Research Article

Bäcklund Transformation and Exact Solutions to a Generalized $(3 + 1)$ -Dimensional Nonlinear Evolution Equation

Yali Shen and Ying Yang

School of Mathematics and Information Technology, Yuncheng University, Yuncheng, Shanxi 044000, China

Correspondence should be addressed to Yali Shen; shenyal422@163.com

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In this article, a generalized $(3 + 1)$ -dimensional nonlinear evolution equation (NLEE), which can be obtained by a multivariate polynomial, is investigated. Based on the Hirota bilinear method, the N -soliton solution and bilinear Bäcklund transformation (BBT) with explicit formulas are successfully constructed. By using BBT, two traveling wave solutions and a mixed solution of the generalized $(3 + 1)$ -dimensional NLEE are obtained. Furthermore, the lump and the interaction solutions for the equation are constructed. Finally, the dynamic properties of the lump and the interaction solutions are described graphically.

1. Introduction

With recent advancements in nonlinear science, the application of nonlinear evolution equations (NLEEs) has gained more prominence amongst mathematicians and physicists. The use of NLEEs is important because NLEEs can be used to describe many interesting nonlinear dynamic behaviors in nonlinear fields like optical fibers, plasma physics, atmospheric science, biologic nerve propagation, and marine science. Although the use of NLEEs is significant, the task of finding an explicit solution for the NLEEs has always been difficult and tedious. However, this task is also an important and meaningful exercise.

Many effective methods for solving NLEEs have been proposed by previous researchers [1–9]. For instance, Zhang et al. obtained the exact solutions for generalized nonlinear diffusion equations by using the separation of variables method [4]. He and Wu also used the Exp-function method to solve the nonlinear wave equations [5]. Xu et al. presented analytic solutions of the discrete PT-symmetric nonlocal nonlinear Schrödinger equation by Darboux transformation [9]. Amongst the proposed methods [1–9], the Hirota bilinear method is one of the most direct and effective methods for obtaining the exact solution of the NLEEs.

The most critical step in using the Hirota bilinear method is to transform the original nonlinear equation into a bilinear equation by introducing appropriate variable transformations. After obtaining the bilinear equation, the multisoliton solution of the original nonlinear equation can then be constructed. Interestingly, Ma et al. showed that multivariable polynomials can also produce Hirota bilinear equations [10]. The necessary criterion for the existence of a linear subspace for the exponential traveling wave solutions of the Hirota bilinear equation was given by Ma et al. [10]. Generally, the Hirota bilinear equations derived from multivariable polynomials have linear subspace solutions.

By considering the multivariable polynomial given by [10]

$$P(x, y, z, t) = c_1 x^4 + c_2 x t + c_3 y^2 + c_4 y z + c_5 z^2, \quad (1)$$

where $c_i (i = 1, \dots, 5)$ are arbitrary constants whose corresponding Hirota bilinear equation is

$$(c_1 D_x^4 + c_2 D_x D_t + c_3 D_y^2 + c_4 D_y D_z + c_5 D_z^2) f \cdot f = 0, \quad (2)$$

where $f = f(x, y, z, t)$ and D is the Hirota bilinear operator [2].

$$D_x^m D_t^n a \cdot b = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^m \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \right)^n a(x_1, t_1) b(x_2, t_2) \Big|_{x_1=x_2, t_1=t_2} \quad (3)$$

If the coefficients c_i are chosen as $c_1 = c_2 = 1$ and $c_3 = c_4 = c_5 = 0$, then (2) can be reduced to the bilinear form of the Korteweg–de Vries (KdV) equation. Furthermore, if the coefficients c_i are set as $c_1 = c_2 = 1$, $c_3 = c_5 = 3$, and $c_4 = 0$, then (2) can be reduced to the bilinear form of (3+1)-dimensional Kadomtsev–Petviashvili (KP) equation.

By introducing the variable transformation given by

$$u = 2(\ln f)_x, \quad (4)$$

(2) is then mapped to

$$c_1 u_{xxxx} + c_2 u_{xt} + c_3 u_{yy} + c_4 u_{yz} + c_5 u_{zz} + 6c_1 u_x u_{xx} = 0, \quad (5)$$

where $f = f(x, y, z, t)$ and $u = u(x, y, z, t)$. Based on our literature review, (5) is a generalized (3+1)-dimensional NLEE which has not been presented by previous authors. Recently, many generalized NLEEs have attracted more attention as a result of their widespread application in many

physical fields [11–14]. By investigating different kinds of generalized NLEEs, more general properties of the equations can be obtained.

This article is divided into sections and it focuses on the generalized (3+1)-dimensional NLEE represented by equation (5). In Section 2, the bilinear Bäcklund transformation (BBT) of equation (5) is constructed based on its Hirota bilinear equation. Armed with the BBT, the traveling wave solution and a mixed solution of (5) are obtained. In Section 3, the one-soliton, the two-soliton, and the three-soliton solutions are obtained, and then the N -soliton solution is derived. In Section 4, the lump and the interaction solutions of (5) are given. The solutions also included some free parameters based on the bilinear form of (5). However, their dynamical behaviors are shown under some selected values of parameters. The article is then concluded with a summary and the implications of this research.

