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Research Article

H^{∞} Filter for Discrete-Time Takagi–Sugeno Fuzzy Systems with Time-Varying Delays via a Novel Wirtinger-Based Inequality

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This paper is devoted to the analysis and design of H^{∞} filtering for discrete-time Takagi–Sugeno (TS) fuzzy systems with time-varying delays. By using the delay-partitioning method, more systematic information is introduced, which can reduce the conservatism of the conclusion. By constructing an appropriate Lyapunov functional and combining with a newly Wirtinger-based summation inequality, the existence condition of the H^{∞} filter is obtained. Thus, utilizing the contract matrix transformation method, the H^{∞} filter design for discrete-time TS fuzzy systems with time-varying delay based on LMI is proposed. Finally, a few numerical analysis results are given to prove the effectiveness of the method.

1. Introduction

In recent years, the analysis and synthesis of nonlinear systems has gradually become one of the research focuses in the domain of automatic control and automatic control technology. Nonlinearity is ubiquitous in chemical processes, robotic systems, automotive systems, and many manufacturing processes, which could introduce severe difficulties in the analysis and synthesis of control systems. Although nonlinearity is more accordant with people's actual production processes, nonlinear systems cannot be simply described by linear differential equations, and the stability of the system not only relies on the structural parameters of the system but also depends on the initial state of the system. These problems have caused considerable dilemmas for researchers to analyze and synthesize the stability of nonlinear systems. With the deepening of research, methods such as output control feedback [1–5], sliding mode control [6–8], optimal control [9, 10], and fuzzy control [11–20] have been proposed. Currently, the TS fuzzy system [21–24] is deemed to be one of the most effective methods for dealing with nonlinear systems.

In recent years, the TS fuzzy system has been widely used. The TS fuzzy model is a nonlinear model described by a series of if-then rules, which is arbitrarily smooth and nonlinear. The system can be approximated by the TS fuzzy model without constant term with any specified accuracy. Up to now, the strict LMI stability sufficient conditions for the robust stability of uncertain continuous TS fuzzy descriptor systems are given [25]. By constructing the augmented Lyapunov–Krasovskii functional and utilizing Wirtinger-based integral inequality, an improved condition for stability of the concerned systems was derived in terms of LMIs [26]. Li et al. [27] designed integral switching function to study the fault detection of sliding mode controller for a class of TS fuzzy singular systems. More research results can be found in [28–30] and the references therein.

Furthermore, time delays are common in any practical system, such as network systems, power systems, hydraulic systems, and so on. Thus, the problems such as poor system stability and degraded performance are unavoidable. Meanwhile, people have made a lot of analysis and research on the delay-dependent and delay-independent TS fuzzy systems [31–33]. At present, delay-partitioning [34, 35] is considered as one of the most effective methods to process the time delay problems in control systems. By artificially dividing the lower bound d_m into m equal-length subintervals, the maximum time delay of the system's asymptotic stability in the guaranteed interval is obtained, thereby reducing the conservativeness of the stability conditions. Compared with the early Jensen inequality [47] and the free weight matrix method [48–50], the recently proposed Wirtinger-based inequality [36–38], Bessel–Legendre inequality [39–42], and reciprocal convex method [43–46] have many improvements in the derivation of quadratic terms. However, to our knowledge, few results can be found in the literature concerning with the discrete-time TS fuzzy delay systems by the delay-partitioning method and Wirtinger-based inequality. Thus, research on the filtering problem for discrete-time TS fuzzy systems with time-varying delay by employing the delay-partitioning method and Wirtinger-based inequality should be of both theoretical and practical importance, which motivates us to carry out this study.

As stated previously, in this paper, we consider the delay-dependent $H\infty$ filter for TS fuzzy discrete-time systems with time-varying delay. Firstly, we will utilize the delay-partitioning method to reduce the effect of system time-varying delays, and the quadratic accumulation term appearing in Lyapunov–Krasovskii function is processed by the improved Wirtinger-based inequality to ensure that the fuzzy system is asymptotically stable. Then, the contract matrix transformation is used to design the $H\infty$ filter. Finally, we will demonstrate the validity and low conservation of our experimental results through some simulation examples.

Notations. Throughout this paper, the superscript "T" represents the matrix transposition, $\mathbb{R}^n$ expresses the n-dimensional Euclidean space, I and 0 signify an identity matrix and a zero matrix with appropriate dimension, respectively, and $P > 0$ is used to denote that the matrix P is positively defined and symmetric. Besides, sym {A} is equivalent to $A + A^T$, diag$\{\ldots\}$ is a block-diagonal matrix, and col$[A, B, C]$ =, *that is*, $[A^T, B^T, C^T]^T$. It should be noted that in the symmetric block matrices, $*$ represent terms caused by symmetry, e.g.,

$$\begin{bmatrix} X_1 & X_2 \\ * & X_3 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}. \tag{1}$$

2. Model Description and Problem Analysis

The description of the TS fuzzy model is represented by the fuzzy IF-THEN rule of the local linear input-output relationship of the nonlinear system. Then, consider a nonlinear system, which can be described as a discrete-time TS fuzzy time-varying delay model as follows.

Rule (i): IF $\vartheta_1(t)$ is F_{i1} and ... and $\vartheta_g(t)$ is F_{ig}, THEN

$$\begin{cases} x(t+1) = A_i x(t) + A_{di} x(t-\tau(t)) + B_i \omega(t), \\ y(t) = C_i x(t) + C_{di} x(t-\tau(t)) + D_i \omega(t), \\ z(t) = L_i x(t) + L_{di} x(t-\tau(t)) + F_i \omega(t), \\ x(t) = \varphi(t), t = -\tau_1, -\tau_1 + 1, \ldots, 0, i = 1, 2, \ldots r, \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ expresses the state vector, $y(t) \in \mathbb{R}^q$ denotes the measured output, $z(t) \in \mathbb{R}^p$ is the signal to be measured, $\omega(t) \in \mathbb{R}^l$ signifies the noise input vector, which belongs to $l_2[0, \infty)$, $t = -\tau_1, -\tau_1 + 1, \ldots, 0$ is given initial condition sequence, and $\tau(t)$ express the time-varying delays (we assume that it satisfies $1 \le \tau_2 \le \tau(t) \le \tau_1, t = 1, 2, \ldots$); besides, τ_2 and τ_1 indicate the known minimum and maximum bounds of delays, respectively. Meanwhile, F_{ij} is the fuzzy set, r represents the number of *IF-THEN* rule, and $\vartheta(t) = [\vartheta_1(t), \vartheta_2(t) \ldots \vartheta_g(t)]$ shows the premise variables vector. $A_i, A_{\mathrm{di}}, B_i, C_i, C_{\mathrm{di}}, D_i, L_i, L_{\mathrm{di}}$, and F_i are known constant systems matrices, and $x(t) = \varphi(t)$ refers to the initial condition.

