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Research Article

Asymptotic Behavior of Weak Solutions of Nonisothermal Flow of Herschel–Bulkley Fluid to Free Boundary

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In this manuscript, the behavior of a Herschel–Bulkley fluid has been discussed in a thin layer in $\mathbb{R}^3$ associated with a nonlinear stationary, nonisothermal, and incompressible model. Furthermore, the limit problem has been considered, and the studied problem in Ω^ε is transformed into another problem defined in Ω^ε without the parameter Ω^ε (ε is the parameter representing the thickness of the layer tend to zero is studied). We also investigated the convergence of the unknowns which are the velocity, pressure, and the temperature of the fluid. In addition, we established the limit problem and the specific Reynolds equation.

1. Introduction

In a recent study of problems for the asymptotic behavior for a problem of continuum mechanics in a thin domain Ω^ε, the problem is transformed into an equivalent problem on a domain Ω independent of the parameter ε. This phenomenon has been presented by many researchers, see, e.g., [1–5]. Specifically, the case of Herschel–Bulkley fluid has been archived in several articles, for instance, [6, 7]. A particularity of Herschel–Bulkley fluid lies in the presence of rigid zones located in the interior of the flow, and as the yield limit increases, the rigid zones become larger and may completely block the flow (see, e.g., [8–10]).

This work is to study the asymptotic behavior for weak solutions of a linked system, including of an incompressible Herschel–Bulkley fluid and the equation of the heat energy, in a three-dimensional bounded domain satisfying Tresca-type fluid solid boundary conditions. The boundary of this thin domain consists of three parts: the bottom, the lateral part, and the top surface.

The article is organized as follows: in Section 2, we present the mechanical problem of the steady-state flow of Herschel–Bulkley fluid in a three-dimensional thin domain. We also introduce some notations, preliminaries, and some function spaces of our coupled problem.

In Section 3, we use the asymptotic analysis, in which the small parameter ε is the height of the domain. We also discuss some estimates, independent on the parameter ε, for the velocity, the pressure, and the temperature. Moreover, we give some convergence results. The main results concerning the limit problem with a specific weak form of the Reynolds equation are established in Section 4. Finally, in Section 5, we include some remarks and conclusions on the work.

2. Statement of the Problem and Variational Formulation

Here, let ϖ be fixed region in plan $s = (s_1, s_2) \in \mathbb{R}^2$. We assume that ϖ has a Lipschitz boundary and is the bottom of the fluid domain. The upper surface Γ_1^ε is defined by $s_3 = \varepsilon h(s)$ where $(0 < \varepsilon < 1)$ is a small parameter that will tend to zero and h a smooth bounded function such that $0 < h_* \le h(s) \le h^*$ for all $(s, 0) \in \varpi$ and Γ_L^ε the lateral surface. We denote by Ω^ε the domain of the following:

$$\Omega^\varepsilon = \left\{(s, s_3) \in \mathbb{R}^3 \colon (s, 0) \in \varpi, 0 < s_3 < \varepsilon h(s)\right\}. \tag{1}$$

The boundary of Ω^ε is Γ^ε where $\Gamma^\varepsilon = \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon \cup \overline{\varpi}$ with $\overline{\Gamma}_L^\varepsilon$ is the lateral boundary. We denote by Σ^ε the deviatoric part and ρ^ε the pressure. The fluid is supposed to be viscoplastic, and the relation between Σ^ε and $D(w^\varepsilon)$ is given by

$$\begin{cases} \Sigma_{ij}^\varepsilon = \widetilde{\Sigma}_{ij}^\varepsilon - \rho^\varepsilon \delta_{ij}, \\ \widetilde{\Sigma}^\varepsilon = g^\varepsilon(T^\varepsilon)\dfrac{D(w^\varepsilon)}{|D_{II}(w^\varepsilon)|} + \Lambda^\varepsilon(T^\varepsilon)|D(w^\varepsilon)|^{\nu-2}D(w^\varepsilon), & \text{if } D(w^\varepsilon) \neq 0, \\ |\widetilde{\Sigma}^\varepsilon| \le g^\varepsilon(T^\varepsilon), & \text{if } D(w^\varepsilon) = 0. \end{cases} \tag{2}$$

For any tensor $D = (d_{ij})$, the notation $|D|$ represents the matrix norm: $|D_{II}| = 1/\sqrt{2}(\sum_{i,j=1}^3 d_{ij}d_{ij})^{1/2}$. Let $n = (n_1, n_2, n_3)$ the unit outward normal vector on the boundary Γ^ε. The normal and the tangential velocity on the boundary Ω^ε are $w_n^\varepsilon = w^\varepsilon.n, w_\tau^\varepsilon = w^\varepsilon - w_n^\varepsilon.n$. Also, Σ^ε is a regular stress tensor field, further let Σ_n^ε and Σ_τ^ε are the normal and tangential components of Σ^ε on the boundary ϖ by $\Sigma_n^\varepsilon = (\Sigma^\varepsilon.n).n$, $\Sigma_\tau^\varepsilon = \Sigma^\varepsilon.n - \Sigma_n^\varepsilon.n$.

Problem 1. Find a velocity field $w^\varepsilon \colon \Omega^\varepsilon \longrightarrow \mathbb{R}^3$, the pressure ρ^ε and a temperature: $\Omega^\varepsilon \longrightarrow \mathbb{R}$ such that

$$\operatorname{div}(\Sigma^\varepsilon) + f^\varepsilon = 0 \, in \Omega^\varepsilon, \tag{3}$$

$$\begin{cases} \widetilde{\Sigma}^\varepsilon = g^\varepsilon(T^\varepsilon)\dfrac{D(w^\varepsilon)}{|D_{II}(w^\varepsilon)|} + \Lambda^\varepsilon(T^\varepsilon)|D(w^\varepsilon)|^{\nu-2}D(w^\varepsilon) & \text{if } D(w^\varepsilon) \neq 0, \\ |\widetilde{\Sigma}^\varepsilon| \le g^\varepsilon(T^\varepsilon) & \text{if } D(w^\varepsilon) = 0, \end{cases} \tag{4}$$

$$\begin{aligned} \operatorname{div}(w^\varepsilon) &= 0 \text{ in } \Omega^\varepsilon, \\ -\nabla.(K^\varepsilon \nabla T^\varepsilon) &= \Lambda^\varepsilon(T^\varepsilon)|D(w^\varepsilon)|^r + \sqrt{2}g^\varepsilon(T^\varepsilon)|D(w^\varepsilon)| \\ &\quad - \alpha^\varepsilon T^\varepsilon \text{ in } \Omega, \\ w^\varepsilon &= 0 \text{ on } \Gamma_1 \cup \Gamma_L, \end{aligned} \tag{5}$$

$$w^\varepsilon \times n = 0 \text{ on } \varpi, \tag{6}$$

$$\begin{aligned} &\begin{cases} |\Sigma_\tau^\varepsilon| < k^\varepsilon \Rightarrow w_\tau^\varepsilon = 0 \\ |\Sigma_\tau^\varepsilon| = k^\varepsilon \Rightarrow \exists \lambda \ge 0, w_\tau^\varepsilon = -\lambda \Sigma_\tau^\varepsilon \end{cases} \text{on } \varpi, \\ &T^\varepsilon = 0 \text{ on} \Gamma_1 \cup \Gamma_L, \end{aligned} \tag{7}$$

$$\frac{\partial T^\varepsilon}{\partial n} = 0 \text{ on} \varpi. \tag{8}$$

where $\operatorname{div}\Sigma = (\Sigma_{ij,j})$ and $\operatorname{div} w = w_{i,i}$. The flow is given by the (3) where the density is assumed equal to one. (4) represents the constitutive law of a Herschel–Bulkley fluid whose the consistency Λ^ε and the yield limit g^ε depend on the temperature, $1 < \nu < 2$ is the power law exponent of the material. (5) represents the incompressibility condition. Equation (5) represents the energy conservation where the specific heat is assumed equal to one, $K^\varepsilon > 0$ is the thermal conductivity and the term $-\alpha^\varepsilon T^\varepsilon$ represents the external heat source with $\alpha^\varepsilon > 0$. (5) gives the velocity on $\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon$. As there is no-flux condition across ϖ, then we have equation (6). Condition (7) represents a Tresca thermal friction law on ϖ, where k^ε is the friction yields coefficient (8) gives the temperature on $\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon$. (8) is a homogeneous Neumann boundary condition on ϖ:

$$\begin{aligned} W^{1,\nu}(\Omega^\varepsilon) &= \left\{\vartheta \in L^\nu(\Omega^\varepsilon)^3 \colon \frac{\partial \vartheta_i}{\partial s_j} \in L^\nu(\Omega^\varepsilon) \text{for} \quad i, j = 1, \ldots, 3\right\}, \\ V^\varepsilon(\Omega^\varepsilon) &= \left\{\vartheta \in W^{1,\nu}(\Omega^\varepsilon)^3 \colon \vartheta = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \vartheta.n = 0 \text{ on } \varpi\right\}, \\ V_{di\,v}^\varepsilon(\Omega^\varepsilon) &= \{\vartheta \in K^\varepsilon \colon \operatorname{div}(\vartheta) = 0\}, \\ L_0^{\nu'}(\Omega^\varepsilon) &= \left\{\vartheta \in L^{\nu'}(\Omega^\varepsilon) \colon \int_{\Omega^\varepsilon} \vartheta ds ds_3 = 0\right\}, \end{aligned} \tag{9}$$

and

$$W_{\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon}^{1,q}(\Omega^\varepsilon) = \left\{\Phi \in W^{1,q}(\Omega^\varepsilon)^3 \colon \Phi = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon\right\}. \tag{10}$$

