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Research Article

Finite-Time Pinning Synchronization Control for Coupled Complex Networks with Time-Varying Delays

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The finite-time pinning synchronization control problem is studied for coupled complex networks with time-varying delays. Based on the finite-time stability theorem, a finite-time tractive synchronous controller is designed. In addition, the selection process of tractive nodes is developed to control as few nodes as possible such that all nodes are synchronized in the network in finite time. At the same time, sufficient conditions of the finite-time constraint synchronization of the drive-response network are obtained using the Lyapunov stability theory and the matrix inequality method. The effectiveness of the proposed controller is verified by numerical simulation. This approach can be applied to large-scale complex networks with time-varying delays.

1. Introduction

Complex networks are ubiquitous, such as the Internet and biological neural, social, and supply chain networks. A large number of nodes, along with the complexity of structural links and temporal and spatial evolution, make these networks a hot research topic in nonlinear science. When some nodes or edges in a network fail, other nodes may fail through the coupling relationship between nodes and propagation dynamics, resulting in a chain reaction and the subsequent collapse of a large subset of nodes or possibly the whole network. For example, in a power network, faults in circuit breakers, transmission lines, and power generation units often lead to large-scale power outages, which can cause significant destruction. Within this propagation mechanism, the interactions between nodes shift the states of different complex dynamic systems slowly toward consistency under different initial conditions [1], which leads to the synchronization of complex networks.

The synchronization behavior of complex networks can explain many complex phenomena in the natural world, such as the resonance of a bridge caused by multiple people crossing, different routers sending messages at the same frequency, and communication satellites staying relatively stationary with respect to the Earth. In 1998, Pecora and Carroll, two scientists from the US naval laboratory, observed the synchronization of two chaotic systems for the first time and proposed a hybrid synchronization method [2]. At present, the synchronization control of coupled, orderless systems involves observations in the field of chaos and control, with practical applications in various fields, including secure communication, a multiagent network coordinated control, and unmanned aerial vehicle (UAV) cooperative operation. This control is especially beneficial in soft robotics, which requires more sensors than rigid robotics because this kind of sensor network presents challenges in cooperation to task completion.

However, the synchronous application of complex networks has disadvantages, such as packet transmission within the Internet. Researchers found that different routers eventually send data synchronously, resulting in network congestion. Therefore, detailed observations on the synchronous control of various natural and unnatural complex dynamic networks are necessary to finally promote the benign synchronization of complex networks in practical projects, establish the positive role of networks, inhibit the malignant synchronization of complex networks, and reduce the negative impact [3]. The challenges of network synchronization reach mathematics, control, genetics, epidemiology and other disciplines [4–6].

The research on synchronization originated from the synchronization behavior of chaotic systems, mainly including internal and external synchronizations. Internal synchronization studies the internal dynamic characteristics of networks, and external synchronization considers multiple systems. Traditional approaches to synchronization use the main stability and Lyapunov functions, which focus on a network's coupling characteristics. When adjusting the coupling characteristics of the network does not improve the system synchronization ability, these methods are no longer applicable. Nishikawa et al. found no apparent relationship between the synchronization ability of complex networks and the average distance between networks. The authors also noted that the distribution of degrees and mediators in the networks could better characterize the synchronization ability of the system. A more uniform distribution results in a stronger synchronization ability of the system. Moreover, a less uniform distribution yields a worse synchronization ability [7]. As far as the problem of discontinuous inertial neural networks (DINNS) with time-varying delays is considered, the periodic solutions are synchronized with a fixed time [8]. Kong analyze the fixed-time synchronization of a class of discontinuous fuzzy inertial neural networks with time-varying delays based on the new improved fixed-time stability lemmas [9].

