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Research Article

Kannan Contraction Maps on the Space of Null Variable Exponent Second-Order Quantum Backward Difference Sequences of Fuzzy Functions and Its Pre-Quasi Ideal

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In this paper, we construct and investigate the space of null variable exponent second-order quantum backward difference sequences of fuzzy functions, which are crucial additions to the concept of modular spaces. The idealization of the mappings has been achieved through the use of extended $s-$ fuzzy functions and this sequence space of fuzzy functions. This new space's topological and geometric properties and the mappings' ideal that corresponds to them are discussed. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated pre-quasi ideal. To demonstrate our findings, we give a number of numerical experiments. There are also some significant applications of the existence of solutions to nonlinear difference equations of fuzzy functions.

1. Introduction

We assume that $\mathscr{N}$ is the set of non-negative integers. Yaying et al. [1] defined quantum second-order backward difference operator, ∇_p^2, where $z_a = 0$, for $a < 0$, and $\nabla_p^2 z_a = z_a - (1 + p)z_{a-1} + p z_{a-2}$, for all $p \in (0,1)$ and $a \in \mathscr{N}$. Note that the operator ∇_p^2 reduces to ∇^2 when $p \longrightarrow 1^-$, which defined and studied in Reference [2]. They proved that the spaces $c_0(\nabla_p^2)$ and $c(\nabla_p^2)$ are Banach spaces linearly isomorphic to c_0 and c, respectively, and obtained their Schauder bases and $\alpha-$, $\beta-$, and $\gamma-$duals. They determined the spectrum, the point spectrum, the continuous spectrum, and the residual spectrum of the operator ∇_p^2 over the Banach space c_0 of null sequences. It is clear to see that

$$c_0 \subsetneqq c_0\left(\nabla_p^2\right) \subsetneqq \left(\nabla^2\right). \tag{1}$$

For the strict inclusion, we have $(1,1,\ldots) \notin c_0$ and $(1,1,\ldots) \in c_0(\nabla_p^2)$, and $(0,1,2,\ldots) \notin c_0(\nabla_p^2)$ and $(0,1,2,\ldots) \in c_0(\nabla^2)$.

The mappings' ideal theory is well regarded in functional analysis. Fixed-point theory, Banach space geometry, normal series theory, approximation theory, and ideal transformations all use mappings' ideal. Using s-numbers is an essential technique. For more background details, see Pietsch [3], Constantin [4], and Tita [5]. Pre-quasi mappings' ideals are more extensive than quasi mappings' ideals, according to Faried and Bakery [6]. Bakery and Elmatty [7] explained a note on Nakano generalized difference sequence space under premodular. Since the booklet of the Banach fixed-point theorem [8], many mathematicians have worked on many developments. For more background and recent works on applicative approach of fixed-point theory, see Ružiĉka [9], Mao et al. [10], and Younis et al. [11–14]. Kannan [15] gave an example of a class of mappings with the same fixed-point actions as contractions, though that fails to be continuous. The only attempt to describe Kannan operators in modular vector spaces was once made in Reference [16]. Bakery and Mohamed [17] explored the concept of the pre-quasi-norm on Nakano sequence space such that its variable exponent belongs to $(0, 1]$. They explained the sufficient conditions on it equipped with the definite pre-quasi-norm to generate pre-quasi Banach. They examined the Fatou property of different pre-quasi-norms on it. Moreover, they showed a fixed point of Kannan pre-quasi-norm contraction maps on it and on the pre-quasi Banach operator ideal constructed by s-numbers that belong to this sequence space.

Zadeh [18] established the concept of fuzzy sets and fuzzy set operations, and many researchers adopted the concept of fuzziness in cybernetics and artificial intelligence as well as in expert systems and fuzzy control. We refer the reader to the following exciting works dealing with Kannan mappings and fuzzy concepts with different applications, see Reference [19–25]. Many researchers in sequence spaces and summability theory studied fuzzy sequence spaces and their properties. In Reference [26], the Nakano sequences of fuzzy integers were defined and analyzed. Bakery and Mohamed [27] introduced the certain space of sequences of fuzzy numbers, in short (cssf), under a certain function to be pre-quasi (cssf). This space and s–numbers have been used to describe the structure of the ideal operators. They defined and studied the weighted Nakano sequence spaces of fuzzy functions. They constructed the ideal generated by extended s– fuzzy functions and the sequence spaces of fuzzy functions. They presented some topological and geometric structures of this class of ideal and multiplication mappings acting on this sequence space of fuzzy functions. Moreover, the existence of Caristi's fixed point was examined. Many fixed-point theorems are effective when applied to a given space because they either enlarge the self-mapping acting on it or expand the space itself. Specifically, in this study, we construct and investigate the space of null variable exponent second-order quantum backward difference sequences of fuzzy functions, which are crucial additions to the concept of modular spaces. The idealization of the mappings has been achieved through the use of extended s– fuzzy functions and this sequence space of fuzzy functions. The topological and geometric properties of this new space and the mappings' ideal that corresponds to them are discussed. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated pre-quasi ideal. Interestingly, several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of nonlinear difference equations of fuzzy functions are introduced.

2. Definitions and Preliminaries

It is worth mentioning that Matloka [28] introduced bounded and convergent fuzzy numbers, investigated some of their properties, and demonstrated that any convergent fuzzy number sequence is bounded. Nanda [29] researched fuzzy number sequences and demonstrated that the set of all convergent fuzzy number sequences forms a complete metric space. Kumar et al. [30] presented the concept limit points and cluster points of sequences of fuzzy numbers. If Ξ is the set of all closed and bounded intervals on the real line $\mathfrak{R}$, then we assume $y = [y_1, y_2]$ and $z = [z_1, z_2]$ in Ξ, let

$$y \le z, \text{if, only if, } y_1 \le z_1, y_2 \le z_2. \tag{2}$$

Clearly, the relation $\le$ is a partial order on Ξ. We define a metric ρ on Ξ by

$$\rho(y, z) = \max\{|y_1 - z_1|, |y_2 - z_2|\}. \tag{3}$$

Matloka [28] proved that ρ is a metric on Ξ and (Ξ, ρ) is a complete metric space.

Definition 1. A fuzzy number z is a fuzzy subset of $\mathfrak{R}$, that is, a mapping $z: \mathfrak{R} \longrightarrow [0, 1]$ that verifies the four conditions:

(a) z is fuzzy convex; that is, for $t_1, t_2 \in \mathfrak{R}$, and $\alpha \in [0, 1]$, $z(\alpha t_1 + (1 - \alpha)t_2) \ge \min\{z(t_1), z(t_2)\}$.

(b) z is normal; that is, there is $t_0 \in \mathfrak{R}$ such that $z(t_0) = 1$.

(c) z is an upper-semicontinuous, that is, for all $\alpha > 0$, $z^{-1}([0, t + \alpha))$, for all $t \in [0, 1]$, is open in the usual topology of $\mathfrak{R}$.

(d) The closure of $z^0: = \{t \in \mathfrak{R}: z(t) > 0\}$ is compact.

The β-level set of a fuzzy real number z, $0 < \beta < 1$, denoted by z^β, is defined as

$$z^\beta = \{t \in \mathfrak{R}: z(t) \ge \beta\}. \tag{4}$$

The set of all upper semicontinuous, normal, convex fuzzy number, and z^β is compact, is marked by $\mathfrak{R}([0, 1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0, 1])$, if we define $r \in \mathfrak{R}([0, 1])$ by

$$\overline{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \ne r. \end{cases} \tag{5}$$

The additive identity and multiplicative identity in $\mathfrak{R}[0, 1]$ are denoted by $\overline{0}$ and $\overline{1}$, respectively. We assume that $y, z \in \mathfrak{R}[0, 1]$and the β-level sets are $[y]^\beta = [y_1^\beta, y_2^\beta]$, $[z]^\beta = [z_1^\beta, z_2^\beta]$, and $\beta \in [0, 1]$. A partial ordering for any $y, z \in \mathfrak{R}[0, 1]$ is as follows: $y \prec z$, if and only if, $y^\beta \le z^\beta$, for all $\beta \in [0, 1]$.

We assume that $\overline{\rho}: \mathfrak{R}[0, 1] \times \mathfrak{R}[0, 1] \longrightarrow \mathfrak{R}^+ \cup \{0\}$ is defined by $\overline{\rho}(y, z) = \sup_{0 \le \beta \le 1} \rho(y^\beta, z^\beta)$.

We recall that

(1) $(\mathfrak{R}[0, 1], \overline{\rho})$ is a complete metric space

(2) $\overline{\rho}(y + x, z + x) = \overline{\rho}(y, z)$ for all $y, z, x \in \mathfrak{R}[0, 1]$

(3) $\overline{\rho}(y + x, z + l) \le \overline{\rho}(y, z) + \overline{\rho}(x, l)$

(4) $\overline{\rho}(\xi y, \xi z) = |\xi| \overline{\rho}(y, z)$, for all $\xi \in \mathfrak{R}$

By c_0, ℓ_∞, and ℓ_r, we denote the space of null, bounded, and r-absolutely summable sequences of real numbers, respectively. We indicate the space of all bounded, finite rank linear mappings from an infinite dimensional Banach space Ω into an infinite dimensional Banach space Λ by $\mathscr{L}(\Omega, \Lambda)$, and $\mathfrak{F}(\Omega, \Lambda)$ and when $\Omega = \Lambda$, we inscribe $\mathscr{L}(\Omega)$ and $\mathfrak{F}(\Omega)$. The space of approximable and compact bounded linear mappings from Ω into Λ will be denoted by

$\Upsilon(\Omega, \Lambda)$ and $\mathscr{L}_c(\Omega, \Lambda)$, and if $\Omega = \Lambda$, we mark $\Upsilon(\Omega)$ and $\mathscr{L}_c(\Omega)$, respectively.

Definition 2 (see [31]). An s-number function is a mapping $s: \mathscr{L}(\Omega, \Lambda) \longrightarrow \mathfrak{R}^{+\mathscr{N}}$ that gives all $V \in \mathscr{L}(\Omega, \Lambda)$ a $(s_d(V))_{d=0}^{\infty}$ holds the following conditions:

(a) $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq \ldots \geq 0$, for every $V \in \mathscr{L}(\Omega, \Lambda)$.
(b) $s_{l+d-1}(V_1 + V_2) \leq s_l((V_1) + s_d(V_2)$, for every $V_1, V_2 \in \mathscr{L}(\Omega, \Lambda)$ and $l, d \in \mathscr{N}$.
(c) $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, for every $W \in \mathscr{L}(\Omega_0, \Omega)$, $Y \in \mathscr{L}(\Omega, \Lambda)$, and $V \in \mathscr{L}(\Lambda, \Lambda_0)$, where Ω_0 and Λ_0 are arbitrary Banach spaces.
(d) Assume $V \in \mathscr{L}(\Omega, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$.
(e) If $\text{rank}(V) \leq d$, then $s_d(V) = 0$, for all $V \in \mathscr{L}(\Omega, \Lambda)$.
(f) $s_{l \geq a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a indicates the unit mapping on the a-dimensional Hilbert space ℓ_2^a.

We give here some examples of s-numbers:

(1) The qth Kolmogorov number, denoted by $d_q(X)$, is marked by $d_q(X) = \inf_{\dim J \leq q} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|$.
(2) The q-th approximation number, indicated by $\alpha_q(X)$, is marked by $\alpha_q(X) = \inf\{\|X - Y\|: Y \in \mathscr{L}(\Omega, \Lambda) \text{ and } \text{rank}(Y) \leq q\}$.

Definition 3 (see [32]). Let $\mathscr{L}$ be the class of all bounded linear operators within any two arbitrary Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is said to be a mappings' ideal, if every $\mathscr{U}(\Omega, \Lambda) = \mathscr{U} \cap \mathscr{L}(\Omega, \Lambda)$ satisfies the following setups:

(i) $I_\Gamma \in \mathscr{U}$, where Γ indicates Banach space of one dimension.
(ii) The space $\mathscr{U}(\Omega, \Lambda)$ is linear over $\mathfrak{R}$.
(iii) If $W \in \mathscr{L}(\Omega_0, \Omega)$, $X \in \mathscr{U}(\Omega, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$, then $YXW \in \mathscr{U}(\Omega_0, \Lambda_0)$.

Notations 1 (see [27]).

$$\begin{aligned}
\overline{\mathscr{F}}_{\mathbf{U}} &:= \left\{\overline{\mathscr{F}}_{\mathbf{U}}(\Omega, \Lambda)\right\}, \text{where } \overline{\mathscr{F}}_{\mathbf{U}}(\Omega, \Lambda) := \left\{V \in \mathscr{L}(\Omega, \Lambda): \left(\left(\overline{s_j(V)}\right)_{j=0}^{\infty} \in \mathbf{U}\right\},\\
\overline{\mathscr{F}^{\alpha}}_{\mathbf{U}} &:= \left\{\overline{\mathscr{F}^{\alpha}}_{\mathbf{U}}(\Omega, \Lambda)\right\}, \text{where } \overline{\mathscr{F}^{\alpha}}_{\mathbf{U}}(\Omega, \Lambda) := \left\{V \in \mathscr{L}(\Omega, \Lambda): \left(\left(\overline{\alpha_j(V)}\right)_{j=0}^{\infty} \in \mathbf{U}\right\},\\
\overline{\mathscr{F}^{d}}_{\mathbf{U}} &:= \left\{\overline{\mathscr{F}^{d}}_{\mathbf{U}}(\Omega, t\Lambda)\right\}, \text{where } \overline{\mathscr{F}^{d}}_{\mathbf{U}}(\Omega, \Lambda) := \left\{V \in \mathscr{L}(\Omega, \Lambda): \left(\left(\overline{d_j(V)}\right)_{j=0}^{\infty} \in \mathbf{U}\right\},
\end{aligned} \tag{6}$$

where

$$\overline{s_j(V)(t)} = \begin{cases} 1, & t = s_j(V), \\ 0, & t \neq s_j(V). \end{cases} \tag{7}$$

Definition 4 (see [6]). A function $H \in [0, \infty)^{\mathscr{U}}$ is said to be a pre-quasi-norm on the ideal $\mathscr{U}$ if the following conditions hold:

(1) Assume $V \in \mathscr{U}(\Omega, \Lambda)$, $H(V) \geq 0$, and $H(V) = 0$, if and only if, $V = 0$
(2) One has $Q \geq 1$ with $H(\alpha V) \leq D|\alpha| H(V)$, for all $V \in \mathscr{U}(\Omega, \Lambda)$ and $\alpha \in \mathfrak{R}$
(3) There are $P \geq 1$ such that $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for all $V_1, V_2 \in \mathscr{U}(\Omega, \Lambda)$
(4) There are $\sigma \geq 1$ so that if $V \in \mathscr{L}(\Omega_0, \Omega)$, $X \in \mathscr{U}(\Omega, \Lambda)$ and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$ then $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$

Theorem 1 (see [6]). *H is a pre-quasi-norm on the ideal $\mathscr{U}$, whenever H is a quasi-norm on the ideal $\mathscr{U}$.*

Lemma 1 (see [33]). *If $\tau_a > 0$ and $v_a, t_a \in \mathfrak{R}$, for all $a \in \mathscr{N}$, then $|v_a + t_a|^{\tau_a} \leq 2^{K-1}(|v_a|^{\tau_a} + |t_a|^{\tau_a})$, where $K = max\{1, sup_a \tau_a\}$.*

3. Some Characteristics of $c_0^F(\nabla_p^2, \tau)$

This section is devoted to provide sufficient criteria for the space of null variable exponent second-order quantum backward difference sequences of fuzzy numbers, $c_0^F(\nabla_p^2, \tau)$, endowed with definite function h, to be pre-quasi Banach. We have examined some algebraic and topological properties such as completeness, solidness, symmetry, and convergence-free. The Fatou property of various pre-quasi-norms h on $c_0^F(\nabla_p^2, \tau)$ has been presented.