A formal application of Green's formula, using (3)–(8) leads to the weak formulation: Find a velocity field $w^\varepsilon \in V_{\text{div}}^\varepsilon$, $\rho^\varepsilon \in L_0^{\nu'}(\Omega^\varepsilon)$ and $T^\varepsilon \in W_{\Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon}^{1,q}(\Omega^\varepsilon)$, $(1 < q < 3/2)$ such that

$$
\begin{aligned}
a\left(w^{\varepsilon},\vartheta-w^{\varepsilon}\right)-\left(\rho^{\varepsilon},\operatorname{div}\vartheta\right)+j\left(T^{\varepsilon},\vartheta\right)-j\left(T^{\varepsilon},w^{\varepsilon}\right)\geq\left(f^{\varepsilon},\vartheta-w^{\varepsilon}\right),\forall\vartheta\in V^{\varepsilon}\left(\Omega^{\varepsilon}\right),\\
b\left(T^{\varepsilon},\Phi\right)=C\left(w^{\varepsilon},T^{\varepsilon},\Phi\right),\forall\Phi\in W^{1,q}_{\Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}}\left(\Omega^{\varepsilon}\right),
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
a\left(w^{\varepsilon},\vartheta-w^{\varepsilon}\right)&=\int_{\Omega^{\varepsilon}}\Lambda^{\varepsilon}\left(T^{\varepsilon}\right)\left|D\left(w^{\varepsilon}\right)\right|^{\nu-2}D\left(w^{\varepsilon}\right)D\left(v^{\varepsilon}\right)\mathrm{d}s\mathrm{d}s_3,\\
\left(\rho^{\varepsilon},\operatorname{div}\vartheta\right)&=\int_{\Omega^{\varepsilon}}\rho^{\varepsilon}\operatorname{div}\vartheta\mathrm{d}s\mathrm{d}s_3,\\
j\left(T^{\varepsilon},v\right)&=\int_{\varpi}k^{\varepsilon}|v|\mathrm{d}s+\sqrt{2}\int_{\Omega^{\varepsilon}}g^{\varepsilon}\left(T^{\varepsilon}\right)|D(v)|\mathrm{d}s\mathrm{d}s_3,\\
\left(f^{\varepsilon},v\right)&=\int_{\Omega^{\varepsilon}}f^{\varepsilon}v\mathrm{d}s\mathrm{d}s_3=\sum_{i=1}^{3}\int_{\Omega^{\varepsilon}}f_i^{\varepsilon}v_i\mathrm{d}s\mathrm{d}s_3,\\
b\left(T^{\varepsilon},\Phi\right)&=\int_{\Omega^{\varepsilon}}K^{\varepsilon}\nabla T^{\varepsilon}\nabla\Phi\mathrm{d}s\mathrm{d}s_3,\\
C\left(w^{\varepsilon},T^{\varepsilon},\Phi\right)&=\int_{\Omega^{\varepsilon}}\Lambda^{\varepsilon}\left(T^{\varepsilon}\right)\left|D\left(w^{\varepsilon}\right)\right|^{\nu}\Phi\mathrm{d}s\mathrm{d}s_3\\
&\quad+\sqrt{2}\int_{\Omega^{\varepsilon}}g^{\varepsilon}\left(T^{\varepsilon}\right)\left|D\left(w^{\varepsilon}\right)\right|\Phi\mathrm{d}s\mathrm{d}s_3+\int_{\Omega^{\varepsilon}}\alpha^{\varepsilon}T^{\varepsilon}\Phi\mathrm{d}s\mathrm{d}s_3.
\end{aligned}
\tag{12}
$$

It is known that this variational problem has a unique solution, see for more details [10–12].

We assume that there exist $\Lambda_*,\Lambda^*,g^*,K_*^{\varepsilon},K_{\varepsilon}^*,\alpha_*^{\varepsilon},\alpha_{\varepsilon}^*$ in $\mathbb{R}$ such that

$$
0\leq\Lambda_*\leq\Lambda^{\varepsilon}\leq\Lambda^*,0\leq g^{\varepsilon}\leq g^*,f^{\varepsilon}\in W^{1,\nu'}\left(\Omega^{\varepsilon}\right)^3,
\tag{13}
$$

and

$$
0\leq K_*^{\varepsilon}\leq K^{\varepsilon}\leq K_{\varepsilon}^*,\quad 0\leq\alpha_*^{\varepsilon}\leq\alpha^{\varepsilon}\leq\alpha_{\varepsilon}^*.
\tag{14}
$$

Following some previous results that are useful in the next sections (cf. [13])

$$
\left\|\nabla w^{\varepsilon}\right\|_{L^{\nu}\left(\Omega^{\varepsilon}\right)}\leq C\left\|D\left(w^{\varepsilon}\right)\right\|_{L^{\nu}\left(\Omega^{\varepsilon}\right)},\ C\text{ is a positive constant independent of }\varepsilon,1<\nu<2\,(\text{Korn inequality}),
\tag{15}
$$

$$
\begin{aligned}
&\left\|w_i^{\varepsilon}\right\|_{L^{\nu}\left(\Omega^{\varepsilon}\right)}\leq\varepsilon h^*\left\|\frac{\partial w_i^{\varepsilon}}{\partial\kappa}\right\|_{L^{\nu}\left(\Omega^{\varepsilon}\right)}\text{ for }\quad i=1,2;h^*=\max(h(s)),\\
&\qquad 1<\nu<2\,(\text{Poincaré inequality}),
\end{aligned}
\tag{16}
$$

$$
ab\leq\frac{a^{\nu}}{\nu}+\frac{b^{\nu'}}{\nu'},\quad\forall(a,b)\in\mathbb{R}^2,\ 1<\nu<2,\ \frac{1}{\nu}+\frac{1}{\nu'}=1\,(\text{Young inequality}),
\tag{17}
$$

$$
\begin{aligned}
&(a+b)^{\rho}\leq(2)^{\rho-1}\left(a^{\rho}+b^{\rho}\right),\forall(a,b)\in\mathbb{R}_+^{*2},\quad\forall\rho>1,\\
&(a+b)^{\rho}\leq\left(a^{\rho}+b^{\rho}\right),\forall(a,b)\in\mathbb{R}_+^{*2},\quad 0<\rho<1.
\end{aligned}
\tag{18}
$$

3. Change of the Domain and Study of Convergence

In this section, we will use the technique of scaling in Ω^{ε} on the coordinate s_3, by introducing the change of the variables $\kappa=s_3/\varepsilon$. We obtain a fixed domain Ω which is independent of ε: $\Omega=\left\{(s,\kappa)\in\mathbb{R}^3:\ (s,0)\in\varpi,0<\kappa<hs\right\}$.

We denote its boundary by $\Gamma=\overline{\Gamma}_1\cup\overline{\Gamma}_L\cup\overline{\varpi}$, also we have

$$
\begin{aligned}
\widehat{w}_i^{\varepsilon}(s,\kappa)&=w_i^{\varepsilon}\left(s,s_3\right),i=1,2,\\
\widehat{w}_3^{\varepsilon}(s,\kappa)&=\varepsilon^{-1}w_3^{\varepsilon}\left(s,s_3\right),\\
\widehat{\rho}^{\varepsilon}(s,\kappa)&=\varepsilon^{\nu}\rho^{\varepsilon}\left(s,s_3\right).
\end{aligned}
\tag{19}
$$

Assume that

$$\left.\begin{aligned}&\widehat{K}(s,\kappa)=\varepsilon^{\beta+\nu-2}K^{\varepsilon}(s,s_3),\widehat{\alpha}(s,\kappa)=\varepsilon^{\beta+\nu}\alpha^{\varepsilon}(s,s_3),\\&\widehat{\Lambda}=\Lambda^{\varepsilon},\widehat{f}(s,\kappa)=\varepsilon^{\nu}f^{\varepsilon}(s,s_3),\widehat{g}=\varepsilon^{\nu-1}g^{\varepsilon},\widehat{k}=\varepsilon^{\nu-1}k^{\varepsilon},\end{aligned}\right\}, \tag{20}$$

with

$$\beta=\frac{3(2-\nu)}{3-\nu}. \tag{21}$$

Let

$$\begin{aligned}
V(\Omega)&=\left\{\widehat{\vartheta}\in\left(W^{1,\nu}(\Omega)\right)^3:\ \widehat{\vartheta}=0\ \text{on}\ \Gamma_1\cup\Gamma_L;\widehat{\vartheta}.n=0\ \text{o}\varpi\right\},\\
V_{\text{div}}(\Omega)&=\left\{\widehat{\vartheta}\in K(\Omega):\ \text{div}\widehat{\vartheta}=0\right\},\\
V_\kappa&=\left\{\widehat{\vartheta}\in\left(L^r(\Omega)\right)^2;\frac{\partial\widehat{\vartheta}_i}{\partial\kappa}\in L^{\nu}(\Omega):\ \widehat{\vartheta}=0\ \text{on}\Gamma_1\cup\Gamma_L\right\},\\
\widetilde{V}_\kappa&=\left\{\widehat{\vartheta}\in V_\kappa:\ \widehat{\vartheta}\ \text{satisfy}\left(\text{D}'\right)\right\},\\
\Pi_\kappa&=\left\{\widehat{\vartheta}\in\left(L^q(\Omega)\right)^2;\frac{\partial\widehat{\vartheta}_i}{\partial\kappa}\in L^q(\Omega)\right\},
\end{aligned} \tag{22}$$

where the condition (D') is given by

$$\int_{\varpi}\left(\widehat{\vartheta}_1\frac{\partial\vartheta}{\partial s_1}+\widehat{\vartheta}_2\frac{\partial\vartheta}{\partial s_2}\right)\text{d}s\text{d}\kappa=0,$$
$$\text{for all}\ \widehat{\vartheta}\in\left(\text{L}^{\nu}(\Omega)\right)^2,\ \vartheta\in\text{C}_0^{\infty}(\Omega).(D). \tag{23}$$

