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Research Article

Decision Analysis Approach Based on 2-Tuple Linguistic m-Polar Fuzzy Hamacher Aggregation Operators

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This research article is devoted to presenting the concept of 2-tuple linguistic m-polar fuzzy sets (2 TL m FSs) and introducing some fundamental operations on them. With 2 TL m FSs, we shall be able to capture imprecise information with high generality. With the appropriate operators, we shall be able to apply 2 TL m FSs in decision-making efficiently. The aggregation operators that we propose are the 2 TL m F Hamacher weighted average (2 TL m FHWA) operator, 2 TL m F Hamacher ordered weighted average (2 TL m FHOWA) operator, 2 TL m Hamacher hybrid average (2 TL m FHHA) operator, 2 TL m F Hamacher weighted geometric (2 TL m HWG) operator, 2 TL m Hamacher ordered weighted geometric (2 TL m HOWG) operator, and 2 TL m F Hamacher hybrid geometric (2 TL m FHHG) operator. We investigate their properties, including the standard cases of monotonicity, boundedness, and idempotency. Then we develop an algorithm to solve multicriteria decision-making problems formulated with 2 TL m F information. The 2 TL m F data in multiattribute decision-making are merged with the help of aggregation operators, and we consider the particular instances of the 2 TL m FHA and 2 TL m FHG operators. The influence of the parameters on the outputs is explored with a numerical simulation. Moreover, a comparative study with existing methods was performed in order to show the applicability of the proposed model and motivate the discussion about its virtues and advantages. The results confirm that the model here developed is reliable for decision-making purposes.

1. Introduction

With its large display of different approaches, multiattribute decision-making (MADM) is a collective enterprise that aspires to deal with complex situations in the presence of multiple attributes. The choice of a decision-making approach plays a vital role in the selection of desirable alternatives. Also, the representation of the information is crucial because the formulation of problems with real-valued attribute endowments is a rarity in decision sciences. For this reason, Zadeh [1] introduced the idea of fuzzy sets (FSs), a mathematical tool that easily tackles MADM, being a mutated form of crisp set theory. Yager [2, 3] presented the less stringent concept of Pythagorean fuzzy sets (PFSs) for which the condition $\mu + \nu \leq 1$ imposed by IFSs is relaxed to $\mu^2 + \nu^2 \leq 1$. FS has a pathbreaking structure that allows it to account for vagueness for the first time. But it is not a totally general framework for the mathematical treatment of partial knowledge. To enlarge its scope, Atanassov [4] introduced intuitionistic fuzzy sets theory (IFSs), a simple modification that can tackle uncertain and vague data more precisely.

Aggregation operators (AOs) play a crucial role in converting different datasets into a single result and dealing with collective decision-making problems. A very popular tool for aggregating data was introduced by Yager [5] under the name of ordered weighted averaging aggregation (OWA) operators. Yager [6] contributed to quantifier guided aggregation using OWA operators. Xu [7] first gave intuitionistic fuzzy set-based aggregation operators. Then Xu and Yager [8] studied geometric aggregation operators and produced some real-life applications. Alcantud et al. [9] first produced AOs that operate on infinitely many intuitionistic fuzzy sets. Improvements to IFSs appeared, and AOs were produced in these new models too. Since there is often a counterpart for each attribute of the alternatives, bipolarity has been used as a conceptual tool for the representation of dual attributes. Wei et al. [10] came forward with hesitant bipolar fuzzy weighted aggregation operators as arithmetic and geometric operators. With the help of hesitant bipolar fuzzy weighted aggregation operators and geometric operators, Xu and Wei [11] gave dual hesitant bipolar fuzzy weighted aggregation operators and geometric operators. In other frameworks, AOs have been studied too. For example, Garg [12] gave a framework for linguistically prioritized aggregation operators.

More sophisticated aggregation operators were developed to improve the accuracy of the subsequent applications. For example, based on algebraic and Einstein t-conorm and t-norm [13], Hamacher t-conorm and t-norm [14] were developed to aggregate data for decision making. Peng and Luo [15] presented decision-making for China's stock market bubble warning. Aggregation operators based on Hamacher operations produce a transparent result in decision-making. Thus, inspired by these operators, Wei et al. [16] developed some induced geometric aggregation operators with intuitionistic fuzzy information and showed their applications to group decision making. Liu [17] proposed aggregation operators for interval-valued intuitionistic fuzzy fields. Further contributions were made by Akram et al. [18] with the introduction of a decision-making model using complex intuitionistic fuzzy Hamacher aggregation operators. Huang [19] put forward the idea of intuitionistic fuzzy Hamacher aggregation operators and illustrated their application in multiattribute decision making. After that, Hamacher aggregation operators were extended so that they could operate on Pythagorean fuzzy sets. Wu and Wei [20] presented the Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Akram et al. [21] used these operators in the complex fuzzy field and developed the idea of a hybrid method for complex Pythagorean fuzzy decision making. Akram et al. [22] designed a decision-making model with the help of complex picture fuzzy Hamacher aggregation operators. After that, bipolar fuzzy Hamacher arithmetic and geometric operators were also developed by Wei et al. [23]. Akram et al. [24] developed q-rung orthopair fuzzy graphs under Hamacher operators.

As many real-life situations contain multiinformation, Chen et al. [25] developed the m-polar fuzzy (m F) set, which enables decision-makers to manipulate multipolar data for the purpose of decision-making. Further, Jana and Pal [26] presented the m-polar fuzzy operators and their application in the multiple-attribute decision-making process. Hwang and Yoon [27] presented the concept of multi-objective decision making-methods with applications. Nevertheless, Hamacher operations were not originally designed to collect information in the form of intuitionistic fuzzy numbers (IFNs), Pythagorean fuzzy numbers (PFNs), bipolar fuzzy numbers [28], or m F numbers (m FNs). Waseem et al. [29] used Hamacher operators to aggregate data in a m-polar fuzzy setting. Akram et al. [30, 31] adapted respective mathematical models to approach decisions in m-polar fuzzy environments.

Most people want to express themselves with common terms like "magnificent," "superb," "best," "better," "poor," and "worst" to gauge some attributes. These assessments of an object's properties should then be used in MADM. Thus, the aggregation operators that collect information in a linguistic form are crucially essential. By using the 2-tuple linguistic (2 TL) tool, we can prevent the loss of data and get more transparent results in decision making. Firstly, Herrera and Martinez [32] introduced the idea of 2 TL representation, which is the most successful tool to take on linguistics decision-making issues. For further notions and applications, the readers are suggested to [33–40]. The main goal of this article is the aggregation of 2-tuple linguistic information by using Hamacher operators and their application in decision-making.

1.1. Motivation and Contribution. The motivation of this work is described as follows:

(i) The justification of any reliable choice in a problem formulated with 2 TL m F information is a highly complicated MADM issue. Nevertheless, the proposed MADM model provides significant results through convincing arguments.

(ii) The field of application of 2 TL m FS is enormous, as this model combines the benefits of both 2 TL and m-polar fuzzy sets. However, the treatment of linguistic techniques with multipolar fuzzy situations, particularly the 2 TL m F MADM approach, remains a challenge that we take up in this article.

(iii) The toolbox that helps us for this purpose includes aggregation operators. Taking this into consideration, aggregation operations are capable of providing valid data combinations in the form of 2 TL m Fs.

(iv) Hamacher aggregation operators are a straightforward tool, easy to apply to real-life MADM problems based on the 2 TL m F environment.

(v) The previously existing techniques, which are designed to take over MCDM problems, are restricted to dealing only with the m-polar fuzzy information. These techniques are unable to take into account linguistic information. So, this may cause a loss of information, which typically leads to undesired results. Thus, existing technical hindrances can be sorted out by using the newly proposed work.

Thus, to choose the best alternative, our 2 TL m F methodology relies on Hamacher AOs. As compared to other plans of action, the developed operators have three major advantages. Firstly, we can make use of 2 TL m F information, which is an asset in decision-making problems as explained above. Secondly, a single parameter suffices to make the methodology flexible while preserving its transparency and simplicity. Further, the decision-making issues are not affected by varying the parameters. Thirdly, the use of Hamacher aggregation operators for 2 TL m F information in MADM produced significant results. To operate in complicated situations with a real background, as in the case of the selection of the best place for a thermal power station, the proposed operators are very affordable.

The major contributions of this research paper are

(i) The generalization of m F Hamacher operators to 2 TL m F Hamacher operators. Some fundamental properties are given, including their proofs and explanations. These operators are more flexible and produce transparent results by aggregating m-polar fuzzy data with linguistic information.

(ii) In order to undertake decision-making problems, an algorithm is developed for 2 TL multipolar information.

(iii) Lastly, the validity, versatility, and traits of the proposed operators are investigated by a comparative study with existing techniques.

1.2. Structure of Paper. The structure of this research work is as follows: in Section 2, we perform a basic revision of some concepts about 2 TL and m F sets, which are properly described. Section 3, contains the study of some operators, namely 2 TL m FHWA, 2 TL m FHOWA, and 2 TL m FHHA, with some basic properties. In Section 4, we investigate additional operators like 2 TL m FHWG, 2 TL m FHOWG, and 2 TL m FHHG. Again, their fundamental properties are studied, and examples are given. In Section 5, a procedure is developed to tackle multicriteria decision-making issues that involve 2 TL m F information. The procedure takes advantage of the 2 TL m FHA and 2 TL m FHG operators. In the next Section 6, numerical work is done for the selection of the best location for a thermal power station by using 2 TL m FHA and 2 TL m FHG operators. It also includes a study about the influence of parameters on the decision. Section 7 contains a comparative study with previously existing methods, which shows the applicability and strength of our method. It also outlines the advantages and limitations of the proposed work. Section 8 contains some concluding remarks with future research directions. The structure of the proposed research article is displayed in Figure 1.

2. Preliminaries

This section reviews some basic definitions that are necessary for this paper.

Definition 1 (see [32, 34]). Let a set $S = \{s_i | i = 0, 1, \ldots, t\}$ of odd numbers of linguist terms, where s_i indicates the probable linguistic term for the linguistic variables. For instance, a linguistic term set S having seven terms can be described as follows:

S = \{s_0 = none, s_1 = very low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very high, s_6 = perfect\}.

If $s_i, s_k \in S$, then the set S meets with the following characteristics:

(i) Ordered set: $s_i > s_k$, if and only if $i > k$.

(ii) Max operator: $\max(s_i, s_k) = s_i$, if and only if $i \geq k$.

(iii) Min operator: $\min(s_i, s_k) = s_i$, if and only if $i \leq k$.

(iv) Negation: Neg $(s_i) = s_k$ such that $k = t - i$.

Herrera and Martinez [32], introduced 2 TL representation model based on the idea of symbolic translation, which is useful for representing the linguistic assessment information by means of a 2-tuple (s_i, ρ_i).

where

(i) s_i is a linguistic label for a predefined linguistic term set S.

(ii) ρ_i is called symbolic translation and $\rho_i \in [-1/2, 1/2)$.

Definition 2 (see [32]). Let φ be the result of an aggregation of the indices of a set of labels assessed in a linguistic term set S, i.e., the result of a symbolic aggregation operation, $\varphi \in [1, t]$, where t is the cardinality of S. Let $i = \text{round}(\varphi)$ and $\rho = \varphi - i$ be two values, such that, $i \in [1, t]$ and $\rho \in [-1/2, 1/2)$ then ρ is called a symbolic translation.

Definition 3 (see [32]). Let $S = \{s_i | i = 1, \ldots, t\}$ be a linguistic term set and $\varphi \in [1, t]$ be a number value representing the aggregation result of linguistic symbolic. Then the function Δ used to obtain the 2-tuple linguistic information equivalent to φ is defined as

$$\Delta\colon [1, t] \longrightarrow S \times \left[-\frac{1}{2}, \frac{1}{2}\right),$$

$$\Delta(\varphi) = \begin{cases} s_i, i = \text{round}(\varphi), \\ \rho = \varphi - i, \rho \in \left[-\frac{1}{2}, \frac{1}{2}\right), \end{cases} \tag{1}$$

Definition 4 (see [32]). Let $S = \{s_i | i = 1, \ldots, t\}$ be a linguistic term set and (s_i, ρ_i) be a 2-tuple, there exists a function Δ^{-1} that restores the 2-tuple to its equivalent numerical value $\varphi \in [1, t] \subset R$, where

$$\Delta^{-1}\colon S \times \left[-\frac{1}{2}, \frac{1}{2}\right) \longrightarrow [1, t],$$
$$\Delta^{-1}(s_i, \rho) = i + \rho = \varphi, \tag{2}$$

Definition 5 (see [32, 34]). Let us consider (s_k, ρ_1) and (s_l, ρ_2) be two 2 TL values. Then,

(1) For $k < l$, we have, (s_k, ρ_1) is less than (s_l, ρ_2)

(2) If $k = l$, then

(i) For $\rho_1 = \rho_2$, implies that (s_l, ρ_1) and (s_k, ρ_2) are same.