Many achievements were made in the research on synchronization modes of complex networks, including $H¥H_{\infty}$ [10], finite-time [11], exponential [12], complete [13], and generalized synchronizations [14]. More recently, scholars aimed to improve the synchronization ability of the network by designing a controller [15] and obtaining sufficient conditions using the Lyapunov method [16]. New techniques are developed in dealing with the stochastic Lyapunov functional, and sufficient conditions are gained to ensure that the resulting dynamics are stochastically stable [17]. Synchronous control methods of complex networks are also emerging, such as intermittent control [18], sampling data control [19], and pulse control [12]. A relatively new approach is to consider pulse synchronization based on an unknown bounded time-varying delay. Each node in the corresponding system can synchronize with the corresponding node in the driving system [15]. The number of nodes in the complex network is large, and the network structure presents different complex characteristics. In fact, the strength of inner link will also be changed under random disturbance [20], which makes all nodes in this networks not necessary to be controlled.

Wang and Chen first proposed the method of pinning control for the synchronization behavior of a scale-free network and achieved good experimental results [21]. Subsequently, boundary control [22] and pinning control [23] have aroused the interest of scholars. Boundary control only needs to select boundary nodes for control. Yang et al. [24] studied the juxtaposed boundary controller with boundary measurement, which solves the cluster synchronization control problem of nonlinear complex spatiotemporal dynamic networks with community structure by placing sensors and actuators at the spatial boundary. Similarly, the network synchronization is controlled through a few nodes. The node selection is more flexible than the boundary control and not limited to nodes along the boundary.

Most research on pinning synchronization control falls into one of two categories. One approach is to design the pinning cluster synchronization controller when the number of network nodes is small (generally less than ten nodes) to control only those directly connected to other clusters, reducing the number of controllers [25–27]. The design aims to provide appropriate pinning feedback controllers such that the nodes converge to a consistent state as well as the equilibrium, periodic orbit, or chaotic orbit of the nonlinear part of the node dynamics. In a simulation of this design [28], a network composed of six nodes is given, and three nodes are selected for control. The other approach assumes that the number of network nodes is large. Shi et al. [29] considered a complex network of 500 nodes, from which 50 nodes are randomly selected for containment control, and better results were obtained. While the pinning node is easier to determine for the network with fewer nodes, efforts to select the appropriate pinning node for a complex network with many nodes are meaningful.

Several achievements have emerged in the research of complex networks and their synchronous control, both in theory and application. The aim is for when time $t \longrightarrow \infty$, error $e(t) \longrightarrow 0$. However, to gain faster control, many scholars have carried out studies on limited-time synchronous control [30]. This approach not only saves significant time and cost but also yields other economic benefits, such as the cooperative work of a UAV network, the information transmission of a communication network, and the coordination between supply chain network enterprises. Traditional work included criteria for finite-time stability of nonlinear systems [31, 32]. Recently, a study [33] introduced combined synchronization in complex networks. Based on the synovial control principle and finite-time stability theory, the network synovial area control input was designed, and sufficient conditions for synchronization were obtained. In [34], an adaptive synchronization controller was designed to realize the finite-time stability of the error system when the network coupling weight was known and unknown. To realize the finite-time control of nonstrict feedback stochastic nonlinear systems with input quantization and full-state constraints, Zhu et al. [35] proposed a semiglobal finite-time uncertain control method in the sense of probability, which accelerated the convergence of the system. The research on single coupling and coupling weight in the above study obtained good results, while the problem of time-delay coupling needs further investigation. In addition, the coupling time delays of complex networks are inevitable phenomena, and especially for networks with many nodes, the time-delay coupling situation is more involved.

Based on actual engineering and management needs and the literature review, a finite-time pinning synchronization control is proposed to solve the synchronization problem of large-scale complex networks with time-varying delay couplings. This control is valuable for practical engineering applications. The main contributions of this paper are as follows:

(1) The designed controller realizes limited-time synchronization of multiple complex networks by controlling some nodes under the consideration of a coupling delay. In other words, the controller not only synchronizes in a limited time but also reduces the number of control nodes, which can present design challenges. The synchronization problem of large-scale time-varying coupled networks can be addressed with this controller, with significant economic benefits.