By injecting the new data and unknown factors in (19) and (20), we prove that $(\widehat{w}^{\varepsilon},\widehat{\rho}^{\varepsilon},\widehat{T}^{\varepsilon})$ is a solution of the following problem:

$$\left\{\begin{aligned}&a_0\left(\widehat{T}^{\varepsilon},\widehat{w}^{\varepsilon},\widehat{\vartheta}-\widehat{w}^{\varepsilon}\right)-\left(\widehat{\rho}^{\varepsilon},\text{div}\left(\widehat{\vartheta}-\widehat{w}^{\varepsilon}\right)\right)+j_0\left(\widehat{T}^{\varepsilon},\widehat{\vartheta}\right)-j_0\left(\widehat{T}^{\varepsilon},\widehat{w}^{\varepsilon}\right)\geq\left(\widehat{f},\widehat{\vartheta}-\widehat{w}^{\varepsilon}\right), && \forall\widehat{\vartheta}\in V(\Omega),\\&b_0\left(\widehat{T}^{\varepsilon},\widehat{\Phi}\right)=C_0\left(\widehat{w}^{\varepsilon},\widehat{T}^{\varepsilon},\widehat{\Phi}\right), && \forall\widehat{\Phi}\in W^{1,q}_{\Gamma_1^{\varepsilon}\cup\Gamma_L^{\varepsilon}}(\Omega),\end{aligned}\right. \tag{24}$$

where

$$\begin{aligned}
a_0\left(\widehat{T}^{\varepsilon},\widehat{w}^{\varepsilon},\widehat{\vartheta}-\widehat{w}^{\varepsilon}\right)&=\sum_{i,j=1}^{2}\int_{\Omega}\left[\varepsilon^2\widehat{\Lambda}\left(\widehat{T}^{\varepsilon}\right)\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right|^{\nu-2}\left(\frac{1}{2}\left(\frac{\partial\widehat{w}_i^{\varepsilon}}{\partial s_j}+\frac{\partial\widehat{w}_j^{\varepsilon}}{\partial s_i}\right)\right)\right]\frac{\partial\left(\widehat{\vartheta}_i-\widehat{w}_i^{\varepsilon}\right)}{\partial s_j}\text{d}s\text{d}\kappa\\
&+\sum_{i=1}^{2}\int_{\Omega}\widehat{\Lambda}\left(\widehat{T}^{\varepsilon}\right)\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right|^{\nu-2}\left(\frac{1}{2}\left(\frac{\partial\widehat{w}_i^{\varepsilon}}{\partial\kappa}+\varepsilon^2\frac{\partial\widehat{w}_3^{\varepsilon}}{\partial s_i}\right)\right)\frac{\partial\left(\widehat{\vartheta}_i-\widehat{w}_i^{\varepsilon}\right)}{\partial\kappa}\text{d}s\text{d}\kappa\\
&+\int_{\Omega}\left(\widehat{\Lambda}\left(\widehat{T}^{\varepsilon}\right)\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right|^{\nu-2}\varepsilon^2\frac{\partial\widehat{w}_3^{\varepsilon}}{\partial\kappa}\right)\frac{\partial\left(\widehat{\vartheta}_3-\widehat{w}_3^{\varepsilon}\right)}{\partial\kappa}\text{d}s\text{d}\kappa+\\
&+\sum_{j=1}^{2}\int_{\Omega}\varepsilon^2\widehat{\Lambda}\left(\widehat{T}^{\varepsilon}\right)\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right|^{\nu-2}\left(\frac{1}{2}\left(\varepsilon^2\frac{\partial\widehat{w}_3^{\varepsilon}}{\partial s_j}+\frac{\partial\widehat{w}_j^{\varepsilon}}{\partial\kappa}\right)\right)\frac{\partial\left(\widehat{\vartheta}_3-\widehat{w}_3^{\varepsilon}\right)}{\partial s_j}\text{d}s\text{d}\kappa,
\end{aligned}$$

$$
\begin{aligned}
&\left(\widehat{\rho}^{\varepsilon}, \operatorname{div}\left(\widehat{\vartheta}-\widehat{w}^{\varepsilon}\right)\right)=\int_{\Omega^{\varepsilon}} \widehat{\rho}^{\varepsilon} \operatorname{div}\left(\widehat{\vartheta}-\widehat{w}^{\varepsilon}\right) ds\, d\kappa, \\
&j_0\left(\widehat{T}^{\varepsilon}, \widehat{\vartheta}\right)=\sqrt{2} \int_{\Omega} \widehat{g}\left(\widehat{T}^{\varepsilon}\right)|\widetilde{D}(\widehat{\vartheta})| \mathrm{d}s\mathrm{d}\kappa+\int_{\varpi} \widehat{k}|\widehat{\vartheta}| \mathrm{d}s, \\
&\left(\widehat{f}^{\varepsilon}, \widehat{\vartheta}-\widehat{w}^{\varepsilon}\right)=\sum_{j=1}^{2} \int_{\Omega} \widehat{f}_i\left(\widehat{\vartheta}_i-\widehat{w}_i^{\varepsilon}\right) \mathrm{d}s\mathrm{d}\kappa+\int_{\Omega} \varepsilon \widehat{f}_3\left(\widehat{\vartheta}_3-\widehat{w}_3^{\varepsilon}\right) \mathrm{d}s\mathrm{d}\kappa, \\
&b_0\left(\widehat{T}^{\varepsilon}, \widehat{\Phi}\right)=\int_{\Omega} \varepsilon^2 \nabla_{\varepsilon} \widehat{T}^{\varepsilon} \nabla_{\varepsilon} \widehat{\Phi} \mathrm{d}s\mathrm{d}\kappa=\sum_{i=1}^{2} \int_{\Omega} \varepsilon^2 \widehat{K} \frac{\partial \widehat{T}^{\varepsilon}}{\partial s_i} \frac{\partial \widehat{\Phi}}{\partial s_i} \mathrm{d}s\mathrm{d}\kappa+\int_{\Omega} \widehat{K} \frac{\partial \widehat{T}^{\varepsilon}}{\partial \kappa} \frac{\partial \widehat{\Phi}}{\partial \kappa} \mathrm{d}s\mathrm{d}\kappa, \\
&C_0\left(\widehat{w}^{\varepsilon}, \widehat{T}^{\varepsilon}, \widehat{\Phi}\right)=\int_{\Omega} \varepsilon^{\beta} \widehat{\Lambda}\left(\widehat{T}^{\varepsilon}\right)\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right|^{\nu} \widehat{\Phi} \mathrm{d}s\mathrm{d}\kappa+\sqrt{2} \int_{\Omega} \varepsilon^{\beta} \widehat{g}\left(\widehat{T}^{\varepsilon}\right)\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right| \widehat{\Phi} \mathrm{d}s\mathrm{d}\kappa-\int_{\Omega} \widehat{\alpha} \widehat{T}^{\varepsilon} \widehat{\Phi} \mathrm{d}s\mathrm{d}\kappa, \\
&\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right|=\left(\frac{1}{4} \sum_{i,j=1}^{2} \varepsilon^2\left(\frac{\partial \widehat{w}_i^{\varepsilon}}{\partial s_j}+\frac{\partial \widehat{w}_j^{\varepsilon}}{\partial s_i}\right)^2+\frac{1}{2} \sum_{i=1}^{2}\left(\frac{\partial \widehat{w}_i^{\varepsilon}}{\partial \kappa}+\varepsilon^2 \frac{\partial \widehat{w}_3^{\varepsilon}}{\partial s_i}\right)^2+\varepsilon^2\left(\frac{\partial \widehat{w}_3^{\varepsilon}}{\partial \kappa}\right)^2\right)^{1/2} .
\end{aligned}
\tag{25}
$$