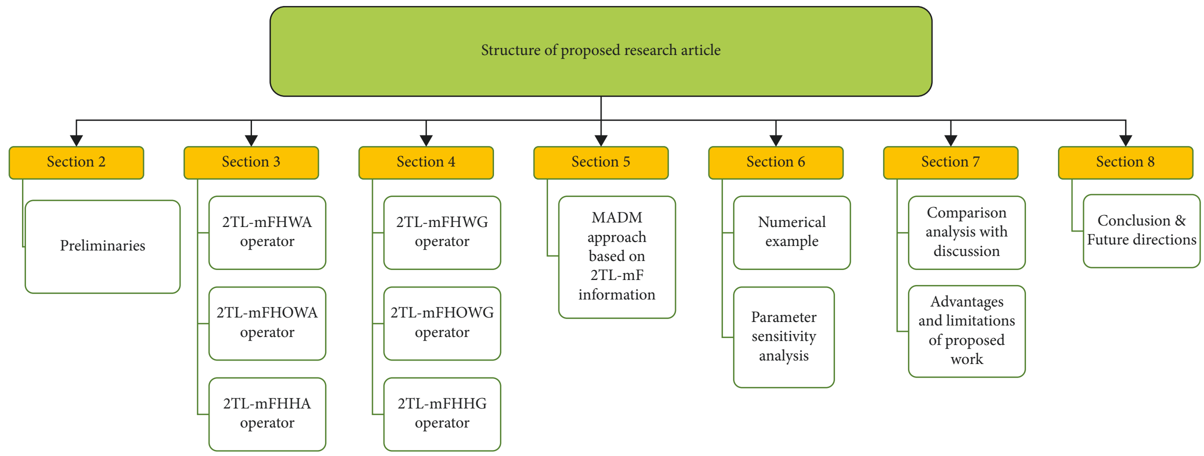

Figure 1: Structure of the proposed research work.

(ii) For $\rho_1 < \rho_2$, implies that (s_l, ρ_1) is less than (s_k, ρ_2)

(iii) For $\rho_1 > \rho_2$, implies that (s_l, ρ_1) is greater than (s_k, ρ_2)

Chen et al. [25] first considered the notion of m-polar fuzzy sets. The membership grade of m-polar fuzzy set belongs to the interval $[0,1]^m$, and it stands for m different divisions of an attribute.

Definition 6 (see [25]). An mF set C on a nonempty set X is defined as a mapping $C: X \longrightarrow [0,1]^m$. The representation of membership value for every element $x \in X$ is denoted as

$$C = \left(p_1{\circ}\mathrm{C}(\mathrm{x}), \mathrm{p}_2{\circ}\mathrm{C}(\mathrm{x}), \ldots, \mathrm{p}_\mathrm{m}{\circ}\mathrm{C}(\mathrm{x})\right). \tag{3}$$

Here $p_i{\circ}\mathrm{C}: [0,1]^\mathrm{m} \longrightarrow [0,1]$ is the i−th projection mapping.

Notice that, $[0,1]^m$ (m-th power of $[0,1]$) is a Poset with the pointwise order $\leq$, where m is an arbitrary ordinal number (we make the convention that $m = \{n | n < m\}$ when $m > 0$), $\leq$ is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$ $(x, y \in [0,1]^m)$, and p_i: $[0,1]^m \longrightarrow [0,1]$ is the i−th projection mapping $(i \in m)$. Where in $[0,1]^m$, the greatest value is $1 = (1, 1, \ldots, 1)$ and the smallest value is $0 = (0, 0, \ldots, 0)$. For convenience, $\widehat{C} = (p_1{\circ}\mathrm{C}, \ldots, \mathrm{p}_\mathrm{m}{\circ}\mathrm{C})$ is the representation of mF number.

Definition 7 (see [25]).The accuracy function H of an mFNs, $\widehat{C} = (p_1{\circ}\mathrm{C}, \ldots, \mathrm{p}_\mathrm{m}{\circ}\mathrm{C})$ is defined as

$$H(\widehat{C}) = \frac{1}{m}\left(\sum_{r=1}^{m} (-1)^r \left(p_r{\circ}\mathrm{C} - 1\right)\right), H(\widehat{C}) \in [0,1], \tag{4}$$

Thus, arbitrarily, for any m-polar fuzzy numbers $\widehat{C}$, $S(\widehat{C}), H(\widehat{C}) \in [0,1]$.

Definition 8 (see [25]). Let $\widehat{C}_1 = (p_1{\circ}\mathrm{C}_1, \ldots, \mathrm{p}_\mathrm{m}{\circ}\mathrm{C}_1)$, and $\widehat{C}_2 = (p_1{\circ}\mathrm{C}_2, \ldots, \mathrm{p}_\mathrm{m}{\circ}\mathrm{C}_2)$ be two m-polar fuzzy numbers. Then

(1) $\widehat{C}_1 < \widehat{C}_2$, if $S(\widehat{C}_1) < S(\widehat{C}_2)$.

(2) $\widehat{C}_1 > \widehat{C}_2$, if $S(\widehat{C}_1) > S(\widehat{C}_2)$.

(3) $\widehat{C}_1 = \widehat{C}_2$, If $S(\widehat{C}_1) = S(\widehat{C}_2)$ and $H(\widehat{C}_1) = H(\widehat{C}_2)$.

(4) $\widehat{C}_1 < \widehat{C}_2$, if $S(\widehat{C}_1) = S(\widehat{C}_2)$, but $H(\widehat{C}_1) < H(\widehat{C}_2)$.

(5) $\widehat{C}_1 > \widehat{C}_2$, if $S(\widehat{C}_1) = S(\widehat{C}_2)$, but $H(\widehat{C}_1) > H(\widehat{C}_2)$.

The nomenclature of the proposed research terms is given in Table 1.

3. 2 TL m F Hamacher Aggregation Operators

We first define the concept of 2-tuple linguistic m-polar fuzzy sets and some basic operations.

Definition 9. A 2TLmFSs Ψ on a nonempty set Y is defined as

$$\widehat{\Psi} = \left\{ < y, \left(\left(s_{\psi_1}(y), \rho_1(y)\right), \left(s_{\psi_2}(y), \rho_2(y)\right), \ldots, \left(s_{\psi_m(y)}, \rho_m(y)\right)\right) >: y \in Y\right\}, \tag{5}$$

where $(s_{\psi_i}(y), \rho_i(y))$, represent the membership degrees, with the conditions $s_{\psi_i}(y) \in \Psi$, $\rho_i(y) \in [-0.5, 0.5)$, $0 \leq \Delta^{-1}(s_{\psi_i}(y), \rho_i(y)) \leq t, i = 1, 2, \ldots, m$. For convenience, we say $\xi = ((s_{\psi_1}, \rho_1), (s_{\psi_2}, \rho_2), \ldots, (s_{\psi_m}, \rho_m))$, a 2-tuple linguistic m-polar fuzzy number, where $0 \leq \Delta^{-1}(s_{\psi_i}, \rho_i) \leq t$, $i = 1, 2, \ldots, m$.

Definition 10. The *score function* S of a 2 TL m-polar fuzzy number $\xi = ((s_{\psi_1}, \rho_1), (s_{\psi_2}, \rho_2), \ldots, (s_{\psi_m}, \rho_m))$, is defined as

$$S(\widehat{\xi}) = \Delta\left(\frac{t}{m}\sum_{r=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\psi_r}, \rho_r\right)}{t}\right)\right), \Delta^{-1}(S(\widehat{\xi})) \in [0, t], \tag{6}$$

Table 1: Nomenclature of proposed terms.

Abbreviation	*Description*
2TL m FN	2-Tuple linguistic m-polar fuzzy number
2TL m FHWA	2-Tuple linguistic m F Hamacher weighted average operators
2TL m FHOWA	2-Tuple linguistic m-polar fuzzy Hamacher ordered weighted average operators
2TL m FHHA	2-Tuple linguistic m-polar fuzzy Hamacher hybrid average operators
2TL m FHWG	2-Tuple linguistic m-polar fuzzy Hamacher weighted geometric operators
2TL m FHOWG	2-Tuple linguistic m-polar fuzzy Hamacher ordered weighted geometric operators
2TL m FHHG	2-Tuple linguistic m-polar fuzzy Hamacher hybrid average geometric
(s_{ψ_m}, ρ_m)	2-Tuple linguistic m-polar fuzzy number
$S(\widehat{\xi})$	Score function of a 2 TL m-polar fuzzy number
$H(\widehat{\xi})$	Accuracy function of a 2 TL m-polar fuzzy number
A_k	Alternatives
ζ_i	Attributes
ϕ_j	Weight of attributes

Definition 11. The *accuracy function* H of a $2TLm$-polar fuzzy number $\xi = ((s_{\psi_1}, \rho_1), (s_{\psi_2}, \rho_2), \ldots, (s_{\psi_m}, \rho_m))$, is defined as

$$H(\widehat{\xi}) = \Delta\left(\frac{t}{m}\sum_{r=1}^{m}(-1)^r\left(\left(\frac{\Delta^{-1}(s_{\psi_r}, \rho_r)}{t}\right) - 1\right)\right), \left(\Delta^{-1}H(\widehat{\xi})\right) \in [0, t]. \tag{7}$$

Definition 12. Let $\xi_1 = ((s_{\psi_1^1}, \rho_1^1), (s_{\psi_2^1}, \rho_2^1), \ldots, (s_{\psi_m^1}, \rho_m^1))$, and $\xi_2 = ((s_{\psi_1^2}, \rho_1^2), (s_{\psi_2^2}, \rho_2^2), \ldots, (s_{\psi_m^2}, \rho_m^2))$, be two 2-tuple linguistic m-polar fuzzy numbers. Then we define operations on 2-tuple linguistic m-polar fuzzy numbers as follows:

(1) $\xi_1 \oplus \xi_2 = (\Delta(t(\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t + \Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t - \Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/\ t.\Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t)), \ldots (\Delta(t(\Delta^{-1}(s_{\psi_m^1},\ \rho_m^1)/t + \Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t - \Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t.\Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t)))$

(2) $\xi_1 \otimes \xi_2 = (\Delta(t(\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t.\Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t)), \ldots, \Delta(t(\Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t.\ \Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t)))$

(3) $\alpha\xi = (\Delta(t(1 - (1 - \Delta^{-1}(s_{\psi_1}, \rho_1)/t)^{\alpha})), \ldots, \quad \Delta(t(1 - (1 - \Delta^{-1}(s_{\psi_m}, \rho_m)/t)^{\alpha}))), \alpha > 0$,

(4) $\xi^{\alpha} = (\Delta(t(\Delta^{-1}(s_{\psi_1}, \rho_1)/t)^{\alpha}), \ldots, \Delta(t(\Delta^{-1}(s_{\psi_m},\ \rho_m)/t)^{\alpha})), \alpha > 0$,

(5) $\xi^{c} = (\Delta(t - \Delta^{-1}(s_{\psi_1}, \rho_1)), \ldots, \Delta(t - \Delta^{-1}(s_{\psi_m}, \rho_m)))$, $\alpha > 0$,

(6) $\xi_1 \subseteq \xi_2$, if and only if $\Delta(t(\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t \leq \Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t)), \ldots, \Delta(t(\Delta^{-1}\ (s_{\psi_m^1}, \rho_m^1)/t \leq \Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t))$,

(7) $\xi_1 \bigcup \xi_2 = \Delta(t(\max(\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t, \Delta^{-1}\quad (s_{\psi_1^2}, \rho_1^2)/t))), \ \ldots, \Delta(t(\max\ (\Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t, \Delta^{-1}(s_{\psi_m^2},\ \rho_m^2)/t)))$,

(8) $\xi_1 \cap \xi_2 = \Delta(t(\min(\Delta^{-1}(s_{\psi_1^1},\ \rho_1^1)/t, \Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t)), \ldots, \Delta(t(\min(\Delta^{-1}\ (s_{\psi_m^1}, \rho_m^1)/t, \Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t))$.

We now define Hamacher operations for 2-tuple linguistic m-polar fuzzy numbers.