(2) The existing literature has not specified a method for pinning node selection. This paper is likely the first effort to determine the number of pinning nodes according to the coupling strength of the network and then select the corresponding pinning nodes through the arrangement of node degrees.

The remainder of this paper is organized as follows. Section 2 presents the node dynamics model of a complex network and related lemmas and assumptions. Section 3 describes the pinning controller design and the finite-time synchronization control theorem and proof. In addition, the procedure for pinning node selection is outlined. In Section 4, the numerical simulation is carried out. Finally, Section 5 concludes the study and briefly explains the application direction and future research.

2. Construction of a Complex Network

Consider the dynamic drive-response model of a complex network composed of N identical nodes, which is described as

$$\begin{aligned}\dot{x}_i(t) &= f(x_i(t)) + c_0 \sum_{j=1}^{N} a_{ij}\Gamma x_j(t) \\ &\quad + c_1 \sum_{j=1}^{N} b_{ij}\Gamma h\big(x_j(t-\tau(t))\big), \quad i = 1,2\ldots,N,\end{aligned} \tag{1}$$

$$\begin{aligned}\dot{y}_i(t) &= f(y_i(t)) + c_0 \sum_{j=1}^{N} a_{ij}\Gamma y_j(t) \\ &\quad + c_1 \sum_{j=1}^{N} b_{ij}\Gamma h\big(y_j(t-\tau(t))\big) + u_i(t), \quad i = 1,2\ldots,N,\end{aligned} \tag{2}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ is the state variable of the first driving network node; $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T \in R^n$ is the state variable of the first node of the response network; i is the network node label; c_0 and c_1 are the coupling strength of complex networks without delay and with delay, respectively; Γ is the internal coupling matrix of network nodes; and $A = (a_{ij}) \in \mathbf{R}^{N\times N}$ and $B = (b_{ij}) \in \mathbf{R}^{N\times N}$ are the delay-free and delay-coupled configuration matrices of the network, respectively. If an uncoupled delay edge exists between nodes i and j, then $a_{ij} > 0\,(i \neq j)$; otherwise, $a_{ij} = 0\,(i \neq j)$. If a coupled time-delay edge is between nodes i and j, then $b_{ij} > 0\,(i \neq j)$; otherwise, $b_{ij} = 0\,(i \neq j)$. Its diagonal elements satisfy $a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}$. Furthermore, $b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij}$. $\tau(t)$ stands for coupled time-delay and continuously differentiable function. Thus, to satisfy $0 \leq \tau(t) \leq \tau < 1$, $u_i(t) \in \mathbf{R}^n$ is the synchronization controller to be designed.

Definition 1. For any initial condition, make $e_i(t) = y_i(t) - x_i(t)$, and there is a certain moment $t_1 > 0$. Under the action of the controller $u_i(t)$, $\lim_{t\longrightarrow t_1}\|e(t)\| = 0$; then, the drive-response network described by equations (1) and (2) realizes synchronization in a finite time t_1.

Remark 1. For the following expressions, symbolic definitions are given: $\|\Gamma\|_2 = \gamma$; $\rho_{\min}$ represents the minimum eigenvalue of matrix $(\Gamma + \Gamma^T)/2$, where $\rho_{\min} \neq 0$; $\widehat{A} = (\widetilde{A}^T + \widetilde{A})/2$ is a symmetric matrix; matrix $\widetilde{A}$ transforms the diagonal element a_{ii} into $(\rho_{\min}/\gamma)a_{ii}$ based on matrix A; $\widehat{A}_l$ is the submatrix of l rows and l columns before $\widehat{A}$ is deleted; l is the number of pinning nodes in the article; and $\lambda_{\max}(A)$ is the maximum eigenvalue of matrix A.