3.1. A Priori Estimates on the Velocity and the Pressure

Theorem 1. *For all $1<\nu<2$ and under assumptions (13) and (14) and (20), there exists a constant $C>0$ independent of ε such that*

$$
\begin{aligned}
&\sum_{i,j=1}^{2}\left\|\varepsilon \frac{\partial \widehat{w}_i^{\varepsilon}}{\partial s_j}\right\|_{L^{\nu}(\Omega)}^{\nu}+\sum_{i=1}^{2}\left(\left\|\frac{\partial \widehat{w}_i^{\varepsilon}}{\partial \kappa}\right\|_{L^{\nu}(\Omega)}^{\nu}+\left\|\varepsilon^2 \frac{\partial \widehat{w}_3^{\varepsilon}}{\partial s_i}\right\|_{L^{\nu}(\Omega)}^{\nu}\right)+\left\|\varepsilon \frac{\partial \widehat{w}_3^{\varepsilon}}{\partial \kappa}\right\|_{L^{\nu}(\Omega)}^{\nu} \leq C, \\
&\left\|\frac{\partial \widehat{\rho}^{\varepsilon}}{\partial s_i}\right\|_{W^{-1,\nu'}(\Omega)} \leq C \quad \text{for } i=1,2, \\
&\left\|\frac{\partial \widehat{\rho}^{\varepsilon}}{\partial \kappa}\right\|_{W^{-1,\nu'}(\Omega)} \leq \varepsilon C.
\end{aligned}
\tag{26}
$$

Proof. Choosing $\vartheta=0$ as test function in inequality (11), we get

$$
a\left(w^{\varepsilon}, w^{\varepsilon}\right)+\sqrt{2} \int_{\Omega^{\varepsilon}} g^{\varepsilon}\left(T^{\varepsilon}\right)\left|D\left(w^{\varepsilon}\right)\right| \mathrm{d}s\mathrm{d}s_3+\int_{\varpi} k^{\varepsilon}\left|w^{\varepsilon}\right| \mathrm{d}s \leq\left(f^{\varepsilon}, w^{\varepsilon}\right),
\tag{27}
$$

from (16) and (17) we have

$$
\begin{aligned}
\left(f^{\varepsilon}, w^{\varepsilon}\right) &\leq \varepsilon h^*\left\|\nabla w^{\varepsilon}\right\|_{L^{\nu}(\Omega)}\left\|f^{\varepsilon}\right\|_{L^{\nu'}(\Omega^{\varepsilon})} \\
&\leq \frac{1}{2} \Lambda_* C_k\left\|\nabla w^{\varepsilon}\right\|_{L^{\nu}(\Omega^{\varepsilon})}^{\nu}+\frac{\left(\varepsilon h^*\right)^{\nu'}}{\nu'\left(1/2 \Lambda_* \nu C_k\right)^{\nu'/\nu}}\left\|f^{\varepsilon}\right\|_{L^{\nu'}(\Omega^{\varepsilon})}^{\nu'} .
\end{aligned}
\tag{28}
$$

From (27) and (28), we deduce

$$
\begin{aligned}
&a\left(w^{\varepsilon}, w^{\varepsilon}\right)+\sqrt{2} \int_{\Omega^{\varepsilon}} g^{\varepsilon}\left(T^{\varepsilon}\right)\left|D\left(w^{\varepsilon}\right)\right| \mathrm{d}s\mathrm{d}s_3+\int_{\varpi} k^{\varepsilon}\left|w^{\varepsilon}\right| \mathrm{d}s \\
&\quad \leq \frac{1}{2} \Lambda_* C_k\left\|\nabla w^{\varepsilon}\right\|_{L^{\nu}(\Omega^{\varepsilon})}^{\nu}+\frac{\left(\varepsilon h^*\right)^{\nu'}}{\nu'\left(1/2 \Lambda_* \nu C_k\right)^{\nu'/\nu}}\left\|f^{\varepsilon}\right\|_{L^{\nu'}(\Omega^{\varepsilon})}^{\nu'} .
\end{aligned}
\tag{29}
$$

We multiply (29) by $\varepsilon^{\nu-1}$, we get

$$
\begin{aligned}
&\varepsilon^{\nu-1} a\left(w^{\varepsilon}, w^{\varepsilon}\right)+\sqrt{2} \int_{\Omega^{\varepsilon}} \widehat{g}(\widehat{T})\left|\widetilde{D}\left(\widehat{w}^{\varepsilon}\right)\right| \mathrm{d}s\mathrm{d}\kappa+\int_{\varpi} \widehat{k}\left|\widehat{w}^{\varepsilon}\right| \mathrm{d}s \\
&\quad \leq \frac{1}{2} \Lambda_* C_k \varepsilon^{\nu-1}\left\|\nabla w^{\varepsilon}\right\|_{L^{\nu}(\Omega^{\varepsilon})}^{\nu}+\varepsilon^{\nu-1} \frac{\left(\varepsilon h^*\right)^{\nu'}}{\nu'\left(1/2 \Lambda_* \nu C_k\right)^{\nu'/\nu}}\left\|f^{\varepsilon}\right\|_{L^{\nu'}(\Omega^{\varepsilon})}^{\nu'} .
\end{aligned}
\tag{30}
$$

As $\varepsilon^{\nu'}\left\|f^{\varepsilon}\right\|_{L^{\nu'}(\Omega^{\varepsilon})}^{\nu'}=\varepsilon^{1-\nu}\|\widehat{f}\|_{L^{\nu'}(\Omega)}^{\nu'}$, we have

$$\varepsilon^{\nu-1}a(w^\varepsilon,w^\varepsilon)+\sqrt{2}\int_\Omega \widehat{g}(\widehat{T}^\varepsilon)|\widetilde{D}(\widehat{w}^\varepsilon)|\mathrm{d}s\mathrm{d}\kappa+\int_\omega \widehat{k}|\widehat{w}^\varepsilon|\mathrm{d}s \leq \frac{1}{2}\Lambda_* C_k \varepsilon^{\nu-1}\|\nabla w^\varepsilon\|^\nu_{L^\nu(\Omega^\varepsilon)}+\frac{(h^*)^{\nu'}}{\nu'(1/2\Lambda_*\nu C_k)^{\nu'/\nu}}\|\widehat{f}\|^{\nu'}_{L^{\nu'}(\Omega)}. \tag{31}$$

From Korn's inequality and (15), there exists a constant C_k independent of ε, such that

$$\frac{1}{2}\Lambda_* C_k \varepsilon^{\nu-1}\|\nabla w^\varepsilon\|^\nu_{L^\nu(\Omega^\varepsilon)}+\sqrt{2}\int_\Omega \widehat{g}(\widehat{T})|\widetilde{D}(\widehat{w}^\varepsilon)|\mathrm{d}s\mathrm{d}\kappa+\int_\omega \widehat{k}|\widehat{w}^\varepsilon|\mathrm{d}s \leq \frac{(h^*)^{\nu'}}{\nu'(1/2\Lambda_*\nu C_k)^{\nu'/\nu}}\|\widehat{f}\|^{\nu'}_{L^{\nu'}(\Omega)} \tag{32}$$

From (32), we deduce (26), with $C=(1/2\Lambda_* C_K)^{-1}(h^*)^{\nu'}/\nu'(1/2\Lambda_*\nu C_k)^{\nu'/\nu}\|\widehat{f}\|^{\nu'}_{L^{\nu'}(\Omega)}$, and

$$\begin{aligned}\varepsilon^{\nu-1}\|\nabla w^\varepsilon\|^\nu_{L^\nu(\Omega^\varepsilon)}&=\|\nabla\widehat{w}^\varepsilon\|^\nu_{L^\nu(\Omega)}\\&=\sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial\widehat{w}_i^\varepsilon}{\partial s_j}\right\|^\nu_{L^\nu(\Omega)}+\left\|\varepsilon\frac{\partial\widehat{w}_3^\varepsilon}{\partial\kappa}\right\|^\nu_{L^\nu(\Omega)}+\sum_{i=1}^{2}\left(\left\|\frac{\partial\widehat{w}_i^\varepsilon}{\partial\kappa}\right\|^\nu_{L^\nu(\Omega)}+\left\|\varepsilon^2\frac{\partial\widehat{w}_3^\varepsilon}{\partial s_i}\right\|^\nu_{L^\nu(\Omega)}\right).\end{aligned} \tag{33}$$

We prove (26) and (26) as in [14]. □

3.2. A Priori Estimates on the Temperature. In this subsection, we look for a priori estimates on the temperature $\widehat{T}^\varepsilon$, for this we need to establish the following result:

Theorem 2. *Assume that the assumptions of Theorem 1 are satisfied. Moreover, assume that there exist K^*, K_*, such that*

$$0<K_*\leq\widehat{K}\leq K^*. \tag{34}$$

Then, there exists a positive constant C_1 independent of ε, such that

$$\begin{aligned}\left\|\frac{\partial\widehat{T}^\varepsilon}{\partial\kappa}\right\|_{W^{1,q}(\Omega)}&\leq C_1,\\ \sum_{i=1}^{2}\left\|\varepsilon\frac{\partial\widehat{T}^\varepsilon}{\partial s_i}\right\|_{W^{1,q}(\Omega)}&\leq C_1.\end{aligned} \tag{35}$$

