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Research Article

Nonlinear Dynamics and Multistability in a Cobweb Model

S. S. Askar

Department of Statistics and Operations Research, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to S. S. Askar; saskar@ksu.edu.sa

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This paper studies the dynamic characteristics of an economic cobweb model whose producers adopt a gradient-based mechanism. The model's producers are taken to be boundedly rational, and then the model is studied as a 1D and 2D discrete time dynamical system. Due to the lack of information in the market regarding the function of demand, the producers depend on the estimation of the marginal profit variations so as to update their prices next period. The equilibrium price point is calculated, and its local stability conditions are discussed through analytical and numerical investigations. In the 2D model, where the memory factor is introduced, the equilibrium price is obtained, and it is shown that it can be destabilized through chaotic behaviors, which are formed due to period doubling and Neimark-Sacker bifurcation. Furthermore, we study the symmetric case when the producers update their prices based on the marginal profit in the past two time periods, and therefore, we investigate the influence of the speed of adjustment parameter on the stability of equilibrium price.

1. Introduction

Complex dynamic characteristics and multistability that arise in many applications such as engineering, biology, and economy have attracted many scholars. In economy, which our work and contributions focus on, many economic models have studied such characteristics and reported their chaotic behaviors that may arise due to different types of bifurcations such as flip and Neimark-Sacker bifurcations. Of those models comes the well-known Cobweb model, which has got more attention from researchers since Ezekiel has developed it in his seminal article [1]. Ezekiel explained the influences of prices in fluctuations of some economic markets. He highlighted certain important features of supply and demand. He stated that the produced quantity must be given based on time and hence producers may observe prices. Therefore, a time lag must exist between demand and supply.

We adhere to this Ezekiel's hypothesis and shall analyze and study a standard cobweb model in the current paper. In this paper, we assume that the producers (or firms) do not possess complete information on market and all they know is the form of demand function. The demand function adopted here is a reciprocal function of price. This assumption with the clearing price condition makes the producer use such local knowledge in order to update its production based on the variation in the marginal profit. Depending on the bounded rationality mechanism that is a gradient-based mechanism, the producer can change the level of production, provided that whether the marginal profit is increased or decreased. The contribution of this paper consists of two parts: in the first part, we study a one-dimensional (1D) discrete map that describes the updating of price at discrete time steps. This map possesses three equilibrium points, and we only focus on the real non-zero equilibrium price point and its stability. Using analytical investigations and numerical simulations, we show the stability/instability conditions of that point. The second part focuses on studying the memory and the speed of adjustment factors and their effects on the stability of the equilibrium point. The memory factor in the proposed model is represented by some weights of the marginal profits in the last two time steps. The numerical simulation experiments show that the memory weights with low and high values make the equilibrium price point lose its stability through two different types of bifurcations, Neimark-Sacker and flip. Using some global investigations, we analyze the dynamics of the map, which include the basin of attraction for some attracting sets and chaotic attractors. The global analysis is pushed further to investigate the effects of the speed parameter on the dynamics. We study the symmetric case on which the memory weights are equal and then show the influence of the speed parameter. The region of stability of the equilibrium point with respect to this parameter is reduced when we increase the marginal cost. Furthermore, we highlight further dynamics of the map such as multistability.

The obtained results in this paper are outlined as follows. In section 2, some relevant works are reported as literature review. The 1D discrete dynamical system is introduced, and its dynamic characteristics using analytical and numerical investigations are discussed in section 3. Section 4 is organized to introduce the memory parameter and study the dynamics of the corresponding system. In section 5, we analyze the impact of speed parameter on the 2D (two-dimensional) system. Finally, in section 6, we outline our results.

2. Literature Review

Literature has reported many works that have studied the cobweb model in several economic contexts. In this section, we report some of these related works. The cobweb phenomenon has been reported in different branches of economic market such as academia [2], bioenergy crops [3], nurses [4], potatoes [5], and real estate [6]. In the presence of nonlinear supply and demand functions, the stability conditions of equilibrium points have been obtained in [7]. In [8], the mechanism of adaptive expectation was introduced to construct more sophisticated models of cobweb phenomena. A nonlinear supply function has been introduced in a traditional cobweb model to show that period doubling feature based on adaptive expectation may exist [9]. In [10], it has been proven based on adaptive expectations that a chaotic behavior in a standard cobweb model may exist when both supply and demand functions are monotonic. The cobweb model under adaptive expectations has been investigated in [11] using different types of demand and supply functions.

The aforementioned works cited above suggested some assumptions where producers (or firms) possessed knowledge about cost of production, and therefore, they can evaluate the function of profit that in turn depends on quantity produced and its price. This helps producers determine the quantity sent to the market as a price-based function. Such hypothesis requires some kind of bounded rationality mechanism and open the gate to many authors to study economic models under such mechanism. For instance, an interesting study on cobweb model based on a gradient-based approach has been described and studied by a discrete map in [12]. In that study, producers possess no complete information about the function of demand, and they instead do some empirical estimation on the marginal profit. More information about the bounded rationality approach can be found elsewhere in literature [13–22]. Other interesting works on the adoption of bounded rationality that requires knowledge and computational capabilities are reported. In [23], the dynamic characteristics of a cobweb map with nonlinear demand and supply functions have been analyzed. In that analysis, producers have used a backward expectation method in order to make forecasts on prices in future time steps. The backward expectation mechanism adopted in [23] depended on forecasting prices in the last two time steps. In [24], the dynamic characteristics of a more general cobweb model based on general demand and cost functions and whose producers adopt naive expectations on future prices have been explored. The literature also contains other works that have suggested that producers may not be rational [25], others works with heterogenous producers [26], and more that considered the replicator dynamics [27].

3. The Cobweb Model

In economic market, the cobweb model is dedicated to explain the influences on prices that might occur due to market periodic fluctuations. The model is adopted so as to describe the cyclical nature of demand and supply in some markets. Supply and demand are an economic model that is used to determine price in a market. Suppose that $D(p)$ is the demand of a consumer, which depends on the price p. The form of demand may be assumed as linear or nonlinear function based on the preferences of consumers. Throughout the current paper, we consider at time t the following demand function:

$$D(p(t)) = q_D(t) = \frac{1}{\sqrt{p(t)}} - a, \quad a > 0,\ p(t) > 0. \tag{1}$$

It is clear that this function is nonlinear. It is known that when a market is not provided by quantities ($q_D(t) = 0$), the consumers will buy a maximum price given in a market with $1/a^2$ per production unit. For (1), the inverse demand function becomes

$$p(t) = D^{-1}(q_D(t)) = \frac{1}{(a + q_D(t))^2}. \tag{2}$$

Providing the market with production next period of time (say at $t+1$) depends on some factors estimated by producers at time t. Due to the lack of market information, producers make some market experiments in order to decide the state of the market. They decide to provide the market with the required amount of quantities depending on the production volume and the gained profits; if their profits are increased, then they increase the amount of production supplied to the market next period of time, while if their profits are decreased, they will reduce the production supplied to the market. The profit of a supplied quantity can be computed as follows:

$$\pi(q_S(t)) = p(t)q_S(t) - TC(q_S(t)), \tag{3}$$

and $q_S(t) \geq 0$ is the quantity supplied to the market at time step t while $TC(q_S(t))$ refers to the total cost and is assumed to be linear as given below.

