Hindawi
Discrete Dynamics in Nature and Society
Volume 2022, Article ID 6468725, 21 pages
https://doi.org/10.1155/2022/6468725

Research Article

Coordination of a Decentralized System with a Mobile Platform and an App by a Bilateral Advertising Contract

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Received 17 January 2022; Revised 10 August 2022; Accepted 20 August 2022; Published 10 October 2022

Academic Editor: A. E. Matouk

A previous study investigates the advertising strategies of the platform and App by assuming that the platform's advertisement will increase the number of the App's users, but the App's advertisement will not increase the number of the platform's users; and the platform overcharges the App and takes the advertising fee as a source of its revenue. However, the existing users of the App may recommend the App to the users who have not used it. As a result, the App's advertisement may increase the number of the platform's users. Additionally, it is not reasonable that the platform takes the advertising fee of the App as a source of its revenue. These motivate us to reanalyze the previous work. This paper reanalyzes the previous work by additionally assuming that the App's advertisement will increase the number of the platform's users and assuming that the platform receives its revenue from its users and shares a proportion of the App's sales revenue and the App receives its revenues from its users and a proportion of its advertising cost subsidized by the platform. We find that these new assumptions have some significant effects on the previous results. We use dynamical optimization approaches to analyze a decentralized system and find the two parties' optimal advertisement efforts and proportions. To achieve the efficiency of the integrated system that is proved to be more efficient than the decentralized system, we design a bilateral advertising contact for the decentralized system and show that there exists a unique contact that can coordinate the decentralized system. We find that both parties are better off under some mild conditions and the proportion that the platform bears the App's advertising cost becomes greater with the contract than without the contract. We have gained some managerial insights.

1. Introduction

Mobile Apps have become an indispensable part of the daily life of users worldwide in the era of mobile Internet. In 2018, the global mobile App revenue has reached $92.1 billion and continues to increase closely to hundreds of billions of dollars (Source: https://newzoo.com/insights/articles/newzoos-2018-global-mobile-market-report-insights-into-the-worlds-3-billion-smartphone-users/.). The huge success of Apps cannot be achieved without two major mobile platforms, i.e., Apple's iOS and Google's Android OS. As a two-sided market connecting users and Apps, the revenue of the platform comes not only from the payment of users to enjoy the functional services provided by the platform but also more importantly from the share of Apps' sales revenue. For example, Apple charges 30% share of sales revenue for all published Apps in iOS App Store (Source: https://developer.apple.com/programs/; https://developer.apple.com/in-app-purchase/.). In addition, the platform owner is willing to provide an advertising subsidy for the App, leading to greater profits for both of them [1].

Both platforms and Apps are constantly facing new challenges and opportunities in occupying market share. [2] consider a decentralized system consisting of two parties: the platform owner and app developer, where the two parties determine their equilibrium advertising efforts and the optimal trajectories of the numbers of users independently by maximizing their separate objectives. Then, they consider an integrated system with the two parties being integrated as one party, where the two parties determine their equilibrium advertising efforts and the optimal trajectories of the numbers of users by maximizing the sum of their objectives. After that, they design a bilateral participation contract to coordinate the decentralized and integrated system. They numerically demonstrate that both the platform owner and app developer can be better off with the contract than without it.

This paper finds that some of the main assumptions and results in the study by Wang et al. [2] are problematic. First, they assume that the platform's advertisement will positively increase the number of the App's users, but the App's advertisement will not increase the number of the platform's users. According to Pedersen and Nysveen [3], the existing users of the App may recommend the App to the users who have not used it yet, i.e., the App's unaware users. This recommendation effect will lead the App's unaware users to have to register as the platform's users before downloading the desired Apps [4]. Thus, the App's advertisement may increase the number of the platform's users. This is ignored in the study by Wang et al. [2] and is captured by a non-negative parameter κ in our model. When $\kappa = 0$, our model reduces to their case, i.e., only positive effect of the platform's advertisement on the number of the App's users is considered. Second, they assume that the platform owner overcharges the App and takes the advertising fee as a source of its revenue in some cases. Clearly, this assumption is not reasonable. The reason is that the platform's revenue should partially come from the App's revenue, but not from the App's cost. Third, they assume that a loss of existing users of the platform (App) negatively influences its users' growth and is captured by the same decay coefficient δ. We think the influences may be the same or different. Fourth, we find that they present wrong expressions of the optimal trajectories of the numbers of platform users and app users for the integrated and decentralized systems (see their Propositions 2 and 4). Thus, their subsequent results and interpretations associated with these expressions are problematic. Finally, as in the study by Wang et al. [2], it is of significant importance to coordinate the two decentralized systems and the integrated system in the sense that the respective optimal advertising efforts of the platform and the App in the new decentralized system are equal to their respective optimal advertising efforts in the integrated system, and the sum of optimal objectives of the platform and the App in the new decentralized system is equal to the optimal objective in the integrated system. Thus, the efficiency is improved with coordination.

This paper aims at reanalyzing the work of Wang et al. [2] by considering the four aspects mentioned above. Our main contributions are shown as follows.

Some of our results for the integrated system, the decentralized system, and the decentralized system with the bilateral participation contract generalize theirs. Specifically, our Propositions 1, 3, and 7 for three systems, respectively, degenerate into their Propositions 1, 3, and 5 when $\kappa = 0$ and $\delta_P = \delta_A = \delta$. However, some of our results for the integrated and decentralized system do not degenerate into theirs: our Propositions 2(i) and 4(i) do not degenerate into their Propositions 2 and 4 because their Propositions 2 and 4 present wrong expressions of the optimal trajectories of the number of platform users and app users for the integrated and decentralized systems. Thus, their subsequent results and interpretations based on their Propositions 2 and 4 are problematic.

We utilize the phase diagram to show that the unique steady-state of the numbers of platform users and App users for the integrated and decentralized systems are all stable nodes and this makes sense in reality. In contrast, they are silent on this.

Finally, we theoretically compare the optimal values of the objective functions for the decentralized systems without the contract and with the contract. Our results are totally different from the related results of Wang et al. [2] and improve their results.

The remainder of this paper is organized as follows. Section 2 reviews related literature. Section 3 establishes a model of an integrated system and a decentralized system consisting of one platform and one App. Section 4 analyzes the two systems. Section 5 designs a bilateral advertising contract to coordinate the decentralized system with the integrated system. Section 6 concludes by discussing the major results and gaining some managerial insights. All proofs are provided in .

2. Literature Review

This paper will contribute to two streams of literature on cooperative advertising and App advertising.

Mathematical modeling of cooperative advertising has always been a focus of operations research. Traditionally, vertical cooperative advertising in the supply chain describes a financial agreement where the manufacturers will share a certain percentage of the retailers' advertising costs [5]. Specifically, the manufacturers set some guidelines for advertising, and the advertising preparation and planning are basically done by the retailers. Then, the retailers can require the manufacturers to provide certain advertising subsidies under the previously determined agreements [6, 7]. Cooperative advertising is therefore essentially an advertising contract or financial agreement about sharing the associated costs [8, 9]. Manufacturers and retailers can have such a cooperative relationship because manufacturers' advertising focuses on brand image, while retailers' advertising focuses on sales [7, 10, 11]. To sum up, cooperative advertising can stimulate the retailers' investment in advertising, so as to achieve the impact of advertising on sales that the manufacturers expect [12]. In addition, the revenue sharing contract and two-part-tariff contract are also introduced to coordinate the supply chain [13, 14]. However, a few studies have proposed bilateral advertising contracts in which retailers may also support their manufacturers' advertising programs [15–17].

Dynamic cooperative advertising can involve a time dimension to characterize the time dependence of advertising decisions. In dynamic models, the crucial state variables, such as brand goodwill, sales, and market share, are usually assumed to change over time, as described by differential equations. In addition, the control variables of the model are not only the advertising effort but also the price, product quality, inventory, and so on [18]. Jørgensen et al. [19] considered the long and short-term effects of advertising on demand and band goodwill. He et al. [20] extended their work to a stochastic dynamic sales model based on the decisions of advertising and prices. De Giovanni [21] introduced the manufacturers' quality improvement efforts, while Zhang et al. [16] considered the impact of advertising on the consumers' reference prices.

Different from the dynamic advertising in the supply chain, in our model setting, the numbers of the platform users and App users are two state variables, which are affected by their respective advertising efforts. Additionally, the platform's user growth is also affected by the App's user base and vice versa. Our model captures a scenario where the platform is willing to provide advertising subsidies to the App given that a revenue-sharing contract exists between the platform and the App. This is in line with the actual economic environment.

The other stream of our research is related to App advertising. For the theoretical modeling of advertising, Zhang et al. [22] modeled the strategic selection of targeted advertising and mass advertising. Sun et al. [23] studied the optimal sequence of fading advertising shown to App users by considering the sojourn and exposure effects. Choi et al. [24] investigated the search-based advertising auction problem, and Yuan et al. [25] added the quality score effect to that issue. Guo et al. [26] conducted an economic analysis of the reward advertising mechanism in the game Apps. However, the abovementioned studies only focus on the App developers' advertising strategies and do not concern the platform's interaction.

In the context of a revenue sharing contract between the platform and the App, Avinadav et al. [27] and Avinadav et al. [28] study the optimal prices and qualities of Apps under three different risk attitudes of developers. Chen et al. [29] analyzed the optimal strategies for developers to provide paid or free Apps in single and two platforms. However, the advertising decisions are not the concern of the above research. Hao et al. [1] established a bilateral market model to determine the platform owner's optimal advertising revenue sharing contract. They found that it is the best way to subsidize the App through advertising channels, which will generate greater profits for both the platform and the App. However, their model setting is static.

For the dynamic advertising model setting, most of the studies are influenced by Kumar and Sethi [30], regarding the dynamic pricing and advertising strategies of web content. However, relatively little research has been done on the dynamic advertising modeling of Apps. Ji et al. [4] studied the joint decisions of the platform and App on the advertising investment and in-App advertising adoption. They showed that revenue sharing and advertising cost sharing should be adopted to coordinate the mobile application system.

Wang et al. [2] explored the dynamic optimal advertising strategies when the platform and the App play multiple roles as sellers, ad publishers, and advertisers. This paper builds on their work, but differs from theirs in three aspects, as follows:

(1) The dynamics of the number of platform users and App users are different

They assume that the number of App's users will not affect the number of the platform's users. Instead, according to Pedersen and Nysveen [3], the recommendation effect will be generated when the App has a large user base. Thus, the current users of the App may recommend the unaware users to register on the platform. Therefore, the App's user base may contribute positively to the platform's user growth to some extent. Therefore, the number of the App's users will positively affect the number of the platform's users, which is considered and captured by a parameter κ in the current paper.

(2) They assume that a loss of existing users of both the platform and App influences their user growth negatively and is captured by the same decay coefficient δ. We assume that the loss should be captured by two different or same decay coefficients of δ_P and δ_A.

When $\kappa = 0$ and $\delta_P = \delta_A = \delta$, our motion equations of the numbers of the platform and App reduce to theirs (see our (4) and (5) and their (2) and (3)).

(3) The flows of revenues and expenditures between the platform and App are different

We assume that the platform is willing to undertake a proportion $\phi(t)$ of the App's advertising cost $C_A(t)$ $(0 \leq \phi(t) \leq 1)$, as the advertising subsidy. The platform undertakes a part of the App' advertising cost (i.e., $\phi(t)\, C_A(t)$) when $0 < \phi(t) < 1$ and does not undertake any of $C_A(t)$ (or equivalently, the App undertakes the whole advertising cost of $C_A(t)$) when $\phi(t) = 0$. In contrast, they define $\phi(t)$ as the platform owner's charge on the app developer's payment rate. By their equation (4), the platform undertakes a part of the App's advertising cost (i.e., $(1 - \phi(t))\, C_A(t)$) when $0 \leq \phi(t) < 1$, does not undertake any of $C_A(t)$ (or equivalently, the App pays the whole advertising cost of $C_A(t)$) when $\phi(t) = 1$, and overcharges the App and takes the advertising fee as a source of revenue (i.e., $(\phi(t) - 1)\, C_A(t)$) when $\phi(t) > 1$.

The abovementioned comparison shows that our $\phi(t)$ $(0 \leq \phi(t) \leq 1)$ has the same meanings as their $1 - \phi(t)$ $(0 \leq \phi(t) \leq 1)$. Clearly, our meaning of $\phi(t)$ is more easily understandable. Their assumption of $\phi(t) > 1$ does not make sense and is not reasonable: it is well known that the platform's revenue should partially come from the App's revenue, but not from the App's cost; they assume that the platform takes the advertising fee of the App as a source of revenue and that the revenue is $(\phi(t) - 1)\, C_A(t)$, which is unreasonable. Accordingly, the flows of revenues and expenditures between the platform and App are different from theirs (see our (2) and (3) and their (4) and (5)). Additionally, the total instantaneous profit of the integrated system is the same as theirs (see our (6) and their (6)).