Proof. Choosing $\Phi=\vartheta(T^\varepsilon)$ in (24), where ϑ is defined by

$$\vartheta(t)=\zeta\operatorname{sign}(t)\int_0^{|t|}\frac{\mathrm{d}\tau}{(1+|t|)^{\zeta+1}}=\operatorname{sign}(t)\left[1-\frac{1}{(1+|t|)^\zeta}\right], \tag{36}$$

We obtain

$$\zeta K_*\int_\Omega\frac{|\nabla\widehat{T}^\varepsilon|^2}{\left(1+|\widehat{T}^\varepsilon|\right)^{\zeta+1}}\leq\Lambda^*\varepsilon^{\beta-2}\int_\Omega|\widetilde{D}(\widehat{w}^\varepsilon)^\nu|\mathrm{d}s\mathrm{d}\kappa+\varepsilon^{\beta-2}\sqrt{2}g^*\int_\Omega|\widetilde{D}(\widehat{w}^\varepsilon)|\mathrm{d}s\mathrm{d}\kappa. \tag{37}$$

On the other hand,

$$\int_\Omega|\widetilde{D}(\widehat{w}^\varepsilon)|^\nu\mathrm{d}s\mathrm{d}\kappa\leq C(\nu)\left[\sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial\widehat{w}_i^\varepsilon}{\partial s_j}\right\|^\nu_{L^\nu(\Omega)}+\sum_{i=1}^{2}\left(\left\|\frac{\partial\widehat{w}_i^\varepsilon}{\partial\kappa}\right\|^\nu_{L^\nu(\Omega)}+\left\|\varepsilon^2\frac{\partial\widehat{w}_3^\varepsilon}{\partial s_i}\right\|^\nu_{L^\nu(\Omega)}\right)^2+\left\|\varepsilon\frac{\partial\widehat{w}_3^\varepsilon}{\partial\kappa}\right\|^\nu_{L^\nu(\Omega)}\right], \tag{38}$$

where $C(\nu)>0$ depends only on ν.

As $\nu>1$ and $0<\varepsilon<1$ then $\varepsilon^{\nu-1}\leq 1$, so using this inequality and (26), we deduce

$$\int_\Omega\frac{|\nabla_\varepsilon\widehat{T}^\varepsilon|^2}{\left(1+|\widehat{T}^\varepsilon|\right)^{\zeta+1}}\leq\frac{1}{\zeta K_*}(\Lambda^* C(\nu)C+\sqrt{2}\,g^*C)\varepsilon^{\beta-2}. \tag{39}$$

Using Holder's inequality with the exponents $2/q$ and $2/2-q$, for $q<3/2$, we obtain

$$\int_\Omega \left|\nabla_\varepsilon \widehat{T}^\varepsilon\right|^q \mathrm{d}s\mathrm{d}\kappa \leq \left(\int_\Omega \frac{\left|\nabla_\varepsilon \widehat{T}^\varepsilon\right|^2}{\left(1+\left|\widehat{T}^\varepsilon\right|\right)^{\zeta+1}}\right)^{q/2} \left(\int_\Omega \left(1+\left|\widehat{T}^\varepsilon\right|\right)^{(\zeta+1)q/2-q}\right)^{2-q/2}, \tag{40}$$

using (39), we get

$$\int_\Omega \left|\nabla_\varepsilon \widehat{T}^\varepsilon\right|^q \mathrm{d}s\mathrm{d}\kappa \leq \left(\frac{1}{\zeta K_*}\left(\Lambda^* C(\nu)C+\sqrt{2}\,g^* C\right)\varepsilon^{\beta-2}\right)^{q/2} \left(\int_\Omega \left(1+\left|\widehat{T}^\varepsilon\right|\right)^{q*}\right)^{2-q/2}, \tag{41}$$

where $q^* = 3q/3-q \geq (\zeta+1)q/2-q$.

By (18) and (18), we find

$$\int_\Omega \left|\nabla_\varepsilon \widehat{T}^\varepsilon\right|^q \mathrm{d}s\mathrm{d}\kappa \leq \left(\frac{1}{\zeta K_*}\left(\Lambda^* C(\nu)C+\sqrt{2}\,g^* C\right)\varepsilon^{\beta-2}\right)^{q/2} 2^{(q^*-1)2-q/2}\left(|\Omega|^{2-q/2}+\left(\int_\Omega \left|\widehat{T}^\varepsilon\right|^{q*}\right)^{2-q/2}\right). \tag{42}$$

Now using the Poincaré-Sobolev inequality, we have

$$\begin{aligned}\left(\int_\Omega \left|\widehat{T}^\varepsilon\right|^{q*} \mathrm{d}s\mathrm{d}\kappa\right)^{1/q^*} &\leq C'\left\|\nabla_\varepsilon \widehat{T}^\varepsilon\right\|_{L^q(\Omega)} \\ &\leq \left(\frac{1}{\zeta K_*}\left(\Lambda^* C(\nu)C+\sqrt{2}\,g^* C\right)\right)^{1/2} \varepsilon^{\beta/2-1} 2^{(q^*-1)2-q/2q} \times C'\left(|\Omega|^{2-q/2q}+\left(\int_\Omega \left|\widehat{T}^\varepsilon\right|^{q*}\right)^{2-q/2q}\right).\end{aligned} \tag{43}$$

On the other hand, for all $a>0, b>0, c>0$ and $0<s<t$, we have the implication:

$$\text{If } a^t \leq b+ca^s \text{ then } a \leq \max\left\{1, (b+c)^{\frac{1}{t-s}}\right\}. \tag{44}$$

Hence from (43) and (44) and the fact that $2-q/2<1$, we deduce

$$\int_\Omega \left|\widehat{T}^\varepsilon\right|^{q*} \mathrm{d}s\mathrm{d}\kappa \leq \xi_\varepsilon, \tag{45}$$

where

$$\xi = \max\left[1, \gamma\varepsilon^{(\beta/2-1)\left(1/q^*-2-q/2q\right)^{-1}2-q/2}\right] = \max\left[1, \gamma\varepsilon^{3(\beta/2-1)(2-q)}\right], \tag{46}$$

and

$$\begin{aligned}\gamma &= \left[\left(\frac{1}{\zeta K_*}\left(\Lambda^* C(\nu)C+\sqrt{2}\,g^* C\right)\right)^{1/2} 2^{(q^*-1)2-q/2q} C'\left(|\Omega|^{2-q/2q}+1\right)\right]^{\left(1/q^*-2-q/2q\right)^{-1}} \\ &= \left[\left(\frac{1}{\zeta K_*}\left(\Lambda^* C(\nu)C+\sqrt{2}\,g^* C\right)\right)^{1/22-q/2q} 2^{(q^*-1)2-q/2q} C'\left(|\Omega|^{2-q/2q}+1\right)\right]^{6}.\end{aligned} \tag{47}$$

As $\beta = 3(2-q/3-q)$, then $3(2-q)(\beta/2-1) < 0$. So for $\varepsilon \leq \beta^{[-3(2-q)(\beta/2-1)]^{-1}}$, we obtain

$$\xi = \gamma \varepsilon^{3(\beta/2-1)(2-q)} \geq 1. \tag{48}$$

From (42) and (45), we get

$$\varepsilon^{q}\int_{\Omega}\left|\nabla_{\varepsilon}\widehat{T}^{\varepsilon}\right|^{q} \mathrm{d}s\mathrm{d}\kappa \leq \left(\frac{1}{\zeta K_{*}}\left(\Lambda^{*}C(\nu)C + \sqrt{2}\,g^{*}C\right)\right)^{q/2} 2^{(q^{*}-1)2-q/2}\left(|\Omega|^{2-q/2}\varepsilon^{\beta/2q} + \gamma\varepsilon^{3(\beta/2-1)(2-q)+\beta/2q}\right), \tag{49}$$

as $3(\beta/2-1)(2-q) + (\beta/2)q = 0$ and $(\beta/2)q > 0$, we obtain

$$\varepsilon^{q}\int_{\Omega}\left|\nabla_{\varepsilon}\widehat{T}^{\varepsilon}\right|^{q} \mathrm{d}s\mathrm{d}\kappa \leq C_1, \tag{50}$$

where

$$C_1 = \left(\frac{1}{\zeta K_{*}}\left(\Lambda^{*}C(\nu)C + \sqrt{2}\,g^{*}C\right)\right)^{q/2} 2^{(q^{*}-1)2-q/2}\left(|\Omega|^{2-q/2} + \gamma\right). \tag{51}$$

where C_1 is a constant independent of ε.Thus, we obtain (35) and (35) □

The following theorem states some immediate estimates of the limit of our initial problem.