Definition 13. Let $\widetilde{\xi_1} = \left\{(s_{\psi_1^1}, \rho_1^1), \ldots, (s_{\psi_m^1}, \rho_m^1)\right\}$, $\widetilde{\xi_2} = \left\{(s_{\psi_1^2}, \rho_1^2), \ldots, (s_{\psi_m^2}, \rho_m^2)\right\}$ and $\widetilde{\xi} = \left\{(s_{\psi_1}, \rho_1), \ldots, (s_{\psi_m}, \rho_m)\right\}$ be 2-tuple linguistic m-polar fuzzy numbers. Then, the basic Hamacher operations for 2-tuple linguistic m-polar fuzzy numbers with $\lambda > 0$ is defined as

(1) $\xi_1 \oplus \xi_2 = (\Delta(t((\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t + \Delta^{-1}\ (s_{\psi_1^2}, \rho_1^2)/t - \Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t.\Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t - (1 - \lambda)\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t.\Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t)/(1 - (1 - \lambda)\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t.\Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t))), \ldots$

$\Delta(t((\Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t + \Delta^{-1}\quad (s_{\psi_m^2}, \rho_m^2)/t - \Delta^{-1}(s_{\psi_m^1}, \rho_m^1)\ /t.\ \Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t - (1 - \lambda)\Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t.\ \Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t)/(1 - (1 - \lambda)\Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t.\Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t))))$,

(2) $\xi_1 \otimes \xi_2 = (\Delta(t((\Delta^{-1}\ (s_{\psi_1^1}, \rho_1^1)/t.\Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t)/(\lambda + (1 - \lambda)(\Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t + \Delta^{-1}(s_{\psi_1^2},\quad \rho_1^2)/t - \Delta^{-1}(s_{\psi_1^1}, \rho_1^1)/t.\Delta^{-1}(s_{\psi_1^2}, \rho_1^2)/t))) \quad \ldots \Delta(t((\Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t.\Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t)/(\lambda + (1 - \lambda)(\Delta^{-1}(s_{\psi_m^1}, \rho_m^1)/t + \Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t - \Delta^{-1}(s_{\psi_m^1},\ \rho_m^1)/t.\Delta^{-1}(s_{\psi_m^2}, \rho_m^2)/t))))$,

(3) $\alpha\xi = (\Delta(t((1 + (\lambda - 1)\Delta^{-1}(s_{\psi_1}, \rho_1)/t)\ ^{\alpha} - (1 - \Delta^{-1}(s_{\psi_1}, \rho_1)/t)^{\alpha}/(1 + (\lambda - 1)\Delta^{-1}(s_{\psi_1}, \rho_1)/t)^{\alpha} + (\lambda - 1)(1 - \Delta^{-1}(s_{\psi_1}, \rho_1)/t)^{\alpha})), \ldots \Delta(t \quad ((1 + (\lambda - 1)\Delta^{-1}(s_{\psi_m},\quad \rho_m)/t)^{\alpha} - (1 - \Delta^{-1}(s_{\psi_m}, \rho_m)/t)^{\alpha}/(1 + (\lambda - 1)\Delta^{-1}(s_{\psi_m}, \rho_m)/t)^{\alpha}\ + (\lambda - 1)(1 - \Delta^{-1}(s_{\psi_m}, \rho_m)/t)^{\alpha})))$, $\alpha > 0$

(4) $\xi^{\alpha} = (\Delta(t(\lambda(\Delta^{-1}(s_{\psi_1}, \rho_1)/t)^{\alpha}/(1 + (\lambda - 1) \quad (1 - \Delta^{-1} \quad (s_{\psi_1}, \rho_1)/t))^{\alpha} + (\lambda - \quad 1)(\Delta^{-1}(s_{\psi_1}, \rho_1)/t)^{\alpha})), \ldots \Delta(t(\lambda(\Delta^{-1} \quad (s_{\psi_m}, \rho_m)/t)^{\alpha}/ \quad (1 + (\lambda - 1)(1 - \Delta^{-1}(s_{\psi_m}, \rho_m)/t))^{\alpha} + (\lambda - 1)\ (\Delta^{-1}(s_{\psi_m}, \rho_m)/t)^{\alpha})))$, $\alpha > 0$

Example 14. Let $\xi_1 = \{(s_3, 0.2), (s_2, 0.5), (s_4, 0.0)\}$ and $\xi_2 = \{(s_3, 0.8), (s_4, 0.0), (s_2, 0.6)\}$ be 2-tuple linguistic 3-polar fuzzy numbers. Then for $\lambda = 3$,

$$\xi_1 \oplus \xi_2 = \left(\Delta\left(4\left(\frac{\Delta^{-1}(s_3,0.2)/4 + \Delta^{-1}(s_3,0.8)/4 - \Delta^{-1}(s_3,0.2)/4.\Delta^{-1}(s_3,0.8)/4 - (1-3)\Delta^{-1}(s_3,0.2)/4.\Delta^{-1}(s_3,0.8)/4}{1-(1-3)\Delta^{-1}(s_3,0.2)/4.\Delta^{-1}(s_3,0.8)/4}\right)\right)\right)$$

$$\Delta\left(4\left(\frac{\Delta^{-1}(s_2,0.5)/4 + \Delta^{-1}(s_4,0.0)/4 - \Delta^{-1}(s_2,0.5)/4.\Delta^{-1}(s_4,0.0)/4 - (1-3)\Delta^{-1}(s_2,0.5)/4.\Delta^{-1}(s_4,0.0)/4}{1-(1-3)\Delta^{-1}(s_2,0.5)/4.\Delta^{-1}(s_4,0.0)/4}\right)\right) \tag{8}$$

$$\Delta\left(4\left(\frac{\Delta^{-1}(s_4,0.0)/4 + \Delta^{-1}(s_2,0.6)/4 - \Delta^{-1}(s_4,0.0)/4.\Delta^{-1}(s_2,0.6)/4 - (1-3)\Delta^{-1}(s_4,0.0)/4.\Delta^{-1}(s_2,0.6)/4}{1-(1-3)\Delta^{-1}(s_4,0.0)/4.\Delta^{-1}(s_2,0.6)/4}\right)\right)$$

$$= \{(s_4,-0.0158), (s_4,0.000), (s_4,0.01)\}.$$

Thus, $\xi_1 \oplus \xi_2$ is again a 2-tuple linguistic 3-polar fuzzy number. So, the closure law is satisfied.

Thus, in a similar pattern, the closure law is verified for all the above-defined Hamacher operations for 2-tuple linguistic m-polar fuzzy numbers.

Definition 14. Let $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ be a set of 2 TL m F numbers, where $j = 1, 2, \ldots, n$. Then, an 2 TL m F Hamacher weighted average operator is a mapping $2TL$ m FHWA: $\widehat{\xi}^n \longrightarrow \widehat{\xi}$, whose domain is the family of 2 TL m F numbers $\widehat{\xi}^n$, which is defined as,

$$2TLm\text{FHWA}_\phi\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = \oplus_{j=1}^n\left(\phi_j \widehat{\xi}_j\right), \tag{9}$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is the weight vector representation for $\widehat{\xi}_j$, for each 'j', $j = 1, 2, \ldots, n$, with $\phi_j > 0$ and $\sum_{j=1}^n \phi_j = 1$.

Example 1. Let $\widehat{\xi}_1 = \{(s_3, 0.2), (s_2, 0.5), (s_4, 0.7), (s_1, 0.3)\}$, $\widehat{\xi}_2 = \{(s_3, 0.8), (s_4, 0.6), (s_2, 0.6), (s_1, 0.4)\}$, and $\widehat{\xi}_3 = \{(s_6, 0), (s_2, 0.2), (s_3, 0.4), (s_1, 0.5)\}$ be 2-tuple linguistic 4-polar fuzzy numbers with a weight vector $\phi = (0.3, 0.5, 0.2)^T$. Then, for $\lambda = 3$,

$$\begin{aligned} 2TLmFWA_\phi\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) &= \oplus_{j=1}^n\left(\phi_j \widehat{\xi}_j\right), \\ &= \left(\Delta\left(t\left(1-\prod_{j=1}^n\left(1-\frac{\Delta^{-1}\left(s_{\psi_1^j}, \rho_1^j\right)}{t}\right)^{\phi_j}\right)\right), \Delta\left(t\left(1-\prod_{j=1}^n\left(1-\frac{\Delta^{-1}\left(s_{\psi_2^j}, \rho_2^j\right)}{t}\right)^{\phi_j}\right)\right), \ldots,\right. \\ &\quad \left.\Delta\left(t\left(1-\prod_{j=1}^n\left(1-\frac{\Delta^{-1}\left(s_{\psi_m^j}, \rho_m^j\right)}{t}\right)^{\phi_j}\right)\right)\right). \end{aligned} \tag{10}$$

Theorem 1. *Let $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ be a set of 2 TL m F numbers, where $j = 1, 2, \ldots, n$. The assembled values of these 2 TL m F numbers using the 2 TL m FHWA operator is also 2 TL m F numbers, given as*

$$2TLm\mathrm{FHWA}_{\phi}\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\oplus_{j=1}^{n}\left(\phi_j\hat{\xi}_j\right),$$

$$=\left(\Delta\left(t\left(\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t^{\phi_j}-\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}/\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t^{\phi_j}+(\lambda-1)\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}\right)\right),\ldots\right.$$
$$\left.\Delta\left(\left(t\left(\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t^{\phi_j}-\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}/\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t^{\phi_j}+(\lambda-1)\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}\right)\right)\right) \tag{11}$$

Proof. We use mathematical induction to prove it. □

Case 1. Let us take $n=1$, by using *Equation* (4), we obtained

$$2TLm\mathrm{FHWA}_{\phi}\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\hat{\xi} \tag{12}$$

Thus, (10) holds for $n=1$.

Case 2. Next, we suppose that the result is true for $n=\mathrm{k}$, where $k\in\mathbb{N}$ (N: natural numbers), we obtain

$$\hat{\xi}^{-}\leq 2TLm\mathrm{FHWA}_{\phi}\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)\leq\hat{\xi}^{+}, \tag{13}$$

for $n=k+1$,

$$2TLm\mathrm{FHWA}_{\phi}\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\widehat{\xi_{k+1}}\right)=\oplus_{j=1}^{k}\left(\phi_j\hat{\xi}_j\right)\oplus\left(\phi_{k+1}\widehat{\xi_{k+1}}\right),$$

$$2TLm\mathrm{FHWA}_{\phi}\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)\leq 2TLm\mathrm{FHWA}_{\phi}\left(\hat{\xi}_1',\hat{\xi}_2',\ldots,\hat{\xi}_n'\right). \tag{14}$$

Thus, (10) holds for $n=k+1$. Conclusively, the result holds for any $n\in\mathbb{N}$.

Remark 1. For $\lambda=1$, 2TLmFHWA operator reduces to 2 TL m F weighted averaging (2 TL m FWA) operator given as follows:

$$2TLmFWA_{\phi}\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\oplus_{j=1}^{n}\left(\phi_j\hat{\xi}_j\right),$$
$$=\left(\Delta\left(t\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)}{t}\right)^{\phi_j}\right)\right),\Delta\left(t\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{\psi_2^j},\rho_2^j\right)}{t}\right)^{\phi_j}\right)\right),\ldots,\Delta\left(t\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)}{t}\right)^{\phi_j}\right)\right)\right). \tag{15}$$

2. For $\lambda=2$, 2 TL m FHWA operator reduces to 2TLmF Einstein weighted averaging (2 TL m FEWA) operator as follows:

$$2TLm\mathrm{FEWA}_{\phi}\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\oplus_{j=1}^{n}\left(\phi_j\hat{\xi}_j\right)$$
$$=\left(\Delta\left(t\left(\frac{\prod_{j=1}^{n}\left(1+\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}-\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}}{\prod_{j=1}^{n}\left(1+\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}+\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}}\right)\right),\ldots\right.$$
$$\left.\Delta\left(t\left(\frac{\prod_{j=1}^{n}\left(1+\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}-\prod_{j=1}^{n}\left(1+\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}}{\prod_{j=1}^{n}\left(1+\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}+\prod_{j=1}^{n}\left(1+\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}}\right)\right)\right). \tag{16}$$

Theorem 2. *Idempotency. Let* $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ *be a set of 2 TL m F numbers, where* $j = 1, 2, \ldots, n$. *If all these numbers are equal, that is,* $\widehat{\xi}_j = \widehat{\xi}$, $\forall$ *'j' varies 1 to n, then we have*

$$2TLm\text{FHWA}_\phi\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = \widehat{\xi}. \tag{17}$$

Proof. Since $2TLm\text{FHWA}_\phi((s_{\psi_1^j}, \rho_1^j), (s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_1^j}, \rho_1^j)) = \widehat{\xi}$, where $j = 1, 2, \ldots, n$, then by using *equation* (4), we get

$$2TLm\text{FEOWA}_w\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = \oplus_{j=1}^n\left(w_j\widehat{\xi_{\sigma(j)}}\right),$$

$$= \left(\Delta\left(t\left(\frac{\prod_{j=1}^n\left(1 + \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j} - \prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^n\left(1 + \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j} + \prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right), \ldots \right. \tag{18}$$

$$\left.\Delta\left(t\left(\frac{\prod_{j=1}^n\left(1 + \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j} - \prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^n\left(1 + \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_{\sigma(j)}} + \prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right).$$

Hence, $2TLm\text{FHWA}_\phi(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n) = \widehat{\xi}$ holds only if we use $\widehat{\xi}_j = \widehat{\xi}$ where $\forall j = 1, 2, \ldots, n$

Further, we will discuss the remaining properties, namely, boundedness and monotonicity, and their proofs are directly followed by definitions. □

Theorem 3. *Let* $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$, *be a set of 2 TL m F numbers, where 'j' varies from 1 to n,* $\widehat{\xi}^- = \cap_{j=1}^n \widehat{\xi}_j$ *and* $\widehat{\xi}^+ = \cup_{j=1}^n \widehat{\xi}_j$, *then*

$$\widehat{\xi}^- \leq 2TLm\text{FHWA}_\phi\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) \leq \widehat{\xi}^+, \tag{19}$$

Theorem 4. *Let* $\widehat{\xi}_j$ *and* $\widehat{\xi}_j'$, $j = 1, 2, \ldots, n$ *be the two sets of 2 TL m F numbers. If* $\widehat{\xi}_j \leq \widehat{\xi}_j'$, *then*

$$2TLm\text{FHWA}_\phi\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) \leq 2TLm\text{FHWA}_\phi\left(\widehat{\xi}_1', \widehat{\xi}_2', \ldots, \widehat{\xi}_n'\right). \tag{20}$$

We now propose the 2 TL m F Hamacher ordered weighted average (2 TL m FHOWA) operator.

