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Research Article

Dynamics for a Type of Differential-Algebraic Complex-Valued Neural Networks with Delay

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In the article, we apply complex-valued neural networks (CVNNs) to differential-algebraic neural networks (DANNs) and establish a new type of differential-algebraic complex-valued neural network (DACVNN) with delay (DDACVNN). First of all, the focus of existence and uniqueness of the solution to DDACVNN is addressed. Additionally, a theorem of global exponential stability (GES) of DDACVNN is investigated. In particular, in the discussion of this article, there is no restriction on whether the activation function requires that the real and imaginary parts can be dissociated. Finally, we will give two examples, namely, the activation function can separate the real and imaginary parts, and the activation function cannot separate the real and imaginary parts, both of which can confirm the truth of the effectiveness of theoretical results.

1. Introduction

Due to the various applications of artificial neural networks (ANNs) in graphics processing, combinatorial optimization, signal processing, and intelligent control, especially, in recently years, as artificial intelligence develops, the research on neural networks has received growing concern. When designing neural networks to solve application problems, it is crucial to guarantee the stability of the model under consideration. In the hardware implementation of neural networks, since the transmission of signals is delayed over time, delays inevitably occur, which may lead to network instability. Hence, the research on time-delay systems is meaningful and necessary. Also many research results can be seen in [1, 2] and its references.

During the past period, due to the diverse applications in multitudinous engineering and technical territories, such as pattern recognition, associative memory, and hole filling, complex-valued neural networks (CVNNs) are becoming increasingly needed in applications, as in [3]. Hence, it can be found initially from [4–6] that numerous CVNNs have been put forward and discussed.

As we all know, the state variables, activation functions, and connection weights of complex-valued systems are all defined in the complex number domain, and the analysis method is very different from that of real-valued systems. Actually, CVNNs have more superior characteristics than real-valued neural networks in complex signal processing and can solve more complex problems. For example, a single real-valued neuron cannot settle the XOR issue and the unearthing symmetry issue, and the application of complex-valued neurons can solve these problems (see [7]). For CVNNs, the staple difficulty is to look for an opposite activation function. Judging by Liouville's theorem, CVNNs cannot select a smooth and bounded function as the activation function, so how to choose a suitable activation function for research also needs to be paid attention to the problem. With the wide application of the neural network, the development of neural network models that can handle complex-valued problems is needed in sundry fields, so the research on CVNNs is meaningful. At present, many researchers have studied the stability of CVNNs, and a large number of related results have been reported (see [8]).

Differential-algebraic systems depend on a combination of differential equations and algebraic equations. Compared with the differential equation system, the differential-algebraic system, as a mathematical model, is more perfect in practical application, such as economic system, chemical process system, and robot system, all of which belong to differential-algebraic systems. Over the past two decades, it has been evidenced that the differential-algebraic (different from the accepted differential geometry) methods could lead to a clearer comprehension of some control concepts and their interrelationships, such as observability, reversibility, decoupling and controller specification form, and observability, existence, and uniqueness of minimal implementations. In [9], the control method problem of a nonlinear differential-algebraic system is considered. In [10], the stability analysis of nonlinear differential-algebraic systems is solved using tools from classical control theory. In [11], a new type of differential-algebraic neural network with delay (DDANN) is proposed. And, one kind of novel mathematical expression combining differential equation and algebraic equation is designed in [12]. In addition to a large number of differential-algebraic systems directly proposed in various disciplines and engineering practices, the theoretical study of differential-algebraic systems also provides new ideas and methods for the study of certain problems in normal systems. For example, optimal control of normal systems, control of specified outputs, and singular perturbations can all be reduced to problems in the theory of differential-algebraic systems. Therefore, the theoretical study of differential-algebraic systems has a wide range of practical engineering background and profound theoretical value. It is important to note that, however, the analysis of differential-algebraic neural networks (DANNs) is still in its early stages.

Based on the above-mentioned argumentation, the target of this article is to investigate a new type of differential-algebraic complex-valued neural network (DACVNN) with delay (DDACVNN). An essential consequent on the global existence and uniqueness of the solution is first proposed. It also shows that under some simple assumptions, we can obtain the unique solution of the system under the compatible initial-value problem. We then derive the global exponential stability of this system. Studying a complex differential-algebraic system has always been a more challenging subject. Approximately stated, the novelties of this article include as follows:

(1) Apply CVNNs to DANN and establish a new type of DDACVNN. In particular, in the discussion in this article, there is no restriction on whether the real and imaginary parts of the activation function are required to be clearly separated.

(2) The sufficient conditions to ensure the existence and uniqueness of DDACVNN solutions are expressed, in terms of Picard existence and uniqueness theorem.

(3) An innovative approach is offered to discuss the stability of DDACVNN; that is, under the present system, global exponential stability (GES) is equivalent to global exponential self-synchronization (GESS), which we also rigorously prove in Lemma 1. Thus, a theorem of GES of DDACVNN is investigated.

The structure of the article is as follows. Section 2 provides the preliminaries, the model description about a DACVNN model, a necessary assumption, and a useful lemma. The main results are given in Section 3. Two illustrative examples are discussed in Section 4 to show the feasibility of the results. At last, some conclusions are dedicated in Section 5.

2. Preliminaries, Model Description, and Hypothesis

2.1. Notation. In the whole article, i signifies the imaginary unit, which is $i = \sqrt{-1}$. For complex number $z = x + iy$, the notation $|z| = \sqrt{x^2 + y^2}$ means the module of z. $\mathbb{R}$ and $\mathbb{R}^n$ stand, respectively, the set of real numbers and all n-dimensional real-valued vectors. $\mathbb{C}$, $\mathbb{C}^n$, and $\mathbb{C}^{n\times m}$ represent, respectively, the set of all complex numbers, the set of all n-dimensional complex-valued vectors, and the set of all $n\times m$ complex-valued matrices. For $z = (z_1, z_2, \ldots z_n)^T \in \mathbb{C}^n$, let $|z| = (|z_1|, |z_2|, \cdots |z_n|)^T$ and $\|z(t)\| = \sqrt{\sum_{k=1}^n |z_k(t)|^2}$. For $A \in \mathbb{C}^{n\times n}$ or $\in \mathbb{R}^{n\times n}$, $A = (a_{ij})_{n\times n}$, we define $|A| = (|a_{ij}|)$. The matrix $A = (a_{ij})_{n\times n}$ is called a nonnegative matrix if $a_{ij} \geq 0$ for all i and j. We use $r_k(A)$ to denote k th row sum of $A = (a_{ij})_{n\times n}$ and let $\|A\|_\infty$ to be infinity norm of a matrix A with $\|A\|_\infty = \max_{1\leq k\leq n} r_k(|A|)$. We use $\lambda_{\max}(A)$ to denote the maximum eigenvalue of a $A \in \mathbb{R}^{n\times n}$. For $A \in \mathbb{C}^{n\times n}$, $\|A\|$ denotes a matrix norm defined by $\|A\| = \sqrt{A^* A}$, where A^* shows the conjugate transpose of complex-valued matrix A.

2.2. Model Description. This section presents a DACVNN model, introduces some functional conceptions, and concludes with a basic hypothesis and two lemmas.

In the first place, a complex-valued singular neural network with time delay can be considered as

$$A\frac{dv(t)}{dt} = -Bv(t) + DF(v(t)) + EG(v(t-\tau)) + I, \tag{1}$$

where matrix $A \in \mathbb{R}^{n\times n}$ may be singular $(0 < \operatorname{rank}(A) = r < n)$; $\tau > 0$ is the time delay; $v(t) = (v_1(t), \ldots, v_n(t))^T \in \mathbb{C}^n$ stands for the complex-valued neuron state vector; $B = \operatorname{diag}\{b_1, \ldots b_n\}$ signifies state feedback coefficient matrix, which $b_k > 0\,(k = 1, 2, \ldots, n)$; $D \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{n\times n}$ correspond to, respectively, the complex-valued connection weight matrix and the complex-valued delayed connection weight matrix; $F(v)$ and $G(v)$ denote complex-valued neuron activation functions; $I \in \mathbb{C}^n$ infers an external complex-valued input signal.