Theorem 3. *Under the same assumptions as in Theorem 1 and Theorem 2, there exist $w^{\star} = (w_1^{\star}, w_2^{\star}) \in \widetilde{V}_{\kappa}$, $\rho^{\star} \in L_0^{r'}(\Omega)$ and $T^{*} \in \Pi_{\kappa}$ such that*

$$\widehat{w}_i^{\varepsilon} \rightharpoonup w_i^{\star}, i = 1, 2 \quad \text{weakly in} \widetilde{V}_{\kappa}, \tag{52}$$

$$\varepsilon\frac{\partial \widehat{w}_i^{\varepsilon}}{\partial s_j} \rightharpoonup 0, i, j = 1, 2 \text{weakly in } L^{\nu}(\Omega), \quad \varepsilon\frac{\partial \widehat{w}_3^{\varepsilon}}{\partial \kappa} \rightharpoonup 0, \text{weakly in } L^{\nu}(\Omega), \tag{53}$$

$$\varepsilon^{2}\frac{\partial \widehat{w}_3^{\varepsilon}}{\partial s_i} \rightharpoonup 0,\ i = 1, 2 \text{weakly in } L^{\nu}(\Omega), \tag{54}$$

$$\varepsilon\widehat{w}_3^{\varepsilon} \rightharpoonup 0,\ \text{weaklyin } L^{\nu}(\Omega), \tag{55}$$

$$\widehat{\rho}^{\varepsilon} \rightharpoonup \rho^{\star}, \text{weakly in } L^{\nu'}(\Omega),\ \rho^{\star} \text{depend only of} s, \tag{56}$$

$$\widehat{T}^{\varepsilon} \rightharpoonup T^{*} \text{ weakly in} \Pi_{\kappa}, \tag{57}$$

$$\frac{\partial \widehat{T}^{\varepsilon}}{\partial s_i} \rightharpoonup 0, i = 1, 2 \text{weakly in} L^{q}(\Omega). \tag{58}$$

Proof. The convergence of (52) to (53) is a direct result of inequality (26). Using (26) and (26), we get (56), while (57) and (58) follow from (35). □

4. Study of the Limit Problem

In this section, we give both the equations satisfied by ρ^{*} and w^{*} in Ω and the inequalities for the trace of the velocity $w^{*}(s, 0)$ and the stress $\partial w^{*}/\partial\kappa(s, 0)$ on ϖ.

Theorem 4. *With the same assumptions of Theorem 3, the solution (w^{*}, ρ^{*}, T^{*}) satisfies the following relations:*

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\widehat{\Lambda}(T^{*})\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\left(\sum_{i=1}^{2}\left(\frac{\partial w_i^{*}}{\partial\kappa}\right)^{2}\right)^{\nu-2/2}\frac{\partial(w_i^{*})}{\partial\kappa}\frac{\partial\left(\widehat{\vartheta}_i - w_i^{*}\right)}{\partial\kappa}\mathrm{d}s\mathrm{d}\kappa \\
&-\int_{\Omega}\rho^{*}(s)\left(\frac{\partial\widehat{\vartheta}_1}{\partial s_1} + \frac{\partial\widehat{\vartheta}_2}{\partial s_2}\right)\mathrm{d}s\mathrm{d}\kappa + \int_{\Omega}\widehat{g}(T^{*})\left(\left|\frac{\partial\widehat{\vartheta}}{\partial\kappa}\right| - \left|\frac{\partial w^{*}}{\partial\kappa}\right|\right)\mathrm{d}s\mathrm{d}\kappa \\
&+\int_{\varpi}\widehat{k}\left(|\widehat{\vartheta}| - |w^{*}|\right)\mathrm{d}s \geq \sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i\left(\widehat{\vartheta}_i - w_i^{*}\right)\mathrm{d}s\mathrm{d}\kappa, \quad \forall\widehat{\vartheta} \in W_{\Gamma_1\cup\Gamma_L},
\end{aligned} \tag{59}$$

$$\int_{\Omega}\widehat{\Lambda}(T^{*})\left(\frac{1}{2}\right)^{\nu/2}\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu}\mathrm{d}s\mathrm{d}\kappa + \int_{\Omega}\widehat{g}(T^{*})\left|\frac{\partial w^{*}}{\partial\kappa}\right|\mathrm{d}s\mathrm{d}\kappa + \int_{\varpi}\widehat{k}|w^{*}|\mathrm{d}s = \int_{\Omega}\widehat{f}w^{*}\mathrm{d}s\mathrm{d}\kappa, \tag{60}$$

$$\int_{\Omega}\widehat{\Lambda}(T^{*})\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-2}\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\widehat{\Phi}}{\partial\kappa}\mathrm{d}s\mathrm{d}\kappa + \int_{\Omega}\widehat{g}(T^{*})\left|\frac{\partial\widehat{\Phi}}{\partial\kappa}\right|\mathrm{d}s\mathrm{d}\kappa + \int_{\varpi}\widehat{k}|\widehat{\Phi}|\mathrm{d}s \geq \int_{\Omega}\widehat{f}\widehat{\Phi}\mathrm{d}s\mathrm{d}\kappa, \quad \forall\widehat{\Phi} \in \Sigma(K), \tag{61}$$

and

$$-\frac{\partial}{\partial\kappa}\left(K\frac{\partial T^*}{\partial\kappa}\right) = -\widehat{\alpha}T^* \text{ in} L^q(\Omega), \quad (62)$$

$$\begin{aligned} T^* &= 0 \text{ in } \Gamma_1 \cup \Gamma_L, \\ \frac{\partial T^*}{\partial n} &= 0 \text{ in } \varpi, \end{aligned} \quad (63)$$

where

$$W_{\Gamma_1\cup\Gamma_L} = \left\{\widehat{\vartheta} = \left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) \in W^{1,\nu}(\Omega)^2, \widehat{\vartheta} = 0 \text{ on } \Gamma_1 \cup \Gamma_L\right\}, \quad (64)$$

and

$$\widetilde{\Sigma}(K) = \left\{\widehat{\Phi} = \left(\widehat{\Phi}_1, \widehat{\Phi}_2\right) \in W^{1,\nu}(\Omega)^2 \colon \widehat{\Phi} \text{ satisfy } (D')\right\}. \quad (65)$$

The proof of this theorem is based on the following lemma.

Lemma 1 (Minty). *Let E be a Banach spaces, $T\colon E \longrightarrow E'$ a monotone and hemicontinuous operator, $J\colon E \longrightarrow]-\infty, +\infty]$ a proper and convex functional. Let $u \in E$ and $f \in E'$. Then the following assertions are equivalent:*

$$\begin{aligned} &\langle Tw; v-w\rangle_{E'\times E} + J(v) - J(w) \geq \langle f; v-w\rangle_{E'\times E'}, \quad \forall v \in E, \\ &\langle Tv; v-w\rangle_{E'\times E} + J(v) - J(w) \geq \langle f; v-w\rangle_{E'\times E}, \quad \forall v \in E. \end{aligned} \quad (66)$$

Proof. By using Minty's Lemma 1 and the fact that $\operatorname{div}(\widehat{w}^\varepsilon) = 0$ in Ω, then (24) is equivalent to

$$\begin{aligned} &a_0\left(\widehat{T}^\varepsilon, \widehat{\vartheta}, \widehat{\vartheta} - \widehat{w}^\varepsilon\right) - \sum_{i=1}^{2}\left(\widehat{\rho}^\varepsilon, \frac{\partial\widehat{\vartheta}_i}{\partial s_i}\right) - \left(\widehat{\rho}^\varepsilon, \frac{\partial\widehat{\vartheta}_3}{\partial\kappa}\right) + j_0\left(\widehat{T}^\varepsilon, \widehat{\vartheta}\right) - j_0\left(\widehat{T}^\varepsilon, \widehat{w}^\varepsilon\right) \\ &\geq \sum_{i=1}^{2}\int_\Omega \widehat{f}_i\left(\widehat{\vartheta}_i - \widehat{w}_i^\varepsilon\right)\mathrm{d}s\mathrm{d}\kappa + \int_\Omega \varepsilon\widehat{f}_3\left(\widehat{\vartheta}_3 - \widehat{w}_3^\varepsilon\right)\mathrm{d}s\mathrm{d}\kappa. \end{aligned} \quad (67)$$

From (57), we have $\widehat{T}^\varepsilon \longrightarrow T^*$ almost everywhere. As $\widehat{\Lambda}$ is continuous function on $\mathbb{R}$, then

$$\widehat{\Lambda}\left(\widehat{T}^\varepsilon\right) \longrightarrow \widehat{\Lambda}(T^*). \quad (68)$$

