, c_1 (market trend), c_2 (policy direction), and c_3 (yearly performance). The experts e_k evaluate stocks x_i relating to the attributes c_j and form the following three PF decision matrices $\mathscr{P}(k) = \left(\mathscr{P}_{ij}{}^{(k)}\right)_{4\times 3}$ in Table 1, weights of experts in Table 2, and attribute weights in Table 3.

For choosing the most attractive alternate(s), we have the following steps:

Step 1: we find the individual measure degree $\xi_k\,(k = 1, 2, 3)$ through weights of experts by using equation (23):

$$\xi_k = (0.0864, 0.6239, 0.8763)^T. \tag{33}$$

Step 2: we determine the normalized vector $\mathrm{S}_k\,(k = 1, 2, 3)$, by using equation (24):

$$\mathrm{S}_k = (0.0545, 0.3932, 0.5523)^T. \tag{34}$$

Step 3.

(a) determine the attributes weights integration $\overline{\text{ш}}_j\,(j = 1, 2, 3)$ by using equation (25):

$$\begin{aligned} \overline{\text{ш}}_1 &= (0.7176, 0.4447), \\ \overline{\text{ш}}_2 &= (0.6320, 0.6266), \\ \overline{\text{ш}}_3 &= (0.6310, 0.5304). \end{aligned} \tag{35}$$

(b) determine the aggregated decision matrix integration by using (26):

Table 1: Pythagorean fuzzy (PF) decision matrices.

	c_1	c_2	c_3
e_1			
x_1	(0.71, 0.26)	(0.79, 0.35)	(0.44, 0.53)
x_2	(0.60, 0.37)	(0.81, 0.38)	(0.63, 0.53)
x_3	(0.54, 0.36)	(0.73, 0.11)	(0.89, 0.13)
x_4	(0.47, 0.23)	(0.57, 0.25)	(0.51, 0.43)
e_2			
x_1	(0.72, 0.43)	(0.55, 0.66)	(0.69, 0.22)
x_2	(0.67, 0.29)	(0.85, 0.35)	(0.91, 0.23)
x_3	(0.58, 0.39)	(0.60, 0.53)	(0.79, 0.45)
x_4	(0.80, 0.47)	(0.83, 0.37)	(0.48, 0.49)
e_3			
x_1	(0.59, 0.49)	(0.52, 0.63)	(0.73, 0.36)
x_2	(0.65, 0.35)	(0.64, 0.62)	(0.79, 0.58)
x_3	(0.73, 0.14)	(0.86, 0.51)	(0.67, 0.29)
x_4	(0.94, 0.16)	(0.76, 0.45)	(0.87, 0.27)

Table 2: Weights of experts.

Experts	PFEs
e_1	(0.78, 0.20)
e_2	(0.87, 0.17)
e_3	(0.92, 0.25)

Table 3: Weights of attributes.

	c_1	c_2	c_3
e_1	(0.83, 0.23)	(0.79, 0.36)	(0.64, 0.54)
e_2	(0.89, 0.37)	(0.87, 0.25)	(0.58, 0.66)
e_3	(0.91, 0.11)	(0.78, 0.53)	(0.87, 0.17)

$$\left(\overline{\mathscr{P}}_{ij}\right)_{4\times 3} = \begin{pmatrix} (0.4788, 0.6976) & (0.4182, 0.7957) & (0.5136, 0.5803) \\ (0.4661, 0.6125) & (0.5916, 0.6984) & (0.6727, 0.6187) \\ (0.4955, 0.4861) & (0.6206, 0.6622) & (0.5652, 0.5949) \\ (0.7376, 0.5096) & (0.5922, 0.6668) & (0.6296, 0.6141) \end{pmatrix}. \quad (36)$$

Step 4:

(a) determine the performance function S_{ij} by using equation (27):

$$\left(S_{ij}\right)_{4\times 3} = \begin{pmatrix} 0.0367 & 0.0000 & 0.1963 \\ 0.2544 & 0.7804 & 0.8040 \\ 0.4562 & 1.0000 & 0.2950 \\ 0.9332 & 0.8453 & 0.7023 \end{pmatrix}. \quad (37)$$

(b) determine preference intensity $PI_j(x_i, x_t)$ by using (28).
Setting attribute threshold function,

$$\Psi_k(d) = \begin{cases} 0.01, & if\ d > 0, \\ 0.00, & if\ d > 0. \end{cases} \quad (38)$$

(c) determine superiority matrix (S. Matrix) by using equation (29):

$$S = \begin{pmatrix} 0.00 & 0.00 & 0.00 \\ 0.01 & 0.01 & 0.03 \\ 0.02 & 0.03 & 0.01 \\ 0.03 & 0.02 & 0.02 \end{pmatrix}. \quad (39)$$

(d) determine inferiority matrix (I. Matrix) by using equation (30):

Step 5: we determine the S-Flow and I-Flow by equations (31) and (32) as shown in Table 4, and they are illustrated in Figure 1.

Step 6.

(a) combine superiority ranking rules with Table 4 that gives

$$Sc\left(\Psi^{>}\left(x_4\right)\right) > Sc\left(\Psi^{>}\left(x_3\right)\right) > Sc\left(\Psi^{>}\left(x_2\right)\right) > Sc\left(\Psi^{>}\left(x_1\right)\right) x_4 \succ x_3 \succ x_2 \succ x_1. \quad (40)$$

Table 4: The PF-SIR flows.