More generally, let $A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. Next, system (1) can compile a DDACVNN indicated as follows:

$$\begin{cases} \dfrac{dz(t)}{dt} = -B_1 z(t) + D_{11}F_1(z(t)) + D_{12}F_2(w(t)) \\ +E_{11}G_1(z(t-\tau)) + E_{12}G_2(w(t-\tau)) + I_1, \\ 0 = -B_2 w(t) + D_{21}F_1(z(t)) + D_{22}F_2(w(t)) \\ +E_{21}G_1(z(t-\tau)) + E_{22}G_2(w(t-\tau)) + I_2, \end{cases} \tag{2}$$

where $\tau > 0$ is the time delay; $z(t) = (z_1(t), \ldots, z_r(t))^T \in \mathbb{C}^r$ and $w(t) = (w_1(t), \ldots, w_{n-r}(t))^T \in \mathbb{C}^{n-r}$; $B_1 = \operatorname{diag}\{b_{11}, \ldots b_{1r}\}$, $B_2 = \operatorname{diag}\{b_{21}, \ldots b_{2r}\}$, b_{ij} are the positive constants; $D_{11} \in \mathbb{C}^{r\times r}$, $D_{12} \in \mathbb{C}^{r\times(n-r)}$, $D_{21} \in \mathbb{C}^{(n-r)\times r}$, and $D_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$; $E_{11} \in \mathbb{C}^{r\times r}$, $E_{12} \in \mathbb{C}^{r\times(n-r)}$, $E_{21} \in \mathbb{C}^{(n-r)\times r}$, and $E_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$; $F_1(z) = (F_{11}(z_1), \ldots, F_{1r}(z_r))^T$, and $F_2(w) = (F_{21}(w_1), \ldots, F_{2(n-r)}(w_{n-r}))^T$; $G_1(z) = (G_{11}(z_1), \ldots, G_{1r}(z_r))^T$, and $G_2(w) = (G_{21}(w_1), \ldots, G_{2(n-r)}(w_{n-r}))^T$; $I_1 = (I_{11}, \ldots, I_{1r})^T$ and $I_2 = (I_{21}, \ldots, I_{2(n-r)})^T$.

We define a Banach space $C_\tau^r = C([-\tau, 0], \mathbb{C}^r)$, which consists of all continuous complex-valued functions $\eta: [-\tau, 0] \longrightarrow \mathbb{C}^r$. $z_t \in C_\tau^r$ denotes $z_t(s) = z(t+s)$ with regard to $s \in [-\tau, 0]$. Towards $\gamma \in \mathbb{C}^r$, we donate $\gamma \in C_\tau^r$, which signifies $\gamma(s) \equiv \gamma$ in $[-\tau, 0]$. Let $\|\eta(s)\|_\tau = \sqrt{\sum_{k=1}^n |\eta_k(s)|_\tau^2}$ for $\eta \in C_\tau^r$, where $|\eta_k(s)|_\tau = \sup_{s\in[-\tau,0]} |\eta_k(s)|$.

Furthermore, we proceed to consider the initial conditions of DDACVNN (2) listed as follows:

$$z_{t_0} = \eta \in C_\tau^r, w_{t_0} = \theta \in C_\tau^{n-r}. \tag{3}$$

The initial conditions (3) of DDACVNN (2) are known as compatible; the necessary and sufficient condition is

$$\begin{aligned} 0 = &-B_2\theta(0) + D_{21}F_1(\eta(0)) + D_{22}F_2(\theta(0)) \\ &+ E_{21}G_1(\eta(-\tau)) + E_{22}G_2(\theta(-\tau)) + I_2, \end{aligned} \tag{4}$$

where $(\eta^T, \theta^T)^T \in C_\tau^r \times C_\tau^{n-r}$ is known as compatible initial value (CIV).

About CVNNs, the foremost difficulty is to look for an opposite activation function. Judging by Liouville's theorem, CVNNs cannot choose a smooth and bounded function as the activation function, so how to choose a suitable activation function for research also needs to pay attention to the problem. Therefore, in order for the following to proceed smoothly, we need to make the following hypothesis:

Hypothesis 1. $F_{1i}(z)$, $F_{2j}(w)$, $G_{1i}(z)$, and $G_{2j}(w)$ $(i = 1, \ldots, r;\ j = 1, \ldots, (n-r))$ satisfy the Lipschitz continuity condition in the complex domain, and there exist positive diagonal matrices $\widehat{F}_1 = \operatorname{diag}\{\widehat{F}_{11}, \ldots, \widehat{F}_{1r}\}$, $\widehat{F}_2 = \operatorname{diag}\{\widehat{F}_{21}, \ldots, \widehat{F}_{2n-r}\}$, $\widehat{G}_1 = \operatorname{diag}\{\widehat{G}_{11}, \ldots, \widehat{G}_{1r}\}$, and $\widehat{G}_2 = \operatorname{diag}\{\widehat{G}_{21}, \ldots, \widehat{G}_{2n-r}\}$ such that

$$\begin{cases} \left|F_{1i}(z) - F_{1i}(\overline{z})\right| \le \widehat{F}_{1i}|z - \overline{z}|, \\ \left|G_{1i}(z) - G_{1i}(\overline{z})\right| \le \widehat{G}_{1i}|z - \overline{z}|, \end{cases} \tag{5}$$

and

$$\begin{cases} \left|F_{2j}(w) - F_{2j}(\overline{w})\right| \le \widehat{F}_{2j}|w - \overline{w}|, \\ \left|G_{2j}(w) - G_{2j}(\overline{w})\right| \le \widehat{G}_{2j}|w - \overline{w}|, \end{cases} \tag{6}$$

for any $z, \overline{z} \in \mathbb{C}^r$ and $w, \overline{w} \in \mathbb{C}^{n-r}$.

Remark 1. In the discussion of the article, we merely require that the complex-valued activation functions globally Lipschitz.

It is worth noting that, in [13, 14], the activation function is required to be manifested by severing the real and imaginary parts as

$$f_j(z_j) = f_j^{\,R}(x_j, y_j) + i f_j^{\,I}(x_j, y_j),\ j = 1, \ldots, n, \tag{7}$$

where $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $f_j^{\,R}(x_j, y_j)$ (or $f_j^{\,R}$) and $f_j^{\,I}(x_j, y_j)$ (or $f_j^{\,I}$) demonstrate, respectively, the real and imaginary parts of $f_j(z_j)$ $(j = 1, \ldots, n)$. Hypothesis 1 indicates that the model discussed in this article is suitable for both the case where the activation function can separate the real and imaginary parts, and the case where it cannot be clearly separated. For instance, $f_j(z_j) = 1 - e^{-z_j}/1 + e^{-z_j}$, $j = 1, \ldots, n$.

Finally, an important lemma is offered, which will be applied in the following sections.

Lemma 1. *Let matrix $A = (a_{ij})_{n\times n}$ have nonpositive off-diagonal elements (i.e., $a_{ij} \le 0, i \ne j$); then, each of the following conditions is equivalent to the statement that A is a nonsingular M matrix.*

(1) All the leading principal minors of A are positive

(2) There are n positive constants $\xi_1, \xi_2, \ldots, \xi_n$ such that

$$\xi_i a_{ii} - \sum_{j=1, j\ne i}^{n} \xi_j \left|a_{ji}\right| > 0, 1 \le i \le n. \tag{8}$$

(3) There are n positive constants $\zeta_1, \zeta_2, \ldots, \zeta_n$ such that

$$\zeta_i a_{ii} - \sum_{j=1, j\ne i}^{n} \zeta_j \left|a_{ij}\right| > 0, 1 \le i \le n. \tag{9}$$

(4) The diagonal elements of A are all positive and there exists a positive vector θ such that $A\theta > 0$ or $A^T\theta > 0$.