Finally, this paper is mostly related to the literature on the coordination of advertising efforts that analyzes the proposed supply chain (SC) using the decentralized models, centralized models, and coordination models. For example, Johari and Hosseini-Motlagh [31] studied a SC with one manufacturer and two retailers facing deterministic effort-dependent market demands, where the supplier shares a fraction of retailers' advertising expenditures and incurs national advertising expenditures. The cooperation is designed by the Nash bargaining model. Ebrahimi et al. [32] investigated the coordination of promotional effort and replenishment decisions in a two-echelon SC including a single supplier and a single retailer, where a stochastic demand influenced by the retailer's promotional effort is assumed and the supplier does not share any fraction of retailers' advertising expenditures. A coordination model is proposed based on a delay in payment contract, where the supplier offers a delay in payment to the retailer as long as the retailer selects the globally optimal solution, which is obtained in the centralized model. Hosseini-Motlagh et al. [33] investigated a competitive SC consisting of one supplier and two retailers, where only the retailers invest in advertising and incur advertising expenditures, the supplier shares a fraction of the retailers' advertising expenditures, and the demand is stochastic and dependent on the effort. The coordination contract is designed in such a way that the supplier considers the trade-off between reducing lead time and paying tax on carbon emissions while providing enough incentives for the competitive retailers.

In contrast, this paper is different from the above three papers in several aspects. First, our decentralized models, centralized models, and coordination models are dynamical, and theirs are static. Second, our demands (the numbers of the users of the platform and App) follow a system of differential equations and do not depend on the efforts, and their demands are stochastic or deterministic and depend on the efforts. Last, our meaning of coordination is different from theirs. By the coordination, we mean the contract designed into the decentralized system in which the respective optimal advertising efforts of the platform and the App are equal to their respective optimal advertising efforts in the integrated system. With the coordination, we expect that the aggregate optimal objective value of the integrated system is equal to that of the decentralized system with the contract.

3. Two Systems

3.1. Costs and Revenues. We consider a system consisting of one platform and one App and respective users. The platform owner grants the App access to the platform, and the users download the desired App only through the platform. For convenience, we refer to the platform owner as "she" and the App developer as "he," respectively. All users are divided into two groups: the platform users and the App users. At time t, there are $x(t)$ users who purchase the related products of the platform, such as value-added services, accessories, and music. Meanwhile, there are $y(t)$ users who purchase the related products of the App, including paid downloads or in-App purchases.

Denote the advertising efforts of the platform and App by $u_P(t)$ and $u_A(t)$, respectively. Exerting advertising efforts will incur the costs. We assume that the costs of advertising efforts of the platform and App take quadratic functions as in the studies by Wang et al. [2]; Zhang et al. [16]; Chintagunta and Jain [34]; and Sethi [35]:

$$C_P(t) = \frac{1}{2}u_P^2(t) \text{ and } C_A(t) = \frac{1}{2}u_A^2(t), \tag{1}$$

which implies an increasing marginal advertising cost.

The App exerts advertising effort not only to increase his number of users but also indirectly benefits the platform through the revenue sharing contract. Similar to the studies by Hao et al. [1] and Wang et al. [2], we assume that the platform is willing to undertake a proportion $\phi(t)$ of the App's advertising cost $C_A(t)$ $(0 \le \phi(t) \le 1)$, as the advertising subsidy, in order to stimulate the App to increase his advertising effort. Accordingly, the total cost of the platform is $C_P(t) + C_A(t)\,\phi(t)$.

The platform and the App provide separate services for their users and then earn sales revenue from their users, respectively. Since the platform provides a user download environment for the App, we assume that the platform will share a proportion λ of App's sales revenue $(0 \le \lambda \le 1)$ when the purchase occurs in the App. Let p_P and p_A denote the marginal sales revenue of the platform and the App, respectively. Then, the App receives a total revenue of $p_A y(t)$, among which $\lambda p_A y(t)$ is paid to the platform. Thus, the platform receives a total revenue of $p_P x(t) + \lambda p_A y(t)$ and the App' total revenue is the sum of the sale net revenue of $(1-\lambda)\,p_A y(t)$ and the advertising subsidy from the platform of $C_A(t)\,\phi(t)$.

Accordingly, the instantaneous profits at time t of the platform and the App are given as follows: (Our two profit functions are different from (4) and (5) assumed by Wang et al. [2]. The differences are presented in the Literature section in a detail, where we also justify why our assumption is reasonable and understandable and theirs are not.)

$$\pi_P(t) = p_P x(t) + \lambda p_A y(t) - \frac{1}{2}u_P^2(t) - \frac{1}{2}u_A^2(t)\,\varphi(t), \tag{2}$$

$$\pi_A(t) = (1-\lambda)p_A y(t) + \frac{1}{2}u_A^2(t)\varphi(t) - \frac{1}{2}u_A^2(t). \tag{3}$$

We summarize the revenue and the expenditure flows of the platform and the App, as shown in Figure 1, where inflows indicate revenues and outflows mean expenditures.

The motion equations of the numbers of both the platform and App are assumed as follows: (Our two motion equations are generalized, but different from Wang et al. [2] (see their (2) and (3)). The coefficients of $x(t)$ and $y(t)$ are identical in the study by Wang et al. [2] $(\delta_P = \delta_A)$ and may be different in our model. They assume that the App can only advertise within the platform and thus ignore the impact of the App's user base on the platform's user growth, i.e., $\kappa = 0$.

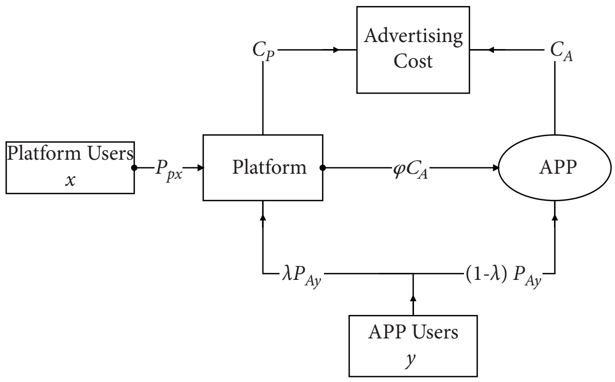

FIGURE 1: Revenue and expenditure flows.

According to Ji et al. [4], the App's user base could also contribute to the platform's user growth, which is captured in our model by a parameter κ. When $\kappa = 0$, our two equations reduced to theirs.)

$$x'(t) = \alpha u_P(t) + \kappa y(t) - \delta_P x(t), x(0) = x_0 > 0, \tag{4}$$

$$y'(t) = \gamma u_A(t) + \beta x(t) - \delta_A y(t), y(0) = y_0 > 0, \tag{5}$$

where the parameters of α, γ, and β have the same meanings as in the (2) and (3) of Wang et al. [2]; δ_P and δ_A have the similar meaning as δ in their (2). We summarize the meanings of all these parameters as follows: (i) The parameters $\alpha, \gamma, \kappa, \beta, \delta_P$, and δ_A are positive. (ii) The constant α (γ) measures the advertising effectiveness of the platform (App) on her (his) user growth. (iii) The parameter κ characterizes the indirect positive impact of the App's user base ($y(t)$) on the platform's user growth $x'(t)$. The reason is that the recommendation effect will be generated when the App has a large user base [2]. It means that the App's existing users will make recommendations to the users who have not yet downloaded the App, namely, the App's unaware users. This leads to the App's unaware users having to register as the platform's users before downloading the desired App. Therefore, the App's user base contributes positively to the platform's user growth to some extent. (iv) The parameter β characterizes the indirect positive impact of platform's user base ($x(t)$) on the App's user growth $y'(t)$. The reason is that the larger the user base of a platform, the more likely the App will be downloaded when the developer advertises it. Thus, the platform's user base will generate a potential market effect for the App [33]. (v) Some of the platform's users may leave the platform for various reasons, such as a lack of good user experience and functional innovation. This is captured by the decay coefficient δ_P. Similarly, this phenomenon may occur to the App's user and is characterized by the decay coefficient δ_A.

In reality, the number of platform users and App users is finite. On the one hand, the contributions to increases in the platform users and App users are characterized by κ and β, respectively. On the other hand, the contributions to decreases in the platform users and App users are characterized by δ_P and δ_A, respectively. Thus, it is reasonable to assume $\delta_P\delta_A > \kappa\beta$ to guarantee the finite numbers of platform users and App users. Otherwise, the number of platform users and App users will approach infinite as $t \longrightarrow +\infty$. Now, we consider two specific systems based on the above assumptions. Later on, we will find that this assumption plays a crucial role in our analysis.

3.2. Integrated System. Suppose that the platform and the App are vertically integrated into one company, which is considered an integrated system. Thus, the total instantaneous profit of the integrated system (denoted by $\pi_I(t)$) is the sum of the profits of the platform and the App. Using (2) and (3), we have

$$\pi_I(t) = \pi_P(t) + \pi_A(t) = p_P x(t) + p_A y(t) - \frac{1}{2}u_P^2(t) - \frac{1}{2}u_A^2(t). \tag{6}$$

We consider an infinite horizon with a positive discount rate ρ. The integrated system's objective is to maximize the present value of (6) over $[0, \infty)$ by choosing the advertising efforts of the platform and App subject to (4) and (5), i.e. (This is the same as (6) in Wang et al. [2]),

$$\begin{aligned} V_I^*(x_0, y_0) &= \max_{u_P(t)\ge 0, u_A(t)\ge 0} V_I \\ &= \int_0^\infty e^{-\rho t}\left[p_P x(t) + p_A y(t) - \frac{1}{2}u_P^2(t) - \frac{1}{2}u_A^2(t)\right]\mathrm{d}t, \end{aligned} \tag{7}$$

s.t. (4) and (5).

3.3. Decentralized System. In the decentralized system, the platform and the App separately maximize the present values of their own profits over $[0, \infty)$ by choosing their own advertising efforts. We model the decentralized system as a Stackelberg game structure. The platform, as a leader, first determines the advertising effort $u_P(x, y)$ and the App's advertising subsidy rate $\phi(x, y)$. After the platform's actions, the App as a follower, then determines the advertising effort $u_A(x, y)$.

Therefore, the platform's optimization problem is

$$V_{DP}^*(x_0, y_0) = \max_{u_P(t)\ge 0, \varphi(t)\ge 0} V_{DP} = \int_0^\infty e^{-\rho t}\left[p_P x(t) + \lambda p_A y(t) - \frac{1}{2}u_P^2(t) - \frac{1}{2}\varphi(t)u_A^2(t)\right]\mathrm{d}t, \tag{8}$$

Table 1: Summary of notation.

t	Time t, $t \geq 0$
Control variables	
$u_P(t)$	Advertising effort by the platform at time t
$u_A(t)$	Advertising effort by the App at time t
$\phi(t)$	App's advertising subsidy rate set by the platform at time t
State variables	
$x(t)$	The number of platform users at time t
$y(t)$	The number of App users at time t
Parameters	
α	The platform's advertising effectiveness on her user growth
γ	The App's advertising effectiveness on his user growth
β	The platform's user base effectiveness on the App's user growth
κ	The App's user base effectiveness on the platform's user growth
δ_P, δ_A	User growth decay parameter of the platform and the App, respectively
ρ	Discount rate
λ	The sharing rate of the App's sales revenue set by the platform
p_P, p_A	Marginal sales revenue of the platform and the App, respectively
$\pi_P(t)$, $\pi_A(t)$, $\pi_I(t)$	Instantaneous profits of the platform, the App, and the integrated system
V_I	Value function of the integrated system
V_{DP}	Value function of the platform in the decentralized system
V_{DA}	Value function of the app in the decentralized system

s.t. (4) and (5). The App's optimization problem is

$$V_{DA}^*(x_0, y_0) = \max_{u_A(t)\geq 0} V_{DA} = \int_0^\infty e^{-\rho t}\left[(1-\lambda)p_A y(t) - \frac{1}{2}u_A^2(t) + \frac{1}{2}\varphi(t)u_A^2(t)\right]\mathrm{d}t, \tag{9}$$

s.t. (4) and (5).

For simplicity, in the rest of this paper, we suppress t in time-dependent variables. We summarize all notations in Table 1.