2. Bilinear Bäcklund Transformation and Its Application

Bäcklund transformation is important in the study of soliton systems. Some bilinear identities are given as follows [2]:

$$(D_x^4 a \cdot a) b^2 - (D_x^4 b \cdot b) a^2 = 2D_x [(D_x^3 a \cdot b) \cdot ab + 3(D_x^2 a \cdot b) \cdot (D_x b \cdot a)], \quad (6)$$

$$(D_x D_t a \cdot a) b^2 - (D_x D_t b \cdot b) a^2 = 2D_x (D_t a \cdot b) \cdot ba, \quad (7)$$

$$(D_x^2 a \cdot a) b^2 - (D_x^2 b \cdot b) a^2 = 2D_x (D_x a \cdot b) \cdot ba, \quad (8)$$

$$D_t (D_x a \cdot b) \cdot ba = D_x (D_t a \cdot b) \cdot ba, \quad (9)$$

$$D_x a \cdot a = 0, \quad (10)$$

where $a = a(x, t)$ and $b = b(x, t)$.

2.1. Bilinear Bäcklund Transformation. The BBT for (2) is then constructed and summarized as follows.

Theorem 1. *If f satisfies (2) as well as the following relations:*

$$B_1 \hat{f} \cdot f = (c_1 D_x^3 + \lambda_1 D_y + \lambda_2 D_z + c_2 D_t + \lambda_3) \hat{f} \cdot f = 0, \quad (11)$$

$$B_2 \hat{f} \cdot f = (3c_1 D_x^2 + \lambda_4 D_x) \hat{f} \cdot f = 0, \quad (12)$$

$$B_3 \hat{f} \cdot f = (c_3 D_y + \lambda_5 D_z + c_4 D_z - \lambda_1 D_x + \lambda_6) \hat{f} \cdot f = 0, \quad (13)$$

$$B_4 \hat{f} \cdot f = (c_5 D_z - \lambda_2 D_x - \lambda_5 D_y + \lambda_7) \hat{f} \cdot f = 0, \quad (14)$$

then $\hat{f}$ also satisfies (2) with (11)–(14), where λ_i ($i = 1, \dots, 7$) are arbitrary constants.

Proof. For the following function,

$$P = (c_1 D_x^4 + c_2 D_x D_t + c_3 D_y^2 + c_4 D_y D_z + c_5 D_z^2) \hat{f} \cdot \hat{f}, \quad (15)$$

we show that $\hat{f}$ satisfying (11)–(14) can make $P = 0$.

$$\begin{aligned}
Pff &= \left[(c_1 D_x^4 + c_2 D_x D_t + c_3 D_y^2 + c_4 D_y D_z + c_5 D_z^2) \hat{f} \cdot \hat{f} \right] f f \\
&\quad - \left[(c_1 D_x^4 + c_2 D_x D_t + c_3 D_y^2 + c_4 D_y D_z + c_5 D_z^2) f \cdot f \right] \hat{f} \hat{f} \\
&= c_1 \left[(D_x^4 \hat{f} \cdot \hat{f}) f^2 - (D_x^4 f \cdot f) \hat{f}^2 \right] + c_2 \left[(D_x D_t \hat{f} \cdot \hat{f}) f^2 - (D_x D_t f \cdot f) \hat{f}^2 \right] \\
&\quad + c_3 \left[(D_y^2 \hat{f} \cdot \hat{f}) f^2 - (D_y^2 f \cdot f) \hat{f}^2 \right] + c_4 \left[(D_y D_z \hat{f} \cdot \hat{f}) f^2 - (D_y D_z f \cdot f) \hat{f}^2 \right] \\
&\quad + c_5 \left[(D_z^2 \hat{f} \cdot \hat{f}) f^2 - (D_z^2 f \cdot f) \hat{f}^2 \right] \\
(6) - (8) &= 2c_1 D_x \left[(D_x^3 \hat{f} \cdot f) \cdot f \hat{f} + 3(D_x^2 \hat{f} \cdot f) \cdot (D_x f \cdot \hat{f}) \right] + 2c_2 D_x (D_t \hat{f} \cdot f) \cdot f \hat{f} \\
&\quad + 2c_3 D_y (D_y \hat{f} \cdot f) \cdot f \hat{f} + 2c_4 D_y (D_z \hat{f} \cdot f) \cdot f \hat{f} + 2c_5 D_z (D_z \hat{f} \cdot f) \cdot f \hat{f} \\
&= 2D_x \left[(c_1 D_x^3 \hat{f} \cdot f + \lambda_1 D_y \hat{f} \cdot f + \lambda_2 D_z \hat{f} \cdot f + \lambda_3 \hat{f} \cdot f) \cdot f \hat{f} + (3c_1 D_x^2 \hat{f} \cdot f + \lambda_4 D_x \hat{f} \cdot f) \cdot (D_x f \cdot \hat{f}) \right] \\
&\quad + 2D_x (c_2 D_t \hat{f} \cdot f) \cdot f \hat{f} \\
&\quad + 2D_y (c_3 D_y \hat{f} \cdot f + \lambda_5 D_z \hat{f} \cdot f + \lambda_6 \hat{f} \cdot f) \cdot f \hat{f} + 2D_y (c_4 D_z \hat{f} \cdot f - \lambda_1 D_x \hat{f} \cdot f) \cdot f \hat{f} \\
&\quad + 2D_z (c_5 D_z \hat{f} \cdot f - \lambda_2 D_x \hat{f} \cdot f - \lambda_5 D_y \hat{f} \cdot f + \lambda_7 \hat{f} \cdot f) \cdot f \hat{f} \\
(11) - (14) &= 2D_x (B_1 \hat{f} \cdot f) \cdot f \hat{f} + 2D_x (B_2 \hat{f} \cdot f) \cdot (D_x f \cdot \hat{f}) + 2D_y (B_3 \hat{f} \cdot f) \cdot f \hat{f} \\
&\quad + 2D_z (B_4 \hat{f} \cdot f) \cdot f \hat{f} \\
&\equiv 0.
\end{aligned} \tag{16}$$