Definition 15. Let $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ be the set of 2 TL m F numbers where $j = 1, 2, \ldots, n$. Then, a 2 TL m FHOWA operator is a mapping $\widehat{\xi}^n \longrightarrow \widehat{\xi}$ with a weight vector

$w = (w_1, w_2, \ldots, w_n)^T$, $w_j \in (0, 1]$ and $\sum_{j=1}^n w_j = 1$. Then,

$$2TLm\text{FHOWA}_\phi\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = \oplus_{j=1}^n\left(w_j\widehat{\xi}_{\sigma(j)}\right), \tag{21}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is the permutation of the indices $j = 1, 2, \ldots, n$, for which $\widehat{\xi}_{\sigma(j-1)} \geq \widehat{\xi}_{\sigma(j)}$, $\forall j = 1, 2, \ldots, n$.

Theorem 6. *Let* $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ *be a set for 2 TL m F numbers, where 'j' varies from 1 to n. Then the assembled values of these 2 TL m F numbers using the 2 TL m FHOWA operator is again 2 TL m F numbers, given as*

$$2TLm\text{FHOWA}_\phi\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = \underset{j=1}{\overset{n}{\oplus}}\left(w_j\widehat{\xi}_{\sigma(j)}\right)$$

$$= \left(\Delta\left(t\left(\frac{\prod_{j=1}^n (1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)^{w_j}/t - \prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^n (1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t^{w_j} + (\lambda - 1)\prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right), \ldots \right. \tag{22}$$

$$\left.\Delta\left(t\left(\frac{\prod_{j=1}^n (1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_{\sigma(j)}\right)/t^{w_j} - \prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^n (1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_{\sigma(j)}\right)/t^{w_j} + (\lambda - 1)\prod_{j=1}^n\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right).$$

Proof. The proof of the theorem is directly followed by the similar arguments as used in Theorem 3.9, as mentioned above. □

Remark. 1. For $\lambda = 1$, 2 TL m FHOWA operator reduce to 2 TL m FOWA operator as follows:

$$2TLm\text{FOWA}_w\left(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n\right) = \mathop{\oplus}_{j=1}^{n}\left(w_j \hat{\xi}_{\sigma(j)}\right)$$

$$= \left(\Delta\left(t\left(1 - \prod_{j=1}^{n} \frac{\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)^{w_j}}{t}\right)\right), \Delta\left(t\left(1 - \prod_{j=1}^{n} \frac{\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)^{w_j}}{t}\right)\right), \ldots, \Delta\left(t\left(1 - \prod_{j=1}^{n} \frac{\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)^{w_j}}{t}\right)\right)\right). \tag{23}$$

2. For $\lambda = 2$, 2 TL m FHOWA operator reduce to 2 TL m FEOWA operator given as follows:

$$2TLm\text{FEOWA}_w\left(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n\right) = \mathop{\oplus}_{j=1}^{n}\left(\widehat{w_j \xi_{\sigma(j)}}\right)$$

$$= \left(\Delta\left(t\left(\frac{\prod_{j=1}^{n}\left(1 + \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j} - \prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}\left(1 + \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j} + \prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right), \ldots \right.$$
$$\left. \Delta\left(t\left(\frac{\prod_{j=1}^{n}\left(1 + \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j} - \prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}\left(1 + \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_{\sigma(j)}} + \prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right). \tag{24}$$

Theorem 7. *Idempotency. Let us consider* $\hat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ *a collection of 2 TL m F numbers, where* $j = 1, 2, \ldots, n$. *For the equality of all these numbers, in other words,* $\hat{\xi}_j = \hat{\xi}$, *where,* $\forall j = 1, 2, \ldots, n$, *then we have*

$$2TLm\text{FHOWA}_w\left(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n\right) = \hat{\xi}. \tag{25}$$

Proof. Since $2TLm\text{FHOWA}_w((s_{\psi_1^j}, \rho_1^j), (s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_1^j}, \rho_1^j)) = \hat{\xi}$, where $j = 1, 2, \ldots, n$. Then by using *Equation* (11), we obtain

$$2TLm\text{FEHA}_{w,\phi}\left(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n\right) = \oplus_{j=1}^{n}\left(w_j \tilde{\hat{\xi}}_{\sigma(j)}\right),$$

$$= \left(\Delta\left(t\left(\frac{\prod_{j=1}^{n}(1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t^{w_j} - \prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}(1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t^{w_j} + .\prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right), \ldots \right.$$
$$\left. \Delta\left(t\left(\frac{\prod_{j=1}^{n}(1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t^{w_j} - \prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}(1 + (\lambda - 1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t^{w_j} + \prod_{j=1}^{n}\left(1 - \Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right) \tag{26}$$

Hence, $2TLm\text{FHOWA}_\phi(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n) = \hat{\xi}$ holds only if $\hat{\xi}_j = \hat{\xi}$, $\forall j = 1, 2, \ldots, n$.

We state boundedness, monotonicity, and commutative properties without their proofs. □

Theorem 8. *Let* $\hat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$, *be a collection of 2 TL m F numbers, where* $j = 1, 2, \ldots, n$, $\hat{\xi}^- = \cap_{j=1}^{n} \hat{\xi}_j$, *and* $\hat{\xi}^+ = \cap_{j=1}^{n} \hat{\xi}_j$, *then*

$$\hat{\xi}^- \leq 2TLm\text{FHOWA}_\phi\left(\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n\right) \leq \hat{\xi}^+. \tag{27}$$

Theorem 9. *Let $\widehat{\xi}_j$ and $\widehat{\xi}'_j$, $j = 1, 2, \ldots, n$ be two sets of 2 TL m F numbers. If $\widehat{\xi}_j \leq \widehat{\xi}'_j$, then*

$$2TLm\text{FHOWA}_{\phi}\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) \leq 2TLm\text{FHOWA}_{\phi} \cdot \left(\widehat{\xi}'_1, \widehat{\xi}'_2, \ldots, \widehat{\xi}'_n\right), \tag{28}$$

Theorem 10. *Let $\widehat{\xi}_j$ and $\widehat{\xi}'_j$, $j = 1, 2, \ldots, n$ be two sets of 2 TL m F numbers, then*

$$2TLm\text{FHOWA}_{\phi}\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = 2TLm\text{FHOWA}_{\phi} \cdot \left(\widehat{\xi}'_1, \widehat{\xi}'_2, \ldots, \widehat{\xi}'_n\right), \tag{29}$$

where $\widehat{\xi}'_j$ is the arbitrary permutation of $\widehat{\xi}_j$.

Remark (i). In Definition 15 and 5, we observe that 2 TL m FHWA operators and 2 TL m FHOWA operators with 2 TL m F numbers and ordered arrangements of 2 TL m F numbers, respectively. (ii) We now propose another operator, namely 2 TL m F Hamacher hybrid averaging operator which combines the qualities of 2 TL m FHWA operator and 2 TL m FHOWA operator.

Definition 16. Let $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ be a set of 2 TL m F numbers, where $j = 1, 2, \ldots, n$. Then, a 2 TL m F Hamacher hybrid averaging (2 TL m FHHA) operator is defined as

$$2TLm\text{FHHA}_{w,\phi}\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = \oplus_{j=1}^{n}\left(w_j \widehat{\xi_{\sigma(j)}}\right), \tag{30}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ represent the associated weight vector of the 2 TL m F numbers $\widehat{\xi}_i$, instead of weighting the experts, for each 'i' varies from 1 to n, with $w_i \in (0,1]$ and $\sum_{i=1}^{n} w_i = 1$, $\widehat{\xi_{\sigma(j)}}$ is the jth biggest 2 TL m F numbers of the $\widehat{\xi}_i\,(i = 1, 2, \ldots, n)$ with $\widetilde{\widehat{\xi}}_{\sigma(i)} = (n\phi_i)\widehat{\xi_{\sigma(i)}}$, $(i = 1, 2, \ldots, n)$, $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is the weight vector for the ordered arguments, with $\phi_j \in (0,1]$, $\sum_{j=1}^{n} \phi_j = 1$ and n serves as the balancing coefficient.

The 2 TL weighted average operators integrate the importance of linguistic arguments. where 2 TL ordered weighted aggregation operators increase the worth of ordered positions of the linguistic arguments. There are different techniques to evaluate the weight vectors. For this, [6] proposed an interesting approach to evaluate the weight vector. In particular, we assign weight values to linguistic terms according to their importance in real-life issues.

Remark. We notice that, if we have $w = (1/n, 1/n, \ldots, 1/n)^T$, then 2 TL m FHHA operator convert into 2 TL m FHWA operator, when $\phi = (1/n, 1/n, \ldots, 1/n)^T$, then 2 TL m FHHA operator degenerates into 2 TL m FHOWA operator. Therefore, 2 TL m FHHA operator is the generalization of the operators, namely, 2 TL m FHWA and 2 TL m FHOWA, which explains the degrees and ordered arguments of the given 2 TL m F values.

Theorem 11. *Let $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ be a set of 2 TL m F numbers, where $j = 1, 2, \ldots, n$. Then the assembled values of these 2 TL m F numbers using the 2 TL m FHHA operator is again a 2 TL m F numbers, which is given as*

$$\begin{aligned}
2TLm\text{FHHA}_{w,\phi}\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) &= \bigoplus_{j=1}^{n}\left(w_j \widetilde{\widehat{\xi}}_{\sigma(j)}\right) \\
= \Bigg(\Delta\Bigg(t\Bigg(&\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t^{w_j} - \prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j} \Big/ \prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t^{w_j} \\
&+ (\lambda-1)\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}}, \rho_1^{\sigma(j)}\right)/t\right)^{w_j}\Bigg)\Bigg)\ldots \\
\cdot\,\Delta\Bigg(t\Bigg(&\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t^{w_j} - \prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j} \Big/ \prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t^{w_j} \\
&+ (\lambda-1)\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}}, \rho_m^{\sigma(j)}\right)/t\right)^{w_j}\Bigg)\Bigg)\Bigg)
\end{aligned} \tag{31}$$

Proof. The proof of the theorem is directly followed by similar arguments as used in Theorem 1. □

Remark 1. For $\lambda = 1$, 2 TL m FHHA operator reduces to $2TL$ m F hybrid averaging (2 TL m FHA) operator as follows:

$$2TLm\text{FHA}_{w,\phi}\left(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n\right) = \oplus_{j=1}^{n}\left(w_j \widetilde{\widehat{\xi}}_{\sigma(j)}\right),$$

$$= \left(\Delta\left(t\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)}{t}\right)^{w_j}\right)\right), \Delta\left(t\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{\psi_2^{\sigma(j)}},\rho_2^{\sigma(j)}\right)}{t}\right)^{w_j}\right)\right),\ldots,\Delta \right.$$
$$\left. \cdot\left(t\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)}{t}\right)^{w_j}\right)\right)\right). \quad (32)$$

2. For $\lambda = 2$, 2 TL m FHHA operator reduce to 2 TL m FEHA operator as follows:

$$2TLm\text{FEHA}_{w,\phi}\left(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n\right) = \underset{j=1}{\overset{n}{\oplus}}\left(w_j\widetilde{\widehat{\xi}}_{\sigma(j)}\right)$$
$$= \left(\Delta\left(t\left(\frac{\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t^{w_j} - \prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t^{w_j} + .\prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right),\ldots\Delta \right.$$
$$\left. \cdot\left(t\left(\frac{\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t^{w_j} - \prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}(1+(\lambda-1))\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t^{w_j} + \prod_{j=1}^{n}\left(1-\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right). \quad (33)$$

4. 2 TL m F Hamacher Geometric Aggregation Operators

We now propose 2 TL m FHWG operators, 2 TL m FHOWG operators, and 2 TL m FHHG operators.