Then, describe the normalized fuzzy-basis function as follows:

$$\hbar_i[\vartheta(t)] = \frac{\prod_{j=1}^{g} F_{ij}\left[\vartheta_j(t)\right]}{\sum_{i=1}^{r} \prod_{j=1}^{g} F_{ij}\left[\vartheta_j(t)\right]} \ge 0, \tag{3}$$

$$\sum_{i=1}^{r} \hbar_i[\vartheta(t)] = 1, \tag{4}$$

with $F_{ij}[\vartheta_j(t)]$ indicating the grade of membership of $\vartheta(t)$ in F_{ij}, and define $\hbar_i = [\hbar_1, \ldots, \hbar_r]$. Through product fuzzy inference and center-of-gravity defuzzification, the following discrete time-varying TS fuzzy dynamic model $\aleph$ can be obtained:

$$\begin{cases} x(t+1) = A(\hbar, t)x(t) + A_d(\hbar, t)x(t-\tau(t)) + B(\hbar, t)\omega(t) \\ y(t) = C(\hbar, t)x(t) + C_d(\hbar, t)x(t-\tau(t)) + D((\hbar, t))\omega(t) \\ z(t) = L(\hbar, t)x(t) + L_d(\hbar, t)x(t-\tau(t)) + F(\hbar, t)\omega(t) \\ x(t) = \varphi(t), t = -\tau_1, -\tau_1 + 1, \ldots, 0, \end{cases} \tag{5}$$

where

$$\begin{aligned} A(\hbar, t) &\triangleq \sum_{i=1}^{r} \hbar_i A_i(t), A_d(\hbar, t) \triangleq \sum_{i=1}^{r} \hbar_i A_{di}(t), B(\hbar, t) \\ &\triangleq \sum_{i=1}^{r} \hbar_i \mathrm{B}_i(t), \\ C(\hbar, t) &\triangleq \sum_{i=1}^{r} \hbar_i C_i(t), C_d(\hbar, t) \triangleq \sum_{i=1}^{r} \hbar_i C_{di}(t), D(\hbar, t) \\ &\triangleq \sum_{i=1}^{r} \hbar_i D_i(t), \\ L(\hbar, t) &\triangleq \sum_{i=1}^{r} \hbar_i L_i(t), L_d(\hbar, t) \triangleq \sum_{i=1}^{r} \hbar_i L_{di}(t), F(\hbar, t) \\ &\triangleq \sum_{i=1}^{r} \hbar_i F_i(t). \end{aligned} \tag{6}$$

Given system (1), we are committed to designing a full-order fuzzy filter for the estimation of $z(t)$ of general structure described by $\aleph_F$:

$$\begin{cases} x_F(t+1) = A_F(\hbar)x_F(t) + B_F(\hbar)y(t), \\ z_F(t) = L_F(\hbar)x_F(t) + D_F(\hbar)y(t), x_F(0) = 0, \end{cases} \tag{7}$$

where $x_F(t) \in \mathbb{R}^n$ is the filter state vector of the system $\aleph_F$ and $A_F(\hbar), B_F(\hbar), L_F(\hbar)$, and $D_F(\hbar)$ are filer matrices to be determined.

Next, define the augmented state vector $\zeta(t){:} = \mathrm{col}[x(t), x_F(t)]$ and the estimation error $e_t{:} = z(t) - z_F(t)$. Thus, we could acquire the filtering error system $\aleph_e$ as follows:

$$\begin{cases} \zeta(t+1) = \overline{A}(\hbar,t)\zeta(t) + \overline{A}_d(\hbar,t)\zeta(t-\tau(t)) + \overline{B}(\hbar,t)\omega(t), \\ e(t) = \overline{L}(\hbar,t)\zeta(t) + \overline{L}_d(\hbar,t)\zeta(t-\tau(t)) + \overline{F}(\hbar,t)\omega(t), \\ \zeta(t) = \left[\phi^T(t), 0\right]^T, t = -\tau_1, -\tau_1+1, \ldots, 0, \end{cases} \tag{8}$$

where

$$\begin{aligned} \overline{A}(\hbar,t) &\triangleq \begin{bmatrix} A(\hbar,t) & 0 \\ B_F(\hbar)C(\hbar,t) & A_F(\hbar) \end{bmatrix}, \\ \overline{A}_d(\hbar,t) &\triangleq \begin{bmatrix} A_d(\hbar,t) \\ B_F(\hbar)C_d(\hbar,t) \end{bmatrix}, \\ \overline{B}(\hbar,t) &\triangleq \begin{bmatrix} B(\hbar,t) \\ B_F(\hbar)D(\hbar,t) \end{bmatrix}, \\ \overline{L}(\hbar,t) &\triangleq \begin{bmatrix} L(\hbar,t) - D_F(\hbar)C(\hbar,t) & -C_F(\hbar) \end{bmatrix}, \\ \overline{L}_d(\hbar,t) &\triangleq \left[L_d(\hbar,t) - D_F(\hbar)C_d(\hbar,t) 0\right], \\ \overline{F}(\hbar,t) &\triangleq F(\hbar,t) - D_F(\hbar)D(\hbar,t). \end{aligned} \tag{9}$$

Remark 1. The $H\infty$ filter is designed as follows. Given a fuzzy system (1), a fuzzy filter such as (6) with filter matrices $[A_F(\hbar), B_F(\hbar), L_F(\hbar), D_F(\hbar)]$ is designed to estimate the signal $z(t)$ to ensure that the filtering error system $\aleph_e$ in (7) is asymptotically stable, and $z_F(t)$ is a good estimate of $z(t)$ with the energy bounded disturbance $\omega(t)$. That means, for a given $\gamma > 0$, the filter error system $\aleph_e$ is asymptotically stable when the $H\infty$ performance index γ is asymptotically stable under the condition of $\omega = 0$, satisfying the following requirements: $\|e\|_2 < \gamma\|\omega\|_2$, for all nonzero $\omega \in l_2[0,\infty)$, where $\|e\|_2 \triangleq \sqrt{\sum_{t=0}^{\infty} e^T(t)e(t)}$.

Remark 2. Since the control input of the original fuzzy system model $\aleph$ is unknown and the system to be estimated must be asymptotically stable, which is also the precondition of asymptotically stable filter error system in $\aleph_e$, we assume that system $\aleph$ is asymptotically stable.

Before entering the main sections, a very vital lemma is introduced for the deduction of our conclusions.