Based on (9) and (10) respectively, it is clear that the coefficients of λ_i ($i = 1, 2, 5$) and λ_i ($i = 3, 4, 6, 7$) are zeros. Hence, the proof of Theorem 1 is complete. $\square$

2.2. Application of BBT. The value $f = 1$ is the solution of (2) and the corresponding original variable is $u = (\ln f)_x = 0$. Due to $D_x^n a \cdot 1 = \partial^n / \partial x^n a$, ($n \geq 1$), BBT (11)–(14) associated with $f = 1$ is reduced to the following equations:

$$\begin{cases} c_1 \hat{f}_{xxx} + \lambda_1 \hat{f}_y + \lambda_2 \hat{f}_z + c_2 \hat{f}_t + \lambda_3 \hat{f} = 0, \\ 3c_1 \hat{f}_{xx} + \lambda_4 \hat{f}_x = 0, \\ c_3 \hat{f}_y + \lambda_5 \hat{f}_z + c_4 \hat{f}_z - \lambda_1 \hat{f}_x + \lambda_6 \hat{f} = 0, \\ c_5 \hat{f}_z - \lambda_2 \hat{f}_x - \lambda_5 \hat{f}_y + \lambda_7 \hat{f} = 0. \end{cases} \tag{17}$$

Different types of solutions to (5) can be obtained by substituting different forms of $\hat{f}$ into (17).

2.2.1. Traveling Wave Solution. First consider the following form:

$$\hat{f} = 1 + \varepsilon e^{kx+ly+mz+\omega t}, \tag{18}$$

where ε , k , l , m , and ω are undetermined constants. By inserting (18) into (17) and assuming $\lambda_3 = \lambda_6 = \lambda_7 = 0$, these parameters can be determined by calculation as follows:

$$\begin{aligned} \lambda_2 &= \frac{c_3 c_5 m + \lambda_5^2 m + c_4 \lambda_5 m - \lambda_1 \lambda_5 k}{c_3 k}, & \lambda_4 &= -3c_1 k, & l &= \frac{\lambda_1 k - \lambda_5 m - c_4 m}{c_3}, \\ \omega &= \frac{2\lambda_1 \lambda_5 m k - c_1 c_3 k^4 + c_4 \lambda_1 m k - \lambda_1^2 k^2 - c_3 c_5 m^2 - \lambda_5^2 m^2 - c_4 \lambda_5 m^2}{c_2 c_3 k}. \end{aligned} \tag{19}$$

Therefore,

$$u = \frac{2ke^{\varepsilon kx + ((\lambda_1 k - \lambda_5 m - c_4 m)/c_3)y + mz + ((2\lambda_1 \lambda_5 mk - c_1 c_3 k^4 + c_4 \lambda_1 mk - \lambda_1^2 k^2 - c_3 c_5 m^2 - \lambda_5^2 m^2 - c_4 \lambda_5 m^2)/c_2 c_3 k)t}}{1 + \varepsilon e^{kx + ((\lambda_1 k - \lambda_5 m - c_4 m)/c_3)y + mz + ((2\lambda_1 \lambda_5 mk - c_1 c_3 k^4 + c_4 \lambda_1 mk - \lambda_1^2 k^2 - c_3 c_5 m^2 - \lambda_5^2 m^2 - c_4 \lambda_5 m^2)/c_2 c_3 k)t}}. \quad (20)$$

This solves (5), where ε , m , λ_5 , λ_7 , and c_i ($i = 1, 4, 5$) are arbitrary constants and $kc_2 c_3 \neq 0$. Solution (20) is a one-stripe soliton solution of (5).

By taking a simple polynomial solution,

$$\hat{f} = kx + ly + mz + \omega t, \quad (21)$$

where k , l , m , and ω are constants to be determined. Similarly, by substituting (21) into (17) and supposing $\lambda_i = 0$ ($i = 3, 6, 7$), a direct computation gets

$$\begin{aligned} \lambda_2 &= \frac{c_3 c_5 m + \lambda_5^2 m + c_4 \lambda_5 m - \lambda_5 \lambda_1 k}{c_3 k}, \\ l &= \frac{\lambda_1 k - \lambda_5 m - c_4 m}{c_3}, \\ \lambda_4 &= 0, \\ \omega &= \frac{2\lambda_1 \lambda_5 mk + c_4 \lambda_1 mk - \lambda_1^2 k^2 - c_3 c_5 m^2 - \lambda_5^2 m^2 - c_4 \lambda_5 m^2}{c_2 c_3 k}, \end{aligned} \quad (22)$$

with $kc_2 c_3 \neq 0$ and m , λ_1 , and λ_5 are arbitrary constants.