Definition 17. Let $\widehat{\xi}_j = ((s_{\psi_1^j},\rho_1^j),\ldots,(s_{\psi_m^j},\rho_m^j))$ be a set of 2 TL m F numbers, where $j = 1,2,\ldots,n$. Then, an 2 TL m FHWG operator is a mapping, $2TL$ m FHWG $: \widehat{\xi}^n \longrightarrow \widehat{\xi}$, whose domain is the set of 2 TL m F numbers, is defined as follows:

$$2TLm\text{FHWG}_{\phi}\left(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n\right) = \otimes_{j=1}^{n}\left(\phi_j\widehat{\xi}_j\right)^{\phi_j}, \quad (34)$$

where $\phi = (\phi_1,\phi_2,\ldots,\phi_n)^T$ represent the weight vector of $\widehat{\xi}_j$, for each 'j' vary from 1 to n, with $\phi_j > 0$ and $\sum_{j=1}^{n}\phi_j = 1$.

Example 2. Let us consider $\widehat{\xi}_1 = \{(s_1,0.5),(s_2,0.3),(s_3,0.2)\}$, $\widehat{\xi}_2 = \{(s_2,0.4),(s_1,0.6),(s_5,0.1)\}$, and $\widehat{\xi}_3 = \{(s_4,0.3),(s_2,0.1),(s_1,0.3)\}$ be 2TL3FNs with the weight vector $\phi = (0.2,0.4,0.1)^T$, and we take $\lambda = 3$. Then, the assembled result can be calculated as follows:

$2TL$ m FHWG$_{\phi}(\xi_1,\xi_2\ldots\xi_n) = \otimes_{j=1}^{3}(\phi_j\xi_j)^{\phi_j}$.

Theorem 12. *Let $\widehat{\xi}_j = ((s_{\psi_1^j},\rho_1^j),\ldots,(s_{\psi_m^j},\rho_m^j))$ be a set of 2 TL m F numbers where 'j' varies from 1 to n. The assembled values of these 2 TL m F numbers using the 2 TL m FHWG operator is again a 2 TL m F numbers, given as follows:*

$$2TLm\text{FHWG}_{\phi}\left(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n\right) = \underset{j=1}{\overset{n}{\otimes}}\left(\phi_j\widehat{\xi}_j\right)^{\phi_j}$$
$$= \left(\begin{array}{l} \Delta\left(t\left(\dfrac{\lambda\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}}{\prod_{j=1}^{n}(1+(\lambda-1))\left(1-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j} + (\lambda-1)\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}}\right)\right),\ldots \\ \Delta\left(t\left(\dfrac{\lambda\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}}{\prod_{j=1}^{n}(1+(\lambda-1))\left(1-\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j} + (\lambda-1)\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}}\right)\right) \end{array} \right) \quad (35)$$

Proof. The proof can be followed easily by using mathematical induction.

We now state idempotency, boundedness, and monotonicity properties for 2 TL m FHG aggregation operators without their proofs. □

Theorem 13. *Let* $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j)$ *be a set of 2 TL m FNs where* $j = 1, 2, \ldots, n$*. For the equality of all these numbers, that is,* $\widehat{\xi}_j = \widehat{\xi}$, $\forall$ *'j' varies from 1 to n, then we have*

$$2TLm\text{FHWG}_\phi(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n) = \widehat{\xi}, \quad (36)$$

Theorem 14. *Let* $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ *be a set of 2 TL m FNs where* $j = 1, 2, \ldots, n$ *and* $\widehat{\xi}^- = \cap_{j=1}^n \widehat{\xi}_j$, $\widehat{\xi}^+ = \cup_{j=1}^n \widehat{\xi}_j$*, then*

$$\widehat{\xi}^- \leq 2TLm\text{FHWG}_\phi(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n) \leq \widehat{\xi}^+, \quad (37)$$

Theorem 15. *Let* $\widehat{\xi}_j$ *and* $\widehat{\xi}'_j$*, where 'j' varies from 1 to n, be a set of 2 TL m FNs. If* $\widehat{\xi}_j \leq \widehat{\xi}'_j$*, then*

$$2TLm\text{FHWG}_\phi(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n) \leq 2TLm\text{FHWG}_\phi \cdot (\widehat{\xi}'_1, \widehat{\xi}'_2, \ldots, \widehat{\xi}'_n), \quad (38)$$

Remark 1. For $\lambda = 1$, 2 TL m FHWG operator reduces to $2TL$ m F weighted geometric (2 TL m FWG) operator as follows:

$$2TLm\text{FWG}_\phi(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n) = \otimes_{j=1}^n (\phi_j \widehat{\xi}_j)^{\phi_j}$$
$$= \left(\Delta\left(t\prod_{j=1}^n \left(\frac{\Delta^{-1}(s_{\psi_1^j}, \rho_1^j)}{t}\right)^{\phi_j}\right), \Delta\left(t\prod_{j=1}^n \left(\frac{\Delta^{-1}(s_{\psi_2^j}, \rho_2^j)}{t}\right)^{\phi_j}\right), \ldots, \Delta\left(t\prod_{j=1}^n \left(\frac{\Delta^{-1}(s_{\psi_m^j}, \rho_m^j)}{t}\right)^{\phi_j}\right)\right). \quad (39)$$

2. For $\lambda = 2$, 2 TL m FHWG operator reduces to 2TLmF Einstein weighted geometric (2 TL m FEWG) operator as follows:

$$2TLm\text{FEWG}_\phi(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n) = \otimes_{j=1}^n (\phi_j \widehat{\xi}_j)^{\phi_j}$$
$$= \left(\Delta\left(t\left(\frac{2\prod_{j=1}^n \left(\Delta^{-1}(s_{\psi_1^j}, \rho_1^j)/t\right)^{\phi_j}}{\prod_{j=1}^n \left(2 - \Delta^{-1}(s_{\psi_1^j}, \rho_1^j)/t\right)^{\phi_j} + \prod_{j=1}^n \left(\Delta^{-1}(s_{\psi_1^j}, \rho_1^j)/t\right)^{\phi_j}}\right)\right)\right), \ldots \quad (40)$$
$$= \left(\Delta\left(t\left(\frac{2\prod_{j=1}^n \left(\Delta^{-1}(s_{\psi_m^j}, \rho_m^j)/t\right)^{\phi_j}}{\prod_{j=1}^n \left(2 - \Delta^{-1}(s_{\psi_m^j}, \rho_m^j)/t\right)^{\phi_j} + \prod_{j=1}^n \left(\Delta^{-1}(s_{\psi_m^j}, \rho_m^j)/t\right)^{\phi_j}}\right)\right)\right).$$

We now propose 2 TL m FHOWG operators.

Definition 18. Let $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ be a set of 2 TL m FNs, where $j = 1, 2, \ldots, n$. Then, an (2 TL m FHOWG) operator is a mapping $\widehat{\xi}^n \longrightarrow \widehat{\xi}$ with the weight vector

$w = (w_1, w_2, \ldots, w_n)^T$, where $w_j \in (0, 1]$ and $\sum_{j=1}^n w_j = 1$. Thus,

$$2TLm\text{FHOWG}_W(\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_n) = \otimes_{j=1}^n (w_j \widehat{\xi}_{\sigma(j)}). \quad (41)$$

Here $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ represent the permutation of the indices 'j' where $j = 1, 2, \ldots, n$, for which $\widehat{\xi}_{\sigma(j-1)} \geq \widehat{\xi}_{\sigma(j)}$,

$\forall j = 1, 2, \ldots, n$.

Theorem 16. *Let* $\widehat{\xi}_j = ((s_{\psi_1^j}, \rho_1^j), \ldots, (s_{\psi_m^j}, \rho_m^j))$ *be a set of 2 TL m F numbers, where* $j = 1, 2, \ldots, n$*. Then the assembled values of these 2 TL m F numbers using the 2 TL m FHOWG operator is also a 2 TL m F numbers, given as follows:*

$$2TLm\text{FHOWG}_W\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\underset{j=1}{\overset{n}{\otimes}}\left(w_j\hat{\xi}_{\sigma(j)}\right)$$
$$=\left(\Delta\left(t\left(\frac{\lambda\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}\left(1+(\lambda-1)\right)\left(1-\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}+(\lambda-1)\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right),\ldots\right.$$
$$\left.\cdot\Delta\left(t\left(\frac{\lambda\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}\left(1+(\lambda-1)\right)\left(1-\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}+((\lambda-1)\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right). \tag{42}$$

Remark 1. For $\lambda=1$, 2 TL m FHOWG operator reduces to $2TL$ m F ordered Weighted geometric (2 TL m FOWG) operator as follows:

$$2TLm\text{FOWG}_w\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\underset{j=1}{\overset{n}{\otimes}}\left(\widehat{\xi_{\sigma(j)}}\right)^{w_j}$$
$$=\left(\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)}{t}\right)^{w_j}\right),\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\psi_2^{\sigma(j)}},\rho_2^{\sigma(j)}\right)}{t}\right)^{w_j}\right),\ldots,\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)}{t}\right)^{w_j}\right)\right). \tag{43}$$

2. For $\lambda=2$, 2 TL m FHOWG operator reduces to 2 TL m F Einstein ordered weighted geometric (2 TL m FEOWG) operator as follows:

$$2TLm\text{FEOWG}_\phi\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\otimes_{j=1}^{n}\left(\hat{\xi}_j\right)^{w_j},=\left(\Delta\left(t\left(\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}\Big/\prod_{j=1}^{n}\left(2-\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}+\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}\right)\right),\ldots,\right.$$
$$\left.\Delta\left(t\left(\frac{\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}\left(2-\Delta^{-1}\left(s_{\psi_m^j},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}+\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right). \tag{44}$$

Theorem 17. *Idempotency. Let $\hat{\xi}_j=((s_{\psi_1^j},\rho_1^j),\ldots,(s_{\psi_m^j},\rho_m^j))$ be a set of 2 TL m FNs, where $j=1,2,\ldots,n$. For the equality of all these numbers, that is, $\hat{\xi}_j=\hat{\xi}$, $\forall j=1,2,\ldots,n$, then the monotonicity property is defined as*

$$2TLm\text{FHOWG}_w\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=\hat{\xi}. \tag{45}$$

The remaining properties, namely, boundedness, monotonicity, and commutativity for the 2 TL m FHOWG operators are defined as

Theorem 18. *Let $\hat{\xi}_j=((s_{\psi_1^j},\rho_1^j),\ldots,(s_{\psi_m^j},\rho_m^j))$, be a set of 2TLmF numbers, where $j=1,2,\ldots,n$, $\hat{\xi}^-=\cup_{j=1}^{n}\hat{\xi}_j$ and $\hat{\xi}^+=\cup_{j=1}^{n}\hat{\xi}_j$ then*

$$\hat{\xi}^-\leq 2TLm\text{FHOWG}_\phi\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)\leq\hat{\xi}^+. \tag{46}$$

Theorem 19. *Let $\hat{\xi}_j$ and $\hat{\xi}'_j$, $j=1,2,\ldots,n$ be a set of 2 TL m F numbers. If $\hat{\xi}_j\leq\hat{\xi}'_j$, then*

$$2TLm\text{FHOWG}_\phi\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)\leq 2TLm\text{FHOWG}_\phi\left(\hat{\xi}'_1,\hat{\xi}'_2,\ldots,\hat{\xi}'_n\right). \tag{47}$$

Theorem 20. *Let $\hat{\xi}_j$ and $\hat{\xi}'_j$, $j=1,2,\ldots,n$ be a set of 2 TL m FNs. If $\hat{\xi}_j\leq\hat{\xi}'_j$, then*

$$2TLm\text{FHOWG}_\phi\left(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n\right)=2TLm\text{FHOWG}_\phi\left(\hat{\xi}'_1,\hat{\xi}'_2,\ldots,\hat{\xi}'_n\right), \tag{48}$$

where $\hat{\xi}'_j$ represents the permutation of $\hat{\xi}_j$, $j=1,2,\ldots,n$.

Now, we propose another operator, namely, 2 TL m F Hamacher hybrid averaging (2 TL m FHHA) operator, which combines 2 TL m FHWG operator and 2 TL m FHOWG operator, respectively.

Definition 19. Let $\hat{\xi}_j=((s_{\psi_1^j},\rho_1^j),\ldots,(s_{\psi_m^j},\rho_m^j))$ be a set of 2 TL m FNs where $j=1,2,\ldots,n$. Then, a 2 TL m F

Hamacher hybrid geometric (2 TL m FHHG) operator is defined as

$$2TLm\text{FHHG}_{w,\phi}\left(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n\right)=\otimes_{j=1}^{n}\left(w_j\widetilde{\widehat{\xi}}_{\sigma(j)}\right), \tag{49}$$

where $w=(w_1,w_2,\ldots,w_n)^T$ represent the associated weight vector of the 2 TL m F numbers $\widehat{\xi}_i$, instead of weighting the experts, for each ‘’ varies from 1 to n, with $w_i\in(0,1]$ and $\sum_{i=1}^{n}w_i=1$, $\widehat{\xi_{\sigma(j)}}$ is the jth biggest 2 TL m F numbers of the $\widehat{\xi}_i\,(i=1,2,\ldots,n)$ with $\widetilde{\widehat{\xi}}_{\sigma(i)}=(n\phi_i)\widehat{\xi_{\sigma(i)}}$, $(i=1,2,\ldots,n)$, $\phi=(\phi_1,\phi_2,\ldots,\phi_n)^T$ is the weight vector for the ordered arguments, with $\phi_j\in(0,1]$, $\sum_{j=1}^{n}\phi_j=1$.