Lemma 1 (see [51, 52]). *For a given symmetric positive definite matrix $\mathfrak{R} \in \mathbb{R}^{n\times n}$, matrices $L_p \in \mathbb{R}^{4n\times n}$, $p = 0, 1, 2$, and any differentiable function ϖ in $[\rho_0, \rho_1] \mapsto \mathbb{R}^n$, the following summation inequality holds:*

$$-\sum_{i=\rho_0}^{\rho_1-1} \varpi^T(i)\mathfrak{R}\varpi(i) \le \Delta, \tag{10}$$

where

$$\begin{aligned} \Delta &= \overline{\delta}^T\left[\Lambda_{(\ell,\mathfrak{R},L_p)} + sym\left\{\chi_{(L_p,I_p)}\right\}\right]\overline{\delta}, \\ \overline{\delta} &= col\left\{x(\rho_1), x(\rho_0), \frac{1}{\ell+1}\Sigma_0, \frac{2}{(\ell+1)(\ell+1)}\Sigma_1\right\}, \\ \ell &= \rho_1 - \rho_0, \Sigma_0 = \sum_{i=\rho_0}^{\rho_1} x(i), \Sigma_1 = \sum_{i=\rho_0}^{\rho_1}\sum_{j=i}^{\rho_1} x(j), \\ \Lambda_{(\ell,\mathfrak{R},L_p)} &= \ell L_0\mathfrak{R}^{-1}L_0^T + \frac{\ell(\ell-1)}{3(\ell+1)}L_1\mathfrak{R}^{-1}L_1^T + \frac{\ell(\ell-1)(\ell-2)}{5(\ell+1)(\ell+2)}L_2\mathfrak{R}^{-1}L_2^T, \\ \chi_{(L_p,I_p)} &= L_0I_0 + L_1I_1 + L_2I_2, \\ I_0 &= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}, \\ I_1 &= \begin{bmatrix} 1 & 1 & -2 & 0 \end{bmatrix}, \\ I_2 &= \begin{bmatrix} 1 & -1 & 6 & -6 \end{bmatrix}. \end{aligned} \tag{11}$$

Remark 3. Introduced by Lemma 1, we can find that $\ell - 1/\ell + 1 < 1$ and $(\ell - 1)(\ell - 2)/(\ell + 1)(\ell + 2) < 1$, and inequality (10) could be abbreviated as follows:

$$\Lambda_{(\ell,\mathfrak{R},L_p)} = \ell\left(L_0\mathfrak{R}^{-1}L_0^T + \frac{1}{3}L_1\mathfrak{R}^{-1}L_1^T + \frac{1}{5}L_2\mathfrak{R}^{-1}L_2^T\right). \quad (12)$$

Furthermore, compared with previous Jensen's inequality, free-weighting matrices method, and Wirtinger-based inequality with single cumulative term, the new summation inequality reduces the conservativeness of the obtained results more effectively when dealing with the quadratic summation term.

3. H∞ Performance Analysis

In this section, based on the fuzzy-basis Lyapunov–Krasovskii function, delay-partitioning technique, and the improved Wirtinger-based inequality, novel delay-dependent stability and stabilization criteria for the fuzzy system $\aleph$ in (4) will be proposed to resolve the problem of robust $H\infty$ filter design mentioned in Section 2. Before beginning, here is an overview of several definitions that will be implemented in this article.

Before we start, in order to simplify, define it as follows:

$$\begin{aligned}&\tau_2 \triangleq m\mu\,(m = 1,\dots,n), \tau_t \triangleq \tau(t), \overline{\tau} \triangleq \tau_2 - \tau_1,\\&\xi(t) \triangleq \operatorname{col}\{x(t), x(t-\mu),\dots,x(t-(m-1)\mu)\}.\end{aligned} \quad (13)$$

Theorem 1. *In consideration of system (1) and fuzzy filter $\aleph_F$ in (6), the filtering error system $\aleph_e$ in (7) is asymptotically stable with $H\infty$ performance γ, if there exist real symmetric positive-definite matrices $P(\hbar)_{(i,t)}, \alpha_{(1i,1l,2i,20)}, R_{(i,v,o)}, S_{(1z,2z)}, T_{(1i,2i)}$, constant matrices $N(\hbar)_j, M(\hbar)_j$, $(j = 0,1,2)$, such that for all i, l, o, t, v, z ∈ (1, 2, . . ., r), and scalar $\gamma > 0$. For given integers $m > 0$ and $\mu > 0$ with satisfying $\tau_2 = m\mu$, the following inequality holds*

$$\Upsilon_{(i,l,o,t,v,z)} < 0, i,l,o,t,v,z \in (1,2,\dots,r), \quad (14)$$

where

$$\begin{aligned}
\Upsilon_{(i,l,o,t,v,z)} &\triangleq \Upsilon_0 + \Upsilon_1 + \Upsilon_2,\\
\Upsilon_0 &\triangleq -\widetilde{\prod}_1^T P_i\widetilde{\prod}_1 + \widetilde{\prod}_2^T\alpha_{1i}\widetilde{\prod}_2 - \widetilde{\prod}_3^T\alpha_{1l}\widetilde{\prod}_3 + e_1^T\alpha_{2i}e_1 - e_{(2+m)+1}^T\alpha_{20}e_{(2+m)+1} + e_1^T(\overline{\tau}+1)R_ie_1\\
&\quad - e_{2+m}^TR_ve_{2+m} - e_{(2+m)+1}^TR_0e_{(2+m)+1} + \operatorname{sym}\left\{\widetilde{\prod}_4^TN(\hbar)_0\widetilde{\prod}_5 + \widetilde{\prod}_4^TN(\hbar)_1\widetilde{\prod}_6 + \widetilde{\prod}_4^TN(\hbar)_2\widetilde{\prod}_7\right\}\\
&\quad + sym\left\{\widetilde{\prod}_8^TM(\hbar)_0\widetilde{\prod}_9 + \widetilde{\prod}_8^TM(\hbar)_1\widetilde{\prod}_{10} + \widetilde{\prod}_8^TM(\hbar)_2\widetilde{\prod}_{11}\right\},\\
\Upsilon_1 &\triangleq \Gamma_0^TP_t\Gamma_0 + \Gamma_1^T(\mu S_{1z} + \overline{\tau}S_{2z})\Gamma_1,\\
\Upsilon_2 &\triangleq \mu\widetilde{\Pi}_4^TN(\hbar)_jT_{1i}^{-1}N(\hbar)_j^T\widetilde{\Pi}4 + \overline{\tau}\widetilde{\Pi}_8^TM(\hbar)_jT_{2i}^{-1}M(\hbar)_j^T\widetilde{\Pi}8\\
\Gamma_0 &\triangleq \left[\overline{A}(\hbar,t)\widetilde{\Pi}4, \overline{A}_d(\hbar,t)e_{2+m}, \overline{B}(\hbar,t)\right]\\
\Gamma_1 &\triangleq \left[\overline{A}(\hbar,t)e_1, A_d(\hbar,t)e_{2+m}, \overline{B}(\hbar,t)\right]\\
e_i &\triangleq \left\{0_{n\times(i-1)n}I_n0_{n\times(9+m-i)n}\right\}, i \in \{1,\dots,9+m\},\\
\widetilde{\Pi}_1 &\triangleq \operatorname{col}\{e_1,e_2\},\\
\widetilde{\Pi}_2 &\triangleq col\{e_1,e_3\},\\
\widetilde{\Pi}_3 &\triangleq col\{e_{2+s}\}, s \in \{1,\dots,m\},\\
\widetilde{\Pi}_4 &\triangleq col\left\{e_1,e_3,e_{(2+m)+2},e_{(2+m)+4}\right\},\\
\widetilde{\Pi}_5 &\triangleq e_1 - e_3,\\
\widetilde{\Pi}_6 &\triangleq e_1 + e_3 - 2e_{(2+m)+1},\\
\widetilde{\Pi}_7 &\triangleq e_1 - e_3 + 6e_{(2+m)+2} - 6e_{(2+m)+4},\\
\widetilde{\Pi}_8 &\triangleq col\left\{e_{2+m},e_{(2+m)+1},e_{(2+m)+3},e_{(2+m)+5}\right\},\\
\widetilde{\Pi}_9 &\triangleq e_{2+m} - e_{(2+m)+1},\\
\widetilde{\Pi}_{10} &\triangleq e_{2+m} + e_{(2+m)+1} - 2e_{(2+m)+3},\\
\widetilde{\Pi}_{11} &\triangleq e_{2+m} - e_{(2+m)+1} + 6e_{(2+m)+3} - 6e_{(2+m)+5}.
\end{aligned} \quad (15)$$