Internet stocks	$\Psi^{>}(x_i)$	$Sc(\Psi^{>}(x_i))$	$(\Psi^{<}(x_i))$	$Sc(\Psi^{<}(x_i))$
x_1 (GAD)	(0.0000, 1.0000)	− 1.0000	(0.2402, 0.9531)	− 0.8507
x_2 (FUT)	(0.1786, 0.9737)	− 0.9162	(0.1632, 0.9788)	− 0.9314
x_3 (NET)	(0.1971, 0.9675)	− 0.8972	(0.1400, 0.9851)	− 0.9508
x_4 (PUM)	(0.2138, 0.9642)	− 0.8840	(0.1120, 0.9885)	− 0.9646

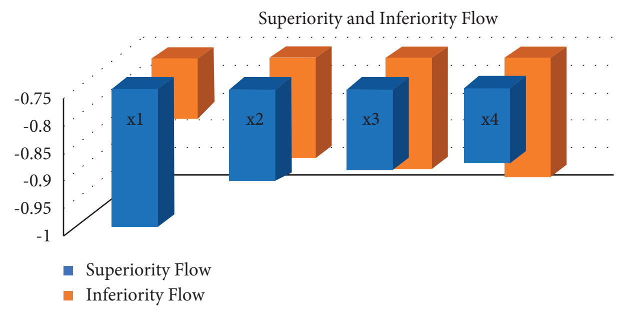

Figure 1: Superiority and inferiority flow.

Table 5: Comparison between proposed and existing work.

	Proposed work		Existing work [6]	
Internet stocks	$Sc(\Psi^{>}(x_i))$	$Sc(\Psi^{<}(x_i))$	$Sc(\Psi^{>}(x_i))$	$Sc(\Psi^{<}(x_i))$
x_1 (GAD)	− 1.0000	− 0.8507	0.0000406	0.0036886
x_2 (FUT)	− 0.9162	− 0.9314	0.0012394	0.0010162
x_3 (NET)	− 0.8972	− 0.9508	0.0020051	0.0005013
x_4 (PUM)	− 0.8840	− 0.9646	0.0022866	0.0003655

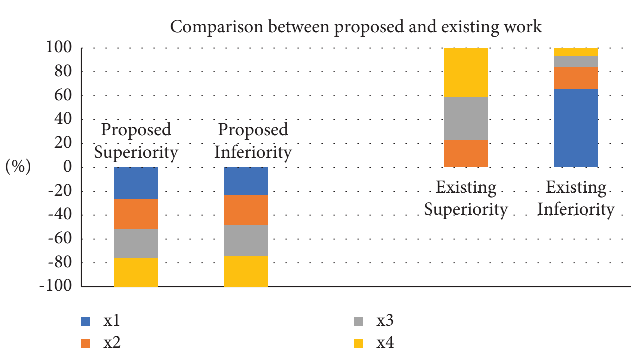

Figure 2: Comparison between proposed and existing work.

(b) combine inferiority ranking rules with Table 4 that gives

$$\begin{aligned}&Ṣc(\Psi^{<}(x_4)) < Ṣc(\Psi^{<}(x_3)) < Ṣc(\Psi^{<}(x_2)) < Ṣc(\Psi^{<}(x_1))\\&x_4 \succ x_3 \succ x_2 \succ x_1\end{aligned}. \quad (41)$$

Step 7: according to the results of SIR rules, the best alternative is x_4 (PUM) for Internet stock investment.

6. Comparative Analysis

In this section, we compare the proposed results with the results already in vogue in [6] which are represented in Table 5, and they are illustrated in Figure 2. It is observed from our numerical example that aggregation operators used in [6] and aggregation operators used in the proposed method give same results. However, the accuracy and authenticity of this approach lie in the fact that $\mathscr{A}\mathscr{A}$ − AOs are established on $\mathscr{A}\mathscr{A}$ − TNs. Hence, these operators are responsible for accurate outcomes. Thus, we found another easier, authentic, and valid method for choosing the best and most attractive alternative for MAGDM for SIR approach.

7. Conclusions

In this research article, we worked on an aggregation operator, namely, the PF-$\mathscr{A}\mathscr{A}$-WA −$\mathscr{C}\mathscr{C}$ operator for PFNs with SIR techniques. Meanwhile, depending on the PF aggregation operator, we examine some properties such as idempotency, boundedness, and monotonicity. This structure of $\mathscr{A}\mathscr{A}$ AO based on t-norm and t-conorm with SIR techniques is more generalized that effectively integrates the complicated problems. Mainly, we used the Pythagorean information and developed the MAGDM approach for the easiness of decision makers. A MAGDM problem for the selection of Internet stocks has been solved to demonstrate the authenticity of the proposed work and measure its validity by comparing its results with the method already in vogue. It is observed that our developed method is also feasible for intuitionistic fuzzy data and fuzzy data which are a very fruitful contribution to the literature.

We will extend our developments on q-rung orthogonal pair data and cubic Pythagorean fuzzy environment in the future. Furthermore, we can spread them to other aggregation operators, such as power mean AOs, Bonferroni mean AOs, Hamacher AOs, Hamy mean AOs, and Dombi AOs with SIR techniques. In the future, there is a lot of potential in machine learning, information retrieval, data mining, artificial intelligence, social network analysis, and many other areas in uncertain scenarios [38–48]. These are all fascinating topics for future studies.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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