Lemma 1. *(see [11]). Assume that function $V(t)$ is continuous positive definite. If a continuous function $\gamma(\cdot)$ exists, $\gamma(\sigma) > 0, \sigma \in (0, +\infty)$ satisfying $V(t) \leq -\gamma(V(t))$, $\int_0^{V(0)} 1/\gamma(\sigma)\mathrm{d}\sigma = t^* < \infty$; then, $V(t) \equiv 0, \forall t \geq t^*$. If $\gamma(\sigma) = d\sigma^q$ and $d > 0, 0 < q < 1$, then the transient synchronization time is formulated as*

$$t^* = \frac{V^{1-q}(0)}{d(1-q)}. \tag{3}$$

Lemma 2. *(see [36]). Given a symmetric matrix $M = \begin{pmatrix} X & Y \\ Y^T & M_l \end{pmatrix} \in \mathbf{R}^{N\times N}$, $D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{R}^{N\times N}$, where $X, D_1 \in \mathbf{R}^{r\times r}\,(1 \leq r \leq N)$, and matrix $D_1 = \mathrm{diag}\{d_1, d_2, \ldots, d_r\}$ is diagonal positive definite, then $M - D < 0$ hold if and only if $M_l < 0, d_i \geq \lambda_{\max}(X - YM_l^{-1}Y^T)$, where $(1 \leq i \leq r)$.*

Lemma 3. *(see [37]). Assume X and Y are arbitrary n-dimensional vectors and $\varepsilon > 0$; the following inequality holds: $2X^TY \leq \varepsilon X^TX + 1/\varepsilon Y^TY$.*

Lemma 4. *(adjoint inequality of Minkowski inequality). If $r > 0$ and $r \neq 1$, then*

$$\begin{aligned}&\sum(a + b + \cdots + l)^r > \sum a^r + \sum b^r + \cdots + \sum l^r, \quad r > 1, \\ &\sum(a + b + \cdots + l)^r < \sum a^r + \sum b^r + \cdots + \sum l^r, \quad 0 < r < 1,\end{aligned} \tag{4}$$

unless all the numbers in each set $a_v, b_v, \ldots, l_v\,(v = 1, 2, \ldots, n)$ except one are zero.

Assumption 1. For arbitrary $x, y \in \mathrm{R}^n$ and the existence constant $L_1 > 0$, the following inequality is true:

$$(y - x)^T(f(y) - f(x)) \leq L_1(y - x)^T(y - x). \tag{5}$$

Assumption 2. Assume that f satisfies the Lipschitz condition, i.e., for any scalars $x, y \in \mathrm{R}^n$, $\xi > 0$, there exist scalars $L_2 > 0$ such that

$$\|f(y) - f(x)\| \leq L_2\|y - x\|. \tag{6}$$

Remark 2. The above assumptions are common Lipschitz conditions in dynamic system research and are widely used in synchronization research. Our assumptions are easy to verify. Most chaotic systems satisfy the above assumptions.

3. Pinning Controller

3.1. Controller Design. Without losing generality, select the first l nodes in the drive-response models (1) and (2) as the constraint nodes. At this time, equation (2) can be further described as

$$\begin{cases} \dot{y}_i(t) = f(y_i(t)) + c_0 \sum_{j=1}^{N} a_{ij}\Gamma y_j(t) + c_1 \sum_{j=1}^{N} b_{ij}\Gamma h\left(y_j(t-\tau(t))\right) + u_i(t), & i = 1, 2, \ldots, l, \\ \dot{y}_i(t) = f(y_i(t)) + c_0 \sum_{j=1}^{N} a_{ij}\Gamma y_j(t) + c_1 \sum_{j=1}^{N} b_{ij}\Gamma h\left(y_j(t-\tau(t))\right), & i = l+1, l+2, \ldots, N. \end{cases} \tag{7}$$

Remark 3. The first l nodes selected here only describe the current complex network dynamic model and do not represent the subsequent selection method of nodes in containment control. The determination method of l is given later in this section.