Let $\omega(F)$ denote the classes of all sequence spaces of fuzzy real numbers. If $\tau = (\tau_a) \in \mathfrak{R}^{+\mathscr{N}}$, where $\mathfrak{R}^{+\mathscr{N}}$ is the space of positive reals. The space of null variable exponent second-order quantum backward difference sequences of fuzzy numbers is defined as follows: $c_0^F(\nabla_p^2, \tau) = \{z = (z_a) \in \omega(F): \lim_{a \longrightarrow \infty} [\overline{\rho}(|\nabla_p^2|\mu z_a||, \overline{0})]^{\tau_a/K} = 0, \text{for some } \mu > 0\}$.

Theorem 2. *If $(\tau_a) \in \ell_\infty$, then*

$$c_0^F\left(\nabla_p^2,\tau\right)=\left\{z=(z_a)\in\omega(F)\colon \lim_{a\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\right|\mu z_a\right\|,\bar{0}\right)\right]^{\tau_a/K}=0,\text{for any }\mu>0\right\}. \tag{8}$$

Proof.

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$$\begin{aligned}
c_0^F\left(\nabla_p^2,\tau\right)&=\left\{z=(z_a)\in\omega(F)\colon \lim_{a\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\right|\mu z_a\right\|,\bar{0}\right)\right]^{(\tau_a/K)}=0,\text{for some }\mu>0\right\},\\
&=\left\{z=(z_a)\in\omega(F)\colon \inf_a|\mu|^{\frac{\tau_a}{K}}\lim_{a\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\right|z_a\right\|,\bar{0}\right)\right]^{\tau_a/K}\leq\lim_{a\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\right|\mu z_a\right\|,\bar{0}\right)\right]^{(\tau_a/K)}=0,\text{for some }\mu>0\right\},\\
&=\left\{z=(z_a)\in\omega(F)\colon \lim_{a\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\right|z_a\right\|,\bar{0}\right)\right]^{\tau_a/K}=0\right\},\\
&=\left\{z=(z_a)\in\omega(F)\colon \lim_{a\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\right|\mu z_a\right\|,\bar{0}\right)\right]^{\tau_a/K}=0,\text{for any }\mu>0\right\}.
\end{aligned} \tag{9}$$

It is clear to see that if $(\tau_a)\in\ell_\infty$, then

$$c_0^F(\tau)\subsetneq c_0^F\left(\nabla_p^2,\tau\right)\subsetneq c_0^F\left(\nabla^2,\tau\right). \tag{10}$$

For the strict inclusion, we have $(\bar{1},\bar{1},\bar{1},\ldots)\notin c_0^F$ and $(\bar{1},\bar{1},\bar{1},\ldots)\in c_0^F(\nabla_p^2,\tau)$, and $(\bar{0},\bar{1},\bar{2},\ldots)\notin c_0^F(\nabla_p^2,\tau)$ and $(\bar{0},\bar{1},\bar{2},\ldots)\in c_0^F(\nabla^2,\tau)$.

For $z=(z_k)$, a given sequence $S(z)$ denotes the set of all permutation of the elements of (z_k); that is, $S(z)=\left\{(z_{\pi(k)})\right\}$. □

Definition 5

(1) A sequence space of fuzzy numbers **U** is said to be symmetric if $S(z)\in\mathbf{U}$, for all $z\in\mathbf{U}$

(2) A sequence space of fuzzy numbers **U** is said to be convergence-free if $(y_k)\in\mathbf{U}$ whenever $(z_k)\in\mathbf{U}$ and $z_k=\bar{0}$ implies $y_k=\bar{0}$

Theorem 3. *If $(\tau_a)\in\ell_\infty$, then the space $(c_0^F(\nabla_p^2,\tau))_h$ is not symmetric.*

Proof. consider $(x_k)=(\overbrace{\bar{1},\ldots,\bar{1}}^{3\text{times}},\overbrace{-\bar{1},\ldots,-\bar{1}}^{3\text{times}},\overbrace{\bar{1},\ldots,\bar{1}}^{6\text{times}},\overbrace{-\bar{1},\ldots,-\bar{1}}^{6\text{times}},\overbrace{\bar{1},\ldots,\bar{1}}^{9\text{times}},\overbrace{-\bar{1},\ldots,-\bar{1}}^{9\text{times}},\ldots)$. Then, $(x_k)\in(c_0^F(\nabla_p^2,\tau))_h$. Now, if (y_k) is the rearrangement of (x_k) defined by $(y_k)=(\bar{1},-\bar{1},\bar{1},-\bar{1},\ldots)$, then $(y_k)\notin(c_0^F(\nabla_p^2,\tau))_h$. Therefore, the space $(c_0^F(\nabla_p^2,\tau))_h$ is not symmetric. □

Theorem 4. *If $(\tau_a)\in\ell_\infty$, then the space $(c_0^F(\nabla_p^2,\tau))_h$ is not convergence-free.*

Proof. consider the sequence $(x_k)=(\bar{1},\bar{1},\ldots)$. Then, $(x_k)\in(c_0^F(\nabla_p^2,\tau))_h$. Again if $(y_k)=(\bar{k})$, then clearly, $(y_k)\notin(c_0^F(\nabla_p^2,\tau))_h$. Hence, the space $(c_0^F(\nabla_p^2,\tau))_h$ is not convergence-free.

Let us mark the space of all functions $h\colon\mathbf{U}\longrightarrow[0,\infty)$ by $[0,\infty)^{\mathbf{U}}$. □

Definition 6 (see [34]). **U** is a vector space. A function $h\in[0,\infty]^{\mathbf{U}}$ is said to be modular if the following conditions hold:

(a) Assume $y\in\mathbf{U}$, $y=\bar{\vartheta}\Leftrightarrow h(y)=0$ with $h(y)\geq 0$, where $\bar{\vartheta}=(\bar{0},\bar{0},\bar{0},\ldots)$

(b) $h(\eta z)=h(z)$ verifies for every $z\in\mathbf{U}$ and $|\eta|=1$

(c) The inequality $h(\alpha y+(1-\alpha)z)\leq h(y)+h(z)$ holds for every $y,z\in\mathbf{U}$ and $\alpha\in[0,1]$

Definition 7 (see [27]). The linear space **U** is called a certain space of sequences of fuzzy numbers (cssf), when

(1) $\{\bar{b}_q\}_{q\in\mathcal{N}}\subseteq\mathbf{U}$, where $\bar{b}_q=(\bar{0},\bar{0},\ldots,\bar{1},\bar{0},\bar{0},\ldots)$, while $\bar{1}$ displays at the q^{th} place

(2) **U** is solid; that is, if $y=(y_q)\in\omega(F)$, $z=(z_q)\in\mathbf{U}$, and $|y_q|\leq|z_q|$ for every $q\in\mathcal{N}$, then $y\in\mathbf{U}$

(3) $(y_{[q/2]})_{q=0}^{\infty}\in\mathbf{U}$, where $[q/2]$denotes the integral part of $q/2$, if $(y_q)_{q=0}^{\infty}\in\mathbf{U}$

Definition 8 (see [27]). A subclass $\mathbf{U}_h$ of **U** is said to be a premodular (cssf), if there is $h\in[0,\infty)^{\mathbf{U}}$, which satisfies the following conditions:

(i) Assume $y\in\mathbf{U}$, $y=\bar{\vartheta}\Leftrightarrow h(y)=0$ with $h(y)\geq 0$, where $\bar{\vartheta}=(\bar{0},\bar{0},\bar{0},\ldots)$.

(ii) One has $Q\geq 1$, the inequality $h(\alpha y)\leq Q|\alpha|h(y)$ holds for all $y\in\mathbf{U}$ and $\alpha\in\mathfrak{R}$.

(iii) One has $P \geq 1$, the inequality $h(y+z) \leq P(h(y)+h(z))$ verifies, for all $y, z \in \mathbf{U}$.

(iv) Suppose $|y_q| \leq |z_q|$, for all $q \in \mathcal{N}$, then $h((y_q)) \leq h((z_q))$.

(v) The inequality, $h((y_q)) \leq h((y_{[q/2]})) \leq P_0 h((y_q))$, verifies, for some $P_0 \geq 1$.

(vi) If E is the space of finite sequences of fuzzy numbers, then the closure of $E = \mathbf{U}_h$.

(vii) One has $\sigma > 0$ with $h(\overline{\alpha}, \overline{0}, \overline{0}, \overline{0}, \ldots) \geq \sigma |\alpha| h(\overline{1}, \overline{0}, \overline{0}, \overline{0}, \ldots)$, where

$$\overline{\alpha}(t) = \begin{cases} 1, & t = \alpha \\ 0, & t \neq \alpha. \end{cases} \tag{11}$$

We note that the notion of premodular vector spaces is more general than modular vector spaces. There are some examples of premodular vector spaces but not modular vector spaces.

Example 1. The function $h(z) = \sup_q [\overline{p}(|\nabla_p^2 |z_q||, \overline{0})]^{4q+1/q+4}$ on the vector space $c_0^F(\nabla_p^2, (4q+1/q+4))$. As for every $z, y \in c_0^F(\nabla_p^2, (4q+1/q+4))$, one has

$$h\left(\frac{z+y}{2}\right) = \sup_q \left[\overline{p}\left(\left|\nabla_p^2 \left|\frac{z_q+y_q}{2}\right|\right|, \overline{0}\right)\right]^{4q+1/q+4} \leq \frac{8}{\sqrt[4]{2}} (h(z)+h(y)). \tag{12}$$

Example 2. The function $h(z) = \sup_q [\overline{p}(|\nabla_p^2 |z_q||, \overline{0})]^{5q+2/q+1}$ on the vector space $c_0^F(\nabla_p^2, (5q+2/q+1))$. As for every $z, y \in c_0^F(\nabla_p^2, (5q+2/q+1))$, one has

$$h\left(\frac{z+y}{2}\right) = \sup_q \left[\overline{p}\left(\left|\nabla_p^2 \left|\frac{z_q+y_q}{2}\right|\right|, \overline{0}\right)\right]^{5q+2/q+1} \leq 4(h(z)+h(y)). \tag{13}$$

Some examples of premodular vector spaces and modular vector spaces are as follows:

Example 3. The function $h(z) = \sup_q [\overline{p}(|\nabla_p^2 |z_q||, \overline{0})]^{q+1/3q+4}$ on the vector space $c_0^F(\nabla_p^2, (q+1/3q+4))$. As for every $z, y \in c_0^F(\nabla_p^2, (q+1/3q+4))$, one has

$$h\left(\frac{z+y}{2}\right) = \sup_q \left[\overline{p}\left(\left|\nabla_p^2 \left|\frac{z_q+y_q}{2}\right|\right|, \overline{0}\right)\right]^{q+1/3q+4} \leq \frac{1}{\sqrt[4]{2}} (h(z)+h(y)). \tag{14}$$

Example 4. The function $h(y) = \inf\{\alpha > 0: \sup_q [\overline{p}(|\nabla_p^2 |y_q/\alpha||, \overline{0})]^{2q+3/q+2} \leq 1\}$ is a premodular (modular) on the vector space $c_0^F(\nabla_p^2, (2q+3/q+2))$.

Definition 9 (see [27]). $\mathbf{U}$ is a cssf. The function $h \in [0,\infty)^{\mathbf{U}}$ is said to be a pre-quasi-norm on $\mathbf{U}$, if it verifies the following settings:

(i) Suppose $y \in \mathbf{U}$, $y = \overline{\vartheta} \Leftrightarrow h(y) = 0$ with $h(y) \geq 0$, where $\overline{\vartheta} = (\overline{0}, \overline{0}, \overline{0}, \ldots)$.

(ii) We have $Q \geq 1$, the inequality $h(\alpha y) \leq Q|\alpha| h(y)$ holds, for all $y \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$.

(iii) One has $P \geq 1$, the inequality $h(y+z) \leq P(h(y)+h(z))$ verifies, for all $y, z \in \mathbf{U}$.

Theorem 5 (see [27]). *We suppose that* $\mathbf{U}$ *is a premodular (cssf), then it is pre-quasi-normed (cssf).*

Theorem 6 (see [27]). $\mathbf{U}$ *is a pre-quasi-normed (cssf), if it is quasi-normed (cssf).*

Definition 10

(a) The function h on $c_0^F(\nabla_p^2, \tau)$ is called h-convex, if

$$h(\alpha y + (1-\alpha)z) \leq \alpha h(y) + (1-\alpha) h(z), \tag{15}$$

for every $\alpha \in [0,1]$ and $y, z \in c_0^F(\nabla_p^2, \tau)$.

(b) $\{y_q\}_{q\in\mathcal{N}} \subseteq (c_0^F(\nabla_p^2, \tau))_h$ is h-convergent to $y \in (c_0^F(\nabla_p^2, \tau))_h$, if and only if, $\lim_{q\longrightarrow\infty} h(y_q - y) = 0$. When the h-limit exists, then it is unique.