3. Main Results

3.1. The Issue of Existence and Uniqueness of Solutions. The purpose of this subsection is to discuss the existence and uniqueness of the solution with the CIV problem (2) and (3).

First, we discuss the following DACVNN:

$$\begin{cases} \dfrac{dz(t)}{dt} = -B_1 z(t) + D_{11}F_1(z(t)) + D_{12}F_2(w(t)) + l_1(t), \\ 0 = -B_2 w(t) + D_{21}F_1(z(t)) + D_{22}F_2(w(t)) + l_2(t), \end{cases} \tag{10}$$

where the functions l_1: $[t_0, +\infty) \longrightarrow \mathbb{C}^r$ and l_2: $[t_0, +\infty) \longrightarrow \mathbb{C}^{n-r}$ are the continuous and differentiable functions.

We consider the initial conditions of DACVNN (10) as follows:

$$z(t_0) = z_0 \in \mathbb{C}^r, w(t_0) = w_0 \in \mathbb{C}^{n-r}. \quad (11)$$

The initial conditions (11) are considered to be compatible; the necessary and sufficient condition is

$$0 = -B_2 w_0 + D_{21} F_1(z_0) + D_{22} F_2(w_0) + l_2(t_0). \quad (12)$$

Theorem 1. *We assume that Hypothesis 1 is satisfied, and $B_2 - |D_{22}|\widehat{F}_2$ is a M matrix. Afterward, for every CIV condition (11) and (12), there is a unique solution $(z^T(t), w^T(t))^T$ on interval $[t_0, t_+)$ of DACVNN (10), where $t_0 < t_+ \leq +\infty$.*

Proof. First, we define a function g: $\mathbb{R} \times \mathbb{C}^n \longrightarrow \mathbb{C}^m$ such that

$$0 = -B_2 g(t, z) + D_{21} F_1(z) + D_{22} F_2(g(t, z)) + l_2(t), \quad (13)$$

for any $t \geq t_0$ and $z \in \mathbb{C}^r$.

In order to achieve the above effect, without loss of generality, we only need to show that, for every $\bar{t} \geq t_0$ and $\bar{z} \in \mathbb{C}^r$, there exists a $\bar{z} \in \mathbb{C}^{n-r}$, of (13) such that

$$0 = -B_2 \overline{w} + D_{21} F_1(\bar{z}) + D_{22} F_2(\overline{w}) + l_2(\bar{t}). \quad (14)$$

Let $D_{ij} = (d_{ijkl})$. On the basis of Lemma 1, there exist $\xi_s > 0\,(s = 1, 2, \ldots, n-r)$ such that

$$\xi_k B_{2k} - \sum_{j=1}^{n-r} d_{22kj} \xi_j \widehat{F}_{2j} > 0, k = 1, \ldots, n-r, \quad (15)$$

as $B_2 - |D_{22}|\widehat{F}_2$ is a M matrix. Thus, we can get

$$r_{\max}\left(\Xi^{-1} B_2^{-1} \left|D_{22}\right| \widehat{F}_2 \Xi\right) < 1, \quad (16)$$

where $\Xi = \operatorname{diag}\{\xi_1, \xi_2, \ldots, \xi_{n-r}\}$.

We define an operator

$$T(\alpha) = \Xi^{-1} B_2^{-1} \left[D_{21} F_1(\bar{z}) + D_{22} F_2(\Xi\alpha) + l_2(\bar{t})\right], \quad (17)$$

where $\alpha \in \mathbb{C}^{n-r}$.

According to the definition of the operator $T(\alpha)$, we can get

$$|T(\beta) - T(\alpha)| \leq \Xi^{-1} B_2^{-1} \left|D_{22}\right| \widehat{F}_2 \Xi |\beta - \alpha|, \quad (18)$$

regarding whichever $\beta, \alpha \in \mathbb{C}^{n-r}$. By calculating the infinity norm in (18), we can get

$$\|T(\beta) - T(\alpha)\|_\infty \leq \nu \|\beta - \alpha\|_\infty, \quad (19)$$

where

$$\nu = r_{\max}\left(\Xi^{-1} B_2^{-1} \left|D_{22}\right| \widehat{F}_2 \Xi\right) < 1. \quad (20)$$

Through (19), we can readily find that T is a contraction operator on $\mathbb{C}^m$. Based on the contraction mapping principle, there exists a unique $\alpha^* \in \mathbb{C}^{n-r}$ so that $T(\alpha^*) = \alpha^*$, which is

$$B_2 \Xi \alpha^* = D_{21} F_1(\bar{z}) + D_{22} F_2(\Xi\alpha^*) + l_2(\bar{t}). \quad (21)$$

Let $\overline{w} = \Xi\alpha^*$, thus (14) satisfies. Therefore, there is a function $g(t, z)$ such that (14) satisfies.

According to (14) and Hypothesis 1, one has

$$\|g(t, \beta) - g(t, \alpha)\| \leq \left\| \left[B_2 - \left|D_{22}\right| \widehat{F}_2\right]^{-1} \left|D_{21}\right| \widehat{F}_1 \right\| \|\beta - \alpha\|, \quad (22)$$

which shows that the function $g(t, \beta)$ is continuous in $\mathbb{R} \times \mathbb{C}^{n-r}$.

Furthermore, from (14) and DACVNN (10), we see that

$$\frac{dz(t)}{dt} = -B_1 z(t) + D_{11} F_1(z(t)) + D_{12} F_2(g(t, z(t))) + l_1(t). \quad (23)$$

Picard existence and uniqueness theorem ensures existence and uniqueness of the solution $z(t)$ of system (23) considered on interval $[t_0, t_+)$ for any initial value $z_0 \in \mathbb{C}^n$. Furthermore, let $w(t) = g(t, z(t))$, and so there exists a unique solution $(z^T(t), w^T(t))^T$ on interval $[t_0, t_+)$ of the DACVNN (10), with the CIV $(z_0^T, g^T(t_0, z_0))^T$.

Furthermore, we consider the extension of t from $[t_0, t_+)$ to $[t_0, +\infty)$. □

Theorem 2. *We presume that Hypothesis 1 is satisfied, if $B_2 - |D_{22}|\widehat{F}_2$ is a M matrix; afterward, for every CIV $(\eta^T, \theta^T)^T$, DDACVNN (2) has a unique solution $(z^T(t), w^T(t))^T$ on $[t_0, +\infty)$.*

Proof. For $t \in [t_0, t_0 + \tau]$, DDACVNN (2) with CIV $(\eta^T, \theta^T)^T$ can be restated as DACVNN (10), in which

$$\begin{cases} l_1(t) = E_{11} G_1(\eta(t-\tau)) + E_{12} G_2(\theta(t-\tau)) + I_1, \\ l_2(t) = E_{21} G_1(\eta(t-\tau)) + E_{22} G_2(\theta(t-\tau)) + I_2. \end{cases} \quad (24)$$

Through Theorem 1, we can know that DDACVNN (2), which is DACVNN (10), has a unique solution on $[t_0, t_0 + \tau]$. Progressively, we are able to prove that there exists a unique solution on $[t_0, t_0 + k\tau]$ for any $k \geq 1$ of DDACVNN (2).