4. Analysis of Two Systems

4.1. Integrated System

Proposition 1. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$ (The condition $\delta_P\delta_A - \kappa\beta > 0$ guarantees that the optimal advertising efforts of the platform and App are positive.). Then, in the integrated system, the respective optimal advertising efforts of the platform and the App are (When $\kappa = 0$ and $\delta_P = \delta_A = \delta$, our two (10) and (11) reduce to (11) and (12) in Wang et al. [2], respectively.)*

$$u_{IP}^* = \alpha\frac{(\rho+\delta_A)p_P + \beta p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \tag{10}$$

$$u_{IA}^* = \gamma\frac{(\rho+\delta_P)p_A + \kappa p_P}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}. \tag{11}$$

By (10), the optimal platform's advertising effort u_{IP}^* increases with the effectiveness of the platform's advertising effort (α) and the effectiveness of the App's user base on the platform's user growth (κ), as well as the platform's marginal sales revenue p_P. Additionally, u_{IP}^* increases with the App's marginal sales revenue p_A, and the effectiveness of the platform's user base on the App's user growth β (Apparently, $\partial u_{IP}^*/\partial\kappa > 0$. Using $\delta_P\delta_A - \kappa\beta > 0$, we have $\partial u_{IP}^*/\partial\beta = \alpha p_A[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta] + \kappa[(\rho+\delta_A)p_P + \beta p_A]/[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]^2 > 0$). Similarly, by (11), the App's optimal advertising effort u_{IA}^* increases with the parameters γ, β, and p_A and also increases with the parameters p_P and κ (Similar to the above proof of $\partial u_{IP}^*/\partial\beta > 0$, we have $\partial u_{IA}^*/\partial\kappa > 0$.). Moreover, by (10) and (11), the higher the user growth decay rate (δ_P and δ_A) and/or discount rate (ρ), the weaker the optimal advertising efforts (u_{IP}^* and u_{IA}^*).

Proposition 2. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$ (The condition $\delta_P\delta_A - \kappa\beta > 0$ guarantees that x_{ISS} and y_{ISS} are positive.). Then, in the integrated system, the following are noted:*

(i) The optimal trajectories of the numbers of platform users and App users are given by (If one substitutes (11) and (12) in Wang et al. [2] into their (2) and (3) and solves the resulting simultaneous differential equation, one can find that the solution is different from (13) and (14) in Proposition 2 of Wang et al. [2]. In other words, they present incorrect expressions of the solutions. Thus, their subsequent analysis based on the solution is problematic.)

$$x^*(t) = Ae^{q_1 t} + Be^{q_2 t} + x_{ISS}, \tag{12}$$

$$y^*(t) = Ce^{q_1 t} + De^{q_2 t} + y_{ISS}, \tag{13}$$

where

$$x_{ISS}=\frac{\alpha\delta_A u_{IP}^{*}+\gamma\kappa u_{IA}^{*}}{\delta_P\delta_A-\kappa\beta}=\frac{p_P\left[\alpha^2\delta_A(\rho+\delta_A)+\gamma^2\kappa^2\right]+p_A\left[\alpha^2\beta\delta_A+\gamma^2\kappa(\rho+\delta_P)\right]}{(\delta_P\delta_A-\kappa\beta)\left[(\rho+\delta_P)(\rho+\delta_A)-\kappa\beta\right]}, \tag{14}$$

$$\begin{aligned}
y_{ISS}&=\frac{\alpha\beta u_{IP}^{*}+\gamma\delta_P u_{IA}^{*}}{\delta_P\delta_A-\kappa\beta}=\frac{p_A\left[\gamma^2\delta_P(\rho+\delta_P)+\alpha^2\beta^2\right]+p_P\left[\gamma^2\kappa\delta_P+\alpha^2\beta(\rho+\delta_A)\right]}{(\delta_P\delta_A-\kappa\beta)\left[(\rho+\delta_P)(\rho+\delta_A)-\kappa\beta\right]},\\
\Delta&=(\delta_P-\delta_A)^2+4\kappa\beta, q_1=\frac{-(\delta_P+\delta_A)+\sqrt{\Delta}}{2}, q_2=\frac{-(\delta_P+\delta_A)-\sqrt{\Delta}}{2},\\
A&=\frac{\left[\sqrt{\Delta}-(\delta_P-\delta_A)\right](x_0-x_{ISS})+2\kappa(y_0-y_{ISS})}{2\sqrt{\Delta}}, B=\frac{\left[\sqrt{\Delta}+(\delta_P-\delta_A)\right](x_0-x_{ISS})-2\kappa(y_0-y_{ISS})}{2\sqrt{\Delta}},\\
C&=\frac{\left[\sqrt{\Delta}+(\delta_P-\delta_A)\right](y_0-y_{ISS})+2\beta(x_0-x_{ISS})}{2\sqrt{\Delta}}, D=\frac{\left[\sqrt{\Delta}-(\delta_P-\delta_A)\right](y_0-y_{ISS})-2\beta(x_0-x_{ISS})}{2\sqrt{\Delta}}.
\end{aligned} \tag{15}$$

(ii) *There exists a unique steady-state of the number of platform users and App users (x_{ISS}, y_{ISS}) that is a stable node*

(iii) *The optimal present value of the total profit, i.e., the optimal objective value is*

$$V_I^{*}(x_0,y_0)=\frac{{u_{IP}^{*}}^{2}+{u_{IA}^{*}}^{2}}{2\rho}+\frac{u_{IP}^{*}}{\alpha}x_0+\frac{u_{IA}^{*}}{\gamma}y_0. \tag{16}$$

The part (ii) of Proposition 2 shows that the optimal trajectories of the number of platform users and App users converge to the steady-state that is the stable node E (x_{ISS}, y_{ISS}). This can be proved by using the phase diagram analysis as follows. By $\delta_P\delta_A-\kappa\beta>0$, we have $\delta_P/\kappa>\beta/\delta_A>0$, implying that the slope of the demarcation curve $x'(t)=0$ is higher than the slope of the demarcation curve $y'(t)=0$. The two demarcation curves (straight lines), intersecting at point E (x_{ISS}, y_{ISS}), divide the phase space into four distinct regions, labeled I through IV as shown in Figure 2. Point E represents the intertemporal equilibrium of the integrated system. Since $\partial x'/\partial x=-\delta_P<0$ and $\partial y'/\partial y=-\delta_A<0$, we have the four groups of xy-arrowheads. Then, we can draw a family of streamlines, or trajectories, to portray the dynamics of the integrated system from any conceivable initial point. As shown in Figure 2, all the streamlines associated with point E flow noncyclically toward it. Thus, point E (x_{ISS}, y_{ISS}) is a stable node.

By (14) and (15), x_{ISS} increases with the parameters α, κ (Since $\partial u_{IP}^{*}/\partial\kappa>0$ and $\partial u_{IA}^{*}/\partial\kappa>0$ implied by Proposition 1, by (14), and $\delta_P\delta_A-\kappa\beta>0$, we h000ave $\partial x_{ISS}/\partial\kappa=\alpha\delta_A\partial u_{IP}^{*}/\partial\kappa+\gamma u_{IA}^{*}+\gamma\kappa\partial u_{IA}^{*}/\partial\kappa/\delta_P\delta_A-\kappa\beta+\beta(\alpha\delta_A u_{IP}^{*}+\gamma\kappa u_{IA}^{*})\ /\ (\delta_P\delta_A-\kappa\beta)^2\ >0$) and p_P, while y_{ISS} increases with the parameters γ, β (Similar to the above proof of $\partial x_{ISS}{}^{*}/\partial\kappa>0$, we have $\partial y_{ISS}/\partial\beta>0$.), and p_A. Furthermore, both x_{ISS} and y_{ISS} decrease in δ_P, δ_A, and ρ. (Since $\delta_P\delta_A-\kappa\beta>0$ and both u_{IP}^{*} and u_{IA}^{*} decrease in ρ, δ_P, and δ_A, we have $\partial x_{ISS}/\partial\delta_A=(\alpha\delta_A\partial u_{IP}^{*}/\partial\delta_A+\gamma\kappa\partial u_{IA}^{*}/\partial\delta_A)(\delta_P\delta_A-\kappa\beta)-\kappa(\alpha\beta u_{IP}^{*}+\gamma\delta_P u_{IA}^{*})/(\delta_P\delta_A-\kappa\beta)^2<0, \partial y_{ISS}/\partial\delta_A=(\alpha\beta\partial u_{IP}^{*}/\partial\delta_A+\gamma\delta_P\partial u_{IA}^{*}/\partial\delta_A)(\delta_P\delta_A-\kappa\beta)-\delta_P(\alpha\beta u_{IP}^{*}+\gamma\delta_P u_{IA}^{*})/(\delta_P\delta_A-\kappa\ \beta)^2<0$. Similarly, we can prove that x_{ISS} and y_{ISS} both decrease in δ_P and ρ.) By (16), V_I^{*} is positively related to the initial number of platform users and App users, i.e., x_0 and y_0. Additionally, V_I^{*} increases with α, γ, β, κ, p_P, and p_A, while decreases in ρ, δ_P, and δ_A (Since u_{IP}^{*} (u_{IA}^{*}) increases in α (γ), p_P, p_A, β, and κ decrease in ρ, δ_P, and δ_A, we can prove V_I^{*} increases in α, γ, p_P, p_A, β, and κ, decreases in ρ, δ_P, and δ_A by (10), (11), and (16).

4.2. Decentralized System

Proposition 3. *Suppose that $\delta_P\delta_A-\kappa\beta>0$ (The condition $\delta_P\delta_A-\kappa\beta>0$ guarantees that u_{DP}^{*} and u_{DA}^{*} are positive.). Then, in the decentralized system, the optimal advertising effort of the platform is as follows:*

$$u_{DP}^{*}=\alpha\frac{(\rho+\delta_A)p_P+\lambda\beta p_A}{(\rho+\delta_P)(\rho+\delta_A)-\kappa\beta}, \tag{17}$$

and the optimal App's advertising subsidy rate set by the platform for the App is as follows:

$$\varphi_D^{*}=\frac{(3\lambda-1)(\rho+\delta_P)p_A+2\kappa p_P}{(1+\lambda)(\rho+\delta_P)p_A+2\kappa p_P}. \tag{18}$$

Furthermore, the optimal advertising effort of App is as follows:

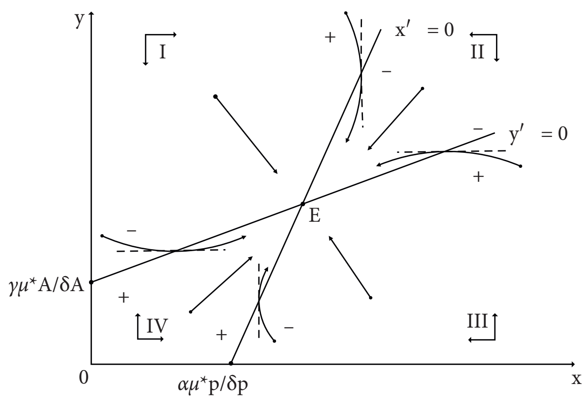

Figure 2: The phase diagram.

$$u_{DA}^* = \gamma \frac{(1+\lambda)(\rho+\delta_P)p_A + 2\kappa p_P}{2[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]}. \quad (19)$$

Next, we perform some comparative static analyses of the results of Proposition 3, leading to some useful managerial insights.

By (17), the platform's optimal advertising effort u_{DP}^* increases with the App's marginal sales revenue p_A and the effectiveness β of the platform's user base on the App's user growth. Clearly, $\partial u_{DP}^*/\partial\lambda > 0$, implying that the more the platform shares the App's sales revenue, the more effort the platform spends on advertising. Additionally, u_{DP}^* increases with α, κ, and p_P and decreases in ρ, δ_P, and δ_A. By (18), we have $\partial\phi_D^*/\partial\kappa > 0$, $\partial\phi_D^*/\partial\lambda > 0$, $\partial\phi_D^*/\partial p_P > 0$, and $\partial\phi^*/\partial p_A < 0$. Thus, the optimal App's advertising subsidy rate set by the platform ϕ_D^* increases with the effectiveness of the App's user base on the platform's user growth (κ), the sharing rate of the App's sales revenue set by the platform (λ), and the platform's marginal sales revenue (p_P), but decreases with the App's marginal sales revenue (p_A).

Remark 1. To guarantee the required condition $0 \le \phi_D^* \le 1$, the inequality of $1/3 - 2\kappa p_P/[3\ (\rho+\delta_P)\ p_A] \le \lambda \le 1$ must hold because these two inequalities are equivalent by (18). It is assumed that the proportion λ that the platform shares from the App's sales revenue is between 0 and 1. Thus, to be meaningful economically, we must add an assumption of $1/3 - 2\kappa p_P/[3\ (\rho+\delta_P)\ p_A] \ge 0$ (i.e., $p_A \ge (2\kappa p_P)/(\rho+\delta_P)$), implying that the platform shares at least a proportion $1/3 - 2\kappa p_P/[3\ (\rho+\delta_P)\ p_A]$ of the App's sales revenue. Under this assumption, we can get some managerial insights as follows:

(i) Clearly, $\lambda = 1$ is equivalent to $\phi_D^* = 1$ by (18). This implies that the platform shares all the sales revenue of the App if and only if it undertakes all the advertising costs of the App.