Using Theorem 4 and the fact j_0 is convex and lower semicontinuous, $(\liminf j_0(\widehat{T}^\varepsilon, \widehat{w}^\varepsilon) \geq j_0(T^*, w^*))$, we find

$$\begin{aligned} &\sum_{i=1}^{2}\int_\Omega \frac{1}{2}\widehat{\Lambda}(T^*)\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\widehat{\vartheta}_i}{\partial\kappa}\right)^2\right)^{\nu-2/2}\frac{\partial\left(\widehat{\vartheta}_i\right)}{\partial\kappa}\frac{\partial\left(\widehat{\vartheta}_i - w_i^*\right)}{\partial\kappa}\mathrm{d}s\mathrm{d}\kappa \\ &-\int_\Omega \rho^*\left(\frac{\partial\widehat{\vartheta}_1}{\partial s_1} + \frac{\partial\widehat{\vartheta}_2}{\partial s_2}\right)\mathrm{d}s\mathrm{d}\kappa - \int_\Omega \rho^*\frac{\partial\widehat{\vartheta}_3}{\partial\kappa}\mathrm{d}s\mathrm{d}\kappa + j_0\left(T^*, \widehat{\vartheta}\right) - j_0(T^*, w^*) \\ &\geq \sum_{j=1}^{2}\int_\Omega \widehat{f}_i\left(\widehat{\vartheta}_i - w_i^*\right)\mathrm{d}s\mathrm{d}\kappa, \end{aligned} \quad (69)$$

and as $\int_\Omega \rho^*(\partial\widehat{\vartheta}_3/\partial\kappa)\mathrm{d}s\mathrm{d}\kappa = 0$, because ρ^* independent of κ, we get

$$\begin{aligned} &\sum_{i=1}^{2}\int_\Omega \frac{1}{2}\widehat{\Lambda}(T^*)\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\widehat{\vartheta}_i}{\partial\kappa}\right)^2\right)^{\nu-2/2}\frac{\partial\left(\widehat{\vartheta}_i\right)}{\partial\kappa}\frac{\partial\left(\widehat{\vartheta}_i - w_i^*\right)}{\partial\kappa}\mathrm{d}s\mathrm{d}\kappa \\ &-\int_\Omega \rho^*\left(\frac{\partial\widehat{\vartheta}_1}{\partial s_1} + \frac{\partial\widehat{\vartheta}_2}{\partial s_2}\right)\mathrm{d}s\mathrm{d}\kappa + j_0\left(T^*, \widehat{\vartheta}\right) - j_0(T^*, w^*) \geq \sum_{j=1}^{2}\int_\Omega \widehat{f}_i\left(\widehat{\vartheta}_i - w_i^*\right)\mathrm{d}s\mathrm{d}\kappa. \end{aligned} \quad (70)$$

Using again Minty's lemma for the second time, thus (61) is equivalent to (59). Now, we can choose $\widehat{\vartheta} = 2w^*$ and $\widehat{\vartheta} = 0$ respectively in (59), we find (60). For (61), we choose $\widehat{\Phi} = \widehat{\vartheta} - w^*$ for all $\widehat{\Phi} \in \Sigma(K)$. Passing to the limit on ε tend to 0 in (24) and using (52)–(54),(57)–(58) we get

$$\int_\Omega \widehat{K} \frac{\partial T^*}{\partial \kappa} \frac{\partial \widehat{\Phi}}{\partial \kappa} \mathrm{d}s\mathrm{d}\kappa = -\int_\Omega \widehat{\alpha} T^* \widehat{\Phi} \mathrm{d}s\mathrm{d}\kappa, \quad \forall \widehat{\Phi} \in W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega), \tag{71}$$

by Green's formula, we obtain

$$-\frac{\partial}{\partial \kappa}\left(\widehat{K} \frac{\partial T^*}{\partial \kappa}\right) = \widehat{\alpha} T^* \ \mathrm{in} W^{-1,q'}(\Omega). \tag{72}$$

□

Theorem 5. *Let us set*

$$\sum^{*} = \widetilde{\sum}^{*} - \nabla \rho^*,$$
$$\widetilde{\sum}^{*} = \left(\frac{1}{2}\right)^{\frac{\nu}{2}} \widehat{\Lambda}(T^*) \left|\frac{\partial w^*}{\partial \kappa}\right|^{\nu-2} \frac{\partial w^*}{\partial \kappa} + \widehat{g}(T^*)\pi, \tag{73}$$

then

$$-\frac{\partial}{\partial \kappa}\left[\frac{1}{2}\widehat{\Lambda}(T^*)\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial w_i^*}{\partial \kappa}\right)^2\right)^{\nu-2/2} \frac{\partial w^*}{\partial \kappa} + \widehat{g}(T^*) \frac{\partial w^*/\partial \kappa}{|\partial w^*/\partial \kappa|}\right] = \widehat{f} - \nabla \rho^*, \mathrm{in} W^{-1,\nu'}(\Omega)^2. \tag{74}$$

where $\pi \in L^\infty(\Omega)^2$ *and* $\|\pi\|_{\Omega,\infty} \le 1$.

Proof. If $\partial w^*/\partial \kappa = 0$, from (73)we get $|\widetilde{\Sigma}^*| < \widehat{g}(T^*)$. For all $\widehat{\Phi} \in \sum(K)$, choosing $\widehat{\Phi} = \widehat{\Phi}$, then $\widehat{\Phi} = -\widehat{\Phi}$ in (61), we obtain

$$\left|F\left(\widehat{k}\widehat{\Phi}, \frac{\partial \widehat{\Phi}}{\partial \kappa}\right)\right| \le \int_\varpi \widehat{k}|\widehat{\Phi}|\mathrm{d}s + \int_\Omega \widehat{g}(T^*)\left|\frac{\partial \widehat{\Phi}}{\partial \kappa}\right| \mathrm{d}s\mathrm{d}\kappa, \tag{75}$$

where

$$F\left(\widehat{k}\widehat{\Phi}, \frac{\partial \widehat{\Phi}}{\partial \kappa}\right) = \int_\Omega \widehat{\Lambda}(T^*)\left(\frac{1}{2}\right)^{\nu/2} \left|\frac{\partial w^*}{\partial \kappa}\right|^{\nu-2} \frac{\partial w^*}{\partial \kappa} \frac{\partial \widehat{\Phi}}{\partial \kappa} \mathrm{d}s\mathrm{d}\kappa - \int_\Omega \widehat{f}\widehat{\Phi}\mathrm{d}s\mathrm{d}\kappa, \tag{76}$$

Now, utilising the Hanh-Banach theorem, then, $\exists(\chi, \pi) \in L^\infty(\varpi)^2 \times L^\infty(\Omega)^2$, with $\|\chi\|_{\varpi,\infty} \le 1 \|\pi\|_{\Omega,\infty} \le 1$, such that

$$F\left(\widehat{k}\widehat{\Phi}, \frac{\partial \widehat{\Phi}}{\partial \kappa}\right) = -\int_\varpi \chi \widehat{k}\widehat{\Phi}\mathrm{d}s - \int_\Omega \pi \widehat{g}(T^*) \frac{\partial \widehat{\Phi}}{\partial \kappa} \mathrm{d}s\mathrm{d}\kappa. \tag{77}$$

In particular, from (60) and (76), we get

$$\int_\varpi \widehat{k}|w^*|\mathrm{d}s + \int_\Omega \widehat{g}(T^*)\left|\frac{\partial w^*}{\partial \kappa}\right| \mathrm{d}s\mathrm{d}\kappa = \int_\varpi \chi \widehat{k} w^* \mathrm{d}s + \int_\Omega \pi \widehat{g}(T^*) \frac{\partial w^*}{\partial \kappa} \mathrm{d}s\mathrm{d}\kappa. \tag{78}$$

Also, from (76) and (77), we have

$$\int_\Omega \widehat{\Lambda}(T^*)\left(\frac{1}{2}\right)^{\nu/2} \left|\frac{\partial w^*}{\partial \kappa}\right|^{\nu-2} \frac{\partial w^*}{\partial \kappa} \frac{\partial \widehat{\Phi}}{\partial \kappa} \mathrm{d}s\mathrm{d}\kappa + \int_\varpi \chi \widehat{k}\widehat{\Phi}\mathrm{d}s + \int_\Omega \pi \widehat{g}(T^*) \frac{\partial \widehat{\Phi}}{\partial \kappa} \mathrm{d}s\mathrm{d}\kappa - \int_\Omega \widehat{f}\widehat{\Phi}\mathrm{d}s\mathrm{d}\kappa = 0. \tag{79}$$

Next using (78), we have

$$\int_\varpi \widehat{k}\left(|w^*| - \chi w^*\right)\mathrm{d}s + \int_{\frac{\partial w^*}{\partial \kappa} \neq 0} \widehat{g}(T^*)\left(\left|\frac{\partial w^*}{\partial \kappa}\right| - \pi \frac{\partial w^*}{\partial \kappa}\right)\mathrm{d}s\mathrm{d}\kappa = 0. \tag{80}$$

As $\|\chi\|_{\varpi,\infty} \le 1, \|\pi\|_{\Omega,\infty} \le 1$, we deduce

$$\left|\frac{\partial w^*}{\partial \kappa}\right| = \pi \frac{\partial w^*}{\partial \kappa},$$
$$|w^*| - \chi w^*. \tag{81}$$

Hence, if $|\partial w^*/\partial \kappa| \neq 0$, by (73), we obtain

$$\widetilde{\Sigma}^* = \left(\frac{1}{2}\right)^{\frac{\nu}{2}} \widehat{\Lambda}(T^*)\left|\frac{\partial w^*}{\partial \kappa}\right|^{\nu-2}\frac{\partial w^*}{\partial \kappa} + \widehat{g}(T^*)\frac{\partial w^*/\partial \kappa}{|\partial w^*/\partial \kappa|}. \tag{82}$$