Theorem 21. *Let $\widehat{\xi}_j=((s_{\psi_1^j},\rho_1^j),\ldots,(s_{\psi_m^j},\rho_m^j))$ be a set of 2TLmFNs where $j=1,2,\ldots,n$. Then the assembled value of these 2 TL m FNs by using the 2 TL m FHHG operator is again a 2 TL m F numbers, given as follows:*

$$\begin{aligned}2TLm\text{FHHG}_{w,\phi}&\left(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n\right)=\bigotimes_{j=1}^{n}\left(w_j\widetilde{\widehat{\xi}}_{\sigma(j)}\right)\\&=\left(\Delta\left(t\left(\frac{\lambda\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}\left(1+(\lambda-1)\right)\left(1-\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}+(\lambda-1)\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right),\ldots\right.\\&\left.\cdot\,\Delta\left(t\left(\frac{\lambda\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}}{\prod_{j=1}^{n}\left(1+(\lambda-1)\right)\left(1-\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}+(\lambda-1)\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)/t\right)^{w_j}}\right)\right)\right).\end{aligned} \tag{50}$$

Proof. In order to prove this theorem, the same steps are followed as discussed above in the theorem. □

Remark 1. For $\lambda=1$, the 2 TL m FHHG operator reduces to *2TL* m *FHG* operator as follows:

$$\begin{aligned}2TLm\text{FHG}_{w,\phi}&\left(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n\right)=\bigotimes_{j=1}^{n}\left(w_j\widetilde{\widehat{\xi}}_{\sigma(j)}\right)\\&=\left(\Delta\left(1-t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\psi_1^{\sigma(j)}},\rho_1^{\sigma(j)}\right)}{t}\right)^{w_j}\right),\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\psi_2^{\sigma(j)}},\rho_2^{\sigma(j)}\right)}{t}\right)^{w_j}\right),\ldots,\right.\\&\left.\cdot\,\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\psi_m^{\sigma(j)}},\rho_m^{\sigma(j)}\right)}{t}\right)^{w_j}\right)\right).\end{aligned} \tag{51}$$

2. For $\lambda=2$, 2 TL m FHHG operator convert to 2 TL m FEHG operator as follows:

$$\begin{aligned}2TLm\text{FEHG}_{w,\phi}&\left(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n\right)=\bigotimes_{j=1}^{n}\left(w_j\widetilde{\widehat{\xi}}_{\sigma(j)}\right)\\&=\left(\Delta\left(t\left(\frac{2\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}}{\prod_{j=1}^{n}\left(2-\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}+\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_1^j},\rho_1^j\right)/t\right)^{\phi_j}}\right)\right),\ldots\right.\\&\left.\cdot\,\Delta\left(t\left(\frac{2\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}}{\prod_{j=1}^{n}\left(2-\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}+\prod_{j=1}^{n}\left(\Delta^{-1}\left(s_{\psi_m^j},\rho_m^j\right)/t\right)^{\phi_j}}\right)\right)\right).\end{aligned} \tag{52}$$

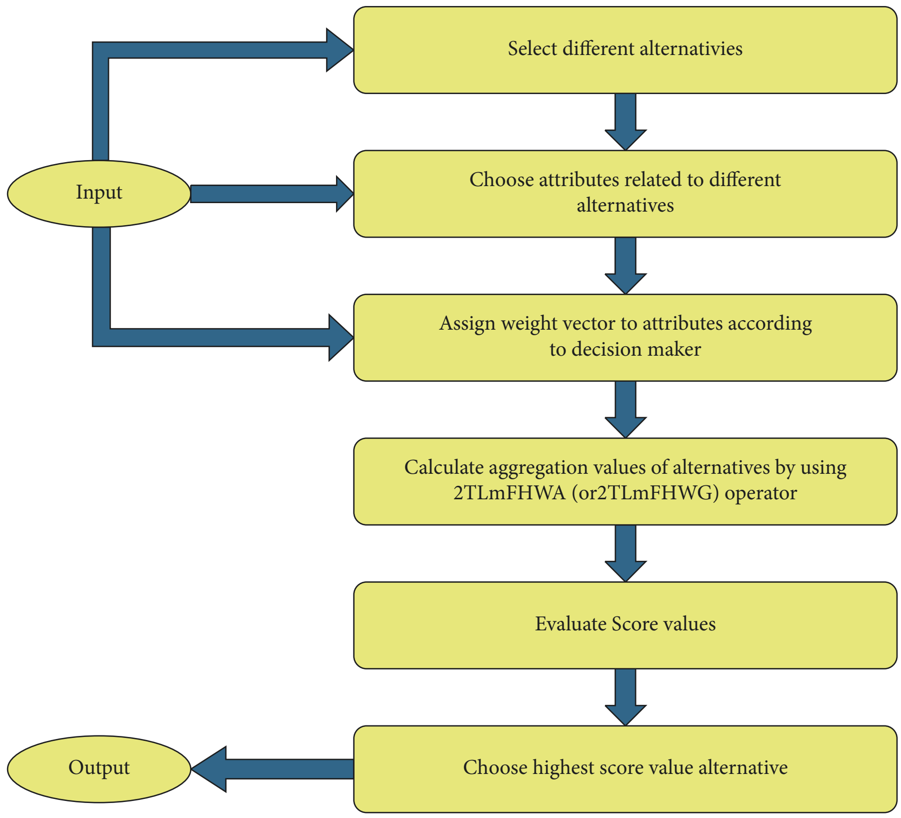

FIGURE 2: Flow chart for decision-making by using the 2TLmFHWA(or2TLmFHWG) operators.

(i) Input:

U, the set of discourse having k alternatives.

ζ be the set having n attributes.

$\phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$, weight vector representation.

(2) In order to calculate the values in 2 TL m F decision matrix $\widehat{\beta}$, we calculate the preference values $\widehat{\beta}$, $i = 1, 2, 3, \ldots, k$, of the objects A_i, by using 2 TL m FHWA operator.

$$\widehat{p}_i = 2TL \text{ m FHWA}_\phi (\widehat{\xi}_{i1}, \widehat{\xi}_{i2}, \ldots, \widehat{\xi}_{in}) = \oplus_{j=1}^{n} (\phi_j \widehat{\xi}_{ij}),$$

$$= (\Delta (t (\prod_{j=1}^{n} (1 + (\lambda - 1))\Delta^{-1} (s_{\psi_1^{ij}}, \rho_1^{ij})/t^{\phi_j} - \prod_{j=1}^{n} (1 - \Delta^{-1} (s_{\psi_1^{ij}}, \rho_1^{ij})/t)^{\phi_j} / \prod_{j=1}^{n} (1 + (\lambda - 1))\Delta^{-1} (s_{\psi_1^{ij}}, \rho_1^{ij})/t^{\phi_j} + (\lambda - 1) \prod_{j=1}^{n} (1 - \Delta^{-1} (s_{\psi_1^{ij}}, \rho_1^{ij})/t)^{\phi_j})), \ldots$$

$$\Delta (t (\prod_{j=1}^{n} (1 + (\lambda - 1))\Delta^{-1} (s_{\psi_m^{ij}}, \rho_m^{ij})/t^{\phi_j} - \prod_{j=1}^{n} (1 - \Delta^{-1} (s_{\psi_m^{ij}}, \rho_m^{ij})/t)^{\phi_j} / \prod_{j=1}^{n} (1 + (\lambda - 1)) \Delta^{-1} (s_{\psi_m^{ij}}, \rho_m^{ij})/t^{\phi_j} + (\lambda - 1) \prod_{j=1}^{n} (1 - \Delta^{-1} (s_{\psi_m^{ij}}, \rho_m^{ij})/t)^{\phi_j}))).$$

Alternatively, if we apply 2 TL m FHWG operators, then

$$\widehat{p}_i = 2TL \text{ m } \text{FHWG}_\phi (\widehat{\xi}_{i1}, \widehat{\xi}_{i2}, \ldots, \widehat{\xi}_{in}) = \otimes_{j=1}^{n} (\phi_j \widehat{\xi}_{ij})^{\phi_j},$$

$$= (\Delta (t (\lambda \prod_{j=1}^{n} (1 - \Delta^{-1} (s_{\psi_1^{ij}}, \rho_1^{ij})/t)^{\phi_j} / \prod_{j=1}^{n} (1 + (\lambda - 1))\Delta^{-1} (s_{\psi_1^{ij}}, \rho_1^{ij})/t^{\phi_j} + (\lambda - 1) \prod_{j=1}^{n} (1 - \Delta^{-1} (s_{\psi_1^{ij}}, \rho_1^{ij})/t)^{\phi_j})), \ldots,$$

$$\Delta(t (\lambda \prod_{j=1}^{n} (\Delta^{-1} (s_{\psi_m^{ij}}, \rho_m^{ij})/t)^{\phi_j} / \prod_{j=1}^{n} (1 + (\lambda - 1) (1 - \Delta^{-1} (s_{\psi_m^{ij}}, \rho_m^{ij})/t))^{\psi_j} + (\lambda - 1) \prod_{j=1}^{n} (\Delta^{-1} (s_{\psi_m^{ij}}, \rho_m^{ij})/t)^{\phi_j}))). \quad (46)$$

(3) We compute the scores $S(\widehat{p}_i)$, $i = 1, 2, 3, \ldots, k$.

(4) By using scores values $S(\widehat{p}_i)$, $i = 1, 2, 3, \ldots, k$, we make ranking for objects. If we have the same score value for two alternatives, then in order to rank the objects, we move toward accuracy function.

Output: an alternative which has the high value in Step (4) will be the decided alternative.

ALGORITHM 1: Procedure to tackle (MADM) problems using 2 TL m FHWA (or 2 TL m FHWG) operators.

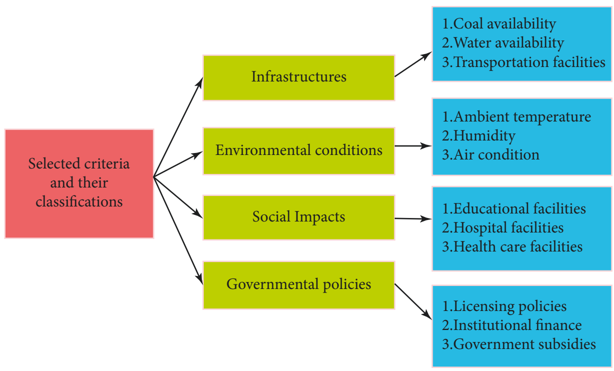

Figure 3: Criteria representation in the selected 3-polar environment.

5. Mathematical Approach for MADM Using 2 TL m F Information

In this section, we handle the multiattribute decision-making issues with 2 TL m F information by applying the 2 TL m FHA operators established in the previous sections.Let $A = \{A_1, A_1, A_2, \ldots, A_k\}$ be the set of alternatives and $\zeta = \{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ be the set of attributes. Let us assume $\phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$, a weight vector for the set of attributes where $\phi_j > 0$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} \phi_j = 1$.Let us take $R = (r_{ij})_{k\times n} = ((s_{\psi_1^{ij}}, \rho_1^{ij}), \ldots, (s_{\psi_m^{ij}}, \rho_m^{ij})_{k\times n}$ be a decision matrix for 2 TL m F information. Here $(s_{\psi_r^{ij}}, \rho_r^{ij})$, $r = 1, 2, \ldots, m$ represent the membership values given by the decision-makers that the alternatives assure with the attributes ζ_j, where $(s_{\psi_r}, \rho_r) \in [0, 1]$, 'r' varies from 1 to m. In order to deal with MADM issues, we explain Algorithm 1 using 2 TL m FHWA (2 TL m FHWG) operators.

Now, we elaborate the above algorithm for decision-making in the form of a flowchart, which is given in Figure 2.

6. Best Location for the Thermal Power Station: Case Study

Thermal power stations are a source of conversion of heat energy into electricity. It is selected anywhere near a water and fuel supply. Thermal power stations use fossil fuels to generate electricity, which produces pollution. By keeping all the circumstances in view, we consider the selection of the best location for a thermal power station as discussed by [26]. The best location selection plays a significant role in the economic operation of the thermal power station and the long-lasting development of the region. So, the company selects five possible areas, which are considered alternatives $A = \{A_1, A_2, A_3, A_4, A_5\}$.