(ii) Since $\phi_D^* = 0$ is equivalent to $\lambda = 1/3 - 2\kappa p_P/[3\ (\rho+\delta_P)\ p_A]$, the platform will not carry any advertising cost of App if and only if it shares a proportion $1/3 - 2\kappa p_P/[3\ (\rho+\delta_P)\ p_A]$ of the App's sales revenue

(iii) Assume that $\kappa = 0$. Then, $0 \le \phi_D^* \le 1$ is equivalent to $1/3 \le \lambda \le 1$ by (18). This means that when the App's users have no impact on the platform's user growth, the platform will share at least a 1/3 proportion of the App's sales revenue. Additionally, with $\kappa = 0$, our decentralized system reduces Wang et al. [2], and $\phi_D^* = (3\lambda - 1)/(1+\lambda) > 0$ by (18), implying that ϕ_D^* only depends on λ and increases with λ. The latter result is contrary to theirs (see their (18)) and is in line with our intuition mentioned above.

(iv) Given that the proportion that the platform shares the App's sales revenue is between the open interval of $(1/3 - 2\kappa p_P/[3\ (\rho+\delta_P)\ p_A],\ 1)$, the platform will carry a proportion $\phi_D^* > 0$ of App's advertising costs.

The App's optimal advertising effort u_{DA}^* increases with the parameters κ, p_P, and λ that have positive effects on the platform's revenue. The reason here is different from that of the integrated system. It is because ϕ_D^* increases with the parameters κ, p_P, and λ. This means that the more revenue the platform earns, the more advertising subsidies she provides to the App, which incentivizes the App to increase his advertising effort. Therefore, the revenue sharing rate will not be a burden for the App to determine his advertising effort. Additionally, u_{DA}^* increases with γ, β, and p_A and decreases in ρ, δ_P, and δ_A.

Proposition 4. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$ (The condition $\delta_P\delta_A - \kappa\beta > 0$ guarantees that x_{DSS} and y_{DSS} are positive.). Then, in the decentralized system:*

(i) The optimal trajectories of the number of platform users and App users are

$$x^*(t) = \overline{A}e^{q_1 t} + \overline{B}e^{q_2 t} + x_{DSS}, \quad (20)$$

$$y^*(t) = \overline{C}e^{q_1 t} + \overline{D}e^{q_2 t} + y_{DSS}, \quad (21)$$

where $\overline{A} = [\sqrt{\Delta} - (\delta_P - \delta_A)](x_0 - x_{DSS}) + 2\kappa(y_0 - y_{DSS})/2\sqrt{\Delta}$, $\overline{B} = [\sqrt{\Delta} + (\delta_P - \delta_A)](x_0 - x_{DSS}) - 2\kappa(y_0 - y_{DSS})/2\sqrt{\Delta}$, $\overline{C} = [\sqrt{\Delta} + (\delta_P - \delta_A)](y_0 - y_{DSS}) + 2\beta(x_0 - x_{DSS})/2\sqrt{\Delta}$, $\overline{D} = [\sqrt{\Delta} - (\delta_P - \delta_A)](y_0 - y_{DSS}) - 2\beta(x_0 - x_{DSS})/2\sqrt{\Delta}$.

$$\begin{aligned} x_{DSS} &= \frac{\alpha\delta_A u_{DP}^* + \gamma\kappa u_{DA}^*}{\delta_P\delta_A - \kappa\beta} \\ &= \frac{2p_P\left[\alpha^2\delta_A(\rho+\delta_A) + \gamma^2\kappa^2\right] + p_A\left[2\lambda\alpha^2\beta\delta_A + \gamma^2\kappa(\rho+\delta_P)(1+\lambda)\right]}{2(\delta_P\delta_A - \kappa\beta)\left[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta\right]}, \end{aligned} \tag{22}$$

$$\begin{aligned} y_{DSS} &= \frac{\alpha\beta u_{DP}^* + \gamma\delta_P u_{DA}^*}{\delta_P\delta_A - \kappa\beta} \\ &= \frac{2p_P\left[\gamma^2\kappa\delta_P + \alpha^2\beta(\rho+\delta_A)\right] + p_A\left[\gamma^2\delta_P(\rho+\delta_P)(1+\lambda) + 2\lambda\alpha^2\beta^2\right]}{2(\delta_P\delta_A - \kappa\beta)\left[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta\right]}. \end{aligned} \tag{23}$$

(ii) There exists a unique steady-state of the number of platform users and App users (x_{DSS}, y_{DSS}) that is a stable node

(iii) The optimal present values of the profits of the platform and the App are given by

$$V_{DP}^*(x_0, y_0) = \frac{u_{DP}^{*\,2} + u_{DA}^{*\,2}}{2\rho} + \frac{\left[(\rho+\delta_A)p_P + \lambda\beta p_A\right]x_0 + \left[\kappa p_P + (\rho+\delta_P)\lambda p_A\right]y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \tag{24}$$

$$V_{DA}^*(x_0, y_0) = \frac{(u_{IP}^* - u_{DP}^*)u_{DP}^* + (u_{IA}^* - u_{DA}^*)u_{DA}^*}{\rho} + \frac{(1-\lambda)p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}\left[\beta x_0 + (\rho+\delta_P)y_0\right]. \tag{25}$$

Furthermore, the sum of $V_{DP}^*(x_0, y_0)$ and $V_{DP}^*(x_0, y_0)$ (denoted by V_D^*) is given by

$$V_D^* \equiv V_{DP}^*(x_0, y_0) + V_{DA}^*(x_0, y_0) = \frac{u_{DP}^*(2u_{IP}^* - u_{DP}^*) + u_{DA}^*(2u_{IA}^* - u_{DA}^*)}{2\rho} + \frac{u_{IP}^*}{\alpha}x_0 + \frac{u_{IA}^*}{\gamma}y_0. \tag{26}$$

Corollary 1. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$. Then, in the decentralized system:*

(i) $\partial V_{DP}^*/\partial\lambda > 0$

(ii) If $\lambda \geq 1/2$, then $\partial V_{DA}^*/\partial\lambda < 0$

(iii) $\partial V_D^*/\partial\lambda > 0$

Part (i) of Corollary 1 shows that a higher sharing rate of App's sales revenue leads to more revenue for the platform. Part (ii) shows that if $\lambda \geq 1/2$, a higher sharing rate of App's sales revenue definitely results in less revenue for the App, regardless of the effects of the other parameters on V_{DA}^*. Thus, the revenue sharing rate should be a crucial reference for App developers to select platforms. Part (iii) shows that the revenue sharing rate has a positive effect on the optimal aggregate present values of profits of the decentralized system (V_D^*). In conclusion, it is necessary to set a reasonable revenue sharing rate to maintain a mutually beneficial relationship between the platform and the App.

Proposition 5. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$. Then, we have*

(i) $u_{DP}^* \leq u_{IP}^*$ and $u_{DA}^* \leq u_{IA}^*$

(ii) $x_{DSS} \leq x_{ISS}$ and $y_{DSS} \leq y_{ISS}$

(iii) V_D^* ($\equiv V_{DP}^* + V_{DA}^*$) $\leq V_I^*$, where V_I^* and V_D^* are given by (16) and (26), respectively.

Proposition 5 shows that the optimal advertising efforts of the platform and App and the steady-state of their user numbers in the decentralized system are always less than or equal to the corresponding ones in the integrated system. Moreover, the optimal aggregate present value of profits in the decentralized system is always less than or equal to the optimal present value of total profit in the integrated system, implying that the decentralized system is less efficient than the integrated system as expected.

5. The Decentralized System with a Bilateral Advertising Contract

In this section, we design a bilateral advertising contract between the platform and App to improve the efficiency of the original decentralized system and expect the sum of the optimal present values of profits of the platform and the App of the decentralized system with the contract to be equal to the optimal present value of total profit of the integrated system (V_I^*). In other words, the contract is designed into the original decentralized system in order to implement the optimal objective value of the integrated system.

By introducing the bilateral advertising contract into the original decentralized system, we mean that the platform bears a proportion ϕ of the App's advertising cost while the App needs to bear a proportion φ of the platform's advertising cost. We denote the bilateral advertising contract as an ordered pair (ϕ, φ) with $0 \leq \phi \leq 1$ and $0 \leq \varphi \leq 1$. Under the contract, we have two new decentralized systems for which the optimization problems of the platform and the App, respectively, become

$$V_{DP}^*(x_0, y_0; \varphi, \phi) = \max_{u_P(t) \geq 0} V_{DP} = \int_0^\infty e^{-\rho t}\left[p_P x(t) + \lambda p_A y(t) - \frac{1}{2}(1-\phi)u_P^2(t) - \frac{1}{2}\varphi u_A^2(t)\right]dt, \tag{27}$$

s.t. (4) and (5),

$$V_{DA}^*(x_0, y_0; \varphi, \phi) = \max_{u_A(t) \geq 0} V_{DA} = \int_0^\infty e^{-\rho t}\left[(1-\lambda)p_A y(t) - \frac{1}{2}(1-\varphi)u_A^2(t) - \frac{1}{2}\phi u_P^2(t)\right]dt, \tag{28}$$

s.t. (4) and (5).

Remark 2. ϕ in (27) and (28) is an undetermined parameter, while ϕ in (8) is a control variable.

Separately solving (27) and (28), we have the optimal advertising efforts of the platform and the App, denoted by u_{DP}^* (ϕ, φ) and u_{DA}^* (ϕ, φ), respectively. We need to determine a particular pair (ϕ^*, φ^*) to coordinate the new decentralized system with the integrated system, where by coordination, we mean that the respective optimal advertising efforts of the platform and the App in the new decentralized system are equal to their respective optimal advertising efforts in the integrated system, i.e., u_{DP}^* $(\phi^*, \varphi^*) = u_{IP}^*$ and u_{DA}^* $(\phi^*, \varphi^*) = u_{IA}^*$. With the coordination, we expect that V_{DP}^* $(x_0, y_0; \phi^*, \varphi^*) + V_{DA}^*$ $(x_0, y_0; \phi^*, \varphi^*) = V_I^*$ (x_0, y_0). The following three propositions present the main results.

Proposition 6. *Consider the decentralized system with a bilateral advertising contract of (ϕ, φ). Suppose that $\delta_P\delta_A - \kappa\beta > 0$ (The condition $\delta_P\delta_A - \kappa\beta > 0$ guarantees that u_{DP}^* (ϕ, φ) and u_{DA}^* (ϕ, φ) are positive.). Then, the optimal advertising efforts of the platform and the App are both constants, i.e.,*

$$u_{DP}^*(\phi, \varphi) = \frac{\alpha}{1-\varphi}\frac{(\rho+\delta_A)p_P + \lambda\beta p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \tag{29}$$

$$u_{DA}^*(\phi, \varphi) = \frac{\gamma}{1-\phi}\frac{(1-\lambda)(\rho+\delta_P)p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}. \tag{30}$$

Proposition 7. *The new decentralized system with the contract of (ϕ^*, φ^*) is coordinated with the integrated system, i.e., $u_{DP}^*(\phi^*, \varphi^*) = u_{IP}^*$ and u_{DA}^* $(\phi^*, \varphi^*) = u_{IA}^*$, where u_{IP}^* and u_{IA}^* are given by (10) and (11), and (ϕ^*, φ^*) is given as*

$$\phi^* = \frac{\lambda(\rho+\delta_P)p_A + \kappa p_P}{(\rho+\delta_P)p_A + \kappa p_P}, \tag{31}$$

$$\varphi^* = \frac{(1-\lambda)\beta p_A}{(\rho+\delta_A)p_P + \beta p_A}. \tag{32}$$

Next, we can gain some managerial insights from Proposition 7. Clearly, by (31), we have $\partial\phi^*/\partial\kappa > 0$, $\partial\phi^*/\partial\lambda > 0$, $\partial\phi^*/\partial p_P > 0$, and $\partial\phi^*/\partial p_A < 0$. These results associated with the bilateral advertising contract are similar to those for the decentralized system without a bilateral advertising contract. Therefore, they have similar insights and are robust.

By (32), we have $\partial\varphi^*/\partial\lambda < 0$, $\partial\varphi^*/\partial p_A > 0$, and $\partial\varphi^*/\partial p_P < 0$. These three inequalities have some managerial insights. The optimal proportion (φ^*) of the platform's advertising costs that the App is willing to bear decreases with the proportion (λ) of the App's sales revenue that the platform shares, increases with the marginal sales revenue of the App (p_A), and decreases with the marginal sales revenue of the platform (p_P).