(c) $\{y_q\}_{q\in\mathcal{N}} \subseteq (c_0^F(\nabla_p^2, \tau))_h$ is h-Cauchy, if $\lim_{q,r\longrightarrow\infty} h(y_q - y_r) = 0$.

(d) $\Gamma \subset (c_0^F(\nabla_p^2, \tau))_h$ is h-closed, when for all h-converges $\{y_q\}_{a\in\mathcal{N}} \subset \Gamma$ to y, then $y \in \Gamma$.

(e) $\Gamma \subset (c_0^F(\nabla_p^2, \tau))_h$ is h-bounded, if $\delta_h(\Gamma) = \sup\{h(y-z): y, z \in \Gamma\} < \infty$.

(f) The h-ball of radius $\varepsilon \geq 0$ and center y, for every $y \in (c_0^F(\nabla_p^2, \tau))_h$, is described as follows:

$$\mathbf{B}_h(y, \varepsilon) = \left\{z \in \left(c_0^F\left(\nabla_p^2, \tau\right)\right)_h: h(y-z) \leq \varepsilon\right\}. \tag{16}$$

(g) A pre-quasi-norm h on $c_0^F(\nabla_p^2, \tau)$ satisfies the Fatou property, if for every sequence $\{z^q\} \subseteq (c_0^F(\nabla_p^2, \tau))_h$ under $\lim_{q \longrightarrow \infty} h(z^q - z) = 0$ and all $y \in (c_0^F(\nabla_p^2, \tau))_h$, one has $h(y - z) \leq \sup_r \inf_{q \geq r} h(y - z^q)$.

We recall that the Fatou property gives the h-closedness of the h-balls. We will denote the space of all increasing sequences of real numbers by $\mathbf{I}$.

Theorem 7. *$(c_0^F(\nabla_p^2, \tau))_h$, where $h(y) = sup_q [\overline{\rho}(\eta_q(q! |\nabla_p^2 | y_q||)^{1/q+1}, \overline{0})]^{\tau_q/K}$, for every $y \in c_0^F(\nabla_p^2, \tau)$, is a premodular (cssf), if the following conditions are satisfied:*

(a) $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$.

(b) ∇_p^2 is an absolute nondecreasing; that is, if $|z_i| \leq |y_i|$, for all $i \in \mathbb{N}$, then $|\nabla_p^2 |z_i|| \leq |\nabla_p^2 |y_i||$.

Proof. Clearly, $h(y) \geq 0$ and $h(y) = 0 \Leftrightarrow y = \overline{\vartheta}$.

(i) Assume $y, z \in c_0^F(\nabla_p^2, \tau)$. We have

$$
\begin{aligned}
h(y + z) &= \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_q + z_q\right||, \overline{0}\right)\right]^{\tau_q/K} \leq \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_q\right||, \overline{0}\right)\right]^{\tau_q/K} + \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | z_q\right||, \overline{0}\right)\right]^{\tau_q/K} \\
&= h(y) + h(z) < \infty.
\end{aligned}
\tag{17}
$$

Then, $y + z \in c_0^F(\nabla_p^2, \tau)$.

(ii) There are $P \geq 1$ with $h(y + z) \leq P(h(y) + h(z))$, for every $y, z \in c_0^F(\nabla_p^2, \tau)$.

(iii) If $\alpha \in \mathfrak{R}$ and $y \in c_0^F(\nabla_p^2, \tau)$, one has

$$
h(\alpha y) = \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | \alpha y_q\right||, \overline{0}\right)\right]^{\tau_q/K} \leq \sup_q |\alpha|^{\tau_q/K} \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_q\right||, \overline{0}\right)\right]^{\tau_q/K} \leq Q |\alpha| h(y) < \infty.
\tag{18}
$$

So, $\alpha y \in c_0^F(\nabla_p^2, \tau)$. From parts (1-i) and (1-ii), we have $c_0^F(\nabla_p^2, \tau)$ is linear. Also, $\overline{b}_p \in c_0^F(\nabla_p^2, \tau)$, for every $p \in \mathcal{N}$, as $h(\overline{b}_p) = \sup_q [\overline{\rho}(|\nabla_p^2 | (\overline{b}_p)_q||, \overline{0})]^{(\tau_q/K)} = 1$.

(iv) One has $Q = \max\left\{1, \sup_q |\alpha|^{\tau_q/K - 1}\right\} \geq 1$ with $h(\alpha y) \leq Q |\alpha| h(y)$, for every $y \in c_0^F(\nabla_p^2, \tau)$ and $\alpha \in \mathfrak{R}$.

(v) If $|y_q| \leq |z_q|$, for every $q \in \mathcal{N}$ and $z \in c_0^F(\nabla_p^2, \tau)$. We obtain

$$
h(y) = \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_q\right||, \overline{0}\right)\right]^{\tau_q/K} \leq \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | z_q\right||, \overline{0}\right)\right]^{\tau_q/K} = h(z) < \infty.
\tag{19}
$$

Then, $y \in c_0^F(\nabla_p^2, \tau)$.

(vi) Evidently, from Reference (2).

(vii) Assume $(y_q) \in c_0^F(\nabla_p^2, \tau)$, one can see

$$
\begin{aligned}
h\left(\left(y_{[q/2]}\right)\right) &= \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_{[q/2]}\right||, \overline{0}\right)\right]^{\tau_q/K} \leq \max\left\{\sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_q\right||, \overline{0}\right)\right]^{\tau_{2q}/K}, \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_q\right||, \overline{0}\right)\right]^{\tau_{2q+1}/K}\right\} \\
&\leq \sup_q \left[\overline{\rho}\left(\left|\nabla_p^2 | y_q\right||, \overline{0}\right)\right]^{\tau_q/K} = h\left(\left(y_q\right)\right).
\end{aligned}
\tag{20}
$$

Then, $(y_{[q/2]}) \in c_0^F(\nabla_p^2, \tau)$. (v) From (3), one has $P_0 = 2 \geq 1$.

(viii) Clearly, the closure of $E = c_0^F(\nabla_p^2, \tau)$.

(ix) One gets $0 < \sigma \leq \sup_q |\alpha|^{\tau_q/K - 1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ with $h(\overline{\alpha}, \overline{0}, \overline{0}, \overline{0}, \ldots) \geq \sigma |\alpha| h(\overline{1}, \overline{0}, \overline{0}, \overline{0}, \ldots)$. □

Theorem 8. *If the conditions of Theorem 7 are satisfied, then $(c_0^F(\nabla_p^2, \tau))_h$ is a pre-quasi Banach (cssf), where $h(y) = sup_q [\overline{\rho}(|\nabla_p^2 | y_q||, \overline{0})]^{\tau_q/K}$, for all $y \in c_0^F(\nabla_p^2, \tau)$.*

Proof. According to Theorem 7 and Theorem 5, the space $(c_0^F(\nabla_p^2, \tau))_h$ is a pre-quasi-normed (cssf). If $y^l = (y_q^l)_{q=0}^\infty$ is a

Cauchy sequence in $(c_0^F(\nabla_p^2,\tau))_h$, hence for all $\varepsilon\in(0,1)$, then $l_0\in\mathcal{N}$ such that for every $l,m\geq l_0$, we have

$$h\left(y^l-y^m\right)=\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\right|y_q^l-y_q^m\right\|,\overline{0}\right)\right]^{\tau_q/K}<\varepsilon. \tag{21}$$

Therefore, $\overline{\rho}(|\nabla_p^2|y_q^l-y_q^m\|,\overline{0})<\varepsilon$. Since $(\mathfrak{R}[0,1],\overline{\rho})$ is a complete metric space, (y_q^m) is a Cauchy sequence in $\mathfrak{R}[0,1]$, for fixed $q\in\mathcal{N}$. This gives $\lim_{m\longrightarrow\infty}y_q^m=y_q^0$, for fixed $q\in\mathcal{N}$. Then, $h(y^l-y^0)<\varepsilon$, for all $l\geq l_0$. As $h(y^0)=h(y^0-y^l+y^l)\leq h(y^l-y^0)+h(y^l)<\infty$. Then, $y^0\in c_0^F(\nabla_p^2,\tau)$. □

Theorem 9. *The function $h(y)=sup_q[\overline{\rho}(|\nabla_p^2|y_q\|,\overline{0})]^{\tau_q/K}$ satisfies the Fatou property, when the conditions of Theorem 7 are satisfied.*

Proof. Let $\{z^r\}\subseteq(c_0^F(\nabla_p^2,\tau))_h$ such that $\lim_{r\longrightarrow\infty}h(z^r-z)=0$. Since $(c_0^F(\nabla_p^2,\tau))_h$ is a pre-quasi closed space, we have $z\in(c_0^F(\nabla_p^2,\tau))_h$. For every $y\in(c_0^F(\nabla_p^2,\tau))_h$, then

$$\begin{aligned}h(y-z)&=\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\right|y_q-z_q\right\|,\overline{0}\right)\right]^{\tau_q/K}\\&\leq\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\right|y_q-z_q^r\right\|,\overline{0}\right)\right]^{\tau_q/K}+\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\right|z_q^r-z_q\right\|,\overline{0}\right)\right]^{\tau_q/K}\leq\sup_m\inf_{r\geq m}h\left(y-z^r\right).\end{aligned} \tag{22}$$

□

Theorem 10. *The function $h(y)=sup_q[\overline{\rho}(|\nabla_p^2|y_q\|,\overline{0})]^{\tau_q}$ does not satisfy the Fatou property, for all $y\in c_0^F(\nabla_p^2,\tau)$, if the conditions of Theorem 7 are satisfied with $\tau_0>1$.*

Proof. Assume $\{z^r\}\subseteq(c_0^F(\nabla_p^2,\tau))_h$ such that $\lim_{r\longrightarrow\infty}h(z^r-z)=0$. As $(c_0^F(\nabla_p^2,\tau))_h$ is a pre-quasi closed space, we have $z\in(c_0^F(\nabla_p^2,\tau))_h$. For all $z\in(c_0^F(\nabla_p^2,\tau))_h$, then

$$\begin{aligned}h(y-z)&=\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\right|y_q-z_q\right\|,\overline{0}\right)\right]^{\tau_q}\leq 2^{\sup_q\tau_q-1}\left(\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\right|y_q-z_q^r\right\|,\overline{0}\right)\right]^{\tau_q}+\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\right|z_q^r-z_q\right\|,\overline{0}\right)\right]^{\tau_q}\right)\\&\leq 2^{\sup_q\tau_q-1}\sup_m\inf_{r\geq m}h\left(y-z^r\right).\end{aligned} \tag{23}$$

□

Example 5. For $(\tau_q)\in[1,\infty)^{\mathcal{N}}$, the function $h(y)=\inf\left\{\alpha>0\colon \sup_q[\overline{\rho}(|\nabla_p^2|y_q/\alpha\|,\overline{0})]^{\tau_q}\leq 1\right\}$ is a norm on $c_0^F(\nabla_p^2,\tau)$.

Example 6. The function $h(y)=\sup_q[\overline{\rho}(|\nabla_p^2|y_q\|,\overline{0})]^{3q+2/q+3}$ is a pre-quasi-norm (not a norm) on $c_0^F(\nabla_p^2,(3q+2/q+3)_{q=0}^{\infty})$.

Example 7. The function $h(y)=\sup_q[\overline{\rho}(|\nabla_p^2|y_q\|,\overline{0})]^{3q+2/q+1}$ is a pre-quasi-norm (not a quasi-norm) on $c_0^F(\nabla_p^2,(3q+2/q+1)_{q=0}^{\infty})$.

4. Structure of Mappings' Ideal

The structure of the mappings' ideal by $(c_0^F(\nabla_p^2,\tau))_h$, where $h(z)=\sup_q[\overline{\rho}(|\nabla_p^2|z_q\|,\overline{0})]^{\tau_q/K}$, for all $z\in c_0^F(\nabla_p^2,\tau)$, and extended $s-$ fuzzy functions have been explained. We study enough setups on $(c_0^F(\nabla_p^2,\tau))_h$ such that the class $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$ is complete. We investigate conditions (not necessary) on $(c_0^F(\nabla_p^2,\tau))_h$ such that the closure of $\mathfrak{F}=\overline{\mathcal{F}^{\alpha}}_{(c_0^F(\nabla_p^2,\tau))_h}$. This gives a negative answer of Rhoades [34] open problem about the linearity of $s-$ type $(c_0^F(\nabla_p^2,\tau))_h$ spaces. We explain enough setups on $(c_0^F(\nabla_p^2,\tau))_h$ such that $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$ is strictly contained for different powers, weights, and backward generalized differences, the class $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$ is simple, and the space of every bounded linear mappings is which sequence of eigenvalues in $(c_0^F(\nabla_p^2,\tau))_h$ equals $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$.

Theorem 11 (see [27]). *If* $\mathbf{U}$ *is a (cssf), then* $\overline{\mathcal{F}}_{\mathbf{U}}$ *is a mappings' ideal.*

In view of Theorem 7 and Theorem 11, one has the following theorem:

Theorem 12. *If the conditions of Theorem 7 are satisfied, then $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$ is a mappings' ideal.*

Theorem 13. *If the conditions of Theorem 7 are satisfied, then the function H is a pre-quasi-norm on $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$, with $H(Z)=sup_q[\overline{\rho}(|\nabla_p^2|\overline{s_q(Z)}\|,\overline{0})]^{\tau_q/K}$, for every $Z\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$.*

Proof

(1) Suppose $X\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$, $H(X)=\sup_q[\overline{\rho}(|\nabla_p^2|\overline{s_q(Z)}\|,\overline{0})]^{\tau_q/K}\geq 0$ and $H(X)=\sup_q[\overline{\rho}(|\nabla_p^2|\overline{s_q(Z)}\|,\overline{0})]^{\tau_q/K}=0$, if and only if, $\overline{s_q(X)}=\overline{0}$, for all $q\in\mathcal{N}$, if and only if, $X=0$.

(2) One has $Q\geq 1$ with $H(\alpha X)=\sup_q[\overline{\rho}(|\nabla_p^2|\overline{s_q(\alpha X)}\|,\overline{0})]^{\tau_q/K}\leq Q|\alpha|H(X)$, for all $X\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$ and $\alpha\in\mathfrak{R}$.