$$TC(q_S(t)) = cq_S(t). \tag{4}$$

while $c = \partial T\ C/\partial q_S > 0$ denotes the marginal cost and is constant. In order to attain equilibrium, the cobweb theory assumes that $q_S(t) = q_D(t) = q(t)$ at any period of time. So, (3) can be expressed as follows:

$$\pi(q(t)) = \frac{q(t)}{(a+q(t))^2} - cq(t). \tag{5}$$

According to (5), positive profit can be obtained, provided that $1/a^2 > c$. This means that the maximum price consumers are willing to pay for a commodity unit must be less that the marginal cost. On the contrary, if $1/a^2 < c$, the profit will not be positive, and the economic meaning of price is useless. Differentiating (5) with respect to $q(t)$ gives the marginal profit:

$$\psi(q(t)) = \frac{\partial \pi(q(t))}{\partial q(t)} = \frac{a-q(t)}{(a+q(t))^3} - c. \tag{6}$$

Now, the amount of quantity that is supplied to the market at the next time step can be described by the following gradient-based mechanism [13]:

$$\begin{aligned} q_S(t+1) &= q(t) + k(q(t))\psi(q(t)) \\ &= q(t) + k\left(\frac{a-q(t)}{(a+q(t))^3} - c\right), \end{aligned} \tag{7}$$

where we assume $k(q(t)) = k$, which is a positive parameter representing the speed of adjustment. Using (1) in (7) the supply quantity can be given by

$$\begin{aligned} S(p(t)) &= q_S(t+1) \\ &= \frac{1}{\sqrt{p(t)}} - a + k[(2a\sqrt{p(t)} - 1)p(t) - c]. \end{aligned} \tag{8}$$

Assuming market clearing price, we get

$$q_S(t+1) = q_D(t+1), \tag{9}$$

which gives

$$\frac{1}{\sqrt{p(t+1)}} - a = \frac{1}{\sqrt{p(t)}} - a + k[(2a\sqrt{p(t)} - 1)p(t) - c]. \tag{10}$$

Simple calculations in (10) show that the dynamics of the price are then described by the one-dimensional difference equation given by

$$\begin{aligned} p(t+1) &= f(p(t)) \\ &= \frac{p(t)}{[1 + k\sqrt{p(t)}((2a\sqrt{p(t)} - 1)p(t) - c)]^2}. \end{aligned} \tag{11}$$

The map (11) has the following market equilibrium price:

$$\begin{aligned} \overline{p} &= \left(\frac{1+\theta+\theta^2}{6a\theta}\right)^2, \\ \theta &= \sqrt[3]{1 + 54ca^2 + a\sqrt{108c\left(1+27ca^2\right)}}, \end{aligned} \tag{12}$$

which is a positive value. Before we analyze the stability of (12) and the complex dynamic characteristics of (11), we highlight some important features between the supply and the demand given in Figure 1(a) and Figure 1(b). In both figures, it is clear that the curves of q_S and q_D intersect in the equilibrium price given in (12). This equilibrium point can not exceed the value of maximum price $1/a^2$ which is held under the condition $1/a^2 > c$; otherwise, negative price may be raised when $(1/a^2) < c$, beginning with $0 < p_o < 1/a^2$ and moving vertically to the blue curve of q_S in order to get the quality in the next time step that is denoted by q_1 in Figure 1(a). Since at each time step one gets equal demand and supply, because the model becomes in an equilibrium state, we move horizontally to the curve of q_D (red curve in Figure 1(a)) in order to get the price p_1 which is used to set the quantity q_2 in the next time step. After setting the quantity q_2 we move horizontally to the red curve in order to get the price p_2. This process is repeated until we reach the equilibrium of the market price $\overline{p}$. Figure 1(b) displays this process for an initial value of price $p_o > 1/a^2$. From Figure 1 one may conclude that the cobweb trajectories may not approach negative values.

Now, we start studying the dynamic characteristics of the map (11) around the fixed point $\overline{p}$ that depends only on the parameters a and c. The profit given in (5) can be presented by

$$\pi(p) = \left(\frac{1}{\sqrt{p}} - a\right)(p - c). \tag{13}$$

Differentiating (13), we get $\mathrm{d}\pi(\overline{p})/\mathrm{d}p = 0$, which means that the profit has a maximum value at the equilibrium price. In addition, $\pi(c) = 0$ and $\pi(p) > 0$ provided that $c < p < 1/a^2$. This means that, before the equilibrium point $\overline{p}$, the variations in the profit $\pi(p)$ due to an increase in the price p in the interval $(c, \overline{p}]$ are positive. They become negative in the interval $(\overline{p}, 1/a^2)$. The explanation for that may be due to the level of production carried out in a monotonic or nonmonotonic way based on how producers can handle the change in profit. It is also clear that the curve of the marginal profit intersects the abscissa axis in the equilibrium price as shown in Figure 2.

Now, we study the dynamic case on which the equilibrium price can be destabilized. The following proposition is given.

Proposition 1. *The dynamics of the equilibrium price $\overline{p}$ have two possibilities:*

(i) It is locally stable, and all prices converge to it if $216a^3\theta^4/(1-\theta+\theta^2)(1+\theta+\theta^2)^3 < k < 432a^3\theta^4/(1-\theta+\theta^2)(1+\theta+\theta^2)^3$.

(ii) It is unstable, and all prices diverge from it if $k > 432a^3\theta^4/(1-\theta+\theta^2)(1+\theta+\theta^2)^3$.