By comparing (31) with (18), we find that the optimal proportion of the App's advertising cost that the platform bears becomes smaller in the original decentralized system (without the contract) than in the new decentralized system with the contract because $\lambda(\rho+\delta_P)p_A + \kappa p_P/(\rho+\delta_P)p_A + \kappa p_P - (3\lambda-1)(\rho+\delta_P)p_A + 2\kappa p_P/(1+\lambda)(\rho+\delta_P)p_A + 2\kappa p_P = (\lambda-1)^2(\rho+\delta_P)^2{p_A}^2/[(\rho+\delta_P)p_A + \kappa p_P]$ $[(1+\lambda)(\rho+\delta_P)p_A + 2\kappa p_P] \geq 0$. This is in line with intuition. Since the contract also requires the App to bear a proportion of the platform's advertising costs, the platform bears a higher proportion of the App's advertising costs compared to the original decentralized system. Furthermore, when $\kappa = 0$ in (31), we have $\phi^* = \lambda$. Namely, when the App's users have no impact on the platform's user growth, the optimal proportion that the platform shares from the App's sales revenue is exactly equal to the proportion of the App's advertising cost that it bears. This result is partially the same as Wang et al. [2] (see their (24)).

Proposition 8. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$. Then, under the unique bilateral advertising contract (ϕ^*, φ^*), the sum of $V^*_{DP}(x_0, y_0; \phi^*, \varphi^*)$ and $V^*_{DA}(x_0, y_0; \phi^*, \varphi^*)$ is equal to the optimal present value of the aggregate profit of the integrated system, $V^*_I(x_0, y_0)$, i.e.,*

$$V^*_{DP}(x_0, y_0; \phi^*, \varphi^*) + V^*_{DA}(x_0, y_0; \phi^*, \varphi^*) = V^*_I(x_0, y_0), \tag{33}$$

where

$$V^*_{DP}(x_0, y_0; \phi^*, \varphi^*) = \frac{u^*_{DP}u^*_{IP} + (2u^*_{DA} - u^*_{IA})u^*_{IA}}{2\rho} + \frac{[(\rho+\delta_A)p_P + \lambda\beta p_A]x_0 + [\kappa p_P + (\rho+\delta_P)\lambda p_A]y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \tag{34}$$

$$V^*_{DA}(x_0, y_0; \phi^*, \varphi^*) = \frac{(u^*_{IP} - u^*_{DP})u^*_{IP} + 2(u^*_{IA} - u^*_{DA})u^*_{IA}}{2\rho} + \frac{(1-\lambda)p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}[\beta x_0 + (\rho+\delta_P)y_0], \tag{35}$$

u^*_{IP} and u^*_{IA} are given by (10) and (11), u^*_{DP} and u^*_{DA} are given by (17) and (19).

Propositions 7 and 8 show that, as expected, the new decentralized system under the contract (ϕ^*, φ^*) not only coordinates with the integrated system (i.e., $u^*_{DP}(\phi^*, \varphi^*) = u^*_{IP}$ and $u^*_{DA}(\phi^*, \varphi^*) = u^*_{IA}$) but also implements the aggregate optimal objective value of the integrated system (see (33)).

Corollary 2. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$. Then, $\partial V^*_{DP}(x_0, y_0; \phi^*, \varphi^*)/\partial\lambda > 0$ and $\partial V^*_{DA}(x_0, y_0; \phi^*, \varphi^*)/\partial\lambda < 0$.*

Compared to Corollary 1 without the bilateral advertising contract, under the unique bilateral advertising contract (ϕ^*, φ^*), $V^*_{DP}(\phi^*, \varphi^*)$ for the platform still increases with the revenue sharing rate λ. However, $V^*_{DA}(\phi^*, \varphi^*)$ for the App always decreases in the revenue sharing rate λ.

Proposition 9. *Suppose that $\delta_P\delta_A - \kappa\beta > 0$. Then,*

(i) Suppose that $\lambda \le 1 - 4\alpha^2\beta^2/[4\alpha^2\beta^2 + \gamma^2(\rho+\delta_P)^2]$ (This condition is applied to guarantee the nonnegativity of the right-hand side in (36).) holds. Then, the necessary and sufficient condition for $V^*_{DP}(x_0, y_0, \phi^*, \varphi^*) \ge V^*_{DP}(x_0, y_0)$ is as follows:

$$(u^*_{IA} - u^*_{DA})^2 \le (u^*_{IP} - u^*_{DP})u^*_{DP} \text{or} \frac{p_P}{p_A} \ge \frac{(1-\lambda)\gamma^2(\rho+\delta_P)^2 - 4\lambda\alpha^2\beta^2}{4\alpha^2\beta(\rho+\delta_A)}. \tag{36}$$

(ii) Suppose that $\lambda \le 1 - 2\alpha^2\beta^2/[4\alpha^2\beta^2 + \gamma^2(\rho+\delta_P)^2]$(This condition is applied to guarantee the nonnegativity of the right-hand side in (37).) holds. Then, the necessary and sufficient condition for $V^*_{DA}(x_0, y_0, \phi^*, \varphi^*) \ge V^*_{DA}(x_0, y_0)$ is as follows:

$$(u^*_{IA} - u^*_{DA})^2 \ge (u^*_{IP} - u^*_{DP})\left(u^*_{DP} - \frac{u^*_{IP}}{2}\right) \text{or} \frac{p_P}{p_A} \le \frac{(1-\lambda)\gamma^2(\rho+\delta_P)^2 - (4\lambda-2)\alpha^2\beta^2}{2\alpha^2\beta(\rho+\delta_A)}. \tag{37}$$

(iii) Suppose that $\lambda \le 1 - 4\alpha^2\beta^2/[4\alpha^2\beta^2 + \gamma^2(\rho+\delta_P)^2]$ holds. Then, the necessary and sufficient condition for $V^*_{DP}(x_0, y_0; \phi^*, \varphi^*) \ge V^*_{DP}(x_0, y_0)$ and $V^*_{DA}(x_0, y_0; \phi^*, \varphi^*) \le V^*_{DA}(x_0, y_0)$ is as follows:

$$\frac{p_P}{p_A} \ge \frac{(1-\lambda)\gamma^2(\rho+\delta_P)^2 - (4\lambda-2)\alpha^2\beta^2}{2\alpha^2\beta(\rho+\delta_A)}. \tag{38}$$

(iv) Suppose that $\lambda \le 1 - 4\alpha^2\beta^2/[4\alpha^2\beta^2 + \gamma^2(\rho+\delta_P)^2]$. Then, the necessary and sufficient condition for $V^*_{DP}(x_0, y_0; \phi^*, \varphi^*) \le V^*_{DP}(x_0, y_0)$ and $V^*_{DA}(x_0, y_0; \phi^*, \varphi^*) \ge V^*_{DA}(x_0, y_0)$ is as follows:

$$\frac{p_P}{p_A} \le \frac{(1-\lambda)\gamma^2(\rho+\delta_P)^2 - 4\lambda\alpha^2\beta^2}{4\alpha^2\beta(\rho+\delta_A)}. \tag{39}$$

(v) Suppose that $\lambda \le 1 - 4\alpha^2\beta^2/[4\alpha^2\beta^2 + \gamma^2(\rho+\delta_P)^2]$. Then, the necessary and sufficient condition for $V^*_{DP}(x_0, y_0; \phi^*, \varphi^*) \ge V^*_{DP}(x_0, y_0)$ and $V^*_{DA}(x_0, y_0; \phi^*, \varphi^*) \ge V^*_{DA}(x_0, y_0)$ is as follows:

$$\frac{(1-\lambda)\gamma^2(\rho+\delta_P)^2 - 4\lambda\alpha^2\beta^2}{4\alpha^2\beta(\rho+\delta_A)} \le \frac{p_P}{p_A} \le \frac{(1-\lambda)\gamma^2(\rho+\delta_P)^2 - (4\lambda-2)\alpha^2\beta^2}{2\alpha^2\beta(\rho+\delta_A)}. \tag{40}$$

(vi) $V^*_{DP}(x_0, y_0; \phi^*, \varphi^*) \le V^*_{DP}(x_0, y_0)$ and $V^*_{DA}(x_0, y_0; \phi^*, \varphi^*) \le V^*_{DA}(x_0, y_0)$ do not hold simultaneously

The first two parts of Proposition 9 identify the respective conditions under which the platform and the App are individually better off with the contract (ϕ^*, φ^*) than without the contract. Parts (iii) and (iv) show that it was possible that, under some conditions, one is better off and the other is worse off. Part (v) shows how to identify the conditions under which both parties are better off—it is a win-win situation all around. Finally, part (vi) shows that it was impossible that both parties are worse off with the contract than without it. In other words, at least one of the two parties will be better off with the contract.

The conditions (38)~(40) in Proposition 9 are mutually exclusive. It is of significance to check whether these conditions are satisfied. Next, we use some numerical examples to verify that each of the above conditions can be satisfied by taking different ranges of the revenue sharing rate λ and fixing the other parameters. Thus, Proposition 9 is valid.

Example 1. The condition (38) can be satisfied. Let $\alpha = \gamma = 5$, $p_P = 20$, $p_A = 15$, $\beta = 0.2$, $\kappa = 0.1$, $\rho = 0.1$, $\delta_P = 0.3$, $\delta_A = 0.2$, $x_0 = 50$, and $y_0 = 30$. Solving (38) for λ yields

$$\lambda \geq \frac{\left[2\alpha^2\beta^2 + \gamma^2(\rho + \delta_P)^2\right]p_A - 2\alpha^2\beta(\rho + \delta_A)p_P}{\left[4\alpha^2\beta^2 + \gamma^2(\rho + \delta_P)^2\right]p_A}. \tag{41}$$

Substituting all the given parameters into the above equation, we have $\lambda \geq 0.25$. Thus, the condition (38) is satisfied for the given parameters when $\lambda \geq 0.25$.

Example 2. The condition (39) can be satisfied. Let $\alpha = 5$, $\gamma = 10$, $p_P = 20$, $p_A = 15$, $\beta = 0.2$, $\kappa = 0.1$, $\rho = 0.1$, $\delta_P = 0.3$, $\delta_A = 0.2$, $x_0 = 50$, and $y_0 = 30$. Solving (39) for λ yields

$$\lambda \leq \frac{\gamma^2(\rho + \delta_P)^2 p_A - 4\alpha^2\beta(\rho + \delta_A)p_P}{\left[4\alpha^2\beta^2 + \gamma^2(\rho + \delta_P)^2\right]p_A}. \tag{42}$$

Substituting all the given parameters into the above equation, we have $\lambda \leq 0.4$. Thus, the condition (39) is satisfied for the given parameters when $\lambda \leq 0.4$.

Example 3. The condition (40) can be satisfied. Let $\alpha = 5$, $\gamma = 10$, $p_P = 20$, $p_A = 15$, $\beta = 0.2$, $\kappa = 0.1$, $\rho = 0.1$, $\delta_P = 0.3$, $\delta_A = 0.2$, $x_0 = 50$, and $y_0 = 30$. Solving (40) for λ yields

$$\begin{aligned} &\frac{\gamma^2(\rho + \delta_P)^2 p_A - 4\alpha^2\beta(\rho + \delta_A)p_P}{\left[4\alpha^2\beta^2 + \gamma^2(\rho + \delta_P)^2\right]p_A} \leq \lambda \\ &\leq \frac{\left[2\alpha^2\beta^2 + \gamma^2(\rho + \delta_P)^2\right]p_A - 2\alpha^2\beta(\rho + \delta_A)p_P}{\left[4\alpha^2\beta^2 + \gamma^2(\rho + \delta_P)^2\right]p_A}. \end{aligned} \tag{43}$$

Substituting all the given parameters into the above equation, we have $0.4 \leq \lambda \leq 0.7$. Thus, the condition (40) is satisfied for the given parameters when $0.4 \leq \lambda \leq 0.7$.

6. Conclusion

This paper reanalyzes the work of Wang et al. [2] by improving some of their assumptions. In particular, we assume that (i) the number of the App's users may increase the number of the platform's users; (ii) a loss of existing users of the platform (App) negatively influences its users' growth differently or equally; and (iii) the platform owner cannot overcharge the app developer and take the advertising fee as a source of revenue, but charge the App's revenue as a source of revenue. Our models and analytical results are different from or are extensions of the work of Wang et al. [2].

We reanalyze the two revised systems (the decentralized system and the integrated system with the platform and App) and design the bilateral advertising contract to coordinate the decentralized system with the integrated system.

First, for the two respective revised systems, we have obtained the two parties' equilibrium advertising efforts and the optimal trajectories of the numbers of users, and we have shown that there exist the unique steady-states of user numbers for the two parties that are shown as the stable nodes by using the phase diagram analysis. In particular, for the decentralized system, we have found the optimal advertising subsidy rate that is set by the platform for the App.

Second, to achieve the efficiency of the integrated system that is proved to be more efficient than the decentralized system, we design a bilateral advertising contact for the decentralized system where we additionally assume that the App bears a proportion of the platform's advertising cost. We show that there exists a unique bilateral advertising contact that can coordinate the decentralized system with the integrated system and simultaneously implement the optimal aggregate objective value of the integrated system.