The decision-maker select the best place for location under the following criteria:

ζ_1: infrastructure

ζ_2: environmental conditions.

ζ_3: social impacts.

ζ_4: governmental policies.

Where each criterion is divided into three components to form a 3-polar fuzzy set.

- Infrastructures: the development of the infrastructure involves different factors, such as the supply of water, extra high cable voltage, gas, roadways, etc. But, here we take three factors, such as the availability of coal, availability of water, and the availability of transportation facilities, to make the 3-polar fuzzy set.
- Environmental conditions: this factor means the state of the environment, including several different natural resources. Let us take the three factors of the environmental conditions which are necessary for the best location selection of the thermal power station; they are ambient temperature, humidity, and air velocities.
- Social impacts: this criterion involves the study of social challenges, which may include both beneficial and adverse effects. We consider three factors, which are education facilities, hospital facilities, and health care facilities.
- Governmental policies: this criterion includes governmental policies, which have been subdivided into three factors, such as licensing policies, institutional finance, and government subsidies. A clear vision of attribute selection in the 3-polar fuzzy environment is shown in Figure 3.

(1) In order to construct the decision matrix, the decision-makers describe their preferences for the best location of a thermal power plant in the form of linguistic terms. But if we proceed with these linguistic terms, then the assembled results may give the same linguistic term against different alternatives. So, to manage this issue, we translate it with zero symbolic translation, which converts the linguistic term data into 2 TL data in which we can rank alternatives with the same linguistic term on the basis of the symbolic translation.

The decision matrix for 2TL3-polar data is given in Table 2.

(2) The weights recommended by the experts are given as $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (0.4{,}0.3{,}0.1{,}0.2)$.

We now proceed to find the most suitable place for the thermal power station. The working procedure is described as follows:

Step 1. Let us take $\lambda = 3$, to assemble the 2 TL m F values. If we choose $\lambda = 1$, then the 2 TL m FHWA(2 TL m FHWG) operator reduces to 2 TL m FWA(2 TL m FWG) operator and for $\lambda = 2$, the 2 TL m FHWA(2 TL m FHWG) operator reduces to 2 TL m FEWA(2 TL m FEWG) operator. So, in the case of $\lambda = 3$, the 2 TL m FHWA(2 TL m FHWG) sustains its own nature. Therefore, $\lambda = 3$ is the most suitable value to deal with the selection of the most suitable place for a thermal power station by using operator 2 TL m FHWA.

Step 2. The 2 TL m FHWA operator is used to evaluate the assembled values, $\widehat{p}_i$ for the thermal power plant location alternatives as given in Table 3.

Step 3. Let us compute the score values $S(\widehat{p}_i)$ for all the 2TL3F numbers $\widehat{p}_i$ as in Table 4.

Table 2: Decision matrix for 2TL3F information.

	A_1	A_2	A_3	A_4	A_5
ζ_1	$((s_4,0),(s_4,0),(s_4,0))$	$((s_5,0),(s_3,0),(s_4,0))$	$((s_4,0),(s_3,0),(s_3,0))$	$((s_4,0),(s_5,0),(s_3,0))$	$((s_5,0),(s_4,0),(s_3,0))$
ζ_2	$((s_4,0),(s_5,0),(s_3,0))$	$((s_4,0),(s_5,0),(s_5,0))$	$((s_4,0),(s_5,0),(s_3,0))$	$((s_3,0),(s_4,0),(s_5,0))$	$((s_4,0),(s_6,0),(s_4,0))$
ζ_3	$((s_4,0),(s_5,0),(s_2,0))$	$((s_4,0),(s_5,0),(s_6,0))$	$((s_4,0),(s_3,0),(s_6,0))$	$((s_4,0),(s_4,0),(s_4,0))$	$((s_3,0),(s_5,0),(s_3,0))$
ζ_4	$((s_4,0),(s_3,0),(s_5,0))$	$((s_6,0),(s_3,0),(s_4,0))$	$((s_3,0),(s_4,0),(s_4,0))$	$((s_5,0),(s_2,0),(s_3,0))$	$((s_3,0),(s_4,0),(s_4,0))$

Table 3: Assembled assessment by using the 2 TL m FHWA operator.

$\widehat{p}_i$	2TL m FHWA
$\widehat{p}_1$	$((s_4,0.00000),(s_4,0.31958),(s_4,-0.1689))$
$\widehat{p}_2$	$((s_6,0.00000),(s_4,-0.0079),(s_6,0.0000))$
$\widehat{p}_3$	$((s_4,-0.1794),(s_4,-0.0499),(s_6,0.0000))$
$\widehat{p}_4$	$((s_4,-0.0039),(s_4,0.18731),(s_4,-0.1398))$
$\widehat{p}_5$	$((s_4,0.24175),(s_6,0.00000),(s_4,-0.4686))$

Table 4: Score values for all the 2TL3F numbers $\widehat{p}_i$.

Score values	2TL m FHWA
$S(\widehat{p}_1)$	$(s_4,0.05022)$
$S(\widehat{p}_2)$	$(s_5,0.33068)$
$S(\widehat{p}_3)$	$(s_5,-0.4098)$
$S(\widehat{p}_4)$	$(s_4,0.01450)$
$S(\widehat{p}_5)$	$(s_5,-0.4089)$

Table 5: Assembled assessment by using the operator.

$\widehat{p}_i$	2TL m FHWG
$\widehat{p}_1$	$((s_4,0.00000),(s_4,0.18858),(s_4,-0.3401))$
$\widehat{p}_2$	$((s_5,-0.18858),(s_4,-0.2284),(s_5,-0.4946))$
$\widehat{p}_3$	$((s_4,-0.2076),(s_4,-0.2232),(s_3,0.48176))$
$\widehat{p}_4$	$((s_4,-0.1146),(s_4,-0.0608),(s_4,-0.3258))$
$\widehat{p}_5$	$((s_4,0.08352),(s_5,-0.2833),(s_3,0.48683))$

Step 4. Alternative rankings according to their score ($S(\widehat{p}_i)$, i = 1,2, . . ., 5) values for all the 2TL3F numbers,
$A_2 > A_5 > A_3 > A_1 > A_4$.

Step 5. Conclusively, A_2 is the best place for the thermal power station.

If we use the 2 TL m FHG operator, the best alternative can be selected in the same pattern as taken above. The procedure is as follows:

Here, we take $\lambda = 3$, in order to select the most suitable place for a thermal power station by using 2 TL m FHWG operator.

Step 1. The 2 TL m FHWG operator is used to assemble the values $\widehat{p}_i$ for the best thermal power plant location alternatives selection as given in Table 5.

Step 2. Evaluate score values $S(\widehat{p}_i)$ for all the 2TL3F numbers $\widehat{p}_i$ as in Table 6.

Table 6: Scores values for all the 2TL3F numbers $\widehat{p}_i$.

Scores values	2TL m FHG operator
$S(\widehat{p}_1)$	$(s_4,-0.0505)$
$S(\widehat{p}_2)$	$(s_4,0.36276)$
$S(\widehat{p}_3)$	$(s_4,-0.3163)$
$S(\widehat{p}_4)$	$(s_4,-0.1671)$
$S(\widehat{p}_5)$	$(s_4,0.09567)$

Table 7: Alternative ranking order.

Operators	Ranking	Best alternative
2TL m FHWA	$A_2 > A_5 > A_3 > A_1 > A_4$	A_2
2TL m FHWG	$A_2 > A_5 > A_1 > A_4 > A_3$	A_2

Step 3. Alternative rankings corresponding to their scores ($S(\widehat{p}_i)$, i = 1,2,\enleadertwodots,5), for all the 2TL3F numbers,

$$A_2 > A_5 > A_1 > A_4 > A_3, \quad (53)$$

Step 4. Thus, A_2 is the best alternative. This calculation shows that A_2 is the most suitable location by using the 2TLmFA and 2TLmFG operators, the order of ranking is given in Table 7.

Now we performed the check work for the influence of the parameter $\lambda \in [0,6]$ on the ranking sequence of alternatives by using 2 TL m FWA and 2 TL m FWG operators.

6.1. Influence of the Parameter λ on Decision-Making Results. The score and ranking sequence for different values of the parameter λ are given in Tables 8 and 9, calculated by using 2 TL m FWA and 2 TL m FWG operators. In the evaluation of 2 TL m FWA, we get the same ranking for different values of λ as $A_2 > A_5 > A_3 > A_1 > A_4$ and in the evaluation of 2 TL m FWG, we get the ranking $A_2 > A_5 > A_3 > A_1 > A_4$. Thus, the proposed MADM problem shows that using different parameter λ values does not show variation in the ranking. Thus, for different parameter values, proposed operators are not much affected.

The parameter λ working influences, on the MADM problem formed on 2 TL m FHWA and 2 TL m FHWG operators are given in Tables 8 and 9. From Table 8, we conclude that when the parameter λ is varying for the 2 TL m FHWA operator, the corresponding ranking orders have no change. So, for $1 \geq \lambda \leq 6$, we have the same ranking orders

Table 8: Score values by varying λ based on the 2TLmFHWA operator.

λ	$S(\hat{p}_1)$	$S(\hat{p}_2)$	$S(\hat{p}_3)$	$S(\hat{p}_4)$	$S(\hat{p}_5)$	Ranking order
1	$(s_4, 0.08293)$	$(s_5, 0.35560)$	$(s_5, -0.38618)$	$(s_4, 0.07355)$	$(s_5, -0.38708)$	$A_2 > A_5 > A_3 > A_1 > A_4$
2	$(s_4, 0.06035)$	$(s_5, 0.33825)$	$(s_5, -0.40259)$	$(s_4, 0.03289)$	$(s_5, -0.40223)$	$A_2 > A_5 > A_3 > A_1 > A_4$
3	$(s_4, 0.05022)$	$(s_5, 0.33068)$	$(s_5, -0.40980)$	$(s_4, 0.01450)$	$(s_5, -0.40895)$	$A_2 > A_5 > A_3 > A_1 > A_4$
4	$(s_4, 0.04441)$	$(s_5, 0.32643)$	$(s_5, -0.41386)$	$(s_4, 0.00392)$	$(s_5, -0.41276)$	$A_2 > A_5 > A_3 > A_1 > A_4$
5	$(s_4, 0.04065)$	$(s_5, 0.32370)$	$(s_5, -0.41648)$	$(s_4, -0.00297)$	$(s_5, -0.41521)$	$A_2 > A_5 > A_3 > A_1 > A_4$
6	$(s_4, 0.03800)$	$(s_5, 0.32180)$	$(s_5, -0.41830)$	$(s_4, -0.00783)$	$(s_5, -0.41692)$	$A_2 > A_5 > A_3 > A_1 > A_4$

Table 9: Score values by varying λ based on the 2 TL m FHWG operator.

λ	$S(\hat{p}_1)$	$S(\hat{p}_2)$	$S(\hat{p}_3)$	$S(\hat{p}_4)$	$S(\hat{p}_5)$	Ranking order
1	$(s_4, -0.09709)$	$(s_4, 0.29230)$	$(s_4, -0.37132)$	$(s_4, -0.25235)$	$(s_4, 0.03175)$	$A_2 > A_5 > A_1 > A_4 > A_3$
2	$(s_4, -0.06776)$	$(s_4, 0.33370)$	$(s_4, -0.33819)$	$(s_4, -0.19843)$	$(s_4, 0.06956)$	$A_2 > A_5 > A_1 > A_4 > A_3$
3	$(s_4, -0.05053)$	$(s_4, 0.36276)$	$(s_4, -0.31638)$	$(s_4, -0.16710)$	$(s_4, 0.09567)$	$A_2 > A_5 > A_1 > A_4 > A_3$
4	$(s_4, -0.03901)$	$(s_4, 0.38489)$	$(s_4, -0.30040)$	$(s_4, -0.14627)$	$(s_4, 0.11544)$	$A_2 > A_5 > A_1 > A_4 > A_3$
5	$(s_4, -0.03070)$	$(s_4, 0.40261)$	$(s_4, -0.28794)$	$(s_4, -0.13131)$	$(s_4, 0.13126)$	$A_2 > A_5 > A_1 > A_4 > A_3$
6	$(s_4, -0.02441)$	$(s_4, 0.41732)$	$(s_4, -0.27779)$	$(s_4, -0.12000)$	$(s_4, 0.14440)$	$A_2 > A_5 > A_1 > A_4 > A_3$

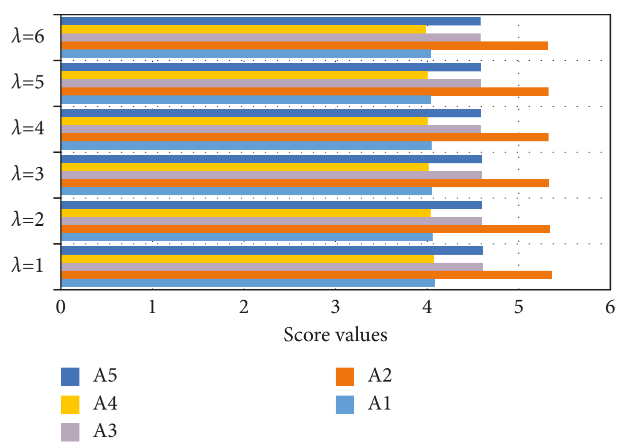

Figure 4: Score values for thermal power stations A_k $(k = 1, 2, \ldots, 5)$ based on the 2 TL m FHWA operator.