Proof. The proof depends on the marginal demand and marginal supply at the equilibrium price. Simple calculations show that

$$\frac{S'(\overline{p})}{D'(\overline{p})} = 1 - \frac{k\left(1-\theta+\theta^2\right)\left(1+\theta+\theta^2\right)^3}{216a^3\theta^4}. \tag{14}$$

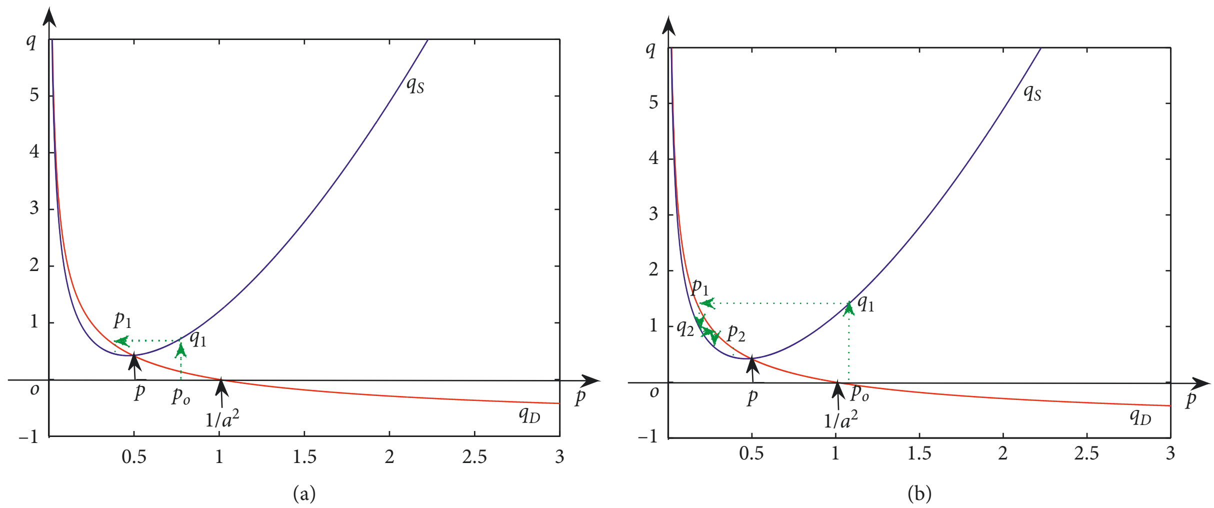


FIGURE 1: The dynamics of price and quantity represented by the cobweb diagram at different initial values of price. The other values are as follows: $a = 1, c = 0.2, k = 1.5$.

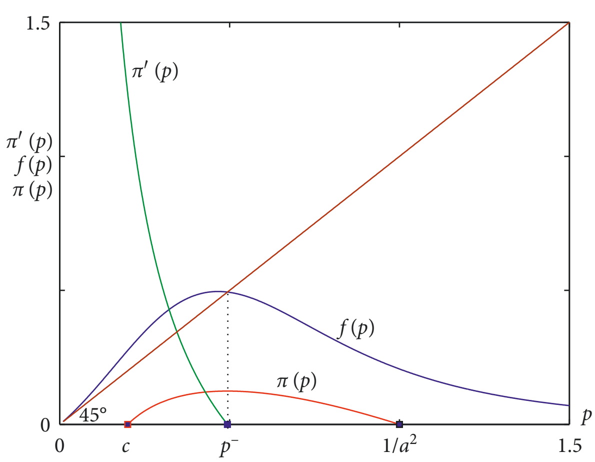


FIGURE 2: The shapes of the functions $\pi'(p)$, $f(b)$, and $\pi(p)$ along with a line with 45°. It is clear that $\pi(c)$ and $\pi(1/a^2)$ have vanished, while $\max \pi(p)$ is attained at a price that is coincided with the equilibrium price $\overline{p}$. The other parameters set is $a = 1, b = 0.2, k = 1.5$.

Recalling the conditions [28] that are given by

(i) The equilibrium point is locally stable if $-1 < S'(\overline{p})/D'(\overline{p}) < 0$

(ii) The equilibrium point gets unstable provided that $S'(\overline{p})/D'(\overline{p}) < -1$

Substituting (14) in these conditions completes the proof.

The above conditions depend on the parameters a, c and k. So, in order to get more insights about the dynamics of the map around the equilibrium price, we perform some numerical simulations. This includes displaying the 1D bifurcation diagram and the basin of attraction for some attracting sets of the map (11). The dynamics of the map are displayed against the parameter k while fixing the other parameter values to $a = 1, c = 0.2, 0.5$. Figure 3(a) consists of two parts: the first part shows that the equilibrium price is locally stable for all the values of k until it approaches the period-2 cycle on which the equilibrium price becomes unstable. We observe that low marginal cost achieves a stable region for the point of equilibrium price with respect to k (as suggested in Figure 3(a)). In the second part and as the marginal cost increases with k as well the stability region reduces, therefore, increasing the marginal cost lead to instability of the equilibrium price (as Figure 3(a) suggests in the green color bifurcation). It is also noted that increasing the speed of adjustment parameter k gives rise to strong reaction from the producers toward both quantity supplied and prices. At the same time, the parameters a and c have an opposite influence on the stability of the equilibrium price. It is clear from Figure 3(b) that increasing the marginal cost c leads to instability of the equilibrium price for low and high values of the speed of adjustment parameter. In Figure 3(c), we see that increasing the parameter a gives an opposite impact of the dynamics of the map (11) and the stability of the equilibrium price may be reached. This discussion gives rise to the fact that the dynamics of equilibrium price and its stability are affected by the map's parameters. For this reason, we give some attracting set for the map around price for different values of the speed parameter. Figure 3(d) shows a monotonic convergence to $\overline{p}$ at $k = 1.5, a = 1, c = 0.2$. As k increases further to 3, a period-2 cycle emerges and is plotted with its basin of attraction in Figure 3(e). Increasing k further while keeping the other parameters fixed gives rise to sequence of period doubling bifurcations till it approaches 4.8, where a chaotic dynamic behavior coexists as shown in Figure 3(f). To end our analysis in this part, we should highlight that any other attracting set for the map (11) around the equilibrium price will have basin of attractions that are bounded by the box $[0, 1/a^2] \times [0, 1/a^2]$ and any initial prices taken out of this box lead to unbounded or negative trajectories that would not have economic meanings. □