The existence and uniqueness of the solutions of complex-valued differential-algebraic systems is one of the most basic problems, and it is the premise and basis for discussing the system. Therefore, it is very important and necessary to study the existence and uniqueness of the solution of this system. For these reasons, this article considers the fundamental concepts of the solution of complex-valued differential-algebraic systems via Picard's theorem. □

3.2. Global Exponential Stability. We will consider the globally exponential stability (GES) and globally exponential self-synchronization (GESS) of DDACVNN (2) in this section.

The $(z^{*T}, w^{*T})^T \in \mathbb{C}^r \times \mathbb{C}^{n-r}$ is an equilibrium point of DDACVNN (2) if and only if

$$\begin{cases} 0 = -B_1 z^* + D_{11} F_1(z^*) + D_{12} F_2(w^*) \\ +E_{11} G_1(z^*) + E_{12} G_2(w^*) + I_1, \\ 0 = -B_2 w^* + D_{21} F_1(z^*) + D_{22} F_2(w^*) \\ +E_{21} G_1(z^*) + E_{22} G_2(w^*) + I_2. \end{cases} \tag{25}$$

Definition 1. If regarding any $t \geq t_0$, and regarding any solution $(z^T(t), w^T(t))^T$ of DDACVNN (2) corresponding to the CIV $(\eta^T, \theta^T)^T$, there exists an equilibrium point $(z^{*T}, w^{*T})^T \in \mathbb{C}^r \times \mathbb{C}^{n-r}$ satisfying

$$\|z(t) - z^*\| + \|w(t) - w^*\| \leq K\left[\|\eta - z^*\| + \|\theta - w^*\|\right] e^{-\varrho(t-t_0)}, \tag{26}$$

where $K \geq 1$ and $\varrho > 0$ are constants, then DDACVNN (2) is called GES.

Definition 2. If for any $t \geq t_0$, and for any solutions $(z^T(t), w^T(t))^T$ and $(\overline{z}^T(t), \overline{w}^T(t))^T$ of the DDACVNN (2) corresponding to the two CIVs $(\eta^T, \theta^T)^T$ and $(\overline{\eta}^T, \overline{\theta}^T)^T$, there are two constants $K \geq 1$ and $\varrho > 0$ satisfying

$$\|z(t) - \overline{z}(t)\| + \|w(t) - \overline{w}(t)\| \leq K[\|\eta - \overline{\eta}\| + \|\theta - \overline{\theta}\|] e^{-\varrho(t-t_0)}, \tag{27}$$

then DDACVNN (2) is called GESS.

Remark 2. Generally speaking, exponential stability and exponential self-synchronization are two completely different concepts. Evidently, exponential stability implies exponential self-synchronization but not vice versa. However, under DDACVNN (2), we have the following claim: GESS can deduce GES.

Lemma 2. *GESS is equivalent to GES in DDACVNN (2).*

Proof. We only need to show that GES can be deduced by GESS. Firstly, let $\Phi = (\eta^T, \theta^T)^T$, $u_t(\Phi) = (z^T(t), w^T(t))^T$ signify a solution of DDACVNN (2) with CIV $u_{t_0} = \Phi \in C_\tau^n$.

Suppose DDACVNN (2) is GESS, namely,

$$\|u_t(\Phi) - u_t(\Psi)\| \leq K\|\Phi - \Psi\| e^{-\varrho(t-t_0)}, \tag{28}$$

for any $\Phi, \Psi \in C_\tau^n$ and $t \geq t_0$, where $K \geq 1$ and $\varrho > 0$ are constants.

We define an operator Γ: $C_\tau^n \longrightarrow C_\tau^n$ with $\Gamma(\Phi) = u_s(\Phi)$, where $s \geq t_0$ so that $\sigma = Ke^{-\varrho(s-t_0)} < 1$. By (14), one has

$$\|\Gamma(\Phi) - \Gamma(\Psi)\| \leq \sigma\|\Phi - \Psi\|, \tag{29}$$

so we can get that Γ is a contraction. Then, through the Banach contraction principle, it can be obtained that Γ has a unique fixed point $\Phi^* \in C_\tau^n$, which is the equilibrium point of DDACVNN (2).

After giving the above lemma, we next introduce the following two lemmas. The introduction of these two lemmas plays a key role in the testimony of the next theorem. □

Lemma 3 (see [15]). *Let* x: $[t_0 - \tau, +\infty) \longrightarrow [0, +\infty)$ *and* J: $[t_0, +\infty) \longrightarrow [0, +\infty)$ *satisfying*

$$\frac{dx(t)}{dt} \leq -\gamma x(t) + \delta x(t-\tau) + J(t), \tag{30}$$

for $t \geq t_0$, *where* $\gamma, \delta, \tau > 0$, *then*

$$\tilde{x}(t) \leq \left(\tilde{x}(t_0) + \int_{t_0}^{t} e^{\alpha(s-t_0)} J(s) ds\right) e^{\alpha(t-t_0)}, \tag{31}$$

for $t \geq t_0$, *where* α *is a constant sufficing* $\alpha - \gamma_1 + \gamma_2 e^{\alpha\tau} \leq 0$ *and* $\tilde{x}(t) = sup_{\theta\in[-\tau,0]} x(t+\theta)$.

Lemma 4 (see [11]). *Let* x, y: $[t_0 - \tau, +\infty) \longrightarrow [0, +\infty)$. $x(t)$ *and* $y(t)$ *are, respectively, continuously differentiable and continuous functions, sufficing*

$$\begin{cases} \dfrac{dx(t)}{dt} & \leq -\gamma_1 x(t) + \gamma_2 x(t-\tau) + \alpha_1 y(t-\tau), \\ y(t) & \leq \delta_1 x(t) + \delta_2 x(t-\tau) + \alpha_2 y(t-\tau). \end{cases} \tag{32}$$

For $t \geq t_0$, where there are positive real numbers $\gamma_1, \gamma_2, \delta_1, \delta_2, \alpha_1$, and α_2. And $x(t) \longrightarrow 0$ and $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$ if $\lambda_{\max}(\Lambda) < 1$, where

$$\Lambda = \begin{pmatrix} e^{-\beta\tau} & \dfrac{\alpha_1}{\beta} \\ \delta_1 e^{-\beta\tau} + \delta_2 & \dfrac{\delta_1\alpha_1 + \beta\alpha_2}{\beta} \end{pmatrix}, \tag{33}$$

where $\beta > 0$ is a constant satisfying $\beta - \gamma_1 + \gamma_2 e^{\beta\tau} \leq 0$, $\tilde{x}(t) = \sup_{\theta\in[-\tau,0]} x(t+\theta)$, and $\tilde{y}(t) = \sup_{\theta\in[-\tau,0]} y(t+\theta)$. In addition, if there exists $\varphi > 0$ satisfying

$$\begin{cases} e^{-\beta\tau} + \varphi\left(\delta_1 e^{-\beta\tau} + \delta_2\right) = \sigma_1 < 1, \\ \dfrac{\alpha_1}{\beta\varphi} + \dfrac{\delta_1\alpha_1 + \beta\alpha_2}{\beta} = \sigma_2 < 1, \end{cases} \tag{34}$$

then

$$x(t) + y(t) \leq K\sigma_{\max}(\tilde{x}(t_0) + \tilde{y}(t_0)) e^{-\varrho(t-t_0)}, \tag{35}$$

for $t \geq t_0$, where $K = \max\{1, \varphi\}/\min\{1, \varphi\}$, $\sigma_{\max} = \max\{\sigma_1, \sigma_2\}$, and $\varrho = -1/\tau \ln\sigma_{\max}$.