Hence,

$$u = \frac{2k}{kx + ly + mz + \omega t} \quad (23)$$

is a rational solution to (5), and parameters k , l , m , and ω satisfy (22).

2.2.2. Mixed Solution. $\hat{f}$ is now taken as a combination of a polynomial function and cosine function:

$$\begin{aligned} \hat{f} &= (k_1 x + l_1 y + m_1 z + \omega_1 t) \\ &+ \varepsilon_1 \cosh(k_2 x + l_2 y + m_2 z + \omega_2 t), \end{aligned} \quad (24)$$

where ε_1 , k_i , l_i , m_i , and ω_i ($i = 1, 2$) are constants to be determined. By inserting (24) into (17), these parameters can be determined by calculation as follows:

$$\begin{aligned} \lambda_2 &= \frac{2c_5 \lambda_1}{c_4 \pm \sqrt{c_4^2 - 4c_3 c_5}}, \\ \lambda_5 &= \frac{-2c_3 c_5}{c_4 \pm \sqrt{c_4^2 - 4c_3 c_5}}, \\ l_1 &= \frac{2\lambda_1 k_1 - m_1 (c_4 \pm \sqrt{c_4^2 - 4c_3 c_5})}{2c_3}, \\ l_2 &= \frac{-m_2 (c_4 \pm \sqrt{c_4^2 - 4c_3 c_5})}{2c_3}, \end{aligned}$$

$$\begin{aligned} \omega_1 &= \frac{\lambda_1 (-\lambda_1 k_1 \pm m_1 \sqrt{c_4^2 - 4c_3 c_5})}{c_2 c_3}, \\ \omega_2 &= \frac{\pm \lambda_1 m_2 \sqrt{c_4^2 - 4c_3 c_5}}{c_2 c_3}, \end{aligned} \quad (25)$$

where ε_1 , λ_1 , k_1 , m_1 , and m_2 are arbitrary constants and $c_2 c_3 \neq 0$ and $c_4^2 \geq 4c_3 c_5$.

Thus,

$$u = \frac{k_1 + \varepsilon_1 k_2 \sinh(k_2 x + l_2 y + m_2 z + \omega_2 t)}{(k_1 x + l_1 y + m_1 z + \omega_1 t) + \varepsilon_1 \cosh(k_2 x + l_2 y + m_2 z + \omega_2 t)}, \quad (26)$$

is a mixed form solution to (5), and k_i , l_i , m_i , and ω_i ($i = 1, 2$) satisfy (25).

3. Soliton Solutions

It is known that many interesting solutions, including breathers, lump, peakon, and rogue solutions, are all special reductions of N -soliton solutions in different situations. This section shows solutions with a few solitons to (5). The standard perturbation method is used to find possible soliton-like solutions. By using a tedious but straightforward calculation, the one-soliton, two-soliton, three-soliton, and N -soliton solutions can be obtained.

3.1. One-Soliton Solution. To obtain the one-soliton solution, we assume that f in (2) is expressed as

$$f = 1 + e^{\eta_1}, \quad (27)$$

with

$$\eta_1 = k_1 x + p_1 y + q_1 z + \omega_1 t + l_1, \quad (28)$$

where k_1 , p_1 , q_1 , l_1 , and ω_1 are arbitrary constants. By substituting (27) into (2) and making the coefficients of all exponential terms zero, the dispersion relation is obtained as follows:

$$\omega_1 = -\frac{c_1 k_1^4 + c_4 p_1 q_1 + c_3 p_1^2 + c_5 q_1^2}{c_2 k_1}. \quad (29)$$

Therefore, by substituting (27)–(29) into (4), the one-soliton solution of (5) can be obtained.

3.2. Two-Soliton Solution. To obtain the two-soliton solution, let f in (2) be expressed as

$$f = 1 + e^{\eta_1} + e^{\eta_2} + \alpha_{12}e^{\eta_1+\eta_2}, \quad (30)$$

with

$$\eta_i = k_i x + p_i y + q_i z + \omega_i t + l_i, \quad (i = 1, 2), \quad (31)$$

where k_i, p_i, q_i, l_i , and ω_i ($i = 1, 2$) are arbitrary constants. By substituting (30) into (2) and making the coefficients of all exponential terms zero and after some reduction, the followings are obtained:

$$\begin{aligned} \omega_i &= -\frac{c_1 k_i^4 + c_4 p_i q_i + c_3 p_i^2 + c_5 q_i^2}{c_2 k_1}, \quad (i = 1, 2), \\ \alpha_{12} &= \frac{3k_1^2 k_2^2 (k_1 - k_2)^2 c_1 - (k_1 p_2 - k_2 p_1)^2 c_3 - (k_1 q_2 - k_2 q_1)(k_1 p_2 - k_2 p_1) c_4 - (k_1 q_2 - k_2 q_1)^2 c_5}{3k_1^2 k_2^2 (k_1 + k_2)^2 c_1 - (k_1 p_2 - k_2 p_1)^2 c_3 - (k_1 q_2 - k_2 q_1)(k_1 p_2 - k_2 p_1) c_4 - (k_1 q_2 - k_2 q_1)^2 c_5}. \end{aligned} \quad (32)$$

By substituting (30)–(32) into (4), the two-soliton solution of (5) can be obtained.