Finally, our analytical results may have some interesting theoretical findings and managerial insights for practice as follows .

(i) In the decentralized system, the more the platform shares the App's sales revenue or the more the platform's marginal sales revenue, the more the platform should undertake the App's advertising cost. However, the more the App's marginal sales revenue, the less the platform should undertake the App's advertising cost. These results are irrespective of whether the bilateral advertising contact is incorporated into the decentralized system or not and thus are robust.

(ii) To guarantee that there exists the optimal advertising subsidy rate in the decentralized system without the contract, the proportion of the App's sales revenue that the platform shares is at least 1/3

(iii) The platform shares all the sales revenue of the App if and only if it undertakes all the advertising costs of the App in the decentralized system without the contract

(iv) When the App's users have no impact on the platform's user growth, as assumed in the study by Wang et al. [2], the platform will share at least a 1/3 proportion of the App's sales revenue in the decentralized system without the contract in order to guarantee the existence of the optimal advertising subsidy rate, and the optimal advertising subsidy rate is different from the study by Wang et al. [2].

(vi) The optimal advertising subsidy rate becomes smaller in the original decentralized system (without the contract) than in the decentralized system with the contract

(vii) The optimal present value of the platform always increases with the revenue sharing rate in both the original decentralized system (without the contract) and the decentralized system with the contract. As the revenue sharing rate increases, the optimal present value of the App decreases in the original decentralized system if the revenue sharing rate is greater than 1/2, and always decreases in the decentralized system with the contract.

(viii) The unique bilateral advertising contact guarantees that at least one of the platforms and the App can benefit from it, compared to the original decentralized system. Furthermore, when the ratio of the platform's marginal sales revenue to the App's marginal sales revenue is within a particular interval, then both the platform and the App can get more benefits from the bilateral advertising contract than from the original decentralized system, achieving a win-win situation.

Our analytical results, including some managerial insights, come from the assumptions that the costs of advertising efforts of the platform and App take quadratic functions and/or there is only one App. What if we assume the other forms of functions, such as cubic functions and several Apps. Can we design a different bilateral participation contract between the platform and the App to coordinate the decentralized system? For example, by the different contract, we mean that the platform shares a proportion of the App's sales revenue while the App shares a proportion of the platform's sales revenue. Can we compare these two contracts? [36].

Appendix

Proof of Proposition A.1. In the integrated system, the platform and the App simultaneously decide their respective advertising efforts u_P and u_A. Thus, the Hamilton–Jacobi–Bellman (HJB) equation [37] for (7) is

$$\rho V_I^*(x,y) = \max_{u_P\geq 0, u_A\geq 0}\left[p_P x + p_A y - \frac{u_P^2+u_A^2}{2} + V_{Ix}^*\left(\alpha u_P + \kappa y - \delta_P x\right) + V_{Iy}^*\left(\gamma u_A + \beta x - \delta_A y\right)\right], \quad (A.1)$$

where $V_{Ix}^* = \partial V_I^*/\partial x$ and $V_{Iy}^* = \partial V_I^*/\partial y$, which can be interpreted as the effects of the platform's user number and the App's user number on the present value of profit of the integrated system, respectively. Apparently, the optimal solution of (A.1) is

$$u_P^* = \alpha V_{Ix}^*, \quad (A.2)$$

$$u_A^* = \gamma V_{Iy}^*. \quad (A.3)$$

Substituting (A.2) and (A.3) into the objective function in (A.1), the HJB equation can be rewritten as

$$\rho V_I^* = \left(p_P + \beta V_{Iy} - \delta_P V_{Ix}\right)x + \left(p_A + \kappa V_{Ix}^* - \delta_A V_{Iy}^*\right)y + \frac{1}{2}\alpha^2\left(V_{Ix}^*\right)^2 + \frac{1}{2}\gamma^2\left(V_{Iy}^*\right)^2. \quad (A.4)$$

We guess a linear value function as

$$V_I^*(x,y) = l_1 x + l_2 y + l_3, \quad (A.5)$$

where l_1, l_2, and l_3 are the undetermined coefficients. Thus, $V_{Ix}^* = l_1$ and $V_{Iy}^* = l_2$. Substituting these two partial derivatives and (A.5) into (A.4), we have

$$\rho l_1 x + \rho l_2 y + \rho l_3 = \left(p_P + \beta l_2 - \delta_P l_1\right)x + \left(p_A + \kappa l_1 - \delta_A l_2\right)y + \frac{1}{2}\alpha^2 l_1^2 + \frac{1}{2}\gamma^2 l_2^2. \quad (A.6)$$

It follows that we have a system, i.e., $\begin{cases}\rho l_1 = p_P + \beta l_2 - \delta_P l_1\\ \rho l_2 = p_A + \kappa l_1 - \delta_A l_2\\ \rho l_3 = 1/2\alpha^2 l_1^2 + 1/2\gamma^2 l_2^2\end{cases}$. Solving this system yields

$$l_1 = V_{Ix}^* = \frac{(\rho+\delta_A)p_P + \beta p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \quad (A.7)$$

$$l_2 = V_{Ix}^* = \frac{(\rho+\delta_P)p_A + \kappa\, p_P}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \quad (A.8)$$

$$l_3 = \frac{1}{2\rho}\left(\alpha^2 l_1^2 + \gamma^2 l_2^2\right). \quad (A.9)$$

Substituting (A.7) into (A.2) and (A.8) into (A.3) yields (10) and (11), respectively. □

Proof of Proposition A.2. Part (i): Substituting (10) into (4) and (11) into (5) yields

$$\begin{cases} x'(t) = \alpha u_{IP}^* + \kappa y(t) - \delta_P x(t), x(0) = x_0 > 0,\\ y'(t) = \gamma u_{IA}^* + \beta x(t) - \delta_A y(t), y(0) = y_0 > 0.\end{cases} \quad (A.10)$$

Solving the above simultaneous differential (A.10), we obtain the optimal trajectories of the numbers of the platform and App users, as shown in (12) and (13). Solving $x'(t) = 0$ and $y'(t) = 0$ yields the unique intertemporal equilibrium (x_{ISS}, y_{ISS}), as shown in (14) and (15).

Part (ii): By $\delta_P\delta_A - \kappa\beta > 0$, we have

$$-(\delta_P+\delta_A) + \sqrt{\Delta} = -(\delta_P+\delta_A) + \sqrt{(\delta_P-\delta_A)^2 + 4\kappa\beta} < -(\delta_P+\delta_A) + \sqrt{(\delta_P-\delta_A)^2 + 4\delta_P\delta_A} = 0. \quad (A.11)$$

This and $-(\delta_P+\delta_A) - \sqrt{\Delta} < 0$ imply that $q_1 < 0$ and $q_2 < 0$. Thus, E (x_{ISS}, y_{ISS}) is a stable node.

Part (iii): Substituting (10)~(13) into the objective function in (7) and simplifying, we have

$$\begin{aligned} V_I{}^* &= \int_0^\infty e^{-\rho t}\left[p_P\left(Ae^{q_1 t} + Be^{q_2 t} + x_{ISS}\right) + p_A\left(Ce^{q_1 t} + De^{q_2 t} + y_{ISS}\right) - \frac{u_{IP}^*{}^2 + u_{IA}^*{}^2}{2}\right]\mathrm{d}t \\ &= \frac{p_P[(A+B)\rho - (Aq_2 + Bq_1)] + p_A[(C+D)\rho - (Cq_2 + Dq_1)]}{(\rho - q_1)(\rho - q_2)} \\ &\quad + \frac{p_P x_{ISS} + p_A y_{ISS}}{\rho} - \frac{u_{IP}^*{}^2 + u_{IA}^*{}^2}{2\rho}. \end{aligned} \tag{A.12}$$

Using the results in part (i), we have

$$\begin{aligned} A + B &= x_0 - x_{ISS}, Aq_2 + Bq_1 = -\delta_A(x_0 - x_{ISS}) - \kappa(y_0 - y_{ISS}), \\ C + D &= y_0 - y_{ISS}, Cq_2 + Dq_1 = -\delta_P(y_0 - y_{ISS}) - \beta(x_0 - x_{ISS}), \\ (\rho - q_1)(\rho - q_2) &= (\rho + \delta_P)(\rho + \delta_A) - \kappa\beta. \end{aligned} \tag{A.13}$$

Substituting the above five equations into (A.11) and simplifying, we obtain

$$\begin{aligned} V_I{}^* &= \frac{[(\delta_P\delta_A - \kappa\beta)p_P + \rho(\delta_P p_P - \beta p_A)]x_{ISS} + [(\delta_P\delta_A - \kappa\beta)p_A + \rho(\delta_A p_A - \kappa p_P)]y_{ISS}}{\rho[(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta]} \\ &\quad + \frac{[p_P(\rho + \delta_A) + p_A\beta]x_0 + [p_P\kappa + p_A(\rho + \delta_P)]y_0}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta} - \frac{u_{IP}^*{}^2 + u_{IA}^*{}^2}{2\rho}. \end{aligned} \tag{A.14}$$

Substituting (14) and (15) into (A.12) and simplifying, we obtain

$$\begin{aligned} V_I{}^* &= \frac{\alpha[(\delta_A + \rho)p_P + \beta p_A]u_{IP}^* + \gamma[\kappa p_P + (\delta_P + \rho)p_A]u_{IA}^*}{\rho[(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta]} \\ &\quad + \frac{[p_P(\rho + \delta_A) + p_A\beta]x_0 + [p_P\kappa + p_A(\rho + \delta_P)]y_0}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta} - \frac{u_{IP}^*{}^2 + u_{IA}^*{}^2}{2\rho}. \end{aligned} \tag{A.15}$$

By using (10) and (11), (A.13) can be simplified as

$$\begin{aligned} V_I{}^* &= \frac{u_{IP}^* u_{IP}^* + u_{IA}^* u_{IA}^*}{\rho} + \frac{u_{IP}^* x_0}{\alpha} + \frac{u_{IA}^* y_0}{\gamma} - \frac{u_{IP}^*{}^2 + u_{IA}^*{}^2}{2\rho} \\ &= \frac{u_{IP}^*{}^2 + u_{IA}^*{}^2}{2\rho} + \frac{u_{IP}^* x_0}{\alpha} + \frac{u_{IA}^* y_0}{\gamma}. \end{aligned} \tag{A.16}$$

This completes the proof. □

Proof of Proposition A.3. The HJB equations for (8) and (9) are, respectively,

$$\rho V_{DP}^* = \max_{u_P \geq 0, \varphi \geq 0}\left[p_P x + \lambda p_A y - \frac{u_P^2 + \phi u_A^2}{2} + V_{DPx}^*(\alpha u_P + \kappa y - \delta_P x) + V_{DPy}^*(\gamma u_A + \beta x - \delta_A y)\right], \tag{A.17}$$

$$\rho V_{DA}^* = \max_{u_A \geq 0}\left[(1 - \lambda)p_A y - \frac{(1 - \phi)u_A^2}{2} + V_{DAx}^*(\alpha u_P + \kappa y - \delta_P x) + V_{DAy}^*(\gamma u_A + \beta x - \delta_A y)\right], \tag{A.18}$$

where $V^*_{DPx} = \partial V^*_{DP}/\partial x$, $V^*_{DPy} = \partial V^*_{DP}/\partial y$, $V^*_{DAx} = \partial V^*_{DA}/\partial x$, and $V^*_{DAy} = \partial V^*_{DA}/\partial y$. We use backward recursion to obtain the stationary Stackelberg equilibrium. We first find the optimal advertising effort of the App (follower). Clearly, the optimal solution to (A.16) is

$$u_A(x, y|u_P, \phi) = \frac{\gamma V^*_{DAy}}{1-\phi}. \quad (A.19)$$

Substituting (A.17) into the objective function in (8) and simplifying, we rewrite the platform's optimization problem as

$$V^*_{DP} = \max_{u_P(t)\geq 0, \varphi(t)\geq 0} \int_0^{\infty} e^{-\rho t}\left[p_P x(t) + \lambda p_A y(t) - \frac{1}{2}u_P^2(t) - \frac{\varphi(t)}{2[1-\varphi(t)]^2}\gamma^2\left(V^*_{DAy}\right)^2\right]\mathrm{d}t, \quad (A.20)$$

s.t. (4) and (5).