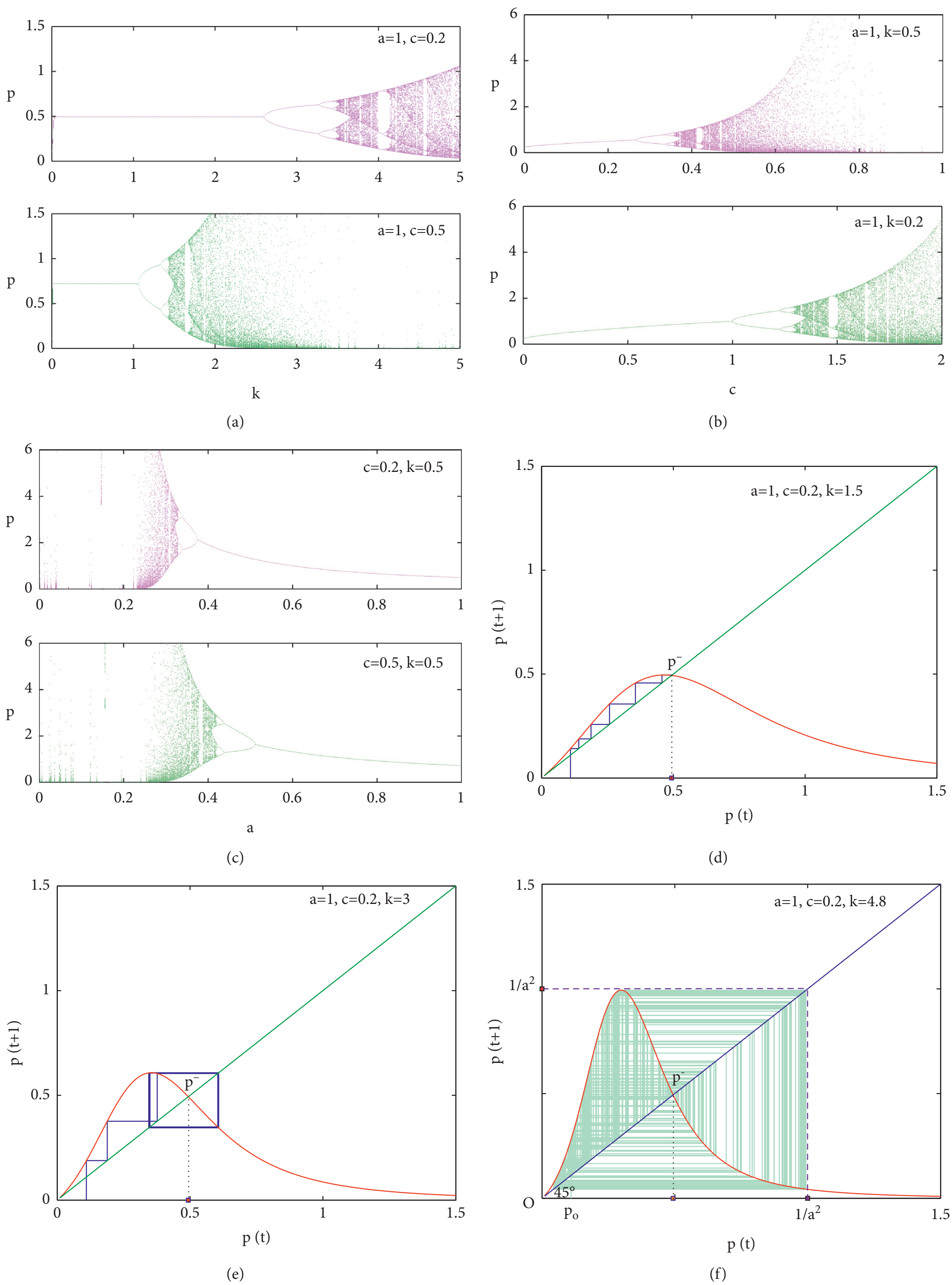


Figure 3: Bifurcation diagrams with respect to p on varying (a) k and $a = 1, c = 0.2, 0.5$, (b) c and $a = 1, k = 0.5, 2$, and (c) a and $k = 0.5, c = 0.2, 0.5$. Basin of attractions for different attracting set of the map (11) at the values (d) $k = 1.5$, (e) $k = 3$, and (f) $k = 4.8$. Other values are $a = 1, c = 0.2$.

4. The Effect of Memory

Let us consider and discuss the 2D version of map (11) by introducing the memory parameter. The memory is used by producers to decide whether they may increase (or decrease) their productions next period. We follow [29] to introduce memory based on a gradient-based mechanism as follows:

$$q_S(t+1) = q(t) + k\left[\omega \frac{\partial \pi(q(t))}{\partial q(t)} + (1-\omega)\frac{\partial \pi(q(t-1))}{\partial q(t-1)}\right], \quad \omega \in [0,1]. \tag{15}$$

The memory here is represented by the producers' estimation for the marginal profits at the periods $t-1$ and t with an average weight ω so that they can decide the supplied quantity at $t+1$. As previously, the supply function can be expressed in terms of price as follows:

$$\begin{aligned} q_S(t+1) &= S(p(t)) \\ &= \frac{1}{\sqrt{p(t)}} - a + k[\omega((2a\sqrt{p(t)} - 1)p(t) - c) \\ &\quad + (1-\omega)((2a\sqrt{p(t-1)} - 1)p(t-1) - c)]. \end{aligned} \tag{16}$$

Now, we impose market clearing $q_D(t+1) = q_S(t+1)$ and then one gets

$$p(t+1) = \frac{p(t)}{(1 + k\sqrt{p(t)}\,[\omega((2a\sqrt{p(t)} - 1)p(t) - c) + (1-\omega)((2a\sqrt{p(t-1)} - 1)p(t-1) - c)])^2}. \tag{17}$$

Putting $p(t) = x_t$ and $p(t-1) = y_t$ in (17) one gets the following two dimensional (2D) map:

$$T: \begin{cases} x_{t+1} = \dfrac{x_t}{\left(1 + k\sqrt{x_t}\left[\omega((2a\sqrt{x_t} - 1)x_t - c) + (1-\omega)((2a\sqrt{y_t} - 1)y_t - c)\right]\right)^2}, \\ \\ y_{t+1} = x_t. \end{cases} \tag{18}$$

The 2D map given in (18) admits an equilibrium point given by $O = (\overline{p}, \overline{p})$ where $\overline{p}$ is defined in (11). Its stability/instability is governed by the eigenvalues of the Jacobian matrix of map (18) or on Jury conditions [30] given in the following two propositions.

Proposition 2. *Suppose that the Jacobian matrix of (18) at O has two eigenvalues λ_1 and λ_2 then we have the following cases:*

(i) The point O is a local stable attracting node if $|\lambda_{1,2}| < 1$

(ii) The point O is an unstable repelling node if $|\lambda_{1,2}| > 1$

(iii) The point O is an unstable saddle point if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$)

(iv) The point O is a nonhyperbolic point if $|\lambda_1| = 1$ and $|\lambda_2| \neq 1$ (or $|\lambda_1| \neq 1$ and $|\lambda_2| = 1$)

Proposition 3. *The equilibrium point O is locally asymptotically stable if the conditions given below are satisfied:*

$$\begin{aligned} g(1) &:= 1 - \tau + \delta > 0, \\ g(-1) &:= 1 + \tau + \delta > 0, \\ \Delta &:= 1 - \delta > 0, \end{aligned} \tag{19}$$

where τ and δ refer to the trace and determinant of the Jacobian matrix of (18) at O. It is also should be noted that if $g(1) = 0$, then O becomes unstable due to transcritical or fold bifurcation. If $g(-1) = 0$, then it gets unstable because of flip (or period-doubling) bifurcation. But if $\Delta = 0$, then O becomes unstable due to the coexistence of Neimark-Sacker bifurcation.