(3) For $X_1,X_2\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$, we have

$$H(X_1+X_2)=\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\left|\overline{s_q(X_1+X_2)}\right|\right|,\overline{0}\right)\right]^{\tau_q/K}\le\left(h\left(\overline{s_{[q/2]}(X_1)}\right)_{q=0}^{\infty}+h\left(\overline{s_{[q/2]}(X_2)}\right)_{q=0}^{\infty}\right)$$
$$\le\left(h\left(\overline{s_q(X_1)}\right)_{q=0}^{\infty}+h\left(\overline{s_q(X_2)}\right)_{q=0}^{\infty}\right). \tag{24}$$

(4) There are $\varrho\ge 1$, if $X\in\mathscr{L}(\Omega_0,\Omega)$, $Y\in\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$ and $Z\in\mathscr{L}(\Lambda,\Lambda_0)$, then

$$H(ZYX)=\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\left|\overline{s_q(ZYX)}\right|\right|,\overline{0}\right)\right]^{\tau_q/K}\le h\left(\|X\|\|Z\|\overline{s_q(Y)}\right)_{q=0}^{\infty}\le\varrho\|X\|H(Y)\|Z\|. \tag{25}$$

In the next theorems, we will use the notation $(\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h},H)$, where $H(V)=h(\overline{(s_q(V))_{q=0}^{\infty}})$, for all $V\in\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}$. □

Theorem 14. *assume that the conditions of Theorem 7 are satisfied, then $(\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h},H)$ is a pre-quasi Banach mappings' ideal.*

Proof. Let $(V_a)_{a\in\mathscr{N}}$ be a Cauchy sequence in $\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$. Since $\mathscr{L}(\Omega,\Lambda)\supseteq S_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$, then

$$H(V_r-V_a)=\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\left|\overline{s_q(V_r-V_a)}\right|\right|,\overline{0}\right)\right]^{\tau_q/K}\ge h\left(\overline{s_0(V_r-V_a)},\overline{0},\overline{0},\overline{0},\ldots\right)\ge\left\|V_r-V_a\right\|^{\tau_0/K}. \tag{26}$$

This implies that $(V_a)_{a\in\mathscr{N}}$ is a Cauchy sequence in $\mathscr{L}(\Omega,\Lambda)$. Since $\mathscr{L}(\Omega,\Lambda)$ is a Banach space, one has $V\in\mathscr{L}(\Omega,\Lambda)$ such that $\lim_{a\longrightarrow\infty}\|V_a-V\|=0$ and as $(\overline{s_q(V_a)})_{q=0}^{\infty}\in(c_0^F(\nabla_p^2,\tau))_h$, for every $a\in\mathscr{N}$, and $(c_0^F(\nabla_p^2,\tau))_h$ is a premodular (cssf). Then, we have

$$H(V)=h\left(\left(\overline{s_q(V)}\right)_{q=0}^{\infty}\right)\le h\left(\left(\overline{s_{[q/2]}(V-V_a)}\right)_{q=0}^{\infty}\right)+h\left(\left(\overline{s_{[q/2]}(V_a)}_{q=0}^{\infty}\right)\right)\le h\left(\left(\|V_a-V\|\overline{1}\right)_{q=0}^{\infty}\right)+h\left(\left(\overline{s_q(V_a)}\right)_{q=0}^{\infty}\right)<\varepsilon. \tag{27}$$

Hence, one has $(\overline{s_q(V)})_{q=0}^{\infty}\in(c_0^F(\nabla_p^2,\tau))_h$, then $V\in\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$. □

Definition 11. A pre-quasi-norm H on the ideal $\overline{\mathscr{I}}_{\mathbf{U}_h}$ satisfies the Fatou property if for all $\{T_q\}_{q\in\mathscr{N}}\subseteq\overline{\mathscr{I}}_{\mathbf{U}_h}(\Omega,\Lambda)$ such that $\lim_{q\longrightarrow\infty}H(T_q-T)=0$ and $M\in\overline{\mathscr{I}}_{\mathbf{U}_h}(\Omega,\Lambda)$, then

$$H(M-T)\le\sup_q\inf_{j\ge q}H\left(M-T_j\right). \tag{28}$$

Theorem 15. *If the conditions of Theorem 7 are satisfied, then $(\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h},H)$ does not satisfy the Fatou property.*

Proof. Let $\{T_q\}_{q\in\mathscr{N}}\subseteq\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$ with $\lim_{q\longrightarrow\infty}H(T_q-T)=0$. Since $\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}$ is a pre-quasi closed ideal, then $T\in\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$; hence, for all $M\in\overline{\mathscr{I}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$, we have

$$H(M-T)=\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\overline{s_q(M-T)}\right|,\overline{0}\right)\right]^{\tau_q/K}$$
$$\le\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\overline{s_{[q/2]}(M-T_i)}\right|,\overline{0}\right)\right]^{\tau_q/K}+\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\overline{s_{[q/2]}(T_i-T)}\right|,\overline{0}\right)\right]^{\tau_q/K}\le\sup_r\inf_{i\ge r}\sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\overline{s_q(M-T_i)}\right|,\overline{0}\right)\right]^{\tau_q/K}. \tag{29}$$

□

Theorem 16. *$\overline{\mathscr{I}}^{\alpha}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)=$ the closure of $\mathfrak{F}(\Omega,\Lambda)$, if the conditions of Theorem 7 are satisfied. But the converse is not necessarily true.*

Proof. As $\overline{b}_m\in(c_0^F(\nabla_p^2,\tau))_h$, for all $m\in\mathscr{N}$ and $(c_0^F(\nabla_p^2,\tau))_h$ is a linear space. If $Z\in\mathfrak{F}(\Omega,\Lambda)$, one has $\overline{(\alpha_m(Z))_{m=0}^{\infty}}\in E$. Then, the closure of $\mathfrak{F}(\Omega,\Lambda)\subseteq\overline{\mathscr{I}}^{\alpha}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$. Suppose

$Z \in \overline{\mathscr{F}^{\alpha}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$, one has $(\overline{\alpha_m(Z)})_{m=0}^{\infty} \in (c_0^F(\nabla_p^2,\tau))_h$. Since $h(\overline{\alpha_m(Z)})_{m=0}^{\infty} < \infty$, if $\rho \in (0,1)$, one has $m_0 \in \mathscr{N} - \{0\}$ so that $h((\overline{\alpha_m(Z)})_{m=m_0}^{\infty}) < (\rho/4)$. As $(\overline{\alpha_m(Z)})_{m=0}^{\infty}$ is decreasing, one gets

$$\sup_{m=m_0+1}^{2m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_{2m_0}(Z)}\right|,\bar{0}\right)\right]^{\tau_m/K} \le \sup_{m=m_0+1}^{2m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m/K} \le \sup_{m=m_0}^{\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m/K} < \frac{\rho}{4}. \tag{30}$$

Then, one has $Y \in \mathfrak{F}_{2m_0}(\Omega,\Lambda)$ such that rank $(Y) \le 2m_0$ and

$$\sup_{m=2m_0+1}^{3m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\|Z-Y\|}\right|,\bar{0}\right)\right]^{\tau_m/K} \le \sup_{m=m_0+1}^{2m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\|Z-Y\|}\right|,\bar{0}\right)\right]^{\tau_m/K} < \frac{\rho}{4}. \tag{31}$$

As $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, we take

$$\sup_{m=0}^{m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\|Z-Y\|}\right|,\bar{0}\right)\right]^{\tau_m/K} < \frac{\rho}{4}. \tag{32}$$

According to inequalities (1–3), then

$$\begin{aligned} d(Z,Y) &= \sup_{m=0}^{\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_m(Z-Y)}\right|,\bar{0}\right)\right]^{\tau_m/K} \\ &\le \sup_{m=0}^{3m_0-1}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_m(Z-Y)}\right|,\bar{0}\right)\right]^{\tau_m/K} + \sup_{m=3m_0}^{\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_m(Z-Y)}\right|,\bar{0}\right)\right]^{\tau_m/K} \le \sup_{m=0}^{3m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\|Z-Y\|}\right|,\bar{0}\right)\right]^{\tau_m/K} \\ &\quad + \sup_{m=m_0}^{\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_{m+2m_0}(Z-Y)}\right|,\bar{0}\right)\right]^{\tau_{m+2m_0}/K} \le \sup_{m=0}^{3m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\|Z-Y\|}\right|,\bar{0}\right)\right]^{\tau_m/K} \\ &\quad + \sup_{m=m_0}^{\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m/K} \le 3\sup_{m=0}^{m_0}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\|Z-Y\|}\right|,\bar{0}\right)\right]^{\tau_m/K} + \sup_{m=m_0}^{\infty}\left[\bar{\rho}\left(\left|\nabla_p^2\overline{\alpha_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m/K} < \rho. \end{aligned} \tag{33}$$

This implies $\overline{\mathscr{F}^{\alpha}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda) \subseteq$ the closure of $\mathfrak{F}(\Omega,\Lambda)$. Contrarily, one has a counterexample as $I_3 \in \overline{\mathscr{F}^{\alpha}}_{(c_0^F(\nabla_p^2,(0,0,1,1,\ldots))}(\Omega,\Lambda)$, but $\tau_0 > 0$ is not satisfied. □

Theorem 17. *assume the conditions of Theorem 7 are satisfied with $\tau_m^{(1)} < \tau_m^{(2)}$, for every $m \in \mathscr{N}$, then*

$$\overline{\mathscr{F}}_{\left(c_0^F\left(\nabla_q^2,\left(\tau_m^{(1)}\right)\right)\right)_n}(\Omega,\Lambda) \subsetneq \overline{\mathscr{F}}_{\left(c_0^F\left(\nabla^2,\left(\tau_m^{(2)}\right)\right)\right)_n}(\Omega,\Lambda) \subsetneq \mathscr{L}(\Omega,\Lambda). \tag{34}$$

Proof. suppose that $Z \in \overline{\mathscr{F}}_{(c_0^F(\nabla_q^2,(\tau_m^{(1)}))_h}(\Omega,\Lambda)$, then $(\overline{s_m(Z)}) \in (c_0^F(\nabla_q^2,(\tau_m^{(1)}))_h$. We have

$$\lim_{m\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla^2\overline{s_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m^{(2)}} = \lim_{m\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_q^2\overline{s_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m^{(1)}} = 0. \tag{35}$$

Then, $Z \in \overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda)$. Next, if we take $(\overline{s_m(Z)})_{m=0}^{\infty} = (\bar{0},\bar{1},\bar{2},\ldots)$, one has $Z \in \mathscr{L}(\Omega,\Lambda)$ so that

$$\begin{aligned} &\lim_{m\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla_q^2\overline{s_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m^{(1)}} \ne 0, \\ &\lim_{m\longrightarrow\infty}\left[\bar{\rho}\left(\left|\nabla^2\overline{s_m(Z)}\right|,\bar{0}\right)\right]^{\tau_m^{(2)}} = 0. \end{aligned} \tag{36}$$

Therefore, $Z \notin \overline{\mathscr{F}}_{(c_0^F(\nabla_q^2,(\tau_m^{(1)})))_h}(\Omega,\Lambda)$ and $Z \in \overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda)$.

Evidently, $\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda) \subset \mathscr{L}(\Omega,\Lambda)$. After, if we choose $(\overline{s_m(Z)})_{m=0}^{\infty}$, then $(\nabla^2\overline{s_m(Z)}) = (\bar{1},\bar{1},\ldots)$. One has $Z \in \mathscr{L}(\Omega,\Lambda)$ such that $Z \notin \overline{\mathscr{F}}_{(\chi_0^F(\nabla_q^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda)$. □

Lemma 2 (see [3]). *If we suppose $B \in \mathscr{L}(\Omega,\Lambda)$ and $B \notin \Upsilon(\Omega,\Lambda)$, then $D \in \mathscr{L}(\Omega)$ and $M \in \mathscr{L}(\Lambda)$ with $MBDe_b = e_b$, with $b \in \mathscr{N}$.*

Theorem 18 (see [3]). *In general, one has*

$$\mathfrak{F}(\Omega)\subsetneq \Upsilon(\Omega)\subsetneq \mathscr{L}_c(\Omega)\subsetneq \mathscr{L}(\Omega). \tag{37}$$

Theorem 19. *If the conditions of Theorem 7 are satisfied with $\tau_m^{(1)} < \tau_m^{(2)}$, for all $m \in \mathscr{N}$, then*

$$\mathscr{L}\left(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda),\overline{\mathscr{F}}_{\left(c_0^F\left(\nabla_q^2,(\tau_m^{(1)})\right)\right)_h}(\Omega,\Lambda)\right) = \Upsilon\left(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda),\overline{\mathscr{F}}_{\left(c_0^F\left(\nabla_q^2,(\tau_m^{(1)})\right)\right)_h}(\Omega,\Lambda)\right). \tag{38}$$

Proof. Let $X \in \mathscr{L}(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda), \overline{\mathscr{F}}_{(c_0^F(\nabla_q^2,(\tau_m^{(1)})))_h}(\Omega,\Lambda))$ and $X \notin \Upsilon(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda), \overline{\mathscr{F}}_{(c_0^F(\nabla_q^2,(\tau_m^{(1)})))_h}(\Omega,\Lambda))$. In view of Lemma 2, one has $Y \in \mathscr{L}(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda))$ and $Z \in \mathscr{L}(\overline{\mathscr{F}}_{(c_0^F(\nabla_q^2,(\tau_m^{(1)})))_h}(\Omega,\Lambda))$ so that $ZXYI_b = I_b$, and then, with $b \in \mathscr{N}$, we have

$$\|I_b\|_{\overline{\mathscr{F}}_{(c_0^F(\nabla_q^2,(\tau_m^{(1)})))_h}(\Omega,\Lambda)} = \sup_m\left[\overline{\rho}\left(\left|\nabla_q^2\overline{s_m(I_b)}\right|,\overline{0}\right)\right]^{\tau_m^{(1)}} \leq \|ZXY\|\|I_b\|_{\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda)} \leq \sup_m\left[\overline{\rho}\left(\left|\nabla^2\overline{s_m(I_b)}\right|,\overline{0}\right)\right]^{\tau_m^{(2)}}. \tag{39}$$

This contradicts Theorem 18. As $X \in \Upsilon(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda), \overline{\mathscr{F}}_{(c_0^F(\nabla_q^2,(\tau_m^{(1)})))_h}(\Omega,\Lambda))$. □

Corollary 1. *suppose that the conditions of Theorem 7 are satisfied with $\tau_m^{(1)} < \tau_m^{(2)}$, for every $m \in \mathscr{N}$, then*

$$\mathscr{L}\left(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda),\overline{\mathscr{F}}_{\left(c_0^F\left(\nabla_q^2,(\tau_m^{(1)})\right)\right)_h}(\Omega,\Lambda)\right) = \mathscr{L}_c\left(\overline{\mathscr{F}}_{(c_0^F(\nabla^2,(\tau_m^{(2)})))_h}(\Omega,\Lambda),\overline{\mathscr{F}}_{\left(c_0^F\left(\nabla_q^2,(\tau_m^{(1)})\right)\right)_h}(\Omega,\Lambda)\right). \tag{40}$$

Proof. Obviously, since $\Upsilon \subset \mathscr{L}_c$. □

Definition 12. [3] A Banach space Ω is said to be simple, if there is only one nontrivial closed ideal in $\mathscr{L}(\Omega)$.