Thus, the platform's HJB equation is

$$\rho V^*_{DP} = \max_{u_P\geq 0, \varphi\geq 0}\left[p_P x + \lambda p_A y - \frac{1}{2}u_P^2 - \frac{\varphi}{2(1-\varphi)^2}\gamma^2\left(V^*_{DAy}\right)^2 + V^*_{DPx}\left(\alpha u_P + \kappa y - \delta_P x\right) + V_{DPy}\left(\frac{\gamma^2 V^*_{DAy}}{1-\varphi} + \beta x - \delta_A y\right)\right]. \quad (A.21)$$

The optimal solution of (A.19) is found to be

$$u_P^* = \alpha V^*_{DPx}, \quad (A.22)$$

$$\varphi_D^* = \frac{2V^*_{DPy} - V^*_{DAy}}{2V^*_{DPy} + V^*_{DAy}}. \quad (A.23)$$

Substituting (A.20) and (A.21) into the objective function in (A.19) yields

$$\rho V^*_{DP} = \left(p_P + \beta V^*_{DPy} - \delta_P V^*_{DPx}\right)x + \left(\lambda p_A + \kappa V^*_{DPx} - \delta_A V^*_{DPy}\right)y + \frac{\alpha^2\left(V^*_{DPx}\right)^2}{2} + \gamma^2 V^*_{DAy}\left[\frac{2\left(1-\varphi_D^*\right)V^*_{DPy} - \varphi_D^* V^*_{DAy}}{2\left(1-\varphi_D^*\right)^2}\right]. \quad (A.24)$$

Substituting (A.17), (A.20), and (A.21) into the objective function in (A.16), we have

$$\rho V^*_{DA} = \left(\beta V^*_{DAy} - \delta_P V^*_{DAx}\right)x + \left[(1-\lambda)p_A + \kappa V^*_{DAx} - \delta_A V^*_{DAy}\right]y + \frac{\gamma^2\left(V^*_{DAy}\right)^2}{2\left(1-\varphi_D^*\right)} + \alpha^2 V^*_{DPx}V^*_{DAx}. \quad (A.25)$$

We guess two linear value functions as follows:

$$V^*_{DP}(x, y) = m_1 x + m_2 y + m_3, \quad (A.26)$$

$$V^*_{DA}(x, y) = v_1 x + v_2 y + v_3, \quad (A.27)$$

where m_1, m_2, m_3, v_1, v_2, and v_3 are the undetermined coefficients. Thus, $V^*_{DPx} = \partial V^*_{DP}/\partial x = m_1$, $V^*_{DPy} = \partial V^*_{DP}/\partial y = m_2$, $V^*_{DAx} = \partial V^*_{DA}/\partial x = v_1$, and $V^*_{DAy} = \partial V^*_{DA}/\partial y = v_2$. Substituting $V^*_{DPx} = m_1$, $V^*_{DPy} = m_2$, $V^*_{DAy} = v_2$, and (A.24) into (A.22) and using the method of undetermined coefficients, we obtain

$$m_1 = V_{DPx} = \frac{\left(\rho+\delta_A\right)p_P + \lambda\beta p_A}{\left(\rho+\delta_P\right)\left(\rho+\delta_A\right) - \kappa\beta}, \quad (A.28)$$

$$m_2 = V_{DPy} = \frac{\lambda\left(\rho+\delta_P\right)p_A + \kappa p_P}{\left(\rho+\delta_P\right)\left(\rho+\delta_A\right) - \kappa\beta}. \quad (A.29)$$

Similarly, substituting $V^*_{DPx} = m_1$, $V^*_{DAx} = v_1$, $V^*_{DAy} = v_2$, and (A.25) into (A.23) yields

$$v_1 = V_{DAx} = \frac{(1-\lambda)\beta p_A}{\left(\rho+\delta_P\right)\left(\rho+\delta_A\right) - \kappa\beta}, \quad (A.30)$$

$$v_2 = V_{DAy} = \frac{(1-\lambda)\left(\rho+\delta_P\right)p_A}{\left(\rho+\delta_P\right)\left(\rho+\delta_A\right) - \kappa\beta}. \quad (A.31)$$

By substituting (A.26) into (A.20), (A.27) and (A.29) into (A.21), and (18) and (A.29) into (A.17), we get (17)–(19). □

Proof of Proposition A.4. The proofs of parts (i) and (ii) are similar to those of parts (i) and (ii) in Proposition 2 (omitted). Next, we show part (iii).

Using Footnote (The condition $\delta_P\delta_A - \kappa\beta > 0$ guarantees that x_{DSS} and y_{DSS} are positive.), substituting (17)~(23) into the objective function in (8) and (9), respectively, and simplifying, we obtain

$$\begin{aligned}
V_{DP}^*(x_0, y_0) &= \int_0^\infty e^{-\rho t}\left[p_P\left(\overline{A}e^{q_1 t} + \overline{B}e^{q_2 t} + x_{DSS}\right) + \lambda p_A\left(\overline{C}e^{q_1 t} + \overline{D}e^{q_2 t} + y_{DSS}\right) - \frac{{u_{DP}^*}^2 + \varphi^* {u_{DA}^*}^2}{2}\right]\mathrm{d}t \\
&= \frac{[(\delta_P\delta_A - \kappa\beta)p_P + \rho(\delta_P p_P - \lambda\beta p_A)]x_{DSS} + [(\delta_P\delta_A - \kappa\beta)\lambda p_A + \rho(\lambda\delta_A p_A - \kappa p_P)]y_{DSS}}{\rho[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]} \\
&\quad + \frac{[p_P(\rho+\delta_A) + \lambda p_A\beta]x_0 + [p_P\kappa + \lambda p_A(\rho+\delta_P)]y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta} - \frac{{u_{DP}^*}^2 + \varphi^*{u_{DA}^*}^2}{2\rho} \\
&= \frac{\alpha[(\delta_A+\rho)p_P + \lambda\beta p_A]u_{DP}^* + \gamma[\kappa p_P + (\delta_P+\rho)\lambda p_A]u_{DA}^*}{\rho[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]} \\
&\quad + \frac{[p_P(\rho+\delta_A) + \lambda p_A\beta]x_0 + [p_P\kappa + \lambda p_A(\rho+\delta_P)]y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta} - \frac{{u_{DP}^*}^2 + \varphi^*{u_{DA}^*}^2}{2\rho} \\
&= \frac{\alpha[(\delta_A+\rho)p_P + \lambda\beta p_A]u_{DP}^*}{\rho[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]} - \frac{{u_{DP}^*}^2}{2\rho} + \gamma\frac{2\kappa p_P + (1+\lambda)(\delta_P+\rho)p_A}{4\rho[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]}u_{DA}^* \\
&\quad + \frac{[p_P(\rho+\delta_A) + \lambda p_A\beta]x_0 + [p_P\kappa + \lambda p_A(\rho+\delta_P)]y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta},
\end{aligned} \tag{A.32}$$

and

$$\begin{aligned}
V_{DA}^*(x_0, y_0) &= \int_0^\infty e^{-\rho t}\left[(1-\lambda)p_A\left(\overline{C}e^{q_1 t} + \overline{D}e^{q_2 t} + y_{DSS}\right) - (1-\varphi^*)\frac{1}{2}{u_{DA}^*}^2\right]\mathrm{d}t \\
&= (1-\lambda)p_A\frac{-\rho\beta x_{DSS} + [(\rho+\delta_P)\delta_A - \kappa\beta]y_{DSS}}{\rho[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]} \\
&\quad + (1-\lambda)p_A\frac{(\rho+\delta_P)y_0 + \beta x_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta} - (1-\varphi^*)\frac{1}{2\rho}{u_{DA}^*}^2 \\
&= (1-\lambda)p_A\frac{\alpha\beta u_{DP}^* + ((\rho+\delta_P))\gamma u_{DA}^*}{\rho[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]} - \frac{(1-\varphi^*){u_{DA}^*}^2}{2\rho} + (1-\lambda)p_A\frac{\beta x_0 + (\rho+\delta_P)y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta} \\
&= \frac{(1-\lambda)p_A[2\alpha\beta u_{DP}^* + \gamma(\rho+\delta_P)u_{DA}^*]}{2\rho[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]} + (1-\lambda)p_A\frac{\beta x_0 + (\rho+\delta_P)y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}.
\end{aligned} \tag{A.33}$$

Subtracting (17) from (10) and (19) from (11), we have

$$u_{IP}^* - u_{DP}^* = \frac{(1-\lambda)\alpha\beta p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \tag{A.34}$$

$$u_{IA}^* - u_{DA}^* = \frac{(1-\lambda)(\rho+\delta_P)\gamma p_A}{2[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta]}. \tag{A.35}$$

Using (1917), (A.32), and (A.33), we can simplify (A.30) and (A.31) as follows:

$$\begin{aligned}
V_{DA}^*(x_0, y_0) &= \frac{{u_{DP}^*}^2}{2\rho} + \frac{{u_{DA}^*}^2}{2\rho} + \frac{[p_P(\rho+\delta_A) + \lambda p_A\beta]x_0 + [p_P\kappa + \lambda p_A(\rho+\delta_P)]y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \\
V_{DA}^*(x_0, y_0) &= \frac{u_{IP}^* - u_{DP}^*}{\rho}u_{DP}^* + \frac{u_{IA}^* - u_{DA}^*}{\rho}u_{DA}^* + (1-\lambda)p_A\frac{(\rho+\delta_P)y_0 + \beta x_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}.
\end{aligned} \tag{A.36}$$

It yields (24) and (25). Clearly, (26) follows from (24) and (25). □ □

Proof of Corollary A.1. Part (i): By the comparative static analysis related to Proposition 3, we have $\partial u^*_{DP}/\partial\lambda > 0$ and $\partial u^*_{DA}/\partial\lambda > 0$. It follows from (24) that we have $\partial V^*_{DP}/\partial\lambda > 0$.

Part (ii): Differentiating V^*_{DA} in (25) with respect to λ, we have

$$\frac{\partial V^*_{DA}}{\partial\lambda} = \frac{(u^*_{IP} - 2u^*_{DP})\partial u^*_{DP}/\partial\lambda + (u^*_{IA} - 2u^*_{DA})\partial u^*_{DA}/\partial\lambda}{\rho} - \frac{p_A[\beta x_0 + (\rho+\delta_P)y_0]}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \tag{A.37}$$

where

$$\begin{aligned} u^*_{IP} - 2u^*_{DP} &= \alpha\frac{-(\rho+\delta_A)p_P + (1-2\lambda)\beta p_A}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \\ u^*_{IA} - 2u^*_{DA} &= \gamma\frac{-\lambda(\rho+\delta_P)p_A - \kappa p_P}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \end{aligned} \tag{A.38}$$

by (10), (17), (11), and (19). Since $\lambda \geq 1/2$, we have $u^*_{IP} - 2u^*_{DP} < 0$. Thus, $\partial V^*_{DA}/\partial\lambda < 0$ by part (i).

Part (iii): Differentiating V^*_D in (26) with respect to λ and simplifying, we obtain

$$\frac{\partial V^*_D}{\partial\lambda} = \frac{\partial u^*_{DP}/\partial\lambda(u^*_{IP} - u^*_{DP}) + \partial u^*_{DA}/\partial\lambda(u^*_{IA} - u^*_{DA})}{\rho}. \tag{A.39}$$

Using (A.32) and (A.33) and $0 < \lambda < 1$, we have $u^*_{IP} - u^*_{DP} > 0$ and $u^*_{IA} - u^*_{DA} > 0$. It follows that we have $\partial V^*_D/\partial\lambda > 0$ by $\partial u^*_{DP}/\partial\lambda > 0$ and $\partial u^*_{DA}/\partial\lambda > 0$. □

Proof of Proposition A.5. Part (i): Comparing (17) with (10) and (19) with (11), we have $u^*_{DP} \leq u^*_{IP}$ and $u^*_{DA} \leq u^*_{IA}$ by $\lambda \leq 1$.

Part (ii): Comparing (22) with (14) and (23) with (15), we have $x_{DSS} \leq x_{ISS}$ and $y_{DSS} \leq y_{ISS}$ by $\lambda \leq 1$.