Design the following controllers, where α, k, and λ are control gains with $\alpha > 0, k > 0$, and $\lambda > 0$. Especially, if $e_i(t) = 0$, then $u_i(t) = 0$.

$$u_i(t) = \begin{cases} -ke_i(t) - \alpha\left(\dfrac{\lambda}{1-\tau}\sum_{i=1}^{l}\int_{t-\tau(t)}^{t} e_i^T(s)e_i(s)\mathrm{d}s\right)^{1+\mu/2}\dfrac{e_i(t)}{\|e(t)\|^2} - \alpha\operatorname{sign}(e_i(t))\left|e_i(t)\right|^{\mu}, & i = 1, 2, \ldots, l \\ 0, \quad i = l+1, l+2, \ldots, N. \end{cases} \tag{8}$$

Remark 4. In the controller design process, we need to consider both finite-time synchronization and containment control, which presents a significant challenge. Moulay et al. [38] proposed a new approach to achieving finite-time stability. However, the Lyapunov function satisfying the assumptions is difficult to find, limiting the practical application of this method. Many controller designs involve symbolic functions, which can cause system state and signal chattering, serious damage, and other impacts on the system [39, 40]. This paper avoids symbolic functions by using the quantitative controller in [41], but containing nodes is difficult. Therefore, weighing the advantages and disadvantages, a controller of equation (7) is designed.

The dynamic equation of the synchronization error can be obtained from equations (1) and (7):

$$\dot{e}_i(t) = \begin{cases} f(y_i(t)) - f(x_i(t)) + c_0 \sum_{j=1}^{N} a_{ij}\Gamma\left(y_j(t) - x_j(t)\right) + c_1 \sum_{j=1}^{N} b_{ij}\Gamma\left[h\left(y_j(t-\tau(t)) - x_j(t-\tau(t))\right)\right] + u_i(t), & i = 1, 2, \ldots, l \\ f(y_i(t)) - f(x_i(t)) + c_0 \sum_{j=1}^{N} a_{ij}\Gamma\left(y_j(t) - x_j(t)\right) + c_1 \sum_{j=1}^{N} b_{ij}\Gamma\left[h\left(y_j(t-\tau(t)) - x_j(t-\tau(t))\right)\right], & i = l+1, l+2, \ldots, N. \end{cases} \tag{9}$$

Theorem 1. *Under Assumptions 1 and 2, the pinning controller equation (8), and the drive of complex network equation (1), the finite time synchronizes the response of the complex network of equation (2). Thus, the synchronization transition time is*

$$t^* = \frac{V^{1-\mu/2}(0)}{\alpha(1-\mu)}, \quad x \in \Omega. \tag{10}$$

If $p + 2\gamma c_0\lambda_{\max}(\widehat{A}_l) < 0$, *where* $l(1 \leq l < N)$, *then* $p = 2L_1 + \lambda/1-\tau + \varepsilon c_1/2\lambda_{\max}(AA^T)\lambda_{\max}(\Gamma\Gamma^T)$, $\lambda > \max\{L_2^2/2\varepsilon, \lambda_{\max}(E - \overline{AQ_1^{-1}A^T})\}$, *and* L_1 *and* L_2 *are positive constant.*

Proof. Construct the Lyapunov functional candidate as

$$V(t) = \sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{\lambda}{1-\tau}\sum_{i=1}^{l}\int_{t-\tau(t)}^{t} e_i^T(s)e_i(s)\mathrm{d}s. \tag{11}$$