Theorem 20. *assume that the conditions of Theorem 7 are verified, then $\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$ is simple.*

Proof. Let $X \in \mathscr{L}_c(\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda))$ and $X \notin \Upsilon(\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda))$. From Lemma 2, there exist $Y, Z \in \mathscr{L}(\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda))$ with $ZXYI_b = I_b$. This implies $I_{\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)} \in \mathscr{L}_c(\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda))$. If $\mathscr{L}(\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)) = \mathscr{L}_c(\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda))$, then $\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}$ is a simple Banach space. □

Notations 2.

$$\begin{aligned}
&\left(\mathscr{F}_{\mathbf{U}}\right)^{\lambda}: =\left\{\left(\mathscr{F}_{\mathbf{U}}\right)^{\lambda}(\Omega,\Lambda);\ \Omega \text{ and } \Lambda \text{ are Banach Spaces}\right\}, \text{ where}\\
&\left(\mathscr{F}_{\mathbf{U}}\right)^{\lambda}(\Omega,\Lambda): =\left\{X \in \mathscr{L}(\Omega,\Lambda):\ \left((\lambda_m(X))_{m=0}^{\infty} \in \mathbf{U} \text{ and } \left\|X-\overline{\rho}(\lambda_m(X),\overline{0})I\right\| \text{is not invertible, with } m \in \mathscr{N}\right\}.
\end{aligned} \tag{41}$$

Theorem 21. *If the conditions of Theorem 7 are satisfied, then*

$$\left(\overline{\mathscr{F}}_{\left(c_0^F\left(\nabla_p^2,\tau\right)\right)_h}\right)^{\lambda}(\Omega,\Lambda) = \overline{\mathscr{F}}_{\left(c_0^F\left(\nabla_p^2,\tau\right)\right)_h}(\Omega,\Lambda). \tag{42}$$

Proof. Let $X \in (\overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h})^{\lambda}(\Omega,\Lambda)$, then $(\lambda_m(X))_{m=0}^{\infty} \in (c_0^F(\nabla_p^2,\tau))_h$ and $\|X-\overline{\rho}(\lambda_m(X),\overline{0})I\| = 0$, for all $m \in \mathscr{N}$. Therefore, $\lim_{m\longrightarrow\infty}[\overline{\rho}(|\nabla_p^2|\lambda_m(X)||,\overline{0})]^{\tau_m/K} = 0$. One has $X = \overline{\rho}(\lambda_m(X),\overline{0})I$, for every $m \in \mathscr{N}$, so

$$\overline{\rho}\left(\overline{s_m(X)},\overline{0}\right) = \overline{\rho}\left(\overline{s_m(\overline{\rho}(\lambda_m(X),\overline{0})I)},\overline{0}\right) = \overline{\rho}(\lambda_m(X),\overline{0}), \tag{43}$$

for every $m \in \mathscr{N}$. Hence, $(\overline{s_m(X)})_{m=0}^{\infty} \in (c_0^F(\nabla_p^2,\tau))_h$, and then, $X \in \overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$. After, we assume $X \in \overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$. Hence, $(\overline{s_m(X)})_{m=0}^{\infty} \in (c_0^F(\nabla_p^2,\tau))_h$. We have

$$\lim_{m\longrightarrow\infty}\left[\overline{\rho}\left(\left|\nabla_p^2\left|\overline{s_m(X)}\right|\right|,\overline{0}\right)\right]^{\tau_m/K} = 0. \tag{44}$$

As ∇_p^2 is continuous, then $\lim_{m\longrightarrow\infty}\overline{\rho}(\overline{s_m(X)},\overline{0}) = 0$. If $\|X-\overline{\rho}(\overline{s_m(X)},\overline{0})I\|^{-1}$ exists, with $m \in \mathscr{N}$, then

$\|X-\overline{\rho}(\overline{s_m(X)},\overline{0})I\|^{-1}$ exists and bounded, for every $m\in\mathcal{N}$. As $\lim_{m\longrightarrow\infty}\|X-\overline{\rho}(\overline{s_m(X)},\overline{0})I\|^{-1}=\|X\|^{-1}$ exists and bounded. As $(\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h},H)$ is a pre-quasi mappings' ideal, one gets

$$I=XX^{-1}\in\overline{\mathcal{F}}_{\left(c_0^F\left(\nabla_p^2,\tau\right)\right)_h}(\Omega,\Lambda)\Rightarrow\left(\overline{s_m(I)}\right)_{m=0}^{\infty}\in c_0^F\left(\nabla_p^2,\tau\right)\Rightarrow\lim_{m\longrightarrow\infty}\overline{\rho}\left(\overline{s_m(I)},\overline{0}\right)=0. \tag{45}$$

We have a contradiction, since $\lim_{m\longrightarrow\infty}\overline{\rho}(\overline{s_m(I)},\overline{0})=1$. Then, $\|X-\overline{\rho}(\overline{s_m(X)},\overline{0})I\|=0$, with $m\in\mathcal{N}$, which proves that $X\in(\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h})^{\lambda}(\Omega,\Lambda)$. □

Theorem 22. *For $s-$ type, $\mathbf{U}_h$: $=\{z=(\overline{s_r(X)})\in\omega(F)$: $X\in\mathcal{L}(\Omega,\Lambda)andh(z)<\infty\}$. If $\overline{\mathcal{F}}_{\mathbf{U}_h}$ is a mappings' ideal, then the following conditions are verified:*

(1) $E\subset s-$ type $\mathbf{U}_h$.

(2) Assume $(\overline{s_r(X_1)})_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$ and $(\overline{s_r(X_2)})_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$, then $(\overline{s_r(X_1+X_2)})_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$.

(3) If $\lambda\in\mathfrak{R}$ and $(\overline{s_r(X)})_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$, then $|\lambda|(\overline{s_r(X)})_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$.

(4) The sequence space $\mathbf{U}_h$ is solid; that is, if $(\overline{s_r(Y)})_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$ and $\overline{s_r(X)}\leq\overline{s_r(Y)}$, for all $r\in\mathcal{N}$ and $X,Y\in\mathcal{L}(\Omega,\Lambda)$, then $(\overline{s_r(X)})_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$.

Proof. If $\overline{\mathcal{F}}_{\mathbf{U}_h}$ is a mappings' ideal.

(i) We have $\mathfrak{F}(\Omega,\Lambda)\subset\overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega,\Lambda)$. Hence, for all $X\in\mathfrak{F}(\Omega,\Lambda)$, we have $(\overline{s_r(X)})_{r=0}^{\infty}\in E$. This gives $(\overline{s_r(X)})_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h$. Hence, $E\subset s-$ type $\mathbf{U}_h$.

(ii) The space $\overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega,\Lambda)$ is linear over $\mathfrak{R}$. Hence, for each $\lambda\in\mathfrak{R}$ and $X_1,X_2\in\overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega,\Lambda)$, we have $X_1+X_2\in\overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega,\Lambda)$ and $\lambda X_1\in\overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega,\Lambda)$. This implies

$$\begin{gathered}\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h\text{ and }\left(\overline{s_r(X_2)}\right)_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h\Rightarrow\left(\overline{s_r(X_1+X_2)}\right)_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h,\\ \lambda\in\mathfrak{R}\text{ and }\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h\Rightarrow|\lambda|\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h.\end{gathered} \tag{46}$$

(iii) If $A\in\mathcal{L}(\Omega_0,\Omega)$, $B\in\overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega,\Lambda)$ and $D\in\mathcal{L}(\Lambda,\Lambda_0)$, then $DBA\in\overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega_0,\Lambda_0)$, where Ω_0 and Λ_0 are arbitrary Banach spaces. Therefore, since $(\overline{s_r(B)})_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h$, then $(\overline{s_r(DBA)})_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h$. Since $\overline{s_r(DBA)}\leq\|D\|\overline{s_r(B)}\|A\|$. By using condition 3, if $(\|D\|\|A\|\overline{s_r(B)})_{r=0}^{\infty}\in\mathbf{U}_h$, we have $(\overline{s_r(DBA)})_{r=0}^{\infty}\in s-\text{type}\,\mathbf{U}_h$. This means $s-\text{type}\,\mathbf{U}_h$ is solid.

In view of Theorem 12 and Theorem 23, we conclude the following properties of the $s-\text{type}(c_0^F(\nabla_p^2,\tau))_h$ space. □

Theorem 23. *If $s-$ type $(c_0^F(\nabla_p^2,\tau))_h$: $=\left\{z=(\overline{s_r(X)})\in\omega(F)\right.$: $X\in\mathcal{L}(\Omega,\Lambda)$and $\left.h(z)<\infty\right\}$, then the following conditions are verified:*

(1) $E\subset s-$ type $(c_0^F(\nabla_p^2,\tau))_h$.

(2) Assume $(\overline{s_r(X_1)})_{r=0}^{\infty}\in s-$ type $(c_0^F(\nabla_p^2,\tau))_h$ and $(\overline{s_r(X_2)})_{r=0}^{\infty}\in s-$ type $(c_0^F(\nabla_p^2,\tau))_h$, then $(\overline{s_r(X_1+X_2)})_{r=0}^{\infty}\in s-$ type $(c_0^F(\nabla_p^2,\tau))_h$.

(3) If $\lambda\in\mathfrak{R}$ and $(\overline{s_r(X)})_{r=0}^{\infty}\in s-$ type $(c_0^F(\nabla_p^2,\tau))_h$, then $|\lambda|(\overline{s_r(X)})_{r=0}^{\infty}\in s-$ type $(c_0^F(\nabla_p^2,\tau))_h$.

(4) The sequence space $(c_0^F(\nabla_p^2,\tau))_h$ is solid; that is, if $(\overline{s_r(Y)})_{r=0}^{\infty}\in s-$ type $(c_0^F(\nabla_p^2,\tau))_h$ and $\overline{s_r(X)}\leq\overline{s_r(Y)}$, for all $r\in\mathcal{N}$ and $X,Y\in\mathcal{L}(\Omega,\Lambda)$, then $(\overline{s_r(X)})_{r=0}^{\infty}\in s-$ type $(c_0^F(\nabla_p^2,\tau))_h$.

Theorem 24. *The space $\overline{l}_0^F(\nabla_p^2,\tau)$ is not mappings' ideal, if the conditions (a) and (c) of Theorem 7 are satisfied*

Proof. If we choose $m=1$, $n=1$, $z_k=\overline{1}$, $y_k=z_k$ for $k=3s$ or $y_k=\overline{0}$, otherwise, for all $s,k\in\mathcal{N}$. We have $|y_k|\leq|z_k|$, for all $k\in\mathcal{N}$, $z\in(c_0^F(\nabla_p^2,\tau))_h$ and $y\notin(c_0^F(\nabla_p^2,\tau))_h$. Hence, the space $(c_0^F(\nabla_p^2,\tau))_h$ is not solid. □

5. Kannan Contraction Mapping on $c_0^F(\boldsymbol{\nabla}_p^2,\boldsymbol{\tau})$

In this section, we look at how to configure $(c_0^F(\nabla_p^2,\tau))_h$ with different h so that there is only one fixed point of Kannan contraction mapping. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated pre-quasi ideal. Interestingly, several numerical experiments are presented to illustrate our results.

Definition 13. An operator V: $\mathbf{U}_h\longrightarrow\mathbf{U}_h$ is said to be a Kannan h-contraction, if one gets $\alpha\in[0,1/2)$ with $h(Vy-Vz)\leq\alpha(h(Vy-y)+h(Vz-z))$, for all $y,z\in\mathbf{U}_h$.

An element $y\in\mathbf{U}_h$ is called a fixed point of V, when $V(y)=y$.

Theorem 25. *If the conditions of Theorem 7 are satisfied, and V: $(c_0^F(\nabla_p^2,\tau))_h\longrightarrow(c_0^F(\nabla_p^2,\tau))_h$ is Kannan h-contraction* *mapping, where* $h(y) = sup_q[\overline{\rho}(|\nabla_p^2|y_q||, \overline{0})]^{\tau_q/K}$, *for all* $y \in c_0^F(\nabla_p^2, \tau)$, *then* V *has a unique fixed point.*

Proof. If $y \in c_0^F(\nabla_p^2, \tau)$, one has $V^p y \in c_0^F(\nabla_p^2, \tau)$. As V is a Kannan h-contraction mapping, one gets

$$\begin{aligned} &h\left(V^{l+1}y - V^l y\right) \le \alpha\left(h\left(V^{l+1}y - V^l y\right) + h\left(V^l y - V^{l-1}y\right)\right) \Rightarrow \\ &h\left(V^{l+1}y - V^l y\right) \le \frac{\alpha}{1-\alpha} h\left(V^l y - V^{l-1}y\right) \le \left(\frac{\alpha}{1-\alpha}\right)^2 h\left(V^{l-1}y - V^{l-2}y\right) \le \cdots \le \left(\frac{\alpha}{1-\alpha}\right)^l h(Vy - y). \end{aligned} \tag{47}$$

So, for all $l, m \in \mathcal{N}$ with $m > l$, one gets

$$h\left(V^l y - V^m y\right) \le \alpha\left(h\left(V^l y - V^{l-1}y\right) + h\left(V^m y - V^{m-1}y\right)\right) \le \alpha\left(\left(\frac{\alpha}{1-\alpha}\right)^{l-1} + \left(\frac{\alpha}{1-\alpha}\right)^{m-1}\right) h(Vy - y). \tag{48}$$

Then, $\{V^l y\}$ is a Cauchy sequence in $(c_0^F(\nabla_p^2, \tau))_h$. As the space $(c_0^F(\nabla_p^2, \tau))_h$ is pre-quasi Banach space. One has $z \in (c_0^F(\nabla_p^2, \tau))_h$ with $\lim_{l\longrightarrow\infty} V^l y = z$ to prove that $Vz = z$. Since h verifies the Fatou property, one obtains

$$h(Vz - z) \le \sup_i \inf_{l\ge i} h\left(V^{l+1}y - V^l y\right) \le \sup_i \inf_{l \ge i} \left(\frac{\alpha}{1-\alpha}\right)^l h(Vy - y) = 0. \tag{49}$$

Then, $Vz = z$. So, z is a fixed point of V to show the uniqueness. Let $y, z \in (c_0^F(\nabla_p^2, \tau))_h$ be two not equal fixed points of V. One has

$$h(y - z) \le h(Vy - Vz) \le \alpha(h(Vy - y) + h(Vz - z)) = 0. \tag{50}$$

So, $y = z$. □

Corollary 2. *If the conditions of Theorem 7 are satisfied, and* $V: (c_0^F(\nabla_p^2, \tau))_h \longrightarrow (c_0^F(\nabla_p^2, \tau))_h$ *is Kannan h-contraction mapping, where* $h(y) = sup_q[\overline{\rho}(|\nabla_p^2|y_q||, \overline{0})]^{\tau_q/K}$, *for all* $y \in c_0^F(\nabla_p^2, \tau)$, *one has unique fixed point* z *of* V *so that* $h(V^l y - z) \le \alpha(\alpha/1-\alpha)^{l-1} h(Vy - y)$.