Proof. Considering the idea of slack variable, now we construct the following fuzzy L-K candidate functions with time-varying delay-partitioning terms:

$$V(t) \triangleq V_0(t) + V_1(t) + V_2(t) + V_3(t), \tag{16}$$

with

$$\begin{aligned}
V_0(t) &\triangleq \zeta^T(t)P(\hbar)\zeta(t),\\
V_1(t) &\triangleq \sum_{i=t-\mu}^{t-1}\xi^T(i)\alpha_1(i)\xi(i) + \sum_{i=t-\tau_1}^{t-1}x^T(i)\alpha_2(i)x(i),\\
V_2(t) &\triangleq \sum_{i=-\tau_1+1}^{-m\mu}\sum_{j=t-1+i}^{t-1}x^T(j)R(j)x(j),\\
V_3(t) &\triangleq \sum_{i=-\mu+1}^{-1}\sum_{j=t+i}^{t-1}\varsigma_d^T(j)S(j)_{1z}\varsigma_d(j) + \sum_{i=-\tau_1}^{-m\mu-1}\sum_{j=t+i}^{t-1}\varsigma_d^T(j)S(j)_{2z}\varsigma_d(j),
\end{aligned} \tag{17}$$

where $\varsigma_d(t) \triangleq x(t+1) - x(t)$ and matrices $P > 0, \alpha_{(1,2)} > 0, R > 0, S > 0$. Then, we can define $\Delta V(t) \triangleq V(t+1) - V(t)$, and along the trajectory of fuzzy error system (7), we have

$$\Delta V_0(t) = \zeta^T(t+1)P(t+1)\zeta(t+1) - \zeta^T(t)P(t)\zeta(t) = \delta^T(t)\left[\Gamma_0^T P_t \Gamma_0 - \widetilde{\prod}_1^T P_t \widetilde{\prod}_1\right]\delta^T, \tag{18}$$

$$\begin{aligned}
\Delta V_1(t) &= \xi^T(t)\alpha_1(t)\xi(t) - \xi^T(t-\mu)\alpha_1(t-\mu)\xi(t-\mu) + x(t)^T\alpha_2(t)x(t) - x^T(t-\tau_1)\alpha_2(t-\tau_1)x(t-\tau_1)\\
&= \delta^T(t)\left[\widetilde{\prod}_2^T\alpha_{1i}\widetilde{\prod}_2 - \widetilde{\prod}_3^T\alpha_{1l}\widetilde{\prod}_3 + e_1^T\alpha_{2i}e_1 - e_5^T\alpha_{2o}\right]\delta(t),
\end{aligned} \tag{19}$$

$$\begin{aligned}
\Delta V_2(t) &= (\tau_1 - m\mu + 1)x^T(t)R(t)x(t) - x^T(t-m\mu)R(t-m\mu)x(t-m\mu) - x^T(t-\tau_1)R(t-\tau_1)x(t-\tau_1)\\
&= \delta^T(t)\left[e_1^T(\tau_1 - m\mu + 1)R_i e_1 - e_4^T R_v e_4 - e_5^T R_o e_5\right]\delta(t),
\end{aligned} \tag{20}$$

$$\begin{aligned}
\Delta V_3(t) &= \varsigma_d^T(t)\left[\left(\mu S_1(t)\right) + (\tau_1 - m\mu)S_2(t)\right]\varsigma_d(t) - \sum_{i=t-\mu}^{t-1}\varsigma_d^T(i)S_1(i)\varsigma_d(t) - \sum_{i=t-\tau_1}^{t-m\mu-1}\varsigma_d^T(i)S_2(i)\varsigma_d(i)\\
&= \delta^T(t)\left\{\Gamma_1^T\left[\mu S_{1z} + (\tau_1 - m\mu)S_{2z}\right]\Gamma_1\right\}\delta(t) - \sum_{i=t-\mu}^{t-1}\varsigma_d^T(i)S_1(i)\varsigma_d(i) - \sum_{i=t-\tau_1}^{t-m\mu-1}\varsigma_d^T(i)S_2(i)\varsigma_d(i),
\end{aligned} \tag{21}$$

where $\delta(t) \triangleq col\{\zeta(t), \xi(t-\mu), 1/\mu + 1\sum_{i=t-\mu}^{t}x(i), 1/\overline{\tau} + 1\sum_{i=t-\tau_1}^{t-m\mu}x(i), 2/(\mu+1)(\mu+2)\sum_{i=t-\mu}^{t}\sum_{j=i}^{t}x(j)2/(\overline{\tau}+1)(\overline{\tau}+2)\sum_{i=t-\tau_1}^{t-m\mu}\sum_{j=i}^{t-m\mu}x(j)\}$, Here we set $\delta_0(t) = col\{x(t), x(t-\mu), 1/\mu + 1\sum_{i=t-\mu}^{t}x(i), 2/(\mu+1)(\mu+2)\sum_{i=t-\mu}^{t}\sum_{j=i}^{t}x(j)\}$. For the summation terms (16) and (17), utilizing (9) and (11) in Lemma 1 and Remark 3, respectively, the following inequality holds:

$$\begin{aligned}
&-\sum_{i=t-\mu}^{t-1}\varsigma_d^T(i)S_1(i)\varsigma_d(i) \le \Delta_1\\
&\Delta_1 \triangleq \delta^T(t)\widetilde{\Pi}_4^T\left[\Lambda_1 + sym\{\chi_1\}\right]\widetilde{\Pi}_4\delta(t),\\
&\Lambda_1 \triangleq \ell\left(M_0S_1^{-1}M_0^T + \frac{1}{3}M_1S_1^{-1}M_1^T + \frac{1}{5}M_2S_1^{-1}M_2^T\right),\\
&\chi_1 \triangleq M_0I_0 + M_1I_1 + M_2I_2.
\end{aligned} \tag{22}$$

Similarly, applying (9) and (11) and setting $\delta(t)_1 = col\{x(t-m\mu), x(t-\tau_1), 1/\overline{\tau}+1\sum_{i=t-\tau_1}^{t-m\mu} x(i), 2/(\overline{\tau}+1)(\overline{\tau}+2)\sum_{i=t-\tau_1}^{t-m\mu}\sum_{j=i}^{t-m\mu} x(j)\}$, we have

$$
\begin{aligned}
&-\sum_{i=t-\tau_1}^{t-m\mu-1} \varsigma_d^T(i) S_2(i)\varsigma_d(i) \le \Delta_2\\
&\Delta_2 \triangleq \delta^T(t)\widetilde{\Pi}_8^T\left[\Lambda_2 + sym\{\chi_2\}\right]\widetilde{\Pi}_8\delta(t),\\
&\Lambda_2 \triangleq \ell\left(N_0S_2^{-1}N_0^T + \frac{1}{3}N_1S_2^{-1}N_1^T + \frac{1}{5}N_2S_2^{-1}N_2^T\right),\\
&\chi_2 \triangleq N_0I_0 + N_1I_1 + N_2I_2.
\end{aligned}
\tag{23}
$$