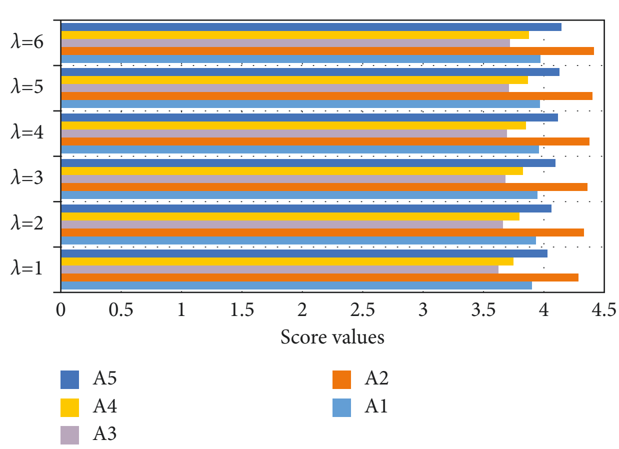

Figure 5: Score values for thermal power stations A_k $(k = 1, 2, \ldots, 5)$ based on the 2 TL m FHWG operator.

Table 10: Comparative analysis of proposed operators with existing ones.

Operators	Ranking	Best alternative
m FDWA [26]	$A_2 > A_5 > A_1 > A_4 > A_3$	A_2
m FDWG [26]	$A_2 > A_5 > A_1 > A_4 > A_3$	A_2
m FHWA [29]	$A_2 > A_5 > A_1 > A_4 > A_3$	A_2
m FHWG [29]	$A_2 > A_5 > A_1 > A_4 > A_3$	A_2
2TL m FHWA(proposed)	$A_2 > A_5 > A_3 > A_1 > A_4$	A_2
2TL m FHWG(proposed)	$A_2 > A_5 > A_1 > A_4 > A_3$	A_2

Table 11: Characteristics comparison of 2 TL m F operators with existing structures.

Operators Existing and proposed	Fusion of linguist data With fuzzy information	Information aggregation more flexible By a parameter λ
m FDWA [26]	×	×
m FDWG [26]	×	✓
m FHWA [29]	×	×
m FHWG [29]	×	×
2TL m FHWA(proposed)	✓	✓
2TL m FHWG(proposed)	✓	✓

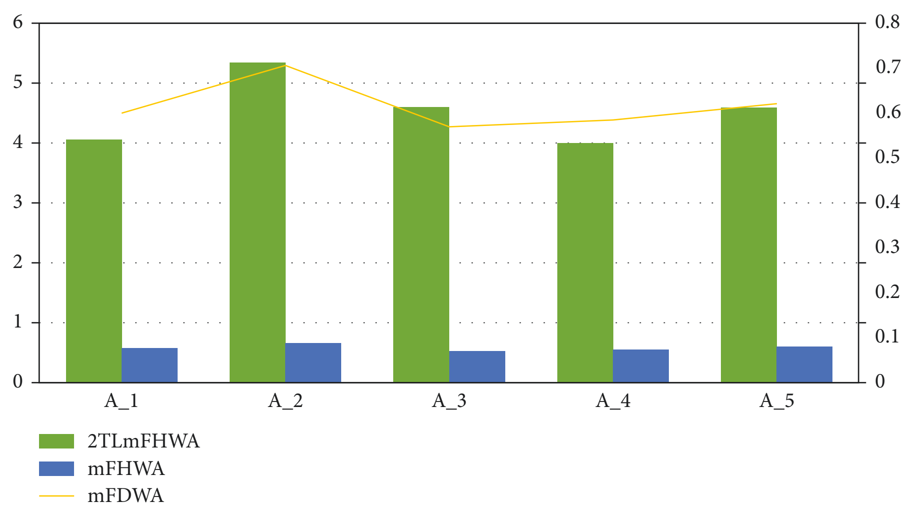

Figure 6: Comparison of proposed operator 2TLmFHWA with existing operators.

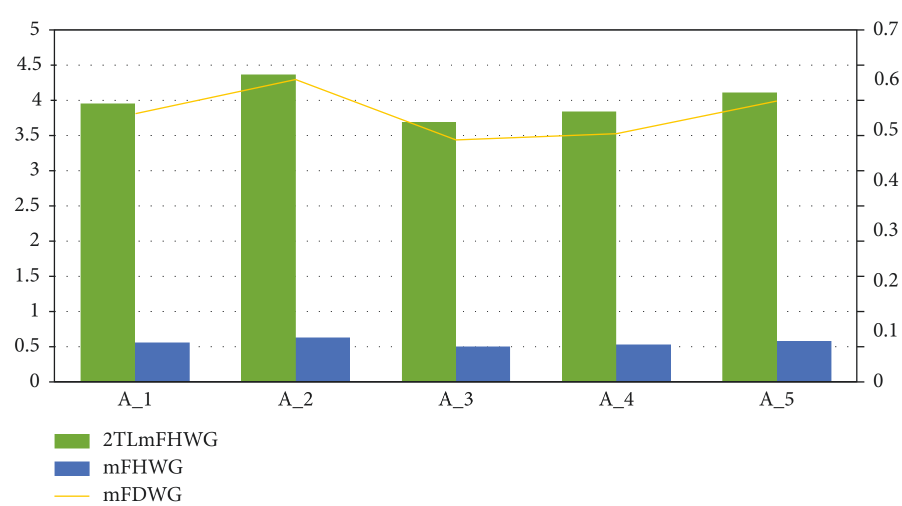

Figure 7: Comparison of proposed operator 2TLmFHWG with existing operators.

$A_2 > A_5 > A_3 > A_1 > A_4$ with the most favorable location is A_2 in all the cases.

In Table 9, when $1 \geq \lambda \leq 6$, then by using the 2 TL m FHWG operator, the order of ranking remains the same as $A_2 > A_5 > A_1 > A_4 > A_3$. Here, A_2 is again the best alternative for a suitable location of the thermal power plant. Thus, in the two situations, the most desirable alternative is A_2 with the same order sequence for $1 \geq \lambda \leq 6$. Where graphical representation of scores variation by changing the parameter λ based on 2 TL m FHWA and 2 TL m FHWG operators are represented in Figures 4 and 5, by applying Δ^{-1} on score function.

Conclusively, the developed MADM problem formed on 2 TL m FHWA and 2 TL m FHWG operators could not change the agnate ranking orders of the alternatives for different parameter values. Thus, the proposed model is reliable and has fewer upshots by λ on multiattribute decision-work.

7. Comparative Analysis

7.1. Comparison with Existing Techniques. This segment contains a comparison survey between the proposed techniques, namely, 2 TL m FHWA and 2 TL m FHWG operators, with existing. We also verify our techniques' applicability and versatility by comparison of proposed models with existing ones. The developed operators, 2 TL m FHWA and 2 TL m FHWG, and their comparison with the existing [26, 29] operators are given in Table 10. We observe that, according to the existing and proposed operators, the best location for a thermal power station is A_2, as in Table 10. Besides this, in the proposed operator 2 TL m FHWG provides the same ranking list $A_2 > A_5 > A_1 > A_4 > A_3$ as compared with the existing [26, 29] operator. But in the case of operator 2 TL m FHWA, the order list is slightly different. The characteristics comparison of 2 TL m F operators with existing structures is in Table 11.

The comparative chart representations ϕ_j of proposed techniques with existing operators are given in Figures 6 and 7.

7.2. Discussion

- The proposed operators 2 TL m FHWA (2 TL m FHWG) consider the interrelationship between the 2 TL and m-polar fuzzy data, which was not the case of the existing operators [26, 29] that only deal with m-polar fuzzy information. Thus, the proposed operators accommodate a greater amount of vagueness and provide more reliable results.
- The techniques in [26, 29], albeit designed to take over MCDM problems, are restricted to deal with m-polar fuzzy information only. They are useless in the presence of linguistic features. So, this may potentially become a cause of loss of information, which typically leads to undesired results. Our new work produces versatile operators that overcome this limitation of previous methods.
- The proposed operators not only operate with 2 TL m F data, but they also have the flexibility to switch from 2 TL m F to m F format by using the Δ^{-1}/t transformation. It is the versatility of the proposed operators, namely, 2 TL m FHWA. Thus, the proposed method is more flexible and transparent than the existing one.

After making the comparison, it has been taken into account that the proposed operators can handle 2 TL m-polar fuzzy information without any complexity. However, our proposed operators quickly describe the fusion of 2 TL and m F information. The new MADM technique for 2 TL m FHWA and 2 TL m FHWG operators improved the resilience of the utilization. Conclusively, the progress made with the help of the m F operators has produced a malleable tool to tackle 2 TL m F information for MADM problems. The ranking order derived from the proposed operators is compatible with the result from the existing operators, so our proposal is reliable and valid for MCDM. Yet more, it is prominent because no loss of data information occurs as in the linguistic information approach. Thus, numeric and linguistic information make the proposed operators more remarkable and adaptable.

7.3. Advantages and Limitations of the Proposed Work. Several aggregation operators have been suggested and put to work in decision-making. Altogether, they enable us to cope with different types of situations, and the decision-makers can select the most suitable operator for their real-world problems. Our proposed methodology is versatile enough to deem it superior as compared to other methods in the 2 TL m F environment. We list some specific advantages as follows:

- The proposed 2 TL m F sets capture descriptions of real-world problems with both quantitative and qualitative components. Because of this versatility, the proposed approach represents many situations more faithfully, and in doing so, it reduces the loss of data.
- The 2 TL m FHWA (resp., 2 TL m FHWG) operator provides more clarity and transparency of information to the decision-makers because of the fusion of 2 TL and multipolar data. This aspect provides an easy way to address MADM risk analysis.
- The proposed approach is more adaptable than other techniques as it completely addresses the interdependence of the linguistic representation with numerical multi-inputs. Therefore, it contains more general information, which gives us more reliable results. Consequently, the proposed operators have a vast range of utilization.
- The preceding strategies are limited to dealing with m-polar fuzzy information. They are rendered ineffective in the presence of linguistic traits. As a result, this might become a source of data loss, which usually results in unfavorable outcomes. Our new approach generated flexible operators that overcame the limitations of earlier methods.

Now, we list some limitations of the proposed work.

- As the proposed 2 TL m F operator collects the 2 TL data with multipolar data, when the number of poles or the subdivision of attributes has increased, this approach seems to be difficult to handle.
- The proposed approach is unable to handle the MADM approach when the decision-makers present their preferences in both positive and negative aspects.

8. Conclusions

Many real situations have a framework that contains 2 TL representations with multipolar data information. As several theoretical frameworks are developed to cover the broader range of complicated situations. Thus, the basic need is the selection of the most suitable MADM approach to tackle the complicated situation in decision making.

Thus, in this research article, we have contributed to the development of the MADM approach in the presence of 2 TL m F data. To overcome the limitations of classical MADM, we have investigated the MADM problem that suitably merges the concepts of 2 TL with Hamacher-type operators and m-polar fuzzy numbers.

Aggregation operators have become a fundamental tool for fusing data from various sources, particularly during the construction of decision-making models. In the present research work, some new aggregation operators are developed that are closely motivated by m F Hamacher operations, namely, the 2 TL m FHWA operator, the 2 TL m FHOWA operator, and the 2 TL m FHHA operator. Moreover, we have introduced the 2 TL m FHWG operator, the 2 TL m FHOWG operator, and the 2 TL m FHHG operator. We have also disclosed the various properties of these operators, namely, monotonicity, idempotency, and the boundedness property. So that, the practitioners may select the version that best serves.

We have applied these aggregation operators to enhance the applicability area of MADM in the m F environment. In the end, we have produced a comparative study of the proposed operators concerning previously existing work. This analysis speaks to the validity of the proposed operators. In our case study for the selection of the best thermal power station location, we have consistently found that A_2 is the best alternative for the location of a thermal power station. Conclusively, the significant contribution of this research article is that it combines the functions of Hamacher aggregation operators as well as the properties of 2 TL m-polar fuzzy numbers. Our proposed model of uncertain knowledge reveals its adaptability in depicting inexact, imperfect facts in complicated settings. The operators show that they are very flexible, allowing them a valuable tool that might be put to other future tasks. In future work, we plan to extend our research work to include outranking decision-making methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest with this study.

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