Substituting equation (9) into the time derivative of $V(t)$,

$$\begin{aligned}\dot{V}(t) &= 2\sum_{i=1}^{N} e_i^T(t)\dot{e}_i(t) + \frac{\lambda}{1-\tau}\sum_{i=1}^{l} e_i^T(t)e_i(t) \\ &\quad - \frac{\lambda}{1-\tau}\sum_{i=1}^{l} e_i^T(t-\tau(t))e_i(t-\tau(t))(1-\dot{\tau}(t)) \\ &\le 2\sum_{i=1}^{N} e_i^T(t)[f(y_i(t)) - f(x_i(t)) + c_0\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) \\ &\quad + c_1\sum_{j=1}^{N} b_{ij}\Gamma e_j h(t-\tau(t))] + 2\sum_{i=1}^{l} e_i^T(t)[-ke_i(t) \\ &\quad - \alpha\left(\frac{\lambda}{1-\tau}\sum_{i=1}^{l}\int_{t-\tau(t)}^{t} e_i^T(s)e_i(s)\mathrm{d}s\right)^{1+\mu/2}\frac{e_i(t)}{\|e(t)\|^2} \\ &\quad - \alpha\operatorname{sign}(e_i(t))|e_i(t)|^{\mu}] \\ &\quad + \frac{\lambda}{1-\tau}\sum_{i=1}^{l} e_i^T(t)e_i(t) - \lambda\sum_{i=1}^{l} e_i^T(t-\tau(t))e_i(t-\tau(t)).\end{aligned} \tag{12}$$

According to the characteristics of coupling matrix $\mathbf{A}$,

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)a_{ij}\Gamma e_j(t) &\le \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma a_{ij}\|e_i(t)\|_2\|e_j(t)\|_2 \\ &\quad + \sum_{i=1}^{N} a_{ii}\rho_{\min} e_i^T(t)e_i(t) = e^T(t)(\gamma\widehat{A})e(t).\end{aligned} \tag{13}$$

From Lemma 3 and the properties of the Kronecker product,

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)b_{ij}\Gamma h\left(e_j(t-\tau(t))\right) \\ &\quad = e^T(t)(B\otimes\Gamma)H(e(t-\tau(t))) \\ &\quad \le \frac{1}{2}\varepsilon e^T(t)\left(BB^T\otimes\Gamma\Gamma^T\right)e(t) \\ &\qquad + \frac{1}{2\varepsilon}H^T(e(t-\tau(t)))H(e(t-\tau(t))) \\ &\quad \le \frac{\varepsilon}{2}\lambda_{\max}\left(BB^T\right)\lambda_{\max}\left(\Gamma\Gamma^T\right)\sum_{i=1}^{N} e_i^T(t)e_i(t) \\ &\qquad + \frac{1}{2\varepsilon}L_2^2\sum_{i=1}^{N} e_i^T(t-\tau(t))e_i(t-\tau(t)),\end{aligned} \tag{14}$$

where L_2 is positive constant, and

$$\begin{aligned}&H(e(t-\tau(t))) \\ &\quad = \left(h^T(e_1(t-\tau(t))), h^T(e_2(t-\tau(t))), \ldots, h^T(e_N(t-\tau(t)))\right)^T.\end{aligned} \tag{15}$$

Substituting equations (13) and (14) into the time derivative of equation (12),

$$\begin{aligned}\dot{V}(t) &\le e^T(t)\left[\left(2L_1 + \frac{\lambda}{1-\tau} + \frac{\varepsilon c_1}{2}\lambda_{\max}\left(BB^T\right)\cdot\lambda_{\max}\left(\Gamma\Gamma^T\right)I_N\right) + 2\gamma c_0\widehat{A} - 2K\right]e(t) \\ &\quad + e^T(t-\tau(t))\left(\frac{1}{2\varepsilon}L_2^2 - \lambda\right)I_N e(t-\tau(t)) - 2\alpha\left(\frac{\lambda}{1-\tau}\sum_{i=1}^{l}\int_{t-\tau(t)}^{t} e_i^T(s)e_i(s)\mathrm{d}s\right)^{1+\mu/2} \\ &\quad - 2\alpha\sum_{i=1}^{l} e_i^T(t)\operatorname{sign}(e_i(t))|e_i(t)|^{\mu},\end{aligned} \tag{16}$$

where $e(t) = (e_1(t), e_2(t), \ldots, e_N(t))^T$ and $K = \operatorname{diag}(\underbrace{k,\ldots,k}_{l}, \underbrace{0,\ldots,0}_{N-l})$. Let

$$Q_1 = \left(2L_1 + \frac{\lambda}{1-\tau}\right)I_N + \frac{\varepsilon c_1}{2}\left(BB^T\otimes\Gamma\Gamma^T\right) + 2\gamma c_0\widehat{A} - 2K. \tag{17}$$