Part (iii): Subtracting (26) from (16) and simplifying, we obtain

$$V^*_I - V^*_D = \frac{(u^*_{IP} - u^*_{DP})^2 + (u^*_{IA} - u^*_{DA})^2}{2\rho} \geq 0. \tag{A.40}$$

□

Proof of Proposition A.6. We begin with solving the App's problem for the optimal advertising effort u_A. Given the platform's any advertising effort u_P and the contract (ϕ, φ), App's HJB equation is

$$\rho V^*_{DA}(x_0, y_0; \phi, \varphi) = \max_{u_A \geq 0}\left[(1-\lambda)p_A y - \frac{1}{2}(1-\varphi)u_A^2 - \frac{1}{2}\phi u_P^2 + V^*_{DAx}(\alpha u_P + \kappa y - \delta_P x) + V^*_{DAy}(\gamma u_A + \beta x - \delta_A y)\right], \tag{A.41}$$

where $V^*_{DAx} = \partial V^*_{DA}/\partial x$ and $V^*_{DAy} = \partial V^*_{DA}/\partial y$. Clearly, the optimal solution to (A.36) is

$$u_A(x, y; \phi, \varphi | u_P) = \frac{\gamma V^*_{DAy}}{1-\varphi}. \tag{A.42}$$

Substituting (A.37) into the objective function in (27), the HJB equation of (27) is

$$\rho V^*_{DP}(x_0, y_0; \phi, \varphi) = \max_{u_P \geq 0}\begin{bmatrix} p_P x + \lambda p_A y - \dfrac{(1-\varphi)}{2}u_P^2 - \dfrac{\psi}{2(1-\phi)^2}\gamma^2(V^*_{DAy})2 \\ +V^*_{DPx}(\alpha u_P + \kappa y - \delta_P x) + V^*_{DPy}\left(\dfrac{\gamma^2 V^*_{DAy}}{1-\phi} + \beta x - \delta_A y\right) \end{bmatrix}, \tag{A.43}$$

where $V^*_{DPx} = \partial V^*_{DP}/\partial x$ and $V^*_{DPy} = \partial V^*_{DP}/\partial y$. The optimal solution to (A.38) can be found to be

$$u^*_P = \frac{\alpha V^*_{DPx}}{1-\phi}. \tag{A.44}$$

Substituting (A.39) into (A.38), we have

$$\rho V_{DP}^{*}(x_0, y_0; \phi, \varphi) = \left(p_P + \beta V_{DPy}^{*} - \delta_P V_{DPx}^{*}\right)x + \left(\lambda p_A + \kappa V_{DPx}^{*} - \delta_A V_{DPy}^{*}\right)y + \frac{\alpha^2 \left(V_{DPx}^{*}\right)^2}{2(1-\varphi)} - \frac{\left[2(1-\phi)V_{DPy}^{*} + \phi V_{DAy}^{*}\right]\gamma^2 V_{DAy}^{*}}{2(1-\phi)^2}. \tag{A.45}$$

Substituting (A.37) and (A.39) into (A.36) yields

$$\rho V_{DA}^{*}(x_0, y_0; \phi, \varphi) = \left(\beta V_{DAy}^{*} - \delta_P V_{DAx}^{*}\right)x + \left[(1-\lambda)p_A + \kappa V_{DAx}^{*} - \delta_A V_{DAy}^{*}\right]y + \frac{\gamma^2 \left(V_{DAy}^{*}\right)^2}{2(1-\phi)} + \frac{\left[2V_{DAx}^{*} - \varphi\left(2V_{DAx}^{*} + 1\right)\right]\alpha^2 V_{DPx}^{*}}{2(1-\varphi)^2}. \tag{A.46}$$

We guess two linear value functions as follows:

$$V_{DP}^{*}(x, y; \phi, \varphi) = s_1 x + s_2 y + s_3, \tag{A.47}$$

$$V_{DA}^{*}(x, y; \phi, \varphi) = w_1 x + w_2 y + w_3, \tag{A.48}$$

where s_1, s_2, s_3, w_1, w_2, and w_3 are the undetermined coefficients. Thus, $V_{DPx}^{*} = s_1$, $V_{DPy}^{*} = s_2$, $V_{DAx}^{*} = w_1$, and $V_{DAy}^{*} = w_2$. Substituting $V_{DPx}^{*} = s_1$, $V_{DPy}^{*} = s_2$, $V_{DAy}^{*} = w_2$, and (A.42) into (A.40) and using the method of undetermined coefficients, we obtain

$$s_1 = \overline{V}_{DPx} = \frac{(\rho + \delta_A)p_P + \lambda\beta p_A}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta}, \tag{A.49}$$

$$m_2 = \overline{V}_{DPy} = \frac{\lambda(\rho + \delta_P)p_A + \kappa p_P}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta}. \tag{A.50}$$

Similarly, substituting $V_{DPx}^{*} = s_1$, $V_{DAx}^{*} = w_1$, $V_{DAy}^{*} = w_2$, and (A.43) into (A.41) yields

$$w_1 = \overline{V}_{DAx} = \frac{(1-\lambda)\beta p_A}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta}, \tag{A.51}$$

$$v_2 = \overline{V}_{DAy} = \frac{(1-\lambda)(\rho + \delta_P)p_A}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta}. \tag{A.52}$$

Substituting (A.44) into (A.39), and (A.47) into (A.37) yields (29) and (30), respectively. □

Proof of Proposition A.7. Let $u_{DP}^{*}(\phi, \varphi) = u_{IP}^{*}$ and $u_{DA}^{*}(\phi, \varphi) = u_{IA}^{*}$, where $u_{DP}^{*}(\phi, \varphi)$ and $u_{DA}^{*}(\phi, \varphi)$ are given by (29) and (30), and u_{IP}^{*} and u_{IA}^{*} given by (10) and (11). Solving this system yields (31) and (32). □

Proof of Proposition A.8. Since $u_{DP}^{*}(\phi^{*}, \varphi^{*}) = u_{IP}^{*}$ and $u_{DA}^{*}(\phi^{*}, \varphi^{*}) = u_{IA}^{*}$ by Proposition 7, we have $x_{DSS}(\phi^{*}, \varphi^{*}) = x_{ISS}$ by (14) and (22) and $y_{DSS}(\phi^{*}, \varphi^{*}) = y_{ISS}$ by (15) and (23). That implies the steady-state of the user number under the contract (ϕ^{*}, φ^{*}) is the same as in the integrated system. Therefore, substituting (10)~(15), (31), and (32) into (27) and (28), respectively, and simplifying, we obtain

$$\begin{aligned}
V_{DP}^{*}(x_0, y_0; \phi^{*}, \varphi^{*}) &= \int_0^{\infty} e^{-\rho t}\left[p_P\left(Ae^{q_1 t} + Be^{q_2 t} + x_{ISS}\right) + \lambda p_A\left(Ce^{q_1 t} + De^{q_2 t} + y_{ISS}\right) - (1-\phi^{*})\frac{1}{2}{u_{IP}^{*}}^2 - \varphi^{*}\frac{1}{2}{u_{IA}^{*}}^2\right]dt \\
&= \frac{\alpha\left[(\delta_A + \rho)p_P + \lambda\beta p_A\right]u_{IP}^{*} + \gamma\left[\kappa p_P + (\delta_P + \rho)\lambda p_A\right]u_{IA}^{*}}{\rho\left[(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta\right]} \\
&\quad + \frac{\left[p_P(\rho + \delta_A) + \lambda p_A\beta\right]x_0 + \left[p_P\kappa + \lambda p_A(\rho + \delta_P)\right]y_0}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta} - \frac{(1-\phi^{*}){u_{IP}^{*}}^2 + \varphi^{*}{u_{IA}^{*}}^2}{2\rho} \\
&= \frac{\alpha\left[(\rho + \delta_A)p_P + \lambda\beta p_A\right]u_{IP}^{*} + \gamma\left[\kappa p_P + \lambda(\rho + \delta_P)p_A\right]u_{IA}^{*}}{2\rho\left[(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta\right]} \\
&\quad + \frac{\left[(\rho + \delta_A)p_P + \lambda\beta p_A\right]x_0 + \left[\kappa p_P + \lambda(\rho + \delta_P)p_A\right]y_0}{(\rho + \delta_P)(\rho + \delta_A) - \kappa\beta},
\end{aligned} \tag{A.53}$$

and

$$\begin{aligned} V_{DA}^{*}\left(x_0, y_0; \varphi^{*}, \varphi^{*}\right) &= \int_0^{\infty} e^{-\rho t}\left[(1-\lambda) p_A\left(C e^{q_1 t} + D e^{q_2 t} + y_{ISS}\right) - \frac{\varphi^{*} u_{IP}^{*\,2} + (1-\phi^{*}) u_{IA}^{*\,2}}{2}\right] \mathrm{d}t \\ &= (1-\lambda) p_A \frac{\alpha\beta u_{IP}^{*} + (\rho+\delta_P)\gamma u_{IA}^{*}}{\rho\left[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta\right]} \\ &\quad + (1-\lambda) p_A \frac{\beta x_0 + (\rho+\delta_P) y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta} - \frac{(1-\phi^{*}) u_{IA}^{*\,2} + \varphi^{*} u_{IP}^{*\,2}}{2\rho} \\ &= \frac{(1-\lambda) p_A\left[\alpha\beta u_{IP}^{*} + (\rho+\delta_P)\gamma u_{IA}^{*}\right]}{2\rho\left[(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta\right]} + (1-\lambda) p_A \frac{\beta x_0 + (\rho+\delta_P) y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}. \end{aligned} \tag{A.54}$$

By (19) and (11), we have

$$2u_{DA}^{*} - u_{IA}^{*} = \gamma \frac{\lambda(\rho+\delta_P) p_A + \kappa p_P}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}. \tag{A.55}$$

Using (17), (A.32), (A.33), and (A.50) to simplify (A.48) and (A.49), we have

$$\begin{aligned} V_{DP}^{*}\left(x_0, y_0; \phi^{*}, \varphi^{*}\right) &= \frac{u_{DP}^{*} u_{IP}^{*}}{2\rho} + \frac{\left(2u_{DA}^{*} - u_{IA}^{*}\right) u_{IA}^{*}}{2\rho} \\ &\quad + \frac{\left[(\rho+\delta_A) p_P + \lambda\beta p_A\right] x_0 + \left[\kappa p_P + \lambda(\rho+\delta_P) p_A\right] y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta}, \end{aligned} \tag{A.56}$$

$$V_{DA}^{*}\left(x_0, y_0; \phi^{*}, \varphi^{*}\right) \tag{A.57}$$

Using (A.51), (A.52), and (16), we have

$$V_{DP}^{*}\left(x_0, y_0; \phi^{*}, \varphi^{*}\right) + V_{DA}^{*}\left(x_0, y_0; \phi^{*}, \varphi^{*}\right) = \frac{u_{IP}^{*\,2} + u_{IA}^{*\,2}}{2\rho} + \frac{u_{IP}^{*}}{\alpha} x_0 + \frac{u_{IA}^{*}}{\gamma} y_0 = V_I^{*}\left(x_0, y_0\right). \tag{A.58}$$

□

Proof of Corollary A.2. By (18) and (19), we have $\partial u_{DP}^{*}/\partial\lambda > 0$ and $\partial u_{DA}^{*}/\partial\lambda > 0$. Thus, we get

$$\begin{aligned} \frac{\partial V_{DP}^{*}\left(x_0, y_0; \phi^{*}, \varphi^{*}\right)}{\partial\lambda} &= \frac{u_{IP}^{*}\partial u_{DP}^{*}/\partial\lambda + 2u_{IA}^{*}\partial u_{DA}^{*}/\partial\lambda}{2\rho} + \frac{\beta p_A x_0 + (\rho+\delta_P) p_A y_0}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta} > 0, \\ \frac{\partial V_{DA}^{*}\left(x_0, y_0; \phi^{*}, \varphi^{*}\right)}{\partial\lambda} &= \frac{-u_{IP}^{*}\partial u_{DP}^{*}/\partial\lambda - 2u_{IA}^{*}\partial u_{DA}^{*}/\partial\lambda}{2\rho} - \frac{p_A\left[\beta x_0 + (\rho+\delta_P) y_0\right]}{(\rho+\delta_P)(\rho+\delta_A) - \kappa\beta} < 0. \end{aligned} \tag{A.59}$$

□

Proof of Proposition A.9. Part (i): subtracting (24) from (34), simplifying and using Remark 2, we have

$$V_{DP}^{*}(x_0, y_0; \phi^{*}, \varphi^{*}) - V_{DP}^{*}(x_0, y_0) = \frac{u_{DP}^{*}(u_{IP}^{*} - u_{DP}^{*}) - (u_{IA}^{*} - u_{DA}^{*})^2}{2\rho}. \tag{A.60}$$

Thus, $V_{DP}^{*}(\phi^{*}, \varphi^{*}) \geq V_{DP}^{*}(\phi^{*}, 0)$ is equivalent to the first inequality of (36). By (10), (11), (17), and (19), the first inequality of (36) can be simplified as the second inequality of (36).

Part (ii): Subtracting (25) from (35) and simplifying, we have

$$V_{DA}^{*}(x_0, y_0; \phi^{*}, \varphi^{*}) - V_{DA}^{*}(x_0, y_0) = \frac{(u_{IP}^{*} - u_{DP}^{*})(u_{IP}^{*} - 2u_{DP}^{*}) + 2(u_{IA}^{*} - u_{DA}^{*})^2}{2\rho}. \tag{A.61}$$

Thus, $V_{DA}^{*}(\phi^{*}, \varphi^{*}) \geq V_{DA}^{*}(\phi^{*}, 0)$ is equivalent to the first inequality of (37). By (10), (11), (17), and (19), the first inequality of (37) can be simplified as the second inequality of (37).

Parts (iii), (iv), (v), and (vi) can be derived directly from parts (i) and (ii), so the relevant proofs are omitted. □ □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was support by the National Natural Science Foundation of China (71571044).

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