Theorem 3. *We assume that hypothesis 1 holds and $B_2 - |D_{22}|\widehat{F}_2$ is a M matrix. Let*

$$\begin{aligned}
\delta_1 &= \max_{1\le l\le m} \frac{r_l\left(|D_{21}|\widehat{F}_1\right)}{r_l\left(B_2 - |D_{22}|\widehat{F}_2\right)}, \\
\delta_2 &= \max_{1\le l\le m} \frac{r_l\left(|E_{21}|\widehat{G}_1\right)}{r_l\left(B_2 - |D_{22}|\widehat{F}_2\right)}, \\
\alpha_2 &= \max_{1\le l\le m} \frac{r_l\left(|E_{22}|\widehat{G}_2\right)}{r_l\left(B_2 - |D_{22}|\widehat{F}_2\right)}, \\
\gamma_1 &= r_{\min}\left(B_1 - |D_{11}|\widehat{F}_1\right) - r_{\max}\left(|D_{12}|\widehat{F}_2\right)\delta_1, \\
\gamma_2 &= r_{\max}\left(|E_{11}|\widehat{G}_1\right) + r_{\max}\left(|D_{12}|\widehat{F}_2\right)\delta_2, \\
\alpha_1 &= r_{\max}\left(|E_{12}|\widehat{G}_2\right) + r_{\max}\left(|D_{12}|\widehat{F}_2\right)\alpha_2,
\end{aligned} \tag{36}$$

and

$$\Lambda = \begin{pmatrix} e^{-\beta\tau} & \dfrac{\alpha_1}{\beta} \\ \delta_1 e^{-\beta\tau} + \delta_2 & \dfrac{\delta_1\alpha_1 + \beta\alpha_2}{\beta} \end{pmatrix}, \tag{37}$$

where $\beta > 0$ is a constant sufficing $\beta - \gamma_1 + \gamma_2 e^{\beta\tau} \le 0$. Then, if $\lambda_{\max}(\Lambda) < 1$, DDACVNN (2) is GES.

Proof. On the basis of Lemma 1, we only need to certify that DDACVNN (2) is able to achieve GESS.

For any two different solutions $(z^T(t), w^T(t))^T$ and $(\overline{z}^T(t), \overline{w}^T(t))^T$ of DDACVNN (2), let $Z(t) = z(t) - \overline{z}(t)$ and $W(t) = w(t) - \overline{w}(t)$. In line with DDACVNN (2), we can obtain that

$$\frac{d}{dt}\|Z(t)\| = \frac{d}{dt}\|z(t) - \overline{z}(t)\| = \frac{d}{dt}\sqrt{\sum_{k=1}^{n}\left|z_k(t) - \overline{z}_k(t)\right|^2}, \tag{38}$$

and

$$B_2\|W(t)\| = B_2\|w(t) - \overline{w}(t)\| = B_{2k}\sqrt{\sum_{k=1}^{m}\left|w_k(t) - \overline{w}_k(t)\right|^2}. \tag{39}$$

Let $|z_i(t) - \overline{z}_i(t)| = \max_{1\le k\le n}\{|z_k(t) - \overline{z}_k(t)|\}$ and $|w_l(t) - \overline{w}_l(t)| = \max_{1\le k\le n}\{|w_k(t) - \overline{w}_k(t)|\}$, then

$$\begin{aligned}
\frac{d}{dt}\|Z(t)\| &\le \sqrt{n}\,\frac{d}{dt}\left|z_i(t) - \overline{z}_i(t)\right| \\
&\le -\sqrt{n}\Bigg\{B_{1i}\left|Z_i(t)\right| + \sum_{j=1}^{m}\left|d_{11ij}\right|\left|F_{1j}\left(z_j(t)\right) - F_{1j}\left(\overline{z}_j(t)\right)\right| \\
&\quad + \sum_{j=1}^{m}\left|d_{12ij}\right|\left|F_{2j}\left(w_j(t)\right) - F_{2j}\left(\overline{w}_j(t)\right)\right| \\
&\quad + \sum_{j=1}^{n}\left|e_{11ij}\right|\left|G_{1j}\left(z_j(t-\tau)\right) - G_{1j}\left(\overline{z}_j(t-\tau)\right)\right|\Bigg\} \\
&\quad + \sqrt{n}\Bigg\{\sum_{j=1}^{m}\left|e_{12ij}\right|\left|G_{2j}\left(w_j(t-\tau)\right) - G_{2j}\left(\overline{w}_j(t-\tau)\right)\right| \\
&\le -B_{1i}\left|Z_i(t)\right| + \sum_{j=1}^{n}\left|d_{11ij}\right|\widehat{F}_{1j}\left|Z_j(t)\right| \\
&\quad + \sum_{j=1}^{m}\left|d_{12ij}\right|\widehat{F}_{2j}\left|W_j(t)\right| + \sum_{j=1}^{n}\left|e_{11ij}\right|\widehat{G}_{1j}\left|Z_j(t-\tau)\right| \\
&\quad + \sum_{j=1}^{m}\left|e_{12ij}\right|\widehat{G}_{2j}\left|W_j(t-\tau)\right|\Bigg\},
\end{aligned} \tag{40}$$

and

$$
\begin{aligned}
B_2\|W(t)\| \le \sqrt{m}B_{2l}\left|W_l(t)\right| \le \sqrt{m}\Bigg\{&\sum_{j=1}^{n}\left|d_{21lj}\right|\widehat{F}_{1j}\left|Z_j(t)\right| \\
&+\sum_{j=1}^{m}\left|d_{22lj}\right|\widehat{F}_{2j}\left|W_j(t)\right| + \sum_{j=1}^{n}\left|e_{21lj}\right|\widehat{G}_{1j}\left|Z_j(t-\tau)\right| \\
&+\sum_{j=1}^{m}\left|e_{22lj}\right|\widehat{G}_{2j}\left|W_j(t-\tau)\right|\Bigg\}.
\end{aligned} \tag{41}
$$

Let $x(t) = |z_i(t) - \overline{z}_i(t)|$ and $y(t) = |w_l(t) - \overline{w}_l(t)|$. Thus, one has

$$
\begin{aligned}
\frac{d}{dt}x(t) \le& \{-B_{1i} + \sum_{j=1}^{n}\left|d_{11ij}\right|\widehat{F}_{1j}\}x(t) \\
&+\sum_{j=1}^{m}\left|d_{12ij}\right|\widehat{F}_{2j}y(t) + \sum_{j=1}^{n}\left|e_{11ij}\right|\widehat{G}_{1j}x(t-\tau) \\
&+\sum_{j=1}^{m}\left|e_{12ij}\right|\widehat{G}_{2j}y(t-\tau) \\
\le& -r_{\min}\left(B_1 - \left|D_{11}\right|\widehat{F}_1\right)x(t) + r_{\max}\left(\left|D_{12}\right|\widehat{F}_2\right)y(t) \\
&+ r_{\max}\left(\left|E_{11}\right|\widehat{G}_1\right)x(t-\tau) + r_{\max}\left(\left|E_{12}\right|\widehat{G}_2\right)y(t-\tau),
\end{aligned} \tag{42}
$$

and

$$
\begin{aligned}
y(t) \le& \frac{1}{b_{2l} - \sum_{j=1}^{m}\left|d_{22lj}\right|\widehat{F}_{2j}}\Bigg\{\sum_{j=1}^{n}\left|d_{21lj}\right|\widehat{F}_{1j}x(t) \\
&+\sum_{j=1}^{n}\left|e_{21lj}\right|\widehat{G}_{1j}x(t-\tau) \\
&+\sum_{j=1}^{m}\left|e_{22lj}\right|\widehat{G}_{2j}y(t-\tau)\Bigg\} \\
\le& \frac{r_l\left(\left|D_{21}\right|\widehat{F}_1\right)}{r_l\left(B_2 - \left|D_{22}\right|\widehat{F}_2\right)}x(t) \\
&+\frac{r_l\left(\left|E_{21}\right|\widehat{G}_1\right)}{r_l\left(B_2 - \left|D_{22}\right|\widehat{F}_2\right)}x(t-\tau) \\
&+\frac{r_l\left(\left|E_{22}\right|\widehat{G}_2\right)}{r_l\left(B_2 - \left|D_{22}\right|\widehat{F}_2\right)}y(t-\tau) \\
\le& \delta_1 x(t) + \delta_2 x(t-\tau) + \alpha_2 y(t-\tau).
\end{aligned} \tag{43}
$$