Afterwards, by combining (14)–(19), under the assumption that the input is zero noise, i.e., $\omega(t)=0$, we can obtain the following conclusions:

$$
\begin{aligned}
\Delta V(t) &= \Delta V_0(t) + \Delta V_1(t) + \Delta V_2(t) + \Delta V_3(t)\\
&\le \delta^T(t)\left[\Upsilon_0 + \Upsilon_1 + \Upsilon_2\right]\delta(t)\\
&= \delta^T(t)\Upsilon_{(i,l,o,t,v,z)}\delta(t).
\end{aligned}
\tag{24}
$$

According to Schur's complement theorem, for all nonzero $\delta(t)$, we have that $\Upsilon_0 + \Upsilon_1 + \Upsilon_2(\tau_t) < 0$ is equivalent to the LMI in (7), that is, $\Delta V(t) < 0$. This condition of the negative definite $\Delta V(t)$ guarantees that the filtering error system $\aleph$ is asymptotically stable.

Furthermore, in order to construct the $H\infty$ performance index, we define $J \triangleq \sum_{t=0}^{\infty}[e^T(t) - \gamma^2\omega^T(t)\omega(t)]$. Also, under zero initial condition $V(0)=0$, $V(\infty)>0$, and $\omega(k)\neq 0$, one obtains

$$
\begin{aligned}
J &\le \sum_{t=0}^{\infty}\left[e^T = \sum_{t=0}^{\infty}\left[e^T(t)e(t) - \gamma^2\omega^T(t)\omega(t)\right] + \Delta V(t)\right]\\
&\le \sum_{t=0}^{\infty}\delta_*^T(t)\left[\Gamma + \Gamma_0^T P_t\Gamma_0 + \Gamma_1^T\widetilde{\Omega}\Gamma_1 + \Gamma_2^T\Gamma_2 + \Gamma_*\right]\delta_*(t)(t)e(t) - \gamma^2\omega^T(t)\omega(t) + V(0) - V(\infty),
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
&\delta_*(t) \triangleq col\{\delta(t), \omega(t)\},\\
&\Gamma \triangleq \begin{bmatrix} \Upsilon_0 & 0\\ * & -\gamma^2 I\end{bmatrix}, \widetilde{\Omega} = \mu S_{1z} + \overline{\tau}S_{2z},\\
&\Gamma_2 \triangleq \left[\overline{L}(\hbar,t)\Pi_1, \overline{L}_d(\hbar,t)e_{(2+m)+1,}\overline{F}(\hbar,t)\right],\\
&\Gamma_* \triangleq \Gamma_M T_{1i}\Gamma_M^T + \Gamma_N T_{2i}\Gamma_N^T,\\
&\Gamma_M \triangleq \mu\widetilde{\Pi}_4^T\left[\, M_0(\hbar)_j \;\; M_1(\hbar)_j \;\; M_2(\hbar)_j \,\right],\\
&\Gamma_N \triangleq \overline{\tau}\widetilde{\Pi}_8^T\left[\, N_0(\hbar)_j \;\; N_1(\hbar)_j \;\; N_2(\hbar)_j \,\right].
\end{aligned}
\tag{26}
$$

Then, using the Schur complement for inequality (21), we can get the following formula:

$$
\begin{bmatrix}
\Gamma & \Gamma_0^T P_t & \Gamma_1^T\widetilde{\Omega} & \Gamma_2^T & \Gamma_M & \Gamma_N\\
* & -P_t & 0 & 0 & 0 & 0\\
* & * & -\widetilde{\Omega} & 0 & 0 & 0\\
* & * & * & -I & 0 & 0\\
* & * & * & * & -T_{1i} & 0\\
* & * & * & * & * & -T_{2i}
\end{bmatrix} < 0,
\tag{27}
$$

where

$$
\begin{aligned}
T_{1i} &\triangleq \mathrm{diag}\{T_{1i}, 3T_{1i}, 5T_{1i}\},\\
T_{2i} &\triangleq \mathrm{diag}\{T_{2i}, 3T_{2i}, 5T_{2i}\}.
\end{aligned}
\tag{28}
$$

Inequality (20) indicates that $J<0$, which is equivalent to the LMI (21). Besides, for all $\omega(t)\in l_2[0,+\infty)$, $||e||_2 < \gamma||\omega||_2$. Then, this completes the proof. □

Remark 4. In this section, the stability analysis method of discrete-time TS fuzzy systems with time-varying delay is mainly based on the TS fuzzy logic theory to model a given mathematical model of a linear system and transform complex martingale linear problems into linear problems. Thus, a bounded real lemma for a discrete-time linear system with time delay is proposed. In addition, in the selection of Lyapunov–Krasovskii functionals, we construct a term that includes time-partitioning division and a double summation term. The double summation term appears during the derivative of the Lyapunov–Krasovskii function, using the novel Wirtinger-based inequality for scaling. As for the common Jensen's inequality, the method of free-weighting matrix, and Wirtinger-based inequality with signal summation term, the system stability conditions obtained by this method are relatively conservative.

4. H_∞ Filter Design

In this section, we are committed to designing a full-order H_∞ fuzzy filter $\aleph_F$ for state delay in system (4).

Theorem 2. *Consider a discrete-time TS fuzzy system with time-varying delay from (4), and assume that the integers $m>0$, $\mu>0$ satisfy $\tau_2 = m\mu$. For a full-order filter of form (6), filter error system (7) is asymptotically stable, with H_∞ performance γ, if there exist matrices with appropriate dimensions $A, B, L, D, W_1, W_2, M_q, N_q$, $(q=0,1,2)$, and the following inequalities hold*

$$\wp > 0, g > 0, Q_{1i} = \begin{bmatrix} \alpha_{1i1} & \alpha_{1i2} \\ * & \alpha_{1i3} \end{bmatrix} > 0, Q_{2ii} > 0,$$
$$S_{1z} = \begin{bmatrix} S_{1z1} & S_{1z2} \\ * & S_{1z3} \end{bmatrix} > 0, S_{2z} = \begin{bmatrix} S_{2z1} & S_{2z2} \\ * & S_{2z3} \end{bmatrix} > 0,$$
$$\chi = \begin{bmatrix} \chi_1 & \chi_2 \\ ast & \chi_3 \end{bmatrix} > 0, R_i > 0, \widetilde{T}_{1i} > 0, \widetilde{T}_{2i} > 0. \tag{29}$$