Proof. In view of Theorem 26, one has a unique fixed point z of V. So

$$h\left(V^l y - z\right) = h\left(V^l y - Vz\right) \le \alpha\left(h\left(V^l y - V^{l-1}y\right) + h(Vz - z)\right) = \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h(Vy - y). \tag{51}$$

□

Example 8. assume $V: (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h \longrightarrow (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$, where $h(z) = \sup_q[\overline{\rho}(|\nabla_p^2|z_q||, \overline{0})]^{2q+3/2q+4}$, for every $z \in c_0^F(\nabla_p^2, (2q + 3/q + 2))$ and

$$V(z) = \begin{cases} \dfrac{z}{4}, & h(z) \in [0,1), \\ \\ \dfrac{z}{5}, & h(z) \in [1,\infty). \end{cases} \tag{52}$$

As for each $x, y \in (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$ with $h(x), h(y) \in [0,1)$, one has

$$h(Vx - Vy) = h\left(\frac{x}{4} - \frac{y}{4}\right) \le \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3x}{4}\right) + h\left(\frac{3y}{4}\right)\right) = \frac{1}{\sqrt[4]{27}}(h(Vx - x) + h(Vy - y)). \tag{53}$$

For all $x, y \in (c_0^F(\nabla_p^2, (2q+3/q+2)))_h$ with $h(x), h(y) \in [1, \infty)$, one has

$$h(Vx - Vy) = h\left(\frac{x}{5} - \frac{y}{5}\right) \leq \frac{1}{\sqrt[4]{64}}\left(h\left(\frac{4x}{5}\right) + h\left(\frac{4y}{5}\right)\right) = \frac{1}{\sqrt[4]{64}}(h(Vx - x) + h(Vy - y)). \quad (54)$$

For all $x, y \in (c_0^F(\nabla_p^2, (2q+3/q+2)))_h$ with $h(x) \in [0, 1)$ and $h(y) \in [1, \infty)$, we get

$$\begin{aligned} h(Vx - Vy) &= h\left(\frac{x}{4} - \frac{y}{5}\right) \leq \frac{1}{\sqrt[4]{27}} h\left(\frac{3x}{4}\right) + \frac{1}{\sqrt[4]{64}} h\left(\frac{4y}{5}\right) \leq \frac{1}{\sqrt[4]{27}}\left(h\left(\frac{3x}{4}\right) + h\left(\frac{4y}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}(h(Vx - x) + h(Vy - y)). \end{aligned} \quad (55)$$

Hence, V is Kannan h-contraction as h satisfies the Fatou property. From Theorem 26, one has V holds one fixed point $\overline{\vartheta} \in (c_0^F(\nabla_p^2, (2q+3/q+2)))_h$.

Definition 14. pick up $\mathbf{U}_h$ be a pre-quasi-normed (cssf), $V: \mathbf{U}_h \longrightarrow \mathbf{U}_h$ and $z \in \mathbf{U}_h$. The operator V is called h-sequentially continuous at z, if and only if, when $\lim_{q\longrightarrow\infty} h(y_q - z) = 0$, then $\lim_{q\longrightarrow\infty} h(Vy_q - Vz) = 0$.

Example 9. suppose that $V: (c_0^F(\nabla_p^2, (q+1/2q+4)))_h \longrightarrow (c_0^F(\nabla_p^2, (q+1/2q+4)))_h$, where $h(z) = \sup_q [\overline{\rho}(|\nabla_p^2 |z_q||, \overline{0})]^{4q+4/2q+4}$, for every $z \in (c_0^F(\nabla_p^2, (q+1/2q+4)))_h$ and

$$V(z) = \begin{cases} \frac{1}{18}\left(\overline{b}_0 + z\right), & z_0(t) \in \left[0, \frac{1}{17}\right), \\ \frac{1}{17}\overline{b}_0, & z_0(t) = \frac{1}{17}, \\ \frac{1}{18}\overline{b}_0, & z_0(t) \in \left(\frac{1}{17}, 1\right]. \end{cases} \quad (56)$$

V is clearly both h-sequentially continuous and discontinuous at $1/17\overline{b}_0 \in (c_0^F(\nabla_p^2, (q+1/2q+4)))_h$.

Example 10. assume that V is defined as in Example 8. Suppose $\{z^{(n)}\} \subseteq (c_0^F(\nabla_p^2, (2q+3/q+2)))_h$ is such that $\lim_{n\longrightarrow\infty} h(z^{(n)} - z^{(0)}) = 0$, where $z^{(0)} \in (c_0^F(\nabla_p^2, (2q+3/q+2)))_h$ with $h(z^{(0)}) = 1$.

As the pre-quasi-norm h is continuous, we have

$$\begin{aligned} \lim_{n\longrightarrow\infty} h\left(Vz^{(n)} - Vz^{(0)}\right) &= \lim_{n\longrightarrow\infty} h\left(\frac{z^{(n)}}{4} - \frac{z^{(0)}}{5}\right) \\ &= h\left(\frac{z^{(0)}}{20}\right) > 0. \end{aligned} \quad (57)$$

Therefore, V is not h-sequentially continuous at $z^{(0)}$.

Theorem 26. *If the conditions of Theorem 7 are satisfied with $\tau_0 > 1$, and $V: (c_0^F(\nabla_p^2, \tau))_h \longrightarrow (c_0^F(\nabla_p^2, \tau))_h$, where $h(y) = \sup_q [\overline{\rho}(|\nabla_p^2 |y_q||, \overline{0})]^{\tau_q}$, for all $y \in c_0^F(\nabla_p^2, \tau)$, then we suppose that*

(1) V is Kannan h-contraction mapping.

(2) V is h-sequentially continuous at $z \in (c_0^F(\nabla_p^2, \tau))_h$.

(3) There is $y \in (c_0^F(\nabla_p^2, \tau))_h$ with $\{V^l y\}$ has $\{V^{l_j} y\}$ converging to z.

Then, $z \in (c_0^F(\nabla_p^2, \tau))_h$ is the only fixed point of V.

Proof. If we assume that z is not a fixed point of V, one has $Vz \neq z$. From parts (2) and (3), we get

$$\begin{aligned} \lim_{l_j\longrightarrow\infty} h\left(V^{l_j} y - z\right) &= 0, \\ \lim_{l_j\longrightarrow\infty} h\left(V^{l_j+1} y - Vz\right) &= 0. \end{aligned} \quad (58)$$

As V is Kannan h-contraction, one obtains

$$\begin{aligned} 0 < h(Vz - z) &= h\left(\left(Vz - V^{l_j+1}y\right) + \left(V^{l_j}y - z\right) + \left(V^{l_j+1}y - V^{l_j}y\right)\right) \\ &\le 2^{2\sup_i \tau_i - 2} h\left(V^{l_j+1}y - Vz\right) + 2^{2\sup_i \tau_i - 2} h\left(V^{l_j}y - z\right) + 2^{\sup_i \tau_i - 1}\alpha\left(\frac{\alpha}{1-\alpha}\right)^{l_j - 1} h(Vy - y). \end{aligned} \tag{59}$$

As $l_j \longrightarrow \infty$, one has a contradiction. Then, z is a fixed point of V to show that the uniqueness. Let $z, y \in (c_0^F(\nabla_p^2, \tau))_h$ be two not equal fixed points of V. One obtains

$$h(z - y) \le h(Vz - Vy) \le \alpha (h(Vz - z) + h(Vy - y)) = 0. \tag{60}$$

Hence, $z = y$. □

Example 11. assume that V is defined as in Example 8. Let $h(z) = \sup_q [\overline{\rho}(|\nabla_p^2 |z_q||, \overline{0})]^{2q+3/q+2}$, for all $z \in (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$. Since for all $x, y \in (c_0^F(\nabla_p^2, (2q+3/q + 2)))_h$ with $h(x), h(y) \in [0, 1)$, one gets

$$h(Vx - Vy) = h\left(\frac{x}{4} - \frac{y}{4}\right) \le \frac{2}{\sqrt{27}}\left(h\left(\frac{3x}{4}\right) + h\left(\frac{3y}{4}\right)\right) = \frac{2}{\sqrt{27}}(h(Vx - x) + h(Vy - y)). \tag{61}$$

For all $x, y \in (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$ with $h(x), h(y) \in [1, \infty)$, one gets

$$h(Vx - Vy) = h\left(\frac{x}{5} - \frac{y}{5}\right) \le \frac{1}{4}\left(h\left(\frac{4x}{5}\right) + h\left(\frac{4y}{5}\right)\right) = \frac{1}{4}(h(Vx - x) + h(Vy - y)). \tag{62}$$

For all $x, y \in (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$ with $h(x) \in [0, 1)$ and $h(y) \in [1, \infty)$, one gets

$$\begin{aligned} h(Vx - Vy) &= h\left(\frac{x}{4} - \frac{y}{5}\right) \le \frac{2}{\sqrt{27}} h\left(\frac{3x}{4}\right) + \frac{1}{4} h\left(\frac{4y}{5}\right) \le \frac{2}{\sqrt{27}}\left(h\left(\frac{3x}{4}\right) + h\left(\frac{4y}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}(h(Vx - x) + h(Vy - y)). \end{aligned} \tag{63}$$

So, V is Kannan$h$-contraction and $V^p(z) = \begin{cases} z/4^p, & h(z) \in [0, 1), \\ z/5^p, & h(z) \in [1, \infty). \end{cases}$
Obviously, V is h-sequentially continuous at $\overline{\vartheta} \in (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$ and $\{V^p z\}$ holds $\{V^{l_j} z\}$ converges to $\overline{\vartheta}$. By Theorem 27, the point $\overline{\vartheta} \in (c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$ is the only fixed point of V.

Definition 15. An operator $V: \overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega, \Lambda) \longrightarrow \overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega, \Lambda)$ is said to be a Kannan H-contraction, if one has $\alpha \in [0, 1/2)$ with $H(VT - VM) \le \alpha (H(VT - T) + H(VM - M))$, for all $T, M \in \overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega, \Lambda)$.

Definition 16. An operator $V: \overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega, \Lambda) \longrightarrow \overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega, \Lambda)$ is said to be H-sequentially continuous at M, where $M \in \overline{\mathcal{F}}_{\mathbf{U}_h}(\Omega, \Lambda)$, if and only if, $\lim_{r \longrightarrow \infty} H(T_r - M) = 0 \Rightarrow \lim_{r \longrightarrow \infty} H(VT_r - VM) = 0$.

Example 12. If$V: \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2, (2q+3/q+2)))_h}(\Omega, \Lambda) \longrightarrow \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2(2q+3/q+2)))_h}(\Omega, \Lambda)$,where $H(T) = \sup_q [\overline{\rho}(|\nabla_p^2 \overline{s_q(T)}|, \overline{0})]^{2q+3/2q+4}$, for every $T \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2, (2q+3/q+2)))_h}(\Omega, \Lambda)$ and

$$V(T) = \begin{cases} \dfrac{T}{6}, & H(T) \in [0, 1), \\ \dfrac{T}{7}, & H(T) \in [1, \infty). \end{cases} \tag{64}$$

Evidently, V is H-sequentially continuous at the zero operator $\Theta \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2, (2q+3/q+2)))_h}(\Omega, \Lambda)$. Let $\{T^{(j)}\} \subseteq \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2, (2q+3/q+2)))_h}(\Omega, \Lambda)$ be such that $\lim_{j \longrightarrow \infty} H(T^{(j)} - T^{(0)}) = 0$, where $T^{(0)} \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2, (2q+3/q+2)))_h}(\Omega, \Lambda)$ with $H(T^{(0)}) = 1$. Since the pre-quasi-norm H is continuous, one gets

$$\lim_{j\longrightarrow\infty} H\left(VT^{(j)} - VT^{(0)}\right) = \lim_{j\longrightarrow\infty} H\left(\frac{T^{(0)}}{6} - \frac{T^{(0)}}{7}\right) = H\left(\frac{T^{(0)}}{42}\right) > 0. \tag{65}$$

Therefore, V is not H-sequentially continuous at $T^{(0)}$.

Theorem 27. *the conditions of Theorem 7 are satisfied and* $V\colon \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda) \longrightarrow \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$, *then we assume that*

(i) V is Kannan H-contraction mapping.

(ii) V is H-sequentially continuous at an element $M \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$.

(iii) There are $G \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$ such that the sequence of iterates $\{V^r G\}$ has a $\{V^{r_m} G\}$ converging to M.

Then, $M \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$ is the unique fixed point of V.