In this case, $|\widetilde{\Sigma}^*| = (1/2)^{\nu/2}\widehat{\Lambda}(T^*)|\partial w^*/\partial \kappa|^{\nu-1} + \widehat{g}(T^*) > \widehat{g}(T^*)$; therefore, we can write

$$\left(\frac{1}{2}\right)^{\nu/2}\widehat{\Lambda}(T^*)\left|\frac{\partial w^*}{\partial \kappa}\right|^{\nu-2}\frac{\partial w^*}{\partial \kappa} = \begin{cases} 0, & \text{if} \left|\widetilde{\Sigma}^*\right| \leq \widehat{\alpha}, \\ \widetilde{\Sigma}^* - \widehat{g}(T^*)\dfrac{\partial w^*/\partial \kappa}{|\partial w^*/\partial \kappa|}, & \text{if} \left|\widetilde{\Sigma}^*\right| > \widehat{\alpha}, \end{cases} \tag{83}$$

for every $\widehat{\Phi} \in \Sigma(K)$ and from (79), there exist $p^* \in L^{r'}(\Omega)^2$ such that

$$\int_\Omega \widehat{\Lambda}(T^*)\left(\frac{1}{2}\right)^{\nu/2}\left|\frac{\partial w^*}{\partial \kappa}\right|^{\nu-2}\frac{\partial w^*}{\partial \kappa}\frac{\partial \widehat{\Phi}}{\partial \kappa}\mathrm{d}s\mathrm{d}\kappa + \int_\varpi \chi\widehat{k}\widehat{\Phi}\mathrm{d}s + \widehat{\alpha}\int_\Omega \pi\widehat{g}(T^*)\frac{\partial \widehat{\Phi}}{\partial \kappa}\mathrm{d}s\mathrm{d}\kappa - \int_\Omega \widehat{f}\widehat{\Phi}\mathrm{d}s\mathrm{d}\kappa = -\int_\Omega \nabla\rho^*\widehat{\Phi}\mathrm{d}s\mathrm{d}\kappa. \tag{84}$$

Using (83) and (84) becomes

$$\int_\Omega \widetilde{\Sigma}^*\frac{\partial \widehat{\Phi}}{\partial \kappa}\mathrm{d}s\mathrm{d}\kappa + \int_\varpi \chi\widehat{k}\widehat{\Phi}\mathrm{d}s = \int_\Omega \widehat{f}\widehat{\Phi}\mathrm{d}s\mathrm{d}\kappa - \int_\Omega \nabla\rho^*\widehat{\Phi}\mathrm{d}s\mathrm{d}\kappa, \tag{85}$$

and choosing $\widehat{\Phi} \in W_0^{1,\nu}(\Omega)^2$ in (85), we find (74). □

The convergence of our problem towards the Reynolds equation given by the following result: Theorem 5.

$$\int_\varpi \left[\frac{h^3}{12}\nabla\rho^* + \widetilde{F} + \int_0^h\int_0^y \widehat{\Lambda}(T^*(s,\zeta))A^*(s,\zeta)\frac{\partial w^*(s,\xi)}{\partial \xi}\mathrm{d}\xi\mathrm{d}y - \frac{h}{2}\int_0^h \widehat{g}(T^*(s,\zeta))\left|\frac{\partial u^*}{\partial \kappa}\right|(s,\xi)\mathrm{d}\xi\right].\nabla\vartheta(s)\mathrm{d}s = 0, \tag{86}$$

for all $\vartheta \in W^{1,\nu}(\varpi)$ where

$$\widetilde{F}(s) = \int_0^h F(s,y)\mathrm{d}y - \frac{h}{2}F(s,h),$$
$$F(s,y) = \int_0^h\int_0^\xi \widehat{f}(s,t)\mathrm{d}t\mathrm{d}\xi, \tag{87}$$
$$A^*(s,\xi) = \frac{1}{2}\left(\frac{1}{2}\,{}^2\sum_{i=1}\left(\frac{\partial w^*}{\partial \kappa}(s,\xi)\right)^2\right)^{\nu-2/2}.$$

Proof. To prove (86), we integrate twice (74) from 0 to κ, then taking $\kappa = h$, we obtain the requested result. □

The uniqueness of the limit velocity and pression are given in the following theorem:

Theorem 6. *The solution* (w^*, T^*, ρ^*) *in* $V_\kappa \times W^{-1,q}(\Omega) \times L_0^{\gamma'}(\varpi)$ *of equality (86) is unique.*

Proof. Let $(w^{*,1}, T^{*,1}, \rho^{*,1})$ and $(w^{*,2}, T^{*,2}, \rho^{*,2})$ be two solutions of (59)–(63) and (86); then $T^{*,1}$ and $T^{*,2}$ solve (62)–(63), so $T = T^{*,1} - T^{*,2}$ satisfies the problem

$$-\frac{\partial}{\partial \kappa}\left(K\frac{\partial T}{\partial \kappa}\right) = -\widehat{\alpha}T,$$
$$T = 0 \text{ in } \Gamma_1 \cup \Gamma_L, \tag{88}$$
$$\frac{\partial T}{\partial \kappa} = 0 \text{ on } \varpi.$$

so $T = 0$, thus $T^{*,1} = T^{*,2}$. Taking $\vartheta = u^{*,2}$ and $\vartheta = w^{*,1}$ respectively, as test function in (59) we get

$$\sum_{i=1}^{2} \int_{\Omega} \widehat{\Lambda}(T^*)\left(\frac{1}{2}\right)^{\nu/2}\left(\sum_{i=1}^{2}\left(\frac{\partial w_i^{*,1}}{\partial \kappa}\right)^2\right)^{\nu-2/2} \frac{\partial w_i^{*,1}}{\partial \kappa}\frac{\partial}{\partial \kappa}\left(w_i^{*,1}-w_i^{*,2}\right)\mathrm{d}s\mathrm{d}\kappa$$
$$\leq {}^2\sum_{i=1} \int_{\Omega} \widehat{\Lambda}(T^*)\left(\frac{1}{2}\right)^{\nu/2}\left({}^2\sum_{i=1}\left(\frac{\partial w_i^{*,2}}{\partial \kappa}\right)^2\right)^{\nu-2/2} \frac{\partial w_i^{*,2}}{\partial \kappa}\frac{\partial}{\partial \kappa}\left(w_i^{*,1}-w_i^{*,2}\right)\mathrm{d}s\mathrm{d}\kappa. \quad (89)$$

Observe that for every $s, y \in \mathbb{R}^n$

$$\left(|s|^{\nu-2}s-|y|^{\nu-2}y, s-y\right) \geq (\nu-1)\left(|s|+|y|\right)^{\nu-2}|s-y|^2, \quad \forall 1<\nu\leq 2, \quad (90)$$

we obtain

$$\int_{\Omega}\left[\left|\frac{\partial w^{*,1}}{\partial \kappa}\right|+\left|\frac{\partial w^{*,2}}{\partial \kappa}\right|\right]^{\nu-2}\left|\frac{\partial w^{*,1}}{\partial \kappa}-\frac{\partial w^{*,2}}{\partial \kappa}\right|^2 \mathrm{d}s\mathrm{d}\kappa = 0, \quad (91)$$

where $|\partial w^{*,j}/\partial \kappa| = (\sum_{i=1}^{2}(\partial w_i^{*,j}/\partial \kappa)^2)^{1/2}$, $j=1,2$ Using Hölder's inequality, we deduce

$$\int_{\Omega}\left[\frac{\partial}{\partial \kappa}\left(w^{*,1}-w^{*,2}\right)\right]^{\nu}\mathrm{d}s\mathrm{d}\kappa \leq C\left(\int_{\Omega}\left[\left|\frac{\partial w^{*,1}}{\partial \kappa}\right|+\left|\frac{\partial w^{*,2}}{\partial \kappa}\right|\right]^{\nu-2}\left|\frac{\partial w^{*,1}}{\partial \kappa}-\frac{\partial w^{*,2}}{\partial \kappa}\right|^2\mathrm{d}s\mathrm{d}\kappa\right)^{\nu/2} \times \left(\int_{\Omega}\left[\left|\frac{\partial w^{*,1}}{\partial \kappa}\right|+\left|\frac{\partial w^{*,2}}{\partial \kappa}\right|\right]^{\nu}\mathrm{d}s\mathrm{d}\kappa\right)^{2-\nu/2}. \quad (92)$$

From (91) and (92), we obtain

$$\left\|\frac{\partial}{\partial \kappa}\left(w^{*,1}-w^{*,2}\right)\right\|_{L^{\nu}(\Omega)} = 0, \quad (93)$$

using Poincare's inequality, we deduce

$$\left\|w^{*,1}-w^{*,2}\right\|_{V_{\kappa}} = 0. \quad (94)$$

Finally, to prove the uniqueness of the pressure, we use (86) with the two pressures $\rho^{*,1}$ and $\rho^{*,2}$, we find

$$\int_{\varpi} h^3/12\nabla\left(\rho^{*,1}-\rho^{*,2}\right)\nabla\vartheta \mathrm{d}s = 0. \quad (95)$$

Taking $\vartheta = \rho^{*,1}-\rho^{*,2}$, and by Poincare's inequality, we deduce $\|\rho^{*,1}-\rho^{*,2}\|_{L^{\nu'}(\varpi)} = 0$. So $\rho^{*,1}=\rho^{*,2}$. □

5. Conclusions

This work studies the asymptotic analysis of an incompressible Herschel–Bulkley fluid in a thin domain with Tresca boundary conditions. The yield stress and the constant viscosity are assumed to vary with respect to the thin layer parameter. Firstly, the problem statement and variational formulation are formulated. We then obtained the estimates for the velocity field and the pressure independently of the parameter. Finally, we gave a specific Reynolds equation associated with variational inequalities and proved the uniqueness [15–23].

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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