3.3. Three-Soliton Solution. To obtain the three-soliton solution, it is assumed that f in (2) is expressed as

$$\begin{aligned} f &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + \alpha_{12}e^{\eta_1+\eta_2} + \alpha_{13}e^{\eta_1+\eta_3} + \alpha_{23}e^{\eta_2+\eta_3} \\ &\quad + \alpha_{123}e^{\eta_1+\eta_2+\eta_3}, \end{aligned} \quad (33)$$

with

$$\eta_i = k_i x + p_i y + q_i z + \omega_i t + l_i, \quad (i = 1, 2, 3), \quad (34)$$

where k_i, p_i, q_i, l_i , and ω_i ($i = 1, 2, 3$) are arbitrary constants. Similar to the above method, but through a more complex process and then reduction, the following results are finally obtained:

$$\begin{aligned} \omega_i &= -\frac{c_1 k_i^4 + c_4 p_i q_i + c_3 p_i^2 + c_5 q_i^2}{c_2 k_1}, \quad (i = 1, 2, 3), \\ \alpha_{ij} &= \frac{3k_i^2 k_j^2 (k_i - k_j)^2 c_1 - (k_i p_j - k_j p_i)^2 c_3 - (k_i q_j - k_j q_i)(k_i p_j - k_j p_i) c_4 - (k_i q_j - k_j q_i)^2 c_5}{3k_i^2 k_j^2 (k_i + k_j)^2 c_1 - (k_i p_j - k_j p_i)^2 c_3 - (k_i q_j - k_j q_i)(k_i p_j - k_j p_i) c_4 - (k_i q_j - k_j q_i)^2 c_5}, \\ \alpha_{123} &= \alpha_{12} \alpha_{13} \alpha_{23}, \quad (1 \leq i < j \leq 3). \end{aligned} \quad (35)$$

By substituting (33)–(35) into (4), the three-soliton solution of (5) can be obtained. When the three-soliton are substituted into (4) for verification, it is shown that (35) needs to satisfy the constraint condition $c_4^2 = 4c_3c_5$.

Figure 1 shows the three-soliton solution of (5) through selecting the appropriate parameters. As it is shown in Figure 1, the wave shapes are kink solitons. Both one and two solitons are also kink solitons, whose graphs are omitted here.

3.4. N-Soliton Solution. (5) admits multiple-soliton solution (4) with f of the following form:

$$f = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \eta_i + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \ln(\alpha_{ij}) \right), \quad (36)$$

where

$$\begin{aligned} \eta_i &= k_i x + p_i y + q_i z + \omega_i t + l_i, \quad \omega_i = -\frac{c_1 k_i^4 + c_4 p_i q_i + c_3 p_i^2 + c_5 q_i^2}{c_2 k_1}, \quad (i = 1, 2, \dots, N), \\ \alpha_{ij} &= \frac{3k_i^2 k_j^2 (k_i - k_j)^2 c_1 - (k_i p_j - k_j p_i)^2 c_3 - (k_i q_j - k_j q_i)(k_i p_j - k_j p_i) c_4 - (k_i q_j - k_j q_i)^2 c_5}{3k_i^2 k_j^2 (k_i + k_j)^2 c_1 - (k_i p_j - k_j p_i)^2 c_3 - (k_i q_j - k_j q_i)(k_i p_j - k_j p_i) c_4 - (k_i q_j - k_j q_i)^2 c_5}, \quad (1 \leq i < j \leq N), \end{aligned} \quad (37)$$

FIGURE 1: The three-soliton solution for equation (5) by choosing suitable parameters: $k_1 = 2, k_2 = 2, k_3 = 3, p_1 = 1, p_2 = 2, p_3 = 1, q_1 = 1, q_2 = 1, q_3 = 1, l_1 = 0, l_2 = 0, l_3 = 0, c_1 = 1, c_2 = 5, c_3 = 1, c_4 = 4$, and $c_5 = 4$. (a) $x = 0, y = 0$; (b) $x = 0, t = 0$; (c) $z = 0, t = 0$.

with k_i, p_i, q_i , and l_i being arbitrary constants, $c_4^2 = 4c_3c_5$, and $\sum_{\mu=0,1}$ is the summation that over all possible combinations of $\mu_i, \mu_j = 0, 1 (i, j = 1, 2, \dots, N)$. By substituting (36) and (37) into (4), the N -soliton solution of (5) can be obtained.

4. Lump and Interaction Solutions

4.1. Lump Solutions. Many studies of the literature have given positive quadratic functions to get lump solutions [15–17]. Based on these studies of the literature, we have the following:

$$f = (m_1x + m_2y + m_3z + m_4t)^2 + (n_1x + n_2y + n_3z + n_4t)^2 + f_0, \quad (38)$$

where $m_i, n_i (i = 1, \dots, 4)$ and f_0 are real parameters to be determined.