Note that Q_1 is symmetric and can be decomposed into $Q_1 = \begin{bmatrix} E-\overline{K} & \overline{B} \\ \overline{B}^T & \overline{Q}_1\end{bmatrix}$, where $\overline{K} = \operatorname{diag}[\underbrace{k,\ldots,k}_{l}]$ and $\overline{Q}_1 = (2L_1 + \lambda/1-\tau + \varepsilon c_1/2\lambda_{\max}(BB^T)\lambda_{\max}(\Gamma\Gamma^T))I_{N-l} + 2\gamma c_0\widehat{A}_{N-l}$ is the minor matrix of Q_1 by removing its first $l\,(1\le l<N)$ row-column pairs. Take appropriate parameters to make $2L_1 + \lambda/1-\tau + \varepsilon c_1/2\lambda_{\max}(BB^T)\lambda_{\max}(\Gamma\Gamma^T) + 2\gamma c_0\lambda_{\max}(\widehat{A}_{N-l}) < 0$.

That is, $\overline{Q}_1 < 0$. At the same time, take $\lambda > \lambda_{\max}(E-\overline{B}\overline{Q}_1^{-1}\overline{B}^T)$, which can be known according to Lemma 2, where $\overline{Q}_1 < 0$ is equivalent to $Q_1 < 0$. For $Q_2 = (1/2\varepsilon L_2^2 - \lambda)I_N$, take $\lambda > L_2^2/2\varepsilon$ to satisfy $Q_2 < 0$. Thus, $\lambda > \max\{L_2^2/2\varepsilon, \lambda_{\max}(E-\overline{B}\overline{Q}_1^{-1}\overline{B}^T)\}$ shows that both Q_1 and Q_2 are less than zero, so we have

$$\begin{aligned}\dot{V}(t) &\le -2\alpha\left(\frac{\lambda}{1-\tau}\sum_{i=1}^{l}\int_{t-\tau(t)}^{t} e_i^T(s)e_i(s)\mathrm{d}s\right)^{1+\mu/2} \\ &\quad - 2\alpha\sum_{i=1}^{l} e_i^T(t)\operatorname{sign}(e_i(t))|e_i(t)|^{\mu},\end{aligned} \tag{18}$$

where

$$
\begin{aligned}
&-2\alpha\sum_{i=1}^{l} e_i^T(t)\operatorname{sign}\left(e_i(t)\right)\left|e_i(t)\right|^{\mu}\\
&\leq -2\alpha\sum_{i=1}^{N}\left|e_i(t)\right|^{1+\mu}\leq -2\alpha\left(\sum_{i=1}^{N}\left|e_i(t)\right|^2\right)^{1+\mu/2}\\
&=-2\alpha\left(\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{1+\mu/2}.
\end{aligned}
\tag{19}
$$

Substituting equation (20) into the time derivative of equation (17), along with Lemma 4,

$$
\begin{aligned}
\dot{V}(t)&\leq -2\alpha\left(\frac{\lambda}{1-\tau}\sum_{i=1}^{l}\int_{t-\tau(t)}^{t} e_i^T(s)e_i(s)\mathrm{d}s\right)^{1+\mu/2}\\
&\quad -2\alpha\left(\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{1+\mu/2}\\
&\leq -2\alpha\left(\frac{\lambda}{1-\tau}\sum_{i=1}^{l}\int_{t-\tau(t)}^{t} e_i^T(s)e_i(s)\mathrm{d}s+\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{1+\mu/2}\\
&=-2\alpha V^{1+\mu/2}(t).
\end{aligned}
\tag{20}
$$

□

Remark 5. Theorem 1 gives sufficient conditions for the drive-response complex network to realize the pinning synchronization in a finite time. In addition, we can calculate the synchronization time. Clearly, the synchronization time is closely related to parameters α and μ in the controller. Therefore, selecting appropriate values α