Proof. Let M be not a fixed point of V; hence, $VM \neq M$. By using parts (ii) and (iii), we get

$$\lim_{r_m\longrightarrow\infty} H\left(V^{r_m} G - M\right) = 0 \text{ and } \lim_{r_m\longrightarrow\infty} H\left(V^{r_m+1} G - VM\right) = 0. \tag{66}$$

Since V is Kannan H-contraction, one obtains

$$\begin{aligned} 0 < H(VM - M) &= H\left(\left(VM - V^{r_m+1}G\right) + \left(V^{r_m}G - M\right) + \left(V^{r_m+1}G - V^{r_m}G\right)\right) \\ &\leq 2H\left(V^{r_m+1}G - VM\right) + 4H\left(V^{r_m}G - M\right) + 4\alpha\left(\frac{\alpha}{1-\alpha}\right)^{r_m-1} H(VG - G). \end{aligned} \tag{67}$$

As $r_m \longrightarrow \infty$, there is a contradiction. Hence, M is a fixed point of V to prove that the uniqueness of the fixed point M. We suppose that one has two not equal fixed points $M, J \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$ of V. So, one gets $H(M - J) \leq H(VM - VJ) \leq \alpha(H(VM - M) + H(VJ - J)) = 0$. Then, $M = J$. □

Example 13. In view of Example 12. Since for all $T_1, T_2 \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2q+3/q+2)))_h}(\Omega,\Lambda)$ with $H(T_1), H(T_2) \in [0,1)$, we have

$$H(VT_1 - VT_2) = H\left(\frac{T_1}{6} - \frac{T_2}{6}\right) \leq \frac{1}{\sqrt[4]{125}}\left(H\left(\frac{5T_1}{6}\right) + H\left(\frac{5T_2}{6}\right)\right) = \frac{1}{\sqrt[4]{125}}\left(H(VT_1 - T_1) + H(VT_2 - T_2)\right). \tag{68}$$

For all $T_1, T_2 \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2q+3/q+2)))_h}(\Omega,\Lambda)$ with $H(T_1), H(T_2) \in [1,\infty)$, we have

$$H(VT_1 - VT_2) = H\left(\frac{T_1}{7} - \frac{T_2}{7}\right) \leq \frac{1}{\sqrt[4]{216}}\left(H\left(\frac{6T_1}{7}\right) + H\left(\frac{6T_2}{7}\right)\right) = \frac{1}{\sqrt[4]{216}}\left(H(VT_1 - T_1) + H(VT_2 - T_2)\right). \tag{69}$$

For all $T_1, T_2 \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2q+3/q+2)))_h}(\Omega,\Lambda)$ with $H[T_1] \in [0,1)$ and $H(T_2) \in [1,\infty)$, we have

$$H(VT_1 - VT_2) = H\left(\frac{T_1}{6} - \frac{T_2}{7}\right) \leq \frac{1}{\sqrt[4]{125}} H\left(\frac{5T_1}{6}\right) + \frac{1}{\sqrt[4]{216}} H\left(\frac{6T_2}{7}\right) \leq \frac{1}{\sqrt[4]{125}}\left(H(VT_1 - T_1) + H(VT_2 - T_2)\right). \tag{70}$$

Hence, V is Kannan H-contraction and $V^r(T) = \begin{cases} (T/6^r), & H(T) \in [0,1), \\ (T/7^r), & H(T) \in [1,\infty). \end{cases}$

Obviously, V is H-sequentially continuous at $\Theta \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2q+3/q+2)))_h}(\Omega,\Lambda)$ and $\{V^r T\}$ has a subsequence $\{V^{r_m} T\}$ converges to Θ. By Theorem 28, Θ is the only fixed point of G.

6. Applications

In this section, some successful applications to the existence of solutions of nonlinear difference equations of fuzzy functions are introduced.

Theorem 28. *consider the summable equation*

$$y_q = R_q + \sum_{r=0}^{\infty} D(q,r)m(r,y_r), \tag{71}$$

which presented by Salimi et al. [35], and assume $V\colon (c_0^F(\nabla_p^2,\tau))_h \longrightarrow (c_0^F(\nabla_p^2,\tau))_h$, where the conditions of Theorem 7 are satisfied and $h(y) = sup_q[\overline{\rho}(|\nabla_p^2|y_q||,\overline{0})]^{\tau_q/K}$, for every $y \in c_0^F(\nabla_p^2,\tau)$, defined by

$$V(y_q)_{q\in\mathcal{N}} = \left(R_q + \sum_{r=0}^{\infty} D(q,r)m(r,y_r)\right)_{q\in\mathcal{N}}. \tag{72}$$

The summable equation (4) has a unique solution in $(c_0^F(\nabla_p^2,\tau))_h$, if $D\colon \mathcal{N}^2 \longrightarrow \mathfrak{R}$, $m\colon \mathcal{N}\times\mathfrak{R}[0,1] \longrightarrow \mathfrak{R}[0,1]$, $R\colon \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$, $z\colon \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$, there is ε so that $sup_q \varepsilon^{(\tau_q/K)} \in [0,0.5)$, and for all $q \in \mathcal{N}$, we have

$$\left|\sum_{r\in\mathcal{N}} D(q,r)(m(r,y_r) - m(r,z_r))\right| \le \varepsilon\left[\left|R_q - y_q + \sum_{r=0}^{\infty} D(q,r)m(r,y_r)\right| + \left|R_q - z_q + \sum_{r=0}^{\infty} D(q,r)m(r,z_r)\right|\right]. \tag{73}$$

Proof. One has

$$\begin{aligned} h(Vy - Vz) &= \sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\left|Vy_q - Vz_q\right|\right|,\overline{0}\right)\right]^{\tau_q/K} = \sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\left|\sum_{r\in\mathcal{N}} D(q,r)(m(r,y_r) - m(r,z_r))\right|\right|,\overline{0}\right)\right]^{\tau_q/K} \\ &\le \sup_q \varepsilon^{\tau_q/K} \sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\left|R_q - y_q + \sum_{r=0}^{\infty} D(q,r)m(r,y_r)\right|\right|,\overline{0}\right)\right]^{\tau_q/K} \\ &\quad + \sup_q \varepsilon^{\tau_q/K} \sup_q\left[\overline{\rho}\left(\left|\nabla_p^2\left|R_q - z_q + \sum_{r=0}^{\infty} D(q,r)m(r,z_r)\right|\right|,\overline{0}\right)\right]^{\tau_q/K} \\ &= \sup_q \varepsilon^{\tau_q/K}(h(Vy - y) + h(Vz - z)). \end{aligned} \tag{74}$$

By Theorem 26, one gets a unique solution of equation (4) in $(c_0^F(\nabla_p^2,\tau))_h$. □

Example 14. suppose $(c_0^F(\nabla_p^2,(2q+3/q+2)))_h$, where $h(y) = \sup_q[\overline{\rho}(|\nabla_p^2|y_q||,\overline{0})]^{2q+3/2q+4}$, for all $y \in c_0^F(\nabla_p^2,(2q+3/q+2))$. We consider the summable equation

$$y_q = R_q + \sum_{r=0}^{\infty} (-1)^{q+r}\left(\frac{y_q}{q^2+r^2+1}\right)^t, \tag{75}$$

with $t > 0$. Let $V\colon c_0^F(\nabla_p^2,(2q+3/q+2)) \longrightarrow c_0^F(\nabla_p^2,(2q+3/q+2))$ defined by

$$V(y_q) = \left(R_q + \sum_{r=0}^{\infty} (-1)^{q+r}\left(\frac{y_q}{q^2+r^2+1}\right)^t\right). \tag{76}$$

Obviously,

$$\begin{aligned} &\left|\sum_{r=0}^{\infty} (-1)^q\left(\frac{y_q}{q^2+r^2+1}\right)^t((-1)^r - (-1)^r)\right| \\ &\le \varepsilon\left[\left|R_q - y_q + \sum_{r=0}^{\infty} (-1)^{q+r}\left(\frac{y_q}{q^2+r^2+1}\right)^t\right| + \left|R_q - z_q + \sum_{r=0}^{\infty} (-1)^{q+r}\left(\frac{z_q}{q^2+r^2+1}\right)^t\right|\right]. \end{aligned} \tag{77}$$

By Theorem 29, the summable equation (75) has a unique solution in $c_0^F(\nabla_p^2, (2q+3/q+2))$.

Example 15. suppose $(c_0^F(\nabla_p^2, (q+3/2q+4)))_h$, where $h(y) = \sup_q[\bar{\rho}(|\nabla_p^2|y_q||, \bar{0})]^{q+3/2q+4}$, for all $y \in c_0^F(\nabla_p^2, (q+3/2q+4))$. We consider the summable equation

$$y_q = R_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t, \tag{78}$$

with $t > 0$. Let $V: c_0^F(\nabla_p^2, (q+3/2q+4)) \longrightarrow c_0^F(\nabla_p^2, (q+3/2q+4))$ defined by

$$V(y_q) = \left(R_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t\right). \tag{79}$$

Obviously,

$$\begin{aligned} &\left|\sum_{r=0}^{\infty} e^{q}\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t (e^r - e^r)\right| \\ &\leq \varepsilon\left[\left|R_q - y_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t\right| + \left|R_q - z_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{z_q^5}{z_q^3 + z_r^2 + \bar{1}}\right)^t\right|\right]. \end{aligned} \tag{80}$$

By Theorem 29, the summable equation (75) has a unique solution in $c_0^F(\nabla_p^2, (q+3/2q+4))$.

Example 16. suppose $(c_0^F(\nabla_p^2, (2q+3/q+2)))_h$, where $h(y) = \sup_q[\bar{\rho}(|\nabla_p^2|y_q||, \bar{0})]^{2q+3/2q+4}$, for every $y \in c_0^F(\nabla_p^2, (2q+3/q+2))$. We consider the nonlinear difference equations:

$$y_q = R_q + \sum_{l=0}^{\infty} (-1)^{q+l}\frac{y_{q-2}^r}{y_{q-1}^w + \overline{l^2 + 1}}, \tag{81}$$

with $r, w > 0$, $y_{-2}(t), y_{-1}(t) > 0$, for all $t \in \mathfrak{R}$, and assume $V: c_0^F(\nabla_p^2, (2q+3/q+2)) \longrightarrow c_0^F(\nabla_p^2, (2q+3/q+2))$, defined by

$$V(y_q)_{q=0}^{\infty} = \left(R_q + \sum_{l=0}^{\infty} (-1)^{q+l}\frac{y_{q-2}^r}{y_{q-1}^w + \overline{l^2 + 1}}\right)_{q=0}^{\infty}. \tag{82}$$

Evidently,

$$\begin{aligned} &\left|\sum_{l=0}^{\infty} (-1)^{q}\frac{y_{q-2}^r}{y_{q-1}^w + \overline{l^2 + 1}}\left((-1)^l - (-1)^l\right)\right| \\ &\leq \varepsilon\left[\left|R_q - y_q + \sum_{l=0}^{\infty} (-1)^{q+l}\frac{y_{q-2}^r}{y_{q-1}^w + \overline{l^2 + 1}}\right| + \left|R_q - z_q + \sum_{l=0}^{\infty} (-1)^{q+l}\frac{z_{q-2}^r}{z_{q-1}^w + \overline{l^2 + 1}}\right|\right]. \end{aligned} \tag{83}$$

By Theorem 29, the nonlinear difference (81) have a unique solution in $c_0^F(\nabla_p^2, (2q+3/q+2))$.

Theorem 29. *consider the summable equation (4) and assume $V: (c_0^F(\nabla_p^2, \tau))_h \longrightarrow (c_0^F(\nabla_p^2, \tau))_h$ is defined by (5), where the conditions of Theorem 7 are satisfied with $\tau_0 > 1$ and $h(y) = sup_q[\bar{\rho}(|\nabla_p^2|y_q||, \bar{0})]^{\tau_q}$, for every $y \in c_0^F(\nabla_p^2, \tau)$. The summable equation (4) has a unique solution $z \in (c_0^F(\nabla_p^2, \tau))_h$, if the following conditions are satisfied:*

(1) If $D: \mathcal{N}^2 \longrightarrow \mathfrak{R}$, $m: \mathcal{N} \times \mathfrak{R}[0,1] \longrightarrow \mathfrak{R}[0,1]$, $R: \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$, $z: \mathcal{N} \longrightarrow \mathfrak{R}[0,1]$, there is ε so that $2^{K-1}sup_q\varepsilon^{\tau_q} \in [0, 0.5)$, and for all $q \in \mathcal{N}$, we have

$$\begin{aligned} &\left|\sum_{r\in\mathcal{N}} D(q,r)\left(m(r,y_r) - m(r,z_r)\right)\right| \\ &\leq \varepsilon\left[\left|R_q - y_q + \sum_{r=0}^{\infty} D(q,r)m(r,y_r)\right| + \left|R_q - z_q + \sum_{r=0}^{\infty} D(q,r)m(r,z_r)\right|\right]. \end{aligned} \tag{84}$$

(2) V is h-sequentially continuous at $z \in (c_0^F(\nabla_p^2, \tau))_h$.

(3) There is $y \in (c_0^F(\nabla_p^2, \tau))_h$ with $\{V^l y\}$ has $\{V^{l_j} y\}$ converging to z.