By substituting f into (2) along with a direct but tedious calculation, 12 sets of solutions are obtained. By removing

the redundant parameter solutions, the following three cases are finally obtained.

Case 1

$$\begin{aligned} m_3 &= \pm n_1^2 \sqrt{\frac{3c_1}{c_5 f_0}}, \\ m_4 &= \mp \frac{(c_4 n_2 + 2c_5 n_3)m_3}{c_2 n_1}, \\ n_4 &= -\frac{(c_5 n_3^2 + c_4 n_3 n_2 + c_3 n_2^2)f_0 + 3c_1 n_1^4}{c_2 n_1 f_0}, \quad m_1 = m_2 = 0, \end{aligned} \quad (39)$$

where $c_2 n_1 \neq 0, c_1 c_5 f_0 < 0$, and n_2, n_3 are arbitrary constants.

By substituting (38) and (39) into (4), the lump solution of (5) is as follows:

$$u = \frac{4n_1(n_1x + n_2y + n_3z + n_4t)}{(m_3z + m_4t)^2 + (n_1x + n_2y + n_3z + n_4t)^2 + f_0}, \quad (40)$$

where m_i and $n_i (i = 1, \dots, 4)$ satisfy (39).

Case 2

$$m_4 = \mp \frac{m_3 n_2 c_4 + 2n_2 m_2 c_3 + 2n_3 m_3 c_5 + n_3 m_2 c_4}{c_2 n_1},$$

$$n_1 = \mp \left(-\frac{(c_5 m_3^2 + c_4 m_3 m_2 + c_3 m_2^2) f_0}{3c_1} \right)^{1/4},$$

$$n_4 = \pm \frac{c_3(m_2^2 - n_2^2) + c_4(m_2 m_3 - n_2 n_3) + c_5(m_3^2 - n_3^2)}{c_2 n_1}, \quad m_1 = 0, \quad (41)$$

where $c_2 \neq 0$, $(c_5 m_3^2 + c_4 m_3 m_2 + c_3 m_2^2) f_0 c_1 < 0$, and n_2, n_3 are arbitrary constants.

By substituting (38) and (41) into (4), the lump solution of (5) is as follows:

$$u = \frac{4n_1(n_1 x + n_2 y + n_3 z + n_4 t)}{(m_2 y + m_3 z + m_4 t)^2 + (n_1 x + n_2 y + n_3 z + n_4 t)^2 + f_0}, \quad (42)$$

$$\Theta_1 = (c_4 n_3 m_1 - c_4 n_1 m_3 + 2c_3 n_2 m_1) f_0,$$

$$\Theta_2 = -12c_1 c_3 c_5 f_0 (n_1^2 + m_1^2)^3 - c_5 f_0^2 (n_3 m_1 - n_1 m_3)^2 (4c_3 c_5 - c_4^2), \quad (44)$$

$$\Theta_3 = -3c_1 c_5 m_1 (n_1^2 + m_1^2)^2 + c_4 c_5^2 n_2 (n_1 m_3 - n_3 m_1) f_0 + c_5^2 (2c_5 m_3 n_3 n_1 - c_5 n_3^2 m_1 - n_2^2 m_1 c_3) f_0,$$

where $c_2 c_3 n_1 f_0 \neq 0$, $\Theta_2 \geq 0$, and m_1, m_3, n_2 , and n_3 are arbitrary constants.

By substituting (38) and (43) into (4), the lump solution of (5) is obtained as follows:

$$u = \frac{4m_1(m_1 x + m_2 y + m_3 z + m_4 t) + 4n_1(n_1 x + n_2 y + n_3 z + n_4 t)}{(m_1 x + m_2 y + m_3 z + m_4 t)^2 + (n_1 x + n_2 y + n_3 z + n_4 t)^2 + f_0}, \quad (45)$$

where m_i and n_i ($i = 1, \dots, 4$) satisfy (43).

It is noteworthy to state that the spatial structures of the lump solutions in Case 1 and Case 3 are like those in Case 2. From the above three cases, the obtained lump waves have the following asymptotic behavior: $\lim_{x \rightarrow \pm\infty} u = 0$, $\lim_{t \rightarrow \infty} u = 0$, $\lim_{y \rightarrow \pm\infty} u = 0$, and $\lim_{z \rightarrow \pm\infty} u = 0$.

4.2. Interaction Solution between Solitary and Lump Waves. To construct interaction solutions, the following form is taken:

$$f = a\xi_1^2 + b\xi_2^2 + kg(\xi_3) + c, \quad (46)$$

with

$$\begin{aligned} \xi_1 &= m_1 x + m_2 y + m_3 z + m_4 t, \\ \xi_2 &= n_1 x + n_2 y + n_3 z + n_4 t, \\ \xi_3 &= k_1 x + k_2 y + k_3 z + k_4 t, \end{aligned} \quad (47)$$

where a, b, k, c, m_i, n_i , and k_i ($i = 1, \dots, 4$) are arbitrary parameters and g is an arbitrary function. By choosing

where m_i and n_i ($i = 1, \dots, 4$) satisfy (41). The first set parameter solution in (41) is then chosen. Figure 2 shows the spatial structures of the lump solution and the contour plots for (42) by selecting the appropriate parameter values.