Thus, the following linear matrix inequality can be obtained:

$$\widetilde{\Upsilon}_{(i,l,o,t,v,z)} = \begin{bmatrix} \widetilde{\Gamma} & \Gamma_M & \Gamma_N \\ * & -\widetilde{T}_{1i} & 0 \\ * & * & -\widetilde{T}_{2i} \end{bmatrix} < 0, \tag{30}$$

where

$$\widetilde{\Gamma} = \begin{bmatrix} \widetilde{\Upsilon}_0 & 0 & \widetilde{\Xi}_1^1 \\ * & -\gamma^2 I & \widetilde{\Xi}_1^2 \\ * & * & \widetilde{\Xi}_1^3 \end{bmatrix}, \widetilde{\Upsilon}_0 = \begin{bmatrix} \Upsilon_0 & 0 \\ 0 & -\chi \end{bmatrix},$$
$$\widetilde{\Xi}_1^1 = col\left[\widetilde{\Theta}_0 \ \ \widetilde{\Theta}_1 \ \ 0_{2\times 4} \ \ \widetilde{\Theta}_2 \ \ \widetilde{\Theta}_3 \ \ 0_{4\times 4} \ \ \widetilde{\Theta}_4 \ \ 0_{2\times 4} \right],$$
$$\widetilde{\Theta}_0 = [\, L_0 \ \ L_1 \ \ L_2 \ \ L_3 \ \ L_4 \ \ L_5 \ \ L_6 \ \ L_7 \ \ L_8 \,], L_0 = A_i^T \wp^T + C_i^T B^T, L_1 = A_i^T g + C_i^T B^T, L_2 = A_i^T W_1 + C_i^T B^T - W_1,$$
$$L_3 = A_i^T W_2 + C_i^T B^T - W_2, L_4 = A_i^T W_1 + C_i^T B^T - W_1, L_5 = A_i^T W_2 + C_i^T B^T - W_2,$$
$$L_6 = A_i^T W_1 + C_i^T B^T - W_1, L_7 = A_i^T W_2 + C_i^T B^T - W_2, L_8 = L_i^T - C_i^T D^T,$$
$$\widetilde{\Theta}_1 = \left[A^T \ \ A^T \ \ A^T - g^T \ \ A^T - g^T \ \ A^T - g^T \ \ A^T - g^T \ \ A^T - g^T \ \ A^T - g^T \ \ -L^T \right],$$
$$\widetilde{\Theta}_2 = \frac{1}{2} [\, M_0 \ \ M_1 \ \ M_2 \ \ M_3 \ \ M_4 \ \ M_5 \ \ M_6 \ \ M_7 \ \ M_8 \,], M_0 = A_{di}^T \wp^T + C_{di}^T B^T, M_1 = A_{di}^T L + C_{di}^T B^T, M_2 = A_{di}^T W_1 + C_{di}^T B^T,$$
$$M_3 = A_{di}^T W_2 + C_{di}^T B^T, M_4 = A_{di}^T W_1 + C_{di}^T B^T, M_5 = A_{di}^T W_2 + C_{di}^T B^T, M_6 = A_{di}^T W_2 + C_{di}^T B^T, M_7 = A_{di}^T W_1 + C_{di}^T B^T, M_8 = A_{di}^T W_2 + C_{di}^T B^T,$$
$$\widetilde{\Theta}_3 = \frac{1}{2} [\, N_0 \ \ N_1 \ \ N_2 \ \ N_3 \ \ N_4 \ \ N_5 \ \ N_6 \ \ N_7 \ \ N_8 \,], N_0 = A_{di}^T \wp^T + C_{di}^T B^T, N_1 = A_{di}^T L + C_{di}^T B^T, N_2 = A_{di}^T W_1 + C_{di}^T B^T,$$
$$N_3 = A_{di}^T W_2 + C_{di}^T B^T, N_4 = A_{di}^T W_1 + C_{di}^T B^T, N_5 = A_{di}^T W_2 + C_{di}^T B^T, N_6 = A_{di}^T W_1 + C_{di}^T BD^T, N_7 = A_{di}^T W_2 + C_{di}^T B^T, N_8 = L_{di}^T - C_{di}^T D^T,$$
$$\widetilde{\Theta}_4 = \frac{\overline{\tau}}{2} [\, P_0 \ \ P_1 \ \ P_2 \ \ P_3 \ \ P_4 \ \ P_5 \ \ P_6 \ \ P_7 \ \ P_8 \,], P_0 = A_{di}^T \wp^T + C_{di}^T B^T, P_1 = A_{di}^T L + C_{di}^T B^T, P_2 = A_{di}^T W_1 + C_{di}^T B^T,$$
$$P_3 = A_{di}^T W_2 + C_{di}^T B^T, P_4 = A_{di}^T W_1 + C_{di}^T B^T, P_5 = A_{di}^T W_2 + C_{di}^T B^T, P_6 = A_{di}^T W_1 + C_{di}^T B^T, P_7 = A_{di}^T W_2 + C_{di}^T B^T, P_8 = L_{di}^T - C_{di}^T D^T,$$
$$\widetilde{\Xi}_1^2 = [\, Q_0 \ \ Q_1 \ \ Q_2 \ \ Q_3 \ \ Q_4 \ \ Q_5 \ \ Q_6 \ \ Q_7 \ \ Q_8 \,], Q_0 = B_i^T \wp^T + D_i^T B^T, Q_1 = B_i^T g^T + D_i^T B^T, Q_2 = B_i^T W_1 + D_i^T B^T,$$
$$Q_3 = B_i^T W_2 + D_i^T B^T, Q_4 = B_i^T W_1 + D_i^T B^T, Q_5 = B_i^T W_2 + D_i^T B^T,$$
$$Q_6 = B_i^T W_1 + D_i^T B^T, Q_7 = B_i^T W_2 + D_i^T B^T, Q_8 = F_i^T - D_i^T D^T,$$
$$\widetilde{\Xi}_1^3 = \begin{bmatrix} -\wp & -g & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -g^T & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \mu^{-1} S_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \overline{\tau}^{-1} S_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Phi & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & \∗ & -\widetilde{T}_{1i} & 0 \\ * & * & * & * & * & * & * & -\widetilde{T}_{2i} \end{bmatrix},$$
$$S_1 = \begin{bmatrix} S_{1z1} - W_1 - W_1^T & S_{1z2} - W_2 - g \\ * & S_{1z3} - g - g^T \end{bmatrix}, S_2 = \begin{bmatrix} S_{2z1} - W_1 - W_1^T & S_{2z2} - W_2 - g \\ * & S_{2z3} - g - g^T \end{bmatrix},$$
$$\Phi = \begin{bmatrix} \chi_1 - W_1 - W_1^T & \chi_2 - W_2 - g \\ * & \chi_3 - g - g^T \end{bmatrix},$$
$$\widetilde{T}_{1i} \triangleq diag\{\mu T_{1i}, 3\mu T_{1i}, 5\mu T_{1i}\}, \widetilde{T}_{2i} \triangleq diag\{\overline{\tau} T_{2i}, 3\overline{\tau} T_{2i}, 5\overline{\tau} T_{2i}\}. \tag{31}$$

Then, the fuzzy filter state space parameters can be implemented as

$$A_F(\hbar) = g^{-1}A, B_F(\hbar) = g^{-1}B, L_F(\hbar) = L, D_F(\hbar) = D. \tag{32}$$