The Jacobian at O is given by

$$J_O = \begin{pmatrix} \dfrac{1 + 2k\omega\overline{p}^{3/2} - 6ka\omega\overline{p}^2}{\left(1 - ck\overline{p}^{1/2} - k\overline{p}^{3/2} + 2ka\overline{p}^2\right)^3} & \dfrac{2k(1-\omega)\left(1 - 3a\overline{p}^{1/2}\right)\overline{p}^{3/2}}{\left(1 - ck\overline{p}^{1/2} - k\overline{p}^{3/2} + 2ka\overline{p}^2\right)^3} \\ \\ 1 & 0 \end{pmatrix}, \tag{20}$$

whose τ and δ become

$$\tau = \frac{1 + 2k\omega\overline{p}^{3/2} - 6ka\omega\overline{p}^2}{\left(1 - ck\overline{p}^{1/2} - k\overline{p}^{3/2} + 2ka\overline{p}^2\right)^3},$$
$$\delta = -\frac{2k(1-\omega)\left(1 - 3a\overline{p}^{1/2}\right)\overline{p}^{3/2}}{\left(1 - ck\overline{p}^{1/2} - k\overline{p}^{3/2} + 2ka\overline{p}^2\right)^3}. \tag{21}$$

Substituting (21) in (19), we get

$$g(1) := 1 - \frac{1 + 2k\overline{p}^3 - 6ka\overline{p}^2}{\left(1 - ck\overline{p}^{1/2} - k\overline{p}^{3/2} + 2ka\overline{p}^2\right)^3},$$
$$g(-1) := 1 + \frac{1 - 2k\overline{p}^3 + 6ka(1-2\omega)\overline{p}^2 + 4k\omega\overline{p}^{3/2}}{\left(1 - ck\overline{p}^{1/2} - k\overline{p}^{3/2} + 2ka\overline{p}^2\right)^3} > 0,$$
$$\Delta := 1 + \frac{2k(1-\omega)\left(1 - 3a\overline{p}^{1/2}\right)\overline{p}^{3/2}}{\left(1 - ck\overline{p}^{1/2} - k\overline{p}^{3/2} + 2ka\overline{p}^2\right)^3} > 0. \tag{22}$$

So, the stability/instability of O is governed by the conditions (22). Furthermore, the types of bifurcations by which O may be destabilized depend on (22) too. Due to the complicated form of O, we do numerical simulation to confirm these results. As one can observe from (22) that both $g(-1)$ and Δ contain the memory parameter ω, breaking any one of those two conditions is responsible for destabilization O due to flip or Neimark-Sacker bifurcation. First, we investigate the influences of ω on the point O and show how this parameter can affect its stability. Let us assume the following parameters' values, $a = 1, c = 0.46, k = 1.4$ and $\omega = 0.7$. The equilibrium point O at this set becomes $O =$ (0.69251, 0.69251), and then Jacobian matrix given in (20) gets

$$J_O \approx \begin{pmatrix} -0.69020 & -0.72423 \\ 1 & 0 \end{pmatrix}. \tag{23}$$

whose complex eigenvalues are $\lambda_{1,2} = -0.3451 \pm 0.0.7779i$ with $|\lambda_{1,2}| \approx 0.85102$. In addition, the trace and determinant of the Jacobian become $\tau = -0.69020$ and $\delta = 0.72423$. We have $\delta < 1$ which means that the map (18) is dissipative. Simple substitution shows that Jury conditions given in (22) are all fulfilled, and hence, the point O is locally asymptotically stable. Figure 4(a) illustrates the impact of memory parameter ω on the point O. The bifurcation diagram is enlarged in Figure 4(b) to display the type of bifurcations coexisting at different intervals for the parameter ω. At this set of parameters, the simulation shows that the point O can be destabilized due to Neimark-Sacker bifurcation provided that $0 \le \omega \le \omega_{ns}$ while it loses its stability due to flip bifurcation when $\omega \in [\omega_f, 1]$. Now, we give some examples of attracting sets for the map (22) around O at different values of the memory parameter ω in order to get more insights on the map's dynamics. At $\omega =$ 0.257536 a cycle of period six (marked by squares) is obtained with unconnected chaotic attractor (red) around the unstable point O under the same values of parameters. It is depicted in Figure 4(c) with its attractive basin marked by two colors: the yellow color refers to the basin of O, while the cyan color denotes the basin of the cycle. Increasing the memory parameter to $\omega = 0.2649$, the basin of attraction of a cycle of period 9 (marked by squares) together with the basin of a closed chaotic attractor (red color) surrounding the unstable equilibrium price O is given in Figure 4(d). Increasing ω further gives disappearance of the period-9 cycle, and we get only the closed chaotic attractor around O which continues to appear until $\omega = 0.2986$, where the basin of attraction of period-9 along with a closed chaotic attractor is obtained in Figure 4(e). Increasing ω further gives rise of chaotic attractor with higher periodic cycles until $\omega = 0.5883$, where the basin of attraction of a period-4 cycle together with a closed invariant curve around O is obtained as given in Figure 4(f). Further increase in the memory parameter to 0.612 period-4 cycle has disappeared, and the closed invariant curve continues to appear until it turns into a spiral. Figure 4(g) depicts a different dynamic behaviors for the map at $\omega = 0.612, 0.613, 0.614$ and $\omega = 0.618$. In Figure 4(h), we plot the region of stability for O at the set of values, $a = 1, c = 0.46$ in the (k, ω) − plane. One can see that at $\omega = 0$ and $k \ge 0.5798161954$ the point O loses its stability because of the existence of Neimark-Sacker bifurcation on varying the parameter k. As ω gets closer to a uniform distribution, the stability interval with respect to k increases. For values of the memory greater than $\omega = 0.7499765981$, this stability region for k shrinks, and the equilibrium price O loses its stability through flip bifurcation. The above discussion makes us display the 2D bifurcation diagram in the (k, ω) − plane. It is plotted in Figure 5 at the set of values, $a = 1$ and $c = 0.46$. As one can see, the periodicity region for period-3 cycle intersects the region of stability of the equilibrium price O which indicates multistability dynamic situations. For instance, we give in Figure 6(a) a dynamic situation of two different chaotic dynamics.