Case 3

$$m_2 = \frac{\Theta_1 \pm \sqrt{\Theta_2}}{2c_3 n_1 f_0},$$

$$m_4 = -\frac{c_4 n_3 (\Theta_1 \pm \sqrt{\Theta_2}) \pm 2c_3 n_2 (\Theta_1 \pm \sqrt{\Theta_2}) + \Theta_3}{2c_2 c_3 n_1^2 f_0},$$

$$n_4 = -\frac{3c_1(m_1^2 + n_1^2)^2 + f_0(c_5 n_3^2 + c_4 n_2 n_3 + c_3 n_2^2)}{f_0 c_2 n_1}, \quad (43)$$

where

different functions for g , abundant interaction solutions can be obtained [18–21].

This section mainly considers the interaction between the lump and the soliton solutions and then discusses their corresponding dynamical behaviors. The following values are chosen for $a = 1$ and $b = c = 0$ in (46) to obtain

$$f = (m_1 x + m_2 y + m_3 z + m_4 t)^2 + k \cosh(k_1 x + k_2 y + k_3 z + k_4 t). \quad (48)$$

By substituting (48) into (2) and combining direct but tedious calculations, 31 sets of solutions are obtained for the above undetermined parameters. To ensure the generality of parametric solutions, the following four cases are obtained.

Case 4

$$k_4 = -\frac{k_2^2 c_3 + c_4 k_2 k_3 + c_5 k_3^2}{c_2 k_1},$$

$$m_4 = -\frac{(c_4 k_2 + 2c_5 k_3 \pm k_2 \sqrt{c_4^2 - 4c_3 c_5})^2 m_1}{4c_2 c_5 k_1^2},$$

$$m_3 = \pm \frac{(c_4 k_2 + 2c_5 k_3 \pm k_2 \sqrt{c_4^2 - 4c_3 c_5}) m_1}{2c_5 k_1}, \quad c_1 = m_2 = 0, \quad (49)$$

where $c_2 c_5 k_1 \neq 0$, $c_4^2 \geq 4c_3 c_5$, and k, k_2, k_3 , and m_1 are arbitrary constants.

FIGURE 2: Spatial structures of lump solution (42) with parameters: $c_1 = -1$ and $c_2 = c_3 = c_4 = c_5 = m_2 = m_3 = n_2 = n_3 = f_0 = 1$. (a) $y = 0, z = 0$; (b) $x = 0, z = 0$; (c) $z = 0, t = 0$. (d-f) Contour plots corresponding to subfigures (a-c), respectively.

By substituting (48) and (49) into (4), the interaction solution of (5) is obtained as follows:

$$u = \frac{4m_1(m_1x + m_3z + m_4t) + 2kk_1 \sinh(k_1x + k_2y + k_3z + k_4t)}{(m_1x + m_3z + m_4t)^2 + k \cosh(k_1x + k_2y + k_3z + k_4t)}, \quad (50)$$

where m_i and k_i ($i = 1, \dots, 4$) satisfy (49) and the first set parameter solution in (49) is chosen. Figure 3 shows the spatial structures of the interaction solution and the contour plots for (50) by selecting the appropriate parameter values. Interestingly, it was found that Figure 3(a) is consistent with the result of the interaction between the kink soliton and the lump waves (see Figures 1 and 2).

Case 5

$$\begin{aligned} k_1 &= \frac{k_2 m_1}{m_2}, \\ k_3 &= \frac{k_2 m_3}{m_2}, \\ k_4 &= \frac{k_2(c_3 m_2^2 + c_4 m_3 m_2 + c_5 m_3^2)}{c_2 m_1 m_2}, \\ m_4 &= \frac{c_3 m_2^2 + c_4 m_3 m_2 + c_5 m_3^2}{c_2 m_1}, \quad c_1 = 0, \end{aligned} \quad (51)$$

where $c_2 m_1 m_2 \neq 0$, and k, k_2, m_3 , and c_i ($i = 3, 4, 5$) are arbitrary constants.

By substituting (48) and (51) into (4), the interaction solution of (5) is as follows:

$$u = \frac{4m_1(m_1x + m_2y + m_3z + m_4t) + 2kk_1 \sinh(k_1x + k_2y + k_3z + k_4t)}{(m_1x + m_2y + m_3z + m_4t)^2 + k \cosh(k_1x + k_2y + k_3z + k_4t)}, \quad (52)$$

where m_i and k_i ($i = 1, \dots, 4$) satisfy (51). Figure 4 shows the spatial structures of the interaction solution and the contour plots for (52) by selecting the appropriate parameter values.