Proof. First, define the matrix $V = \begin{bmatrix} V_1 & V_2 \\ * & V_3 \end{bmatrix}$, where $V_1 \in \mathbb{R}^{n\times n}, V_3 \in \mathbb{R}^{n\times n}$ are positive-definite symmetric matrices. Here, suppose $V_2 \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, and define the following nonsingular matrix:

$$\begin{gathered}
f = \begin{bmatrix} I & 0 \\ 0 & V_3^{-1}V_2^T \end{bmatrix}, W = \begin{bmatrix} W_1 & W_2V_2^{-T}V_3 \\ V_2^T & V_3 \end{bmatrix}, \\
g \triangleq V_2V_3^{-1}V_2^T, \wp \triangleq V_1, \\
Q_{1i} \triangleq g^{-T}\widetilde{Q}_{1i}g^{-1}, Q_{2ii} \triangleq g^{-T}\widetilde{Q}_{2i}g^{-1}, \\
S_{1z} \triangleq g^{-T}\widetilde{S}_{1z}g^{-1}, S_{2z} \triangleq g^{-T}\widetilde{S}_{2z}g^{-1}, \\
\widetilde{T}_{1i} \triangleq g^{-T}\widetilde{T}_{1i}g^{-1}, \widetilde{T}_{2i} \triangleq g^{-T}\widetilde{T}_{2i}g^{-1}, \\
M_q \triangleq g^{-T}\widetilde{M}\mathrm{q}_q g^{-1}, N_q \triangleq g^{-T}\widetilde{N}_q g^{-1}, \\
\chi \triangleq g^{-T}\widetilde{\chi}\mathrm{q}_q g^{-1}.
\end{gathered} \tag{33}$$

Given that $V > 0$, linear matrix inequality (22) can be obtained:

$$\begin{bmatrix} A_F(\hbar) & B_F(\hbar) \\ L_F(\hbar) & D_F(\hbar) \end{bmatrix} \triangleq \begin{bmatrix} V_2^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A_F(\hbar) & B_F(\hbar) \\ L_F(\hbar) & D_F(\hbar) \end{bmatrix}\begin{bmatrix} V_3^{-1}V_2^T & 0 \\ 0 & I \end{bmatrix}. \tag{34}$$

Then, perform contract matrix transformation on equation (12):

$$J_1^T\Upsilon_{(i,l,o,t,v,z)}J_1 > 0, \tag{35}$$

where $J_1 = diag\{ f, f, f, f, I, f, f, f, f, I\}$, and by combining (24) and (25), we can get linear matrix inequality (22). Here, note that equation (22) can be equivalent to equation (27):

$$\begin{bmatrix} A_F(\hbar) & B_F(\hbar) \\ L_F(\hbar) & D_F(\hbar) \end{bmatrix} \triangleq \begin{bmatrix} V_2^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ L & D \end{bmatrix}\begin{bmatrix} V_2^{-T}V_3 & 0 \\ 0 & I \end{bmatrix} \triangleq \begin{bmatrix} \left(V_2^{-T}V_3\right)^{-1}g^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ L & D \end{bmatrix}\begin{bmatrix} V_2^{-T}V_3 & 0 \\ 0 & I \end{bmatrix}. \tag{36}$$

Therefore, the state space implementation $[A_F(\hbar), B_F(\hbar), L_F(\hbar), D_F(\hbar)]$ of the desired filter as defined in (6) can be obtained from (22). Then, this completes the proof. □

Remark 5. Regarding the design process of the $H\infty$ filter, first, expand (21) to form an augmented matrix in the form of an LMI. It is found here that the filter parameter matrices $[A_F(\hbar), B_F(\hbar), L_F(\hbar), D_F(\hbar)]$ can be written as (36),which implies that V2–T V3 can be viewed as a similarity transformation on the state-space realization of the filter. Therefore, the contract matrix transformation method is used to decouple the matrix to be sought from the Lyapunov–Krasovskii matrix, so that the filter parameter matrix can be systematically replaced and obtained.

5. Numerical Examples

Example 1 (see [21, 53]). Firstly, we consider discrete-time system (4) with time-varying delay:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -0.291 & 1 \\ 0 & 0.95 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.012 & 0.014 \\ 0 & 0.015 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -0.1 & 0 \\ 1 & -0.2 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.01 & 0 \\ 0.01 & 0.015 \end{bmatrix},
\end{aligned} \tag{37}
$$

where τ_t represents time-varying delay and the number of delay-partitioning segment $m = 3$. Furthermore, the upper bound of the delay is given in combination with the method in Theorem 1. Table 1 shows a detailed comparison. It can be seen that the method proposed in this paper is superior to the results published in the literature [20, 22, 53].

Example 2 (see [21]). Then, we consider the following time-varying delay Henon mapping system:

$$
\begin{cases}
x_1(t+1) = -[Cx_1(t) + (1-C)x_1(t-\tau(t))]^2 + 0.3x_2 + \omega(t), \\
x_2(t+1) = Cx_1(t) + (1-C)x_1(t-\tau(t)), \\
y(t) = Cx_1(t) + (1-C)x_1(t-\tau(t)) + \omega(t), \\
z(t) = x_1(t),
\end{cases} \tag{38}
$$

where $\omega(t)$ is the disturbance input and $C \in [0, 1]$ is a constant representing the delay coefficient. Next, we define $\vartheta(t) = \coprod x_1(t) + (1 - \coprod)x_1(t-\tau)$, and suppose that $\vartheta(t) \in [-M, M], M > 0$. By utilizing the same procedure as in paper [54], the nonlinear term $\vartheta^2(t)$ could be expressed as $\vartheta^2(t) = h_1(\vartheta(t))(-M)\vartheta(t) + h_2(\vartheta(t))(M)\vartheta(t)$, where $h_1(\vartheta(t)), h_2(\vartheta(t)) \in [0, 1]$, and $h_1(\vartheta(t)) + h_2(\vartheta(t)) = 1$. Then, by solving the equations, the membership functions $h_1(\vartheta(t))$ and $h_2(\vartheta(t))$ can be obtained as

$$
\begin{aligned}
h_1(\vartheta(t)) &= \frac{1}{2}\left(1 - \frac{\vartheta(t)}{M}\right), \\
h_2(\vartheta(t)) &= \frac{1}{2}\left(1 + \frac{\vartheta(t)}{M}\right).
\end{aligned} \tag{39}
$$

It can be concluded from the above expression that $h_1(\vartheta(t)) = 1$, $h_2(\vartheta(t)) = 0$ when $\vartheta(t) = -M$, and $h_1(\vartheta(t)) = 0$, $h_2(\vartheta(t)) = 1$ when $\vartheta(t) = M$. Thus, nonlinear system (30) can be expressed as the following approximate TS fuzzy model.

Plant From.