5. The Effect of the Speed of Adjustment

Now, we consider the case of symmetry on which $\omega = 1/2$ that reduces the map (18) into the following form:

$$T: \begin{cases} x_{t+1} = \dfrac{x_t}{\left(1 + k\sqrt{x_t}\left[0.5\left((2a\sqrt{x_t} - 1)x_t - c\right) + 0.5\left((2a\sqrt{y_t} - 1)y_t - c\right)\right]\right)^2}, \\ \\ y_{t+1} = x_t. \end{cases} \tag{24}$$

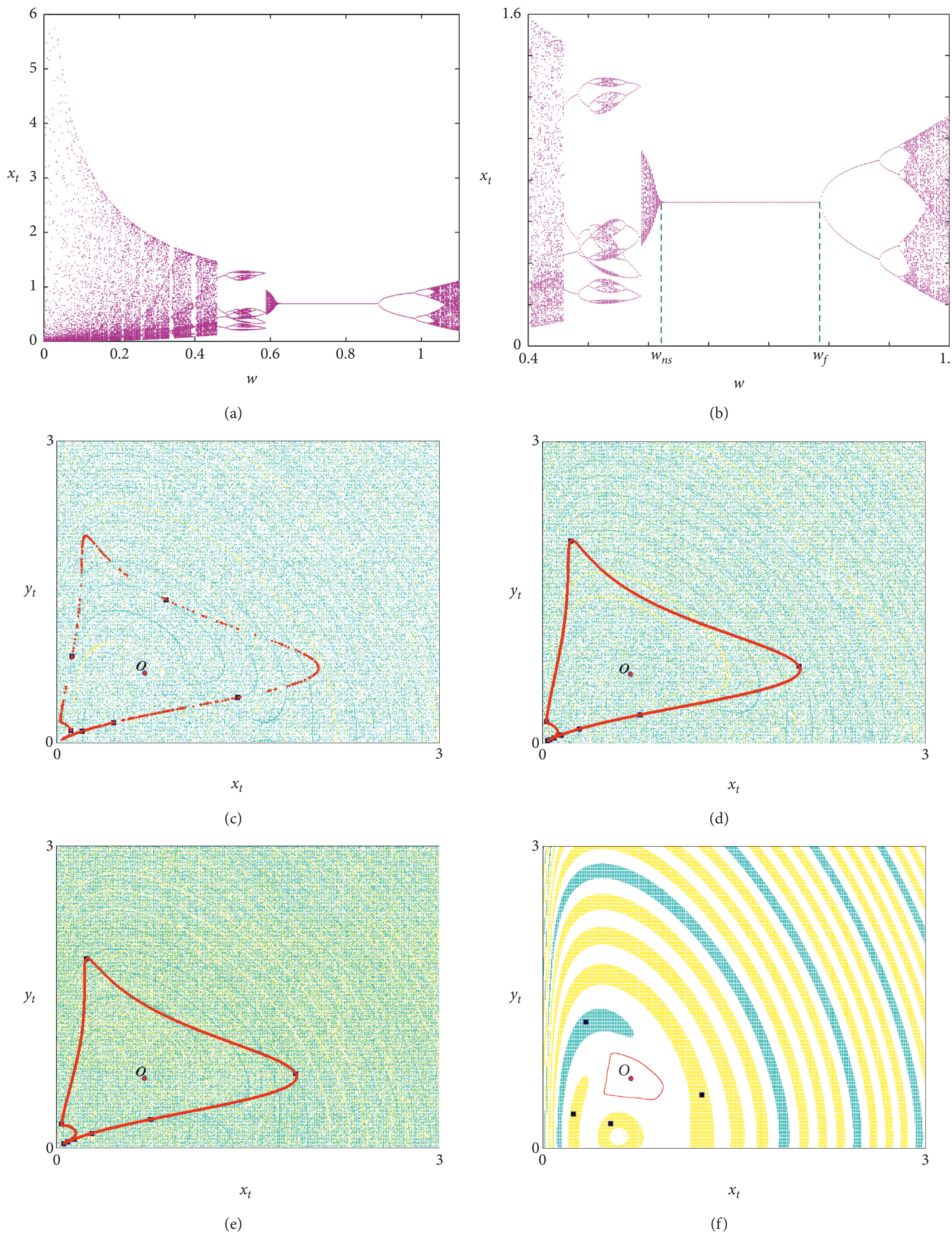
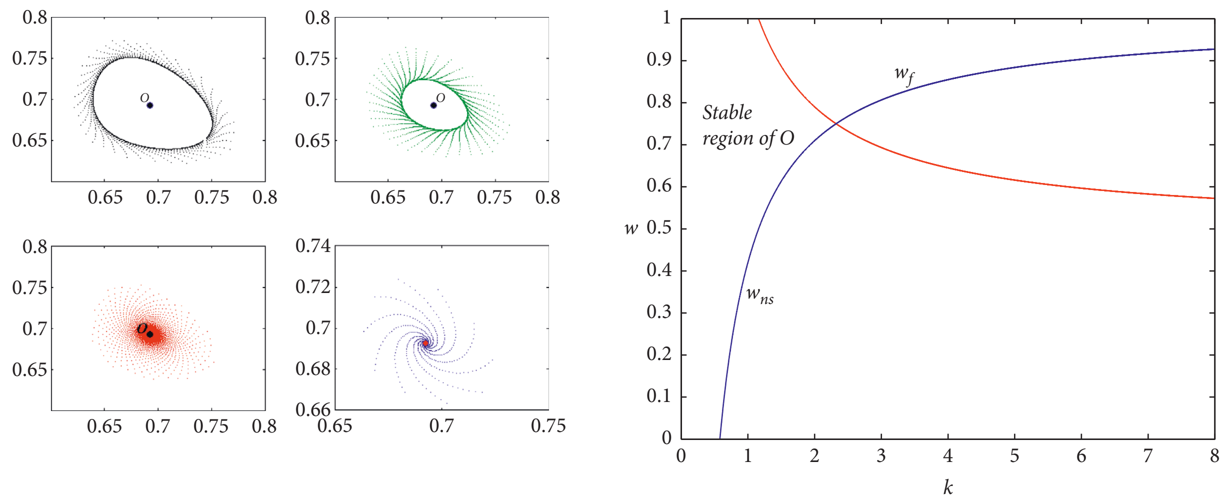


Figure 4: Continued.



(g) (h)

Figure 4: (a) Bifurcation diagram on varying the parameter of memory ω at $k = 1.4$. (b) Enlargement of the bifurcation diagram in a for different intervals of ω. (c) The basin of attraction of the period-6 cycle and the chaotic attractor at $\omega = 0.257536$ and $k = 1.4$. (d) The basin of attraction of the period-9 cycle appearing with a closed chaotic attractor at $\omega = 0.2649$ and $k = 1.4$. (e) The basin of attraction of the period-8 cycle appearing with a closed chaotic attractor at $\omega = .2986$ and $k = 1.4$. (f) The basin of attraction of the period-4 cycle appearing with a closed invariant curve at $\omega = 0.5883$ and $k = 1.4$. (g) The phase plane of different dynamic behaviors at $\omega = 0.6.12, 0.613, 0.614, 0.618$ and $k = 1.4$. (h) The stability region of the equilibrium price O and the routes to chaos through flip and Neimark-Sacker using the values $a = 1$ and $c = 0.46$.