Case 6

$$\begin{aligned} k_4 &= \frac{c_3 k_2^2 + c_4 k_3 k_2 + c_5 k_3^2}{c_2 k_1}, \\ m_3 &= \frac{m_2(-c_4 \pm \sqrt{c_4^2 - 4c_3 c_5})}{2c_5}, \\ m_4 &= \frac{m_2 \sqrt{c_4^2 - 4c_3 c_5} (k_2 \sqrt{c_4^2 - 4c_3 c_5} \mp (c_4 k_2 + 2c_5 k_3))}{2c_2 c_5 k_1}, \\ c_1 &= m_1 = 0, \end{aligned} \quad (53)$$

FIGURE 3: Spatial structures of interaction solution (50) with parameters: $c_2 = 1, c_3 = 1, c_4 = 4, c_5 = 4$, and $m_1 = k_1 = k_2 = k_3 = k_4 = 1$. (a) $z = 0, t = 0$; (b) $y = 0, z = 0$; (c) $x = 0, y = 0$. (d-f) Contour plots corresponding to subfigures (a-c), respectively.

FIGURE 4: Continued.

FIGURE 4: Spatial structures of interaction solution (52) with parameters: $c_2 = c_3 = c_4 = c_5 = k_1 = k_2 = 1, m_1 = 1, m_2 = 2$, and $m_3 = 3$. (a) $y = 0, z = 0$; (b) $x = 0, z = 0$; (c) $z = 0, t = 0$. (d-f) Contour plots corresponding to subfigures (a-c), respectively.

FIGURE 5: Spatial structures of interaction solution (54) with parameters: $c_2 = c_3 = c_5 = k_1 = k_2 = k_3 = k = 1, c_4 = 3$, and $m_2 = 2$. (a) $y = 0, z = 0$; (b) $x = 0, z = 0$; (c) $z = 0, t = 0$. (d-f) Contour plots corresponding to subfigures (a-c), respectively.

where $c_2 c_5 k_1 \neq 0$, $c_4^2 \geq 4c_3 c_5$, and k, k_2 , and k_3 are arbitrary constants.

By substituting (48) and (53) into (4), the interaction solution of (5) is obtained as follows:

$$u = \frac{2kk_1 \sinh(k_1 x + k_2 y + k_3 z + k_4 t)}{(m_2 y + m_3 z + m_4 t)^2 + k \cosh(k_1 x + k_2 y + k_3 z + k_4 t)}, \quad (54)$$

FIGURE 6: Spatial structures of interaction solution (56) with parameters: $c_2 = c_3 = c_5 = k_1 = k_2 = k_3 = k = m_1 = 1$, $c_4 = 3$, and $m_2 = 2$. (a) $y = 0, z = 0$; (b) $x = 0, z = 0$; (c) $y = 0, t = 0$. (d–f) Contour plots corresponding to subfigures (a–c), respectively.

where $m_i, k_i (i = 1, \dots, 4)$ satisfy (53). The first set parameter solution in (53) is then chosen. Figure 5 shows the spatial structures of the interaction solution and the contour plots for (54) by selecting the appropriate parameter values.

Case 7

$$\begin{aligned} k_3 &= \frac{-c_4 k_2 m_1 \pm \sqrt{c_4^2 - 4c_3 c_5} (2k_1 m_2 - k_2 m_1)}{2c_5 m_1}, \\ k_4 &= \frac{m_2 (-c_4^2 + 4c_3 c_5) (k_1 m_2 - k_2 m_1)}{c_2 c_5 m_1^2}, \\ m_3 &= \frac{m_2 (-c_4 \pm \sqrt{c_4^2 - 4c_3 c_5})}{2c_5}, \quad c_1 = m_4 = 0, \end{aligned} \quad (55)$$

where $c_2 c_5 m_1 \neq 0$, $c_4^2 \geq 4c_3 c_5$, and k, k_2, m_1 , and m_2 are arbitrary constants.

By substituting (48) and (55) into (4), the interaction solution of (5) is obtained as follows:

$$u = \frac{4m_1 (m_1 x + m_2 y + m_3 z) + 2kk_1 \sinh(k_1 x + k_2 y + k_3 z + k_4 t)}{(m_1 x + m_2 y + m_3 z)^2 + k \cosh(k_1 x + k_2 y + k_3 z + k_4 t)}, \quad (56)$$

where $m_i, k_i (i = 1, \dots, 4)$ satisfies (55). The first set parameter solution in (55) is then chosen. Figure 6 shows the spatial structures of the interaction solution and the contour plots for (56) by selecting the appropriate parameter values.

5. Conclusion

In this article, a generalized (3+1)-dimensional NLEE whose bilinear equation is obtained by a multivariable polynomial was discussed. Based on the bilinear equation, the BBT is given and it consists of four bilinear equations involving seven arbitrary parameters except for the parameters of the equation itself. By using the BBT, two traveling wave solutions and a mixed solution were obtained. Also, the N -soliton solutions were constructed and the dynamic properties of the soliton solutions were shown graphically. Furthermore, the lump and the interaction solutions were constructed. The properties of the explicit analytic solutions were shown by using figures. The general expression of the interaction solution was also given. By determining the different forms of the arbitrary function, more abundant interaction solutions will be obtained. Hence, the interaction between the lump and other solutions, including periodic solution and kink solution, can further be explored.

Finally, this article not only presents a new example for $(3 + 1)$ -dimensional NLEE but also provides a reference for finding the lump and the interaction solutions of other generalized multidimensional NLEEs.

Data Availability

The Maple data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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