Rule 1. IF $\vartheta(t)$ is $-M$, THEN

$$
\begin{cases}
x(t+1) = A_1 x(t) + A_{d1} x(t-\tau(t)) + B_1\omega(t), \\
y(t) = C_1 x(t) + C_{d1} x(t-\tau(t)) + D_1\omega(t), \\
z(t) = C_1 x(t).
\end{cases} \tag{40}
$$

Rule 2. IF $\vartheta(t)$ is M, THEN

$$
\begin{cases}
x(t+1) = A_2 x(t) + A_{d2} x(t-\tau(t)) + B_2\omega(t), \\
y(t) = C_2 x(t) + C_{d2} x(t-\tau(t)) + D_2\omega(t), \\
z(t) = L_2 x(t),
\end{cases} \tag{41}
$$

where

$$
\begin{aligned}
&A_1 = \begin{bmatrix} CM & 0.3 \\ C & 0 \end{bmatrix}, A_{d1} = \begin{bmatrix} (1-C)M & 0 \\ 1-C & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -CM & 0.3 \\ C & 0 \end{bmatrix}, A_{d2} = \begin{bmatrix} -(1-C)M & 0 \\ 1-C & 0 \end{bmatrix}, \\
&B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_1 = [\,C \;\; 0\,], C_{d1} = [\,1-C \;\; 0\,], C_2 = [\,C \;\; 0\,], C_{d2} = [\,1-C \;\; 0\,], \\
&D_1 = 1, L_1 = [\,1 \;\; 0\,], D_2 = 0.5, L_2 = [\,1 \;\; 0\,].
\end{aligned} \tag{42}
$$

Here, $x(t) = \mathrm{col}\{x_1(t), x_2(t)\}$, $C = 0.8, M = 0.2$, m = 2, and $1 \le \tau(t) \le 3$ means time-varying state delay. Then, by solving the conditions in Theorem 2, the minimum upper bound of the H^∞ performance level obtained is $\gamma = 0.04574$. It can be seen that the result obtained in this section is smaller than the result obtained in reference [53], $\gamma = 2.0403$. The parameters of the full-order fuzzy filter are as follows:

$$
\begin{aligned}
A_F(\hbar) &= \begin{bmatrix} 0.7032 & 0.0516 \\ 0.0019 & 0.6709 \end{bmatrix}, B_F(\hbar) = \begin{bmatrix} -1.1930 \\ -0.0536 \end{bmatrix}, \\
L_F(\hbar) &= [\,-2.0658 \times 10^{-5} \;\; -2.5756 \times 10^{-5}\,], \\
D_F(\hbar) &= -6.4756 \times 10^{-5}.
\end{aligned} \tag{43}
$$

Then, in the case where the initial condition is zero, that is $x(0) = 0$, assume that the disturbance input $\omega(t)$ is

Table 1: The upper bound of τ_1 with various τ_2 (Example 1).

$\tau 2$	3	12
[20, Theorem 1]	14	21
[22,Theorem 3]	$23(m=3)$	$32(m=3)$
[53,Corollary 1]	$100(m=3)$	$109(m=3)$
Theorem 2	$110(m=3)$	$130(m=3)$

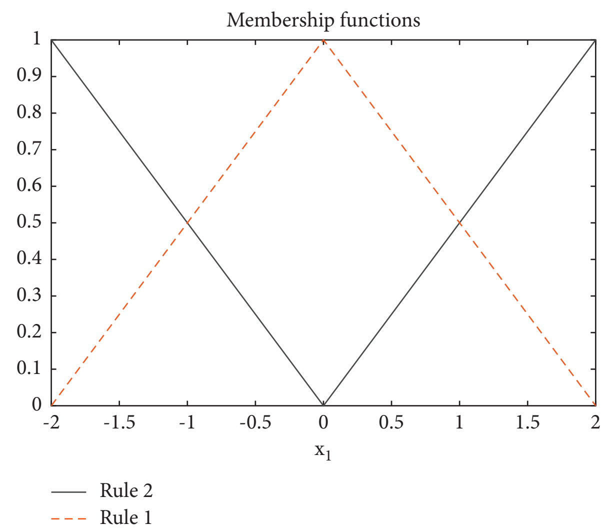

Figure 1: Membership functions (Example 2).

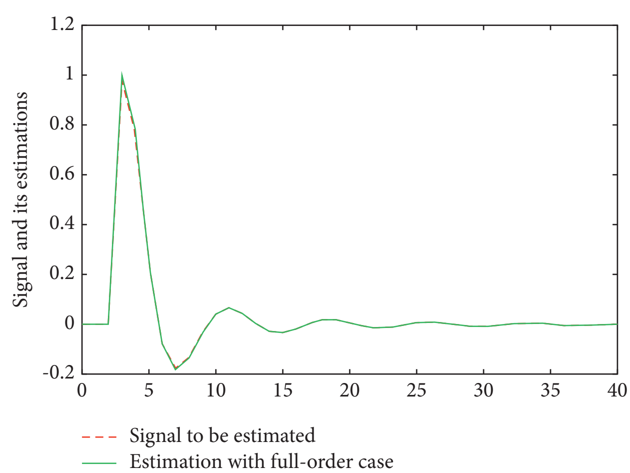

Figure 2: Signal z(t) and its estimation $z_F(t)$ of the filter (Example 2).

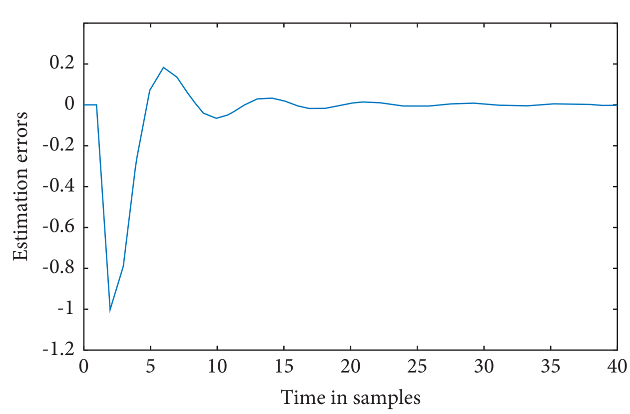

Figure 3: Estimation error $e(t)$ (Example 2).

$$\omega(t) = \frac{3\sin(0.85t)}{(0.55t)^2 + 1}. \quad (44)$$

Utilize the member function shown in Figure 1. Also, it can be concluded from the simulation results that Figures 2 and 3 reveal the signal $z(t)$ with its estimation $z_F(t)$ of the full-order filters and estimation error $e(t)$, respectively.

6. Conclusions

The main content of this paper is the analysis and design of $H\infty$ filter for discrete-time TS fuzzy systems with time-varying delay. In this paper, the bounded real lemma based on the improved Wirtinger-based inequality proposed in Section 3 is generalized and applied to the analysis and synthesis of nonlinear systems. For a given nonlinear system with time-varying delays and external disturbances, TS fuzzy logic is used to model it, and the nonlinear problem is transformed into a linear system analysis and synthesis problem. Based on the bounded real lemma and Wirtinger-based inequality with quadratic terms, $H\infty$ performance analysis is performed on the discrete-time TS fuzzy system with time-varying delay, and the coupling terms of the filter matrix and the Lyapunov matrix are uniformly replaced using the contract matrix transformation method. Furthermore, the design method of $H\infty$ filter for TS fuzzy system is given, and the bounded real lemma proposed in this paper is successfully extended to the analysis and synthesis of nonlinear systems. Finally, simulation examples are given to prove the validity and low conservativeness of the conclusion.

Data Availability

The data used to support the findings of this study are included within the article. Because it is a numerical simulation example, readers can get the same results as this article by using the LMI toolbox of Matlab and the theorem given in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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