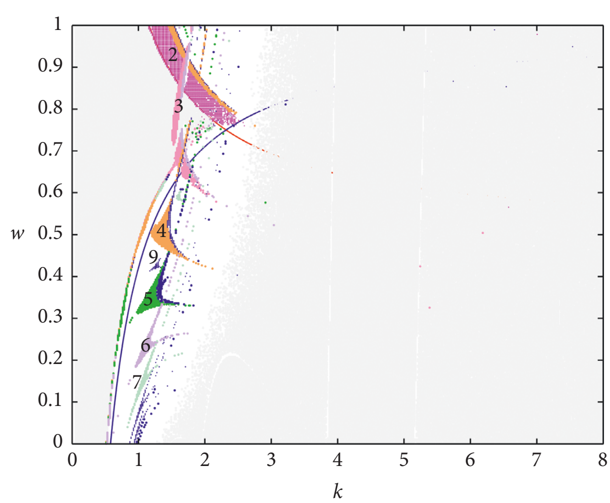


Figure 5: The diagram of 2D bifurcation. It is plotted in the (k, ω) − plane showing the regions where different period cycles are born using the values $a = 1$ and $c = 0.46$.

It means that competitors consider the average change in marginal profits at the past two periods in order to set price to the next time. For the map (24), we study now the effects of the speed parameter k on its dynamic characteristics. This requires performing a global stability analysis that includes the shape of attraction basins of some attracting sets that can occur under certain parameters' values. Such global analysis is important in order to get more insights and satisfaction about trajectories' long run behaviors. Since the map (24) contains three parameters that are a, c and k, we start our global analysis by showing the relation between the two parameters c and k when fixing the parameter a via plotting the 2D bifurcation diagram between them. Assuming $a = 1$, the 2D bifurcation is given in Figure 7(a). As one can see, there are some basins of periodic cycles that intersect the stable region of O. Assuming the parameters $a = 1, c = 0.15$, we display in Figure 7(b) the 1D bifurcation diagram on varying the bifurcation parameter k. At those parameters, the equilibrium price $O = (0.44633, 0.44633)$ becomes stable for any values of k until it loses its stability through Neimark-Sacker bifurcation. Increasing k to 3.44 a period-4 cycle emerges around the unstable O. It is shown in Figure 7(c) the basin of attraction of this cycle and the equilibrium price and the white color refer to infeasible

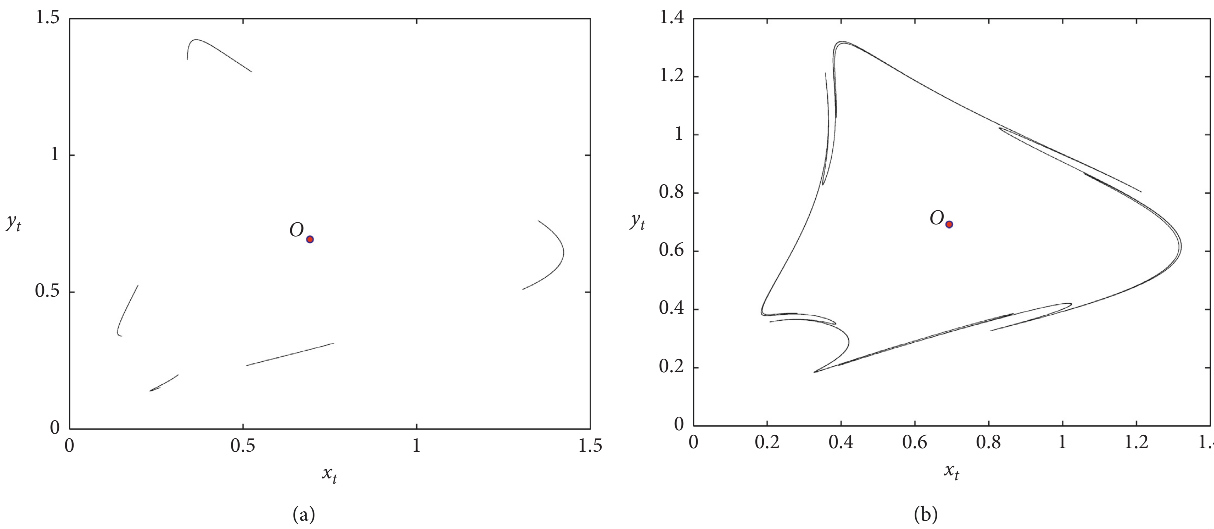


FIGURE 6: It represents two different dynamic situations, (a) Five chaotic attractors at $k = 1.4$ and $\omega = 0.413$. (b) A strange attractor $k = 1.4$ and $\omega = 0.449$. Other parameters are $a = 1, c = 0.46$.