By (42) and (43), it follows that

$$
\begin{cases}
\dfrac{dx(t)}{dt} & \le -\gamma_1 x(t) + \gamma_2 x(t-\tau) + \alpha_1 y(t-\tau), \\
y(t) & \le \delta_1 x(t) + \delta_2 x(t-\tau) + \alpha_2 y(t-\tau).
\end{cases} \tag{44}
$$

According to Lemma 1 for the reason that $I_m - \Lambda$ is a M matrix since $\lambda_{\max}(\Lambda) < 1$. According to Lemma 2, we can deduce that there exists $\varphi > 0$ so that

$$
\begin{cases}
e^{-\beta\tau} + \varphi\left(\delta_1 e^{-\beta\tau} + \delta_2\right) = \sigma_1 < 1, \\
\dfrac{\alpha_1}{\beta\varphi} + \dfrac{\delta_1\alpha_1 + \beta\alpha_2}{\beta} = \sigma_2 < 1.
\end{cases} \tag{45}
$$

According to Lemma 2, and by (44) and (45), one has

$$
x(t) + y(t) \le K\sigma_{\max}\tilde{x}(t_0) + \tilde{y}(t_0))e^{-\varrho(t-t_0)}, \tag{46}
$$

for $t \ge t_0$, where $K = \max\{1,\varphi\}/\min\{1,\varphi\}, \sigma_{\max} = \max\{\sigma_1,\sigma_2\}$, and $\varrho = -(1/\tau)\ln\sigma_{\max}$. From (46), one has

$$
\|z(t) - \overline{z}(t)\| + \|w(t) - \overline{w}(t)\| \le K'[\|\eta - \overline{\eta}\| + \|\theta - \overline{\theta}\|]e^{-\varrho(t-t_0)}, \tag{47}
$$

where $K' = K\max\{\sqrt{n},\sqrt{m}\}$, $\|\eta - \overline{\eta}\| \ge \tilde{x}(t_0)$ and $\|\theta - \overline{\theta}\| \ge \tilde{y}(t_0)$, which signifies that DDACVNN (2) is GESS. In accordance with Lemma 1, DDACVNN (2) is GES. □

Remark 3. In this article, we innovatively combine complex values with DANN and do not study complex-valued in real-valued methods (i.e., divide complex values into real and imaginary parts). And the requirements for the activation function are also reduced a lot (only the condition of global Lipschitz needs to be satisfied), so many restrictions on the activation function are eliminated, making the results of this article more effective.

Remark 4. Above, we propose a new paradigm and skill to discuss the stability of DDACVNN (2). It is strictly proved that, in Lemma 1, the DDACVNN (2) is GES if it can achieve GESS, which is able to reduce the argument of exponential stability in DDACVNN (2). Using a differential-algebraic inequality introduced, the GESS problem is studied, and the GES theorem of DDACVNN is given. Finally, in Theorem 1, it is successfully studied whether DDACVNN (2) can achieve GESS and thus achieve GES.

Remark 5. On the basis of this article, we can also discuss the case of time-varying delays. In fact, in the case where the time delays is time-varying delays, the results of this article still hold. We will continue to consider this issue in future research.

Remark 6. In addition, we can also consider nonlinear differential-algebraic systems. Objectively speaking, there are many essential differences between nonlinear differential-algebraic systems and ordinary differential equations and linear differential-algebraic systems. The most fundamental difference is that of nonzero exponents. In general, in an exponential sense, ordinary differential equations can be thought of as zero-exponential systems, a special case of differential-algebraic systems. The differential-algebraic system we mentioned, generally, refers to the nonzero exponential system, and the transition from the zero-exponential system to the nonzero exponential system is a leap, just like the transition from finite-dimensional space to infinite-dimensional space in mathematical analysis. Therefore, it will inevitably lead to qualitative changes in many issues. All the above have led to the very slow development of the research on nonlinear differential-algebraic systems.

4. Examples

At last, two numerical examples are proposed to demonstrate the above achievements in this section. Respectively, the activation function can separate the real and imaginary parts, and the other is that they cannot be clearly separated.

Example 1. We consider DDACVNN (2), taking $r = 1$ and $n = 3$. We allow $\tau = 0.1, I_1 = 1 + i$ and $I_2 = (0, 0)^T$; $F_{1k}(\theta) = G_{1k}(\theta) = F_{2j}(\theta) = G_{2j}(\theta) = 0.5x + i0.5y$ where $\theta = x + iy$, for $k = 1$ and $j = 1, 2$,

$$\begin{aligned} &D_{11} = (1+i)_{1\times1}, E_{11} = (i)_{1\times1}, D_{12} = (-i,0)_{1\times2}, E_{12} = (0, 2+i)_{1\times2},\\ &D_{21} = \begin{pmatrix} 1-i \\ 0 \end{pmatrix}_{2\times1}, E_{21} = \begin{pmatrix} 1-i \\ 2-i \end{pmatrix}_{2\times1},\\ &D_{22} = \begin{pmatrix} 1-3i & 2+i \\ -1+i & 3+i \end{pmatrix}_{2\times2}, E_{22} = \begin{pmatrix} 1+2i & 0 \\ -1+i & 3-i \end{pmatrix}_{2\times2}. \end{aligned} \tag{48}$$

$B_1 = (10\sqrt{2})$ and $B_2 = \operatorname{diag}\{8\sqrt{10}, 8\sqrt{10}\}$.

We obviously see that Hypothesis 1 holds and $\widehat{F}_1 = \widehat{G}_1 = (0.5)$ and $\widehat{F}_2 = \widehat{G}_2 = \operatorname{diag}\{0.5, 0.5\}$. By calculation, we have

$$\begin{aligned} &|D_{11}| = (\sqrt{2}), |E_{11}| = (1), |D_{12}| = (1,0), |E_{12}| = (0,\sqrt{5}),\\ &|D_{21}| = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, |E_{21}| = \begin{pmatrix} \sqrt{2} \\ \sqrt{5} \end{pmatrix}, |D_{22}| = \begin{pmatrix} \sqrt{10} & \sqrt{5} \\ \sqrt{2} & \sqrt{10} \end{pmatrix}, |E_{22}| = \begin{pmatrix} \sqrt{5} & 0 \\ \sqrt{2} & \sqrt{10} \end{pmatrix}. \end{aligned} \tag{49}$$

And

$$B_2 - |D_{22}|\widehat{F}_2 = \begin{pmatrix} \dfrac{15\sqrt{10}}{2} & -\sqrt{5} \\ -\sqrt{2} & \dfrac{15\sqrt{10}}{2} \end{pmatrix}, \tag{50}$$

which is a M matrix. On the basis of Theorem 1, we can conclude that for any CIV $(\eta^T, \theta^T)^T \in C_\tau^1 \times C_\tau^2$, DDACVNN (2) has a unique solution.