Proof. One has

$$\begin{aligned} h(Vy - Vz) &= \sup_q\left[\bar{\rho}\left(\left|\nabla_p^2\right|Vy_q - Vz_q\right\|, \bar{0}\right)\right]^{\tau_q} = \sup_q\left[\bar{\rho}\left(\left|\nabla_p^2\right|\sum_{r\in\mathcal{N}} D(q,r)\left(m(r,y_r) - m(r,z_r)\right)\right\|, \bar{0}\right)\right]^{\tau_q} \\ &\le 2^{K-1}\sup_q \varepsilon^{\tau_q}\sup_q\left[\bar{\rho}\left(\left|\nabla_p^2\right|R_q - y_q + \sum_{r=0}^{\infty} D(q,r)m(r,y_r)\right\|, \bar{0}\right)\right]^{\tau_q} \\ &\quad + 2^{K-1}\sup_q \varepsilon^{\tau_q}\sup_q\left[\bar{\rho}\left(\left|\nabla_p^2\right|R_q - z_q + \sum_{r=0}^{\infty} D(q,r)m(r,z_r)\right\|, \bar{0}\right)\right]^{\tau_q} \\ &= 2^{K-1}\sup_q \varepsilon^{\tau_q}\left(h(Vy - y) + h(Vz - z)\right). \end{aligned} \tag{85}$$

By Theorem 27, one gets a unique solution $z \in (c_0^F(\nabla_p^2, \tau))_h$ of (4). □

Example 17. suppose $(c_0^F(\nabla_p^2, (2q + 3/q + 2)))_h$, where $h(y) = \sup_q[\bar{\rho}(|\nabla_p^2|y_q\|, \bar{0})]^{2q+3/q+2}$, for all $y \in c_0^F(\nabla_p^2, (2q + 3/q + 2))$. We consider the summable equation

$$y_q = R_q + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{y_q}{q^2 + r^2 + 1}\right)^t, \tag{86}$$

with $t > 0$. Let $V: c_0^F(\nabla_p^2, (2q + 3/q + 2)) \longrightarrow c_0^F(\nabla_p^2, (2q + 3/q + 2))$ defined by

$$V(y_q) = \left(R_q + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{y_q}{q^2 + r^2 + 1}\right)^t\right). \tag{87}$$

We assume V is h-sequentially continuous at $z \in (c_0^F(\nabla_p^2, \tau))_h$, and there is $y \in (c_0^F(\nabla_p^2, \tau))_h$ with $\{V^l y\}$ has $\{V^{l_j} y\}$ converging to z. Obviously,

$$\begin{aligned} &\left|\sum_{r=0}^{\infty}(-1)^q\left(\frac{y_q}{q^2 + r^2 + 1}\right)^t\left((-1)^r - (-1)^r\right)\right| \\ &\quad \le \varepsilon\left[\left|R_q - y_q + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{y_q}{q^2 + r^2 + 1}\right)^t\right| + \left|R_q - z_q + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{z_q}{q^2 + r^2 + 1}\right)^t\right|\right]. \end{aligned} \tag{88}$$

By Theorem 29, the summable equation (86) has a unique solution $z \in c_0^F(\nabla_p^2, (2q + 3/q + 2))$.

Example 18. suppose $(c_0^F(\nabla_p^2, (5q + 3/q + 1)))_h$, where $h(y) = \sup_q[\bar{\rho}(|\nabla_p^2|y_q\|, \bar{0})]^{5q+3/q+1}$, for all $y \in c_0^F(\nabla_p^2, (5q + 3/q + 1))$. We consider the summable equation

$$y_q = R_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t, \tag{89}$$

with $t > 0$. Let $V: c_0^F(\nabla_p^2, (5q + 3/q + 1)) \longrightarrow c_0^F(\nabla_p^2, (5q + 3/q + 1))$ defined by

$$V(y_q) = \left(R_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t\right). \tag{90}$$

We assume that V is h-sequentially continuous at $z \in (c_0^F(\nabla_p^2, \tau))_h$, and there is $y \in (c_0^F(\nabla_p^2, \tau))_h$ with $\{V^l y\}$ has $\{V^{l_j} y\}$ converging to z. Obviously,

$$\begin{aligned} &\left|\sum_{r=0}^{\infty} e^q\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t\left(e^r - e^r\right)\right| \\ &\quad \le \varepsilon\left[\left|R_q - y_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{y_q^5}{y_q^3 + y_r^2 + \bar{1}}\right)^t\right| + \left|R_q - z_q + \sum_{r=0}^{\infty} e^{q+r}\left(\frac{z_q^5}{z_q^3 + z_r^2 + \bar{1}}\right)^t\right|\right]. \end{aligned} \tag{91}$$

By Theorem 29, the summable equation (89) has a unique solution $z \in c_0^F(\nabla_p^2, (5q+3/q+1))$.

Example 19. suppose $(c_0^F(\nabla_p^2, (2q+3/q+2)))_h$, where $h(y) = \sup_q [\overline{\rho}(|\nabla_p^2|y_q||, \overline{0})]^{2q+3/q+2}$, for every $y \in c_0^F(\nabla_p^2, (2q+3/q+2))$. We consider the nonlinear difference equation:

$$y_q = R_q + \sum_{l=0}^{\infty} (-1)^{q+l} \frac{y_{q-2}^r}{y_{q-1}^w + \overline{l^2+1}}, \tag{92}$$

with $r, w > 0$, $y_{-2}(t), y_{-1}(t) > 0$, for all $t \in \mathfrak{R}$, and assume $V: c_0^F(\nabla_p^2, (2q+3/q+2)) \longrightarrow c_0^F(\nabla_p^2, (2q+3/q+2))$, defined by

$$V(y_q)_{q=0}^{\infty} = \left(R_q + \sum_{l=0}^{\infty} (-1)^{q+l} \frac{y_{q-2}^r}{y_{q-1}^w + \overline{l^2+1}} \right)_{q=0}^{\infty}. \tag{93}$$

We suppose that V is h-sequentially continuous at $z \in (c_0^F(\nabla_p^2, \tau))_h$, and there is $y \in (c_0^F(\nabla_p^2, \tau))_h$ with $\{V^l y\}$ has $\{V^{l_j} y\}$ converging to z. Evidently,

$$\begin{aligned} &\left| \sum_{l=0}^{\infty} (-1)^q \frac{y_{q-2}^r}{y_{q-1}^w + \overline{l^2+1}} \left((-1)^l - (-1)^l \right) \right| \\ &\leq \varepsilon \left[\left| R_q - y_q + \sum_{l=0}^{\infty} (-1)^{q+l} \frac{y_{q-2}^r}{y_{q-1}^q + \overline{l^2+1}} \right| + \left| R_q - z_q + \sum_{l=0}^{\infty} (-1)^{q+l} \frac{z_{q-2}^r}{z_{q-1}^w + \overline{l^2+1}} \right| \right]. \end{aligned} \tag{94}$$

By Theorem 29, the nonlinear difference equation (18) has a unique solution $z \in c_0^F(\nabla_p^2, (2q+3/q+2))$.

In this part, we search for a solution to nonlinear matrix (81) at $D \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda)$, where Ω and Λ are Banach spaces, the conditions of Theorem 7 are satisfied, and

$\Psi(G) = \sup_q [\overline{\rho}(|\nabla_p^2|\overline{s_q(G)}||, \overline{0})]^{\tau_q/K}$, for all $G \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda)$. We consider the summable equation

$$\overline{s_a(G)} = \overline{s_a(P)} + \sum_{m=0}^{\infty} A(a,m) f(m, s_m(G)). \tag{95}$$

And we suppose that $W: \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda) \longrightarrow \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda)$ is defined by

$$W(G) = \left(\overline{s_a(P)} + \sum_{m=0}^{\infty} A(a,m) f(m, s_m(G)) \right) I. \tag{96}$$

Theorem 30. *The summable equation (18) has one solution $D \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda)$, if the following conditions are satisfied:*

(a) $A: \mathcal{N}^2 \longrightarrow \mathfrak{R}$, $f: \mathcal{N} \times \mathfrak{R}[0,1] \longrightarrow \mathfrak{R}[0,1]$, $P \in \mathcal{L}(\Omega, \Lambda)$, $T \in \mathcal{L}(\Omega, \Lambda)$, and for every $a \in \mathcal{N}$, there is κ so that $sup_a \kappa^{\tau_a} \in [0, 0.5)$, with

$$\begin{aligned} &\left| \sum_{m \in \mathcal{N}} A(a,m) \left(f\left(m, \overline{s_m(G)}\right) - f\left(m, \overline{s_m(T)}\right) \right) \right| \\ &\leq \kappa^K \left[\left| \overline{s_a(P)} - \overline{s_a(G)} + \sum_{m \in \mathcal{N}} A(a,m) f\left(m, \overline{s_m(G)}\right) \right| + \left| \overline{s_a(P)} - \overline{s_a(T)} + \sum_{m \in \mathcal{N}} A(a,m) f\left(m, \overline{s_m(T)}\right) \right| \right]. \end{aligned} \tag{97}$$

(b) W is Ψ-sequentially continuous at a point $D \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda)$.

(c) There is $B \in \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda)$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to D.

Proof. suppose the settings are verified. We consider the mapping $W: \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda) \longrightarrow \overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega, \Lambda)$ defined by (19). We have

$$\begin{aligned}\Psi(WG-WT) &= \sup_a\left[\overline{\rho}\left(\left|\nabla_p^2\left|\sum_{m\in\mathcal{N}} A(a,m)\left(f\left(m,\overline{s_m(G)}\right)-f\left(m,\overline{s_m(T)}\right)\right)\right|\right|,\overline{0}\right)\right]^{\tau_a/K}\\ &\leq \sup_a\kappa^{\tau_a}\sup_a\left[\overline{\rho}\left(\left|\nabla_p^2\left|\overline{s_a(P)}-\overline{s_a(G)}+\sum_{m\in\mathcal{N}}A(a,m)f\left(m,\overline{s_m(G)}\right)\right|\right|,\overline{0}\right)\right]^{\tau_a/K}\\ &\quad+\sup_a\kappa^{\tau_a}\sup_a\left[\overline{\rho}\left(\left|\nabla_p^2\left|\overline{s_a(T)}-\overline{s_a(G)}+\sum_{m\in\mathcal{N}}A(a,m)f\left(m,\overline{s_m(T)}\right)\right|\right|,\overline{0}\right)\right]^{\tau_a/K}\\ &= \sup_a\kappa^{\tau_a}\left(\Psi(WG-G)+\Psi(WT-T)\right).\end{aligned} \tag{98}$$

In view of Theorem 27, one obtains a unique solution of equation (18) at $D\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,\tau))_h}(\Omega,\Lambda)$. □

Example 20. We assume the class $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(a+1/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$, where $\Psi(G)=\sup_a[\overline{\rho}(|\nabla_p^2|\overline{s_a(G)}||,\overline{0})]^{a+1/a+2}$, for all $G\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(a+1/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$.

We consider the nonlinear difference equations:

$$\overline{s_a(G)}=\overline{e^{-(2a+3)}}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-a)\cos^r\left|\overline{s_{a-2}(G)}\right|}{\sinh^q\left|\overline{s_{a-1}(G)}\right|+\overline{\sin ma}+\overline{1}}, \tag{99}$$

where $a\geq 2$ and $r,q>0$ and let $W:\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(a+1/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)\longrightarrow\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(a+1/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$ be defined as

$$W(G)=\left(\overline{e^{-(2a+3)}}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-a)\cos^r\left|\overline{s_{a-2}(G)}\right|}{\sinh^q\left|\overline{s_{a-1}(G)}\right|+\overline{\sin ma}+\overline{1}}\right)I. \tag{100}$$

We suppose W is Ψ-sequentially continuous at a point $D\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(a+1/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$, and there is $B\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(a+1/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$ so that the sequence of iterates $\{W^aB\}$ has a subsequence $\{W^{a_i}B\}$ converging to D. It is easy to see that

$$\begin{aligned}&\left|\sum_{m=0}^{\infty}\frac{\cosh(3m-a)\cos^r\left|\overline{s_{a-1}(G)}\right|}{\sinh^q\left|\overline{s_{a-1}(G)}\right|+\overline{\sin ma}+\overline{1}}(\tan(2m+1)-\tan(2m+1))\right|^{(a+1)/(a+2)}\\ &\leq\frac{1}{5}\left|\overline{e^{-(2a+3)}}-\overline{s_a(G)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-a)\cos^r\left|\overline{s_{a-2}(G)}\right|}{\sinh^q\left|\overline{s_{a-1}(G)}\right|+\overline{\sin ma}+\overline{1}}\right|^{(a+1)/(a+2)}\\ &\quad+\frac{1}{5}\left|\overline{e^{-(2a+3)}}-\overline{s_a(T)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-a)\cos^r\left|\overline{s_{a-2}(T)}\right|}{\sinh^q\left|\overline{s_{a-1}(T)}\right|+\overline{\sin ma}+\overline{1}}\right|^{(a+1)/(a+2)}.\end{aligned} \tag{101}$$

By Theorem 30, the nonlinear difference (99) has one solution $D\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,a+1/a+2_{a=0}^{\infty}))_h}(\Omega,\Lambda)$.

Example 21. assume the class $\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2a+3/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$, where $\Psi(G)=\sup_a[\overline{\rho}(|\nabla_p^2|\overline{s_a(G)}||,\overline{0})]^{2a+3/2a+4}$, for all $G\in\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2a+3/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$. We consider the nonlinear difference equation (20) and let $W:\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2a+3/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)\longrightarrow\overline{\mathcal{F}}_{(c_0^F(\nabla_p^2,(2a+3/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$ be defined as (21). Suppose W is Ψ-sequentially continuous at a point

$D \in \overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,(2a+3/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$, and there is $B \in \overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,(2a+3/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$ so that the sequence of iterates $\{W^a B\}$ has a subsequence $\{W^{a_i} B\}$ converging to D. It is easy to see that

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3m-a)\cos^r\overline{|s_{a-2}(G)|}}{\sinh^q\overline{|s_{a-1}(G)|}+\overline{\sin ma}+\overline{1}}(\tan(2m+1)-\tan(2m+1))\right|^{(2a+3)/(2a+4)}$$
$$\leq\frac{1}{25}\left|\overline{e^{-(2a+3)}}-\overline{s_a(G)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-a)\cos^r\overline{|s_{a-2}(G)|}}{\sinh^q\overline{|s_{a-1}(G)|}+\overline{\sin ma}+\overline{1}}\right|^{(2a+3)/(2a+4)} \tag{102}$$
$$+\frac{1}{5}\left|\overline{e^{-(2a+3)}}-\overline{s_a(T)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-a)\cos^r\overline{|s_{a-2}(T)|}}{\sinh^q\overline{|s_{a-1}(T)|}+\overline{\sin ma}+\overline{1}}\right|^{(2a+3)/(2a+4)}.$$

By Theorem 30, the nonlinear difference (99) has one solution $D \in \overline{\mathscr{F}}_{(c_0^F(\nabla_p^2,(2a+3/a+2)_{a=0}^{\infty}))_h}(\Omega,\Lambda)$.

7. Conclusion

In this paper, we have explained sufficient settings of the space $c_0^F(\nabla_p^2,\tau)$ equipped with definite function h to be pre-quasi Banach. The Fatou property of various pre-quasi-norms h on $c_0^F(\nabla_p^2,\tau)$ has been investigated. The geometric and topological structures of the mappings' ideal by this space and extended $s-$ fuzzy functions have been explained. We construct the existence of a fixed point of Kannan contraction mapping acting on this space and its associated pre-quasi ideal. Interestingly, several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of nonlinear difference equations of fuzzy functions are introduced. As a future project, we can build the domain of second-order quantum backward difference in Nakano sequences of fuzzy functions space and look at its properties.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declared that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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