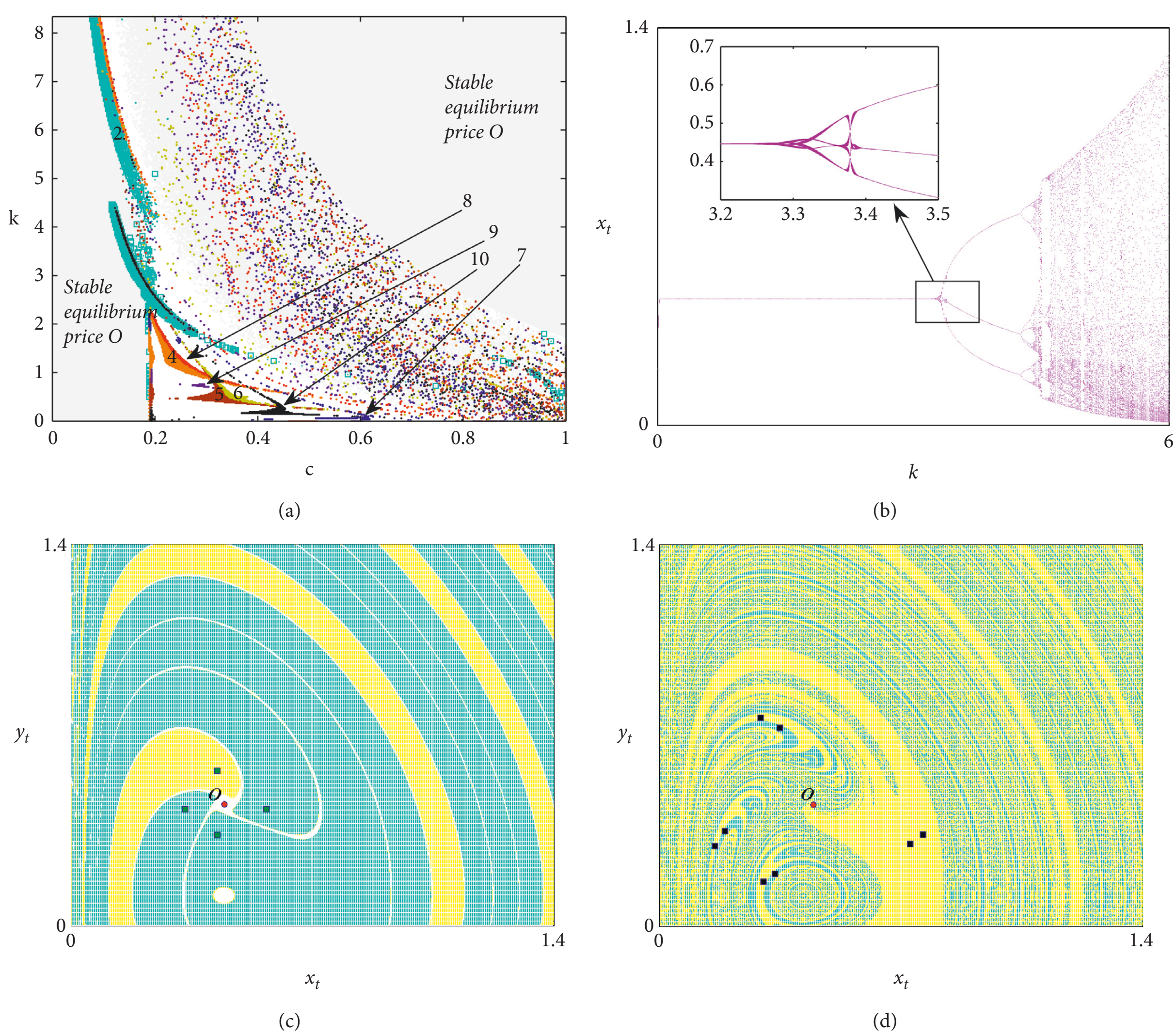
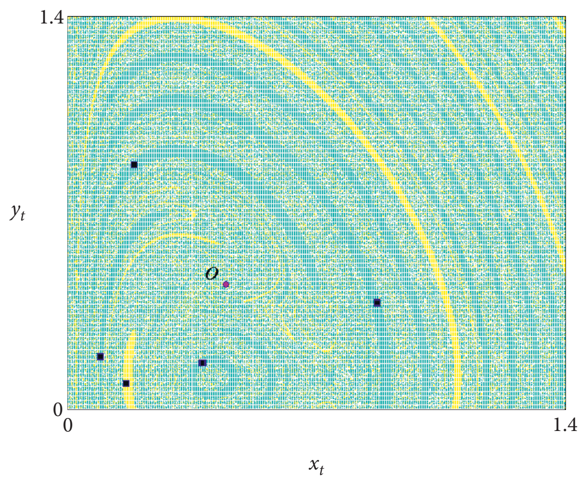
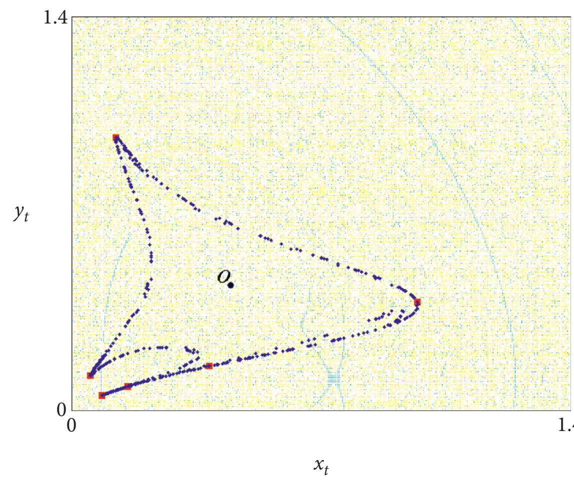


FIGURE 7: Continued.



(e)



(f)

FIGURE 7: (a) The 2D bifurcation diagram in the (k, c) − plane at $a = 1$ showing different period cycles with different colors. (b) Bifurcation diagram on varying the parameter of memory k at $a = 1$ and $c = 0.15$. (c) The basin of attraction of the period-4 cycle at $a = 1, c = 0.15$ and $k = 3.44$. (d) The basin of attraction of the period-8 cycle at $a = 1, c = 0.15$ and $k = 4.3$. (e) The basin of attraction of the period-5 cycle at $a = 1, c = 0.15$ and $k = 4.543$. (f) The basin of attraction of the period-6 cycle appearing with a chaotic attractor at $a = 1, c = 0.15$ and $k = 4.97122$.

points. This cycle continues to occur until it turns into a period-8 cycle at $k = 4.3$ as shown in Figure 7(d). It shows a complex basin of attraction for that periodic cycle. Increasing k further gives rise to a chaotic attractor, which is changed into a period-5 cycle at $k = 4.543$ as shown in Figure 7(e). We end the simulation in this part by giving in Figure 7(f) a dynamic situation consisting of a period-6 cycle that occurred with a chaotic attractor. This dynamic behavior is obtained at $k = 4.97122$. We should highlight here that as the parameter c increases, the region of stability of O is decreased on varying k. On the other hand, the parameter a has an opposite effect on the stability of O. It is observed through numerical experiments that as a increases, the region of stability of O with respect to k increases, and vice versa.

6. Conclusion

Here, in this manuscript, we have studied an economic cobweb model with producers possessing no perfect knowledge about the economic market and updating their prices based on the well-known mechanism of bounded rationality. They try to estimate the marginal profit by observing the small variations occurring in it at the beginning of the production line. We have shown that the 1D discrete cobweb model can generate complex and chaotic behaviors for the price as it has been taken as the model's variable. Those behaviors have resulted in due to the producers' responses about the small variations in the marginal profit. The results obtained have concluded that the equilibrium price may lose its stability due to an increasing in the marginal cost with low and high values of the speed parameter. Furthermore, for relatively small values of the speed parameter, the equilibrium price of the one-dimensional model gets locally stable. Higher values for that parameter does not guarantee price stability, and hence, cycles of high periods and chaotic attractors for the model's dynamics may be raised. All those observations have been supported by numerical simulation experiments.

Another contribution for this paper is the introduction of memory parameter in the model. Including memory in the one-dimensional model can be used to convert it to a two-dimensional one. As many studies in literature, the dynamics of 2D models may possess interesting chaotic fluctuations and require intensive analysis on the dynamics and the basin of attractions of some attracting sets. The memory parameter has been included in the model by adopting a convex combination between the marginal profits at the two time steps $t - 1$ and t. The later combination has been adopted by several authors in literature; see for instance [23,29]. Such combination has added another parameter known as the memory's parameter into the model's system. Our obtained results have shown that the memory parameter has qualitatively affected the equilibrium price and its stability. Analytical and numerical investigations have discussed and analyzed the conditions by which the equilibrium price can be destabilized via two types of bifurcations, that is, Flip and Neimark-Sacker. In those types, the memory parameter has been taken as the bifurcation's parameter. The global analysis carried out through numerical simulation has exhibited the coexistence of attracting sets whose basin of attractions is complex. Furthermore, some multistability situations have been reported. Moreover, the qualitative impact of the speed parameter has been analyzed in the symmetric case. We have shown through numerical simulation that the equilibrium price can be destabilized through Neimark-Saker and flip bifurcations.

Data Availability

The raw data required to reproduce the above findings are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

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