Moreover, we can continue to calculate the following:

$$\begin{aligned} \delta_1 &= \max_{1\le l\le m} \frac{r_l(|D_{21}|\widehat{F}_1)}{r_l(B_2 - |D_{22}|\widehat{F}_2)} = 0.0317,\\ \delta_2 &= \max_{1\le l\le m} \frac{r_l(|E_{21}|\widehat{G}_1)}{r_l(B_2 - |D_{22}|\widehat{F}_2)} = 0.0501,\\ \alpha_2 &= \max_{1\le l\le m} \frac{r_l(|E_{22}|\widehat{G}_2)}{r_l(B_2 - |D_{22}|\widehat{F}_2)} = 0.1343,\\ \gamma_1 &= r_{\min}(B_1 - |D_{11}|\widehat{F}_1) - r_{\max}(|D_{12}|\widehat{F}_2)\delta_1 = 13.4192,\\ \gamma_2 &= r_{\max}(|E_{11}|\widehat{G}_1) + r_{\max}(|D_{12}|\widehat{F}_2)\delta_2 = 0.5159,\\ \alpha_1 &= r_{\max}(|E_{12}|\widehat{G}_2) + r_{\max}(|D_{12}|\widehat{F}_2)\alpha_2 = 1.6852, \end{aligned} \tag{51}$$

and

$$\Lambda=\begin{pmatrix} e^{-\beta\tau} & \dfrac{\alpha_1}{\beta} \\ \delta_1 e^{-\beta\tau}+\delta_2 & \dfrac{\delta_1\alpha_1+\beta\alpha_2}{\beta}\end{pmatrix}=\begin{pmatrix}0.3263 & 0.1505\\ 0.0604 & 0.1391\end{pmatrix}, \tag{52}$$

where $\beta=11.2>0$ such that $\beta-\gamma_1+\gamma_2 e^{\beta\tau}=-0.6380<0$; then, $\lambda(\Lambda)=0.3663, 0.0991, \lambda_{\max}(\Lambda)=0.3663<1$.

In the example, all the conditions of Theorem 2 are fully satisfied, so in the light of Theorem 2, DDACVNN (2) is GES.

Numerical simulations of Example 1 are exhibited in Figure 1.

Example 2. We consider DDACVNN (2), taking $r=1$ and $n=3$. We allow $\tau=0.1, I_1=1+i$ and $I_2=(0,0)^T$; $F_{1k}(\theta)=G_{1k}(\theta)=F_{2j}(\theta)=G_{2j}(\theta)=\ 1-e^{-|\theta|/1+|\theta|}/1+e^{-|\theta|/1+|\theta|}$, for $k=1$ and $j=1,2$,

$$\begin{aligned}&D_{11}=(1-i)_{1\times1}, E_{11}=(1-i)_{1\times1}, D_{12}=(i,0)_{1\times2}, E_{12}=(0,2+i)_{1\times2},\\ &D_{21}=\begin{pmatrix}1+i\\ -i\end{pmatrix}_{2\times1}, E_{21}=\begin{pmatrix}1+2i\\ 1-i\end{pmatrix}_{2\times1},\\ &D_{22}=\begin{pmatrix}3+i & 0\\ -1+i & 1+3i\end{pmatrix}_{2\times2}, E_{22}=\begin{pmatrix}1+3i & -1+i\\ 0 & 2-i\end{pmatrix}_{2\times2}.\end{aligned} \tag{53}$$

$B_1=(10\sqrt{2})$ and $B_2=\mathrm{diag}\{10\sqrt{10}, 9\sqrt{10}\}$.

We obviously see that hypothesis 1 holds and $\widehat{F}_1=\widehat{G}_1=(0.5)$ and $\widehat{F}_2=\widehat{G}_2=\mathrm{diag}\{0.5,0.5\}$. By calculation, we have

$$\begin{aligned}&|D_{11}|=(\sqrt{2}), |E_{11}|=(\sqrt{2}), |D_{12}|=(1,0), |E_{12}|=(0,\sqrt{5}),\\ &|D_{21}|=\begin{pmatrix}\sqrt{2}\\ 1\end{pmatrix}, |E_{21}|=\begin{pmatrix}\sqrt{5}\\ \sqrt{2}\end{pmatrix}, |D_{22}|=\begin{pmatrix}\sqrt{10} & 0\\ \sqrt{2} & \sqrt{10}\end{pmatrix}, |E_{22}|=\begin{pmatrix}\sqrt{1}0 & \sqrt{2}\\ 0 & \sqrt{5}\end{pmatrix}.\end{aligned} \tag{54}$$

And

$$B_2-|D_{22}|\widehat{F}_2=\begin{pmatrix}\dfrac{19\sqrt{10}}{2} & 0\\ -\sqrt{2} & \dfrac{17\sqrt{10}}{2}\end{pmatrix}, \tag{55}$$

which is a M matrix. On the basis of Theorem 1, we can conclude that for any CIV $(\eta^T,\theta^T)^T\in C_\tau^1\times C_\tau^2$, DDACVNN (2) has a unique solution.

Moreover, we can continue to calculate the following:

$$\begin{aligned}&\delta_1=\max_{1\le l\le m}\frac{r_l\left(|D_{21}|\widehat{F}_1\right)}{r_l\left(B_2-|D_{22}|\widehat{F}_2\right)}=0.1664,\\ &\delta_2=\max_{1\le l\le m}\frac{r_l\left(|E_{21}|\widehat{G}_1\right)}{r_l\left(B_2-|D_{22}|\widehat{F}_2\right)}=0.0744,\\ &\alpha_2=\max_{1\le l\le m}\frac{r_l\left(|E_{22}|\widehat{G}_2\right)}{r_l\left(B_2-|D_{22}|\widehat{F}_2\right)}=0.4985,\\ &\gamma_1=r_{\min}\left(B_1-|D_{11}|\widehat{F}_1\right)-r_{\max}\left(|D_{12}|\widehat{F}_2\right)\delta_1=13.3518,\\ &\gamma_2=r_{\max}\left(|E_{11}|\widehat{G}_1\right)+r_{\max}\left(|D_{12}|\widehat{F}_2\right)\delta_2=0.7443,\\ &\alpha_1=r_{\max}\left(|E_{12}|\widehat{G}_2\right)+r_{\max}\left(|D_{12}|\widehat{F}_2\right)\alpha_2=1.3673,\end{aligned} \tag{56}$$

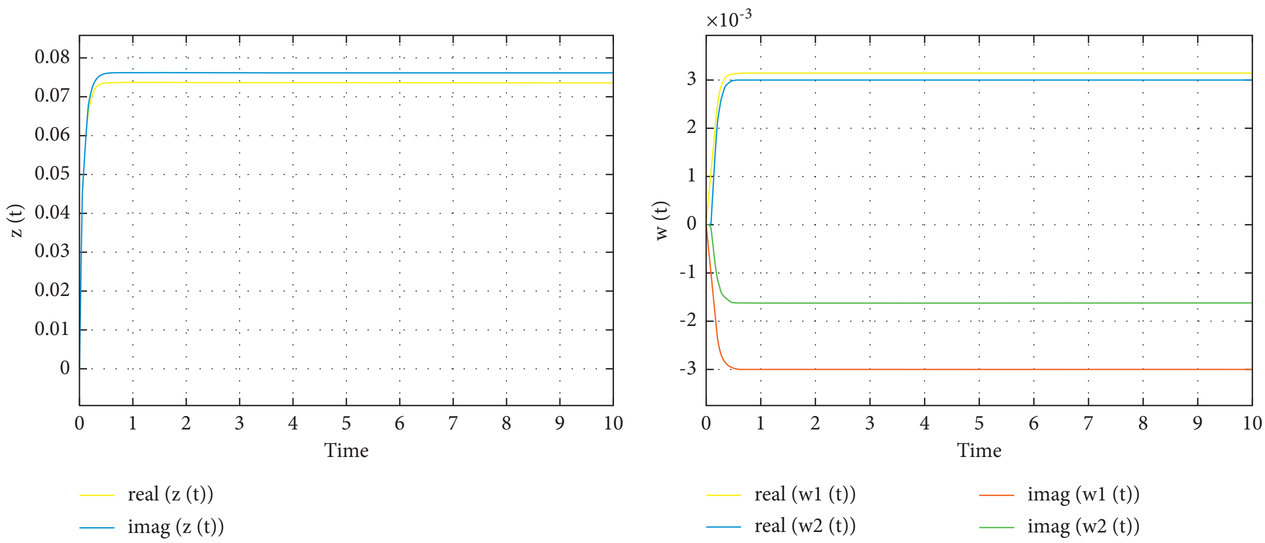

FIGURE 1: Curves of the real and imaginary parts of $z(t)$ and $w(t)$ in example 1.

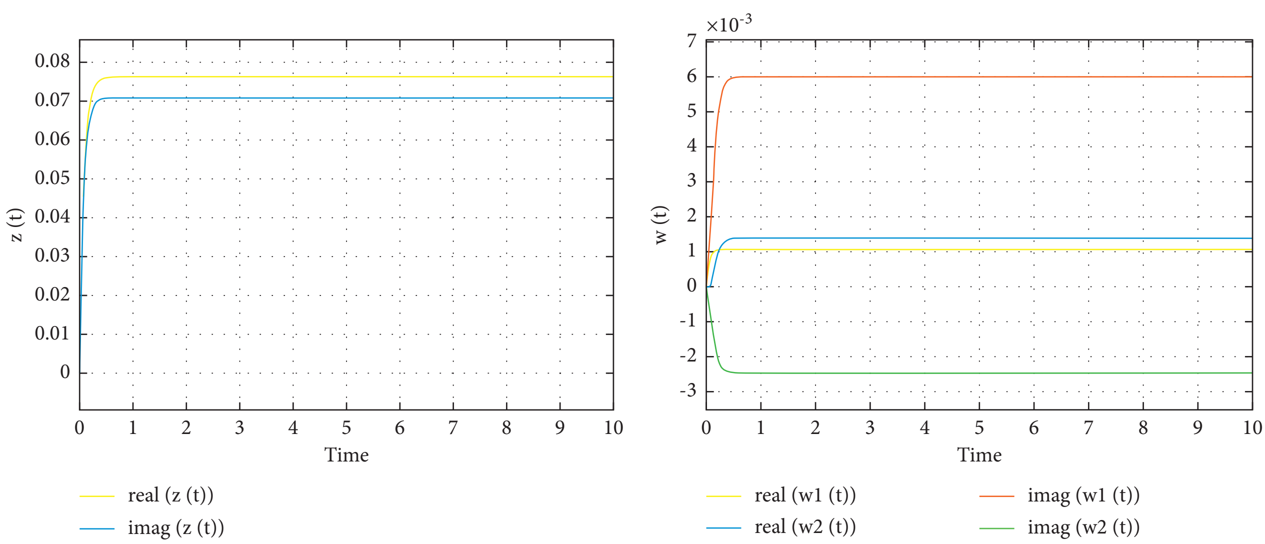

FIGURE 2: Curves of the real and imaginary parts of $z(t)$ and $w(t)$ in Example 2.

and

$$\Lambda = \begin{pmatrix} e^{-\beta\tau} & \dfrac{\alpha_1}{\beta} \\ \delta_1 e^{-\beta\tau} + \delta_2 & \dfrac{\delta_1\alpha_1 + \beta\alpha_2}{\beta} \end{pmatrix} = \begin{pmatrix} 0.3325 & 0.1242 \\ 0.1297 & 0.5192 \end{pmatrix}, \tag{57}$$

where $\beta = 11.01 > 0$ such that $\beta - \gamma_1 + \gamma_2 e^{\beta\tau} = -0.1036 < 0$; then, $\lambda(\Lambda) = 0.2683, 0.5834$, $\lambda_{\max}(\Lambda) = 0.5834 < 1$.

In the example, all the conditions of Theorem 2 are fully satisfied, so on the basis of Theorem 2, DDACVNN (2) is GES.

Numerical simulations of Example 2 are exhibited in Figure 2.

Remark 7. In the above, we have obtained that the activation function of the system considered in the article is feasible regardless of whether the real and imaginary parts can be departed, and the above two examples also prove the correctness of the derived conclusions.

5. Conclusion

Looked through the article, we establish a new type of DDACVNN. First, the existence and uniqueness of the DDACVNN solution is addressed. In addition, the GES theorem of DDACVNN is also studied. In particular, we verify that our activation function does not have the restriction that the real and imaginary parts must be separated. Finally, we also give two examples to confirm the validity of the theoretical results.

For the sake of simplicity, we only consider a simpler form of the differential-algebraic system. And studying a more complex differential-algebraic system has always been a more challenging subject. For example, we can conduct a series of studies similar to traditional neural networks on differential-algebraic systems, such as multiple stability, synchronicity, and periodic oscillations. At the same time, we can also study a differential-algebraic system with a high degree of exponential, or a differential-algebraic system with variable delays. In addition, we can also consider nonlinear differential-algebraic systems. And these issues will be considered in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

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References

[1] P. Pepe, "Discrete-time systems with constrained time delays and delay-dependent Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 65, no. 4, pp. 1724–1730, 2020.

[2] D. Han, "Improved delay-derivative-dependent stability criteria for linear systems with time-varying delay," in *Proceedings of the 32nd Chinese Control Conference*, pp. 1374–1379, Xi'an, China, Oct. 2013.

[3] C. M. Marcus and R. M. Westervelt, "Stability of analog neural networks with delay," *Physical Review A*, vol. 39, no. 1, pp. 347–359, 1989.

[4] A. Hirose, "Complex-valued neural networks: theories and applications," *World Scientific Publishing Company*, vol. 5, 2003.

[5] T. Nitta, "Orthogonality of decision boundaries in complex-valued neural networks," *Neural Computation*, vol. 16, no. 1, pp. 73–97, 2004.

[6] M. F. Amin and K. Murase, "Single-layered complex-valued neural network for real-valued classification problems," *Neurocomputing*, vol. 72, no. 4-6, pp. 945–955, Jan. 2009.

[7] T. Nitta, "Solving the XOR problem and the detection of symmetry using a single complex-valued neuron," *Neural Networks*, vol. 16, no. 8, pp. 1101–1105, 2003.

[8] Q. Song, Z. Zhao, Y. Liu, and F. E. Alsaadi, "Mean-square input-to-state stability for stochastic complex-valued neural networks with neutral delay," *Neurocomputing*, vol. 470, pp. 269–277, 2022.

[9] W.-T. Wang and G.-M. Li, "Nonlinear control methods for a class of differential-algebraic systems," in *Proceedings of the 2009 Chinese Control and Decision Conference*, pp. 2900–2904, Guilin, China, Aug. 2009.

[10] P. Di Franco, G. Scarciotti, and A. Astolfi, "Stability of nonlinear differential-algebraic systems via additive identity," *IEEE/CAA Journal of Automatica Sinica*, vol. 7, no. 4, pp. 929–941, 2020.

[11] J. Chen, B. Chen, and Z. Zeng, "Basic theorem and global exponential stability of differential-algebraic neural networks with delay," *Neural Networks*, vol. 140, pp. 336–343, Aug. 2021.

[12] Q. Liu and J.-E. Zhang, "Evolution properties of complex-valued memristive differential-algebraic neural networks," in *Proceedings of the 2019 IEEE Symposium Series on Computational Intelligence (SSCI)*, pp. 1255–1262, Xiamen, China, Dec. 2019.

[13] J. Hu and J. Wang, "Global stability of complex-valued recurrent neural networks with time-delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, no. 6, pp. 853–865, 2012.

[14] X.-H. Xu, J.-Y. Zhang, and J.-Z. Shi, "Exponential stability of complex-valued neural networks with mixed delays," *Neurocomputing*, vol. 128, pp. 483–490, 2014.

[15] J. Chen, B. Chen, Z. Zeng, and P. Jiang, "Event-based synchronization for multiple neural networks with time delay and switching disconnected topology," *IEEE Transactions on Cybernetics*, vol. 51, no. 12, pp. 5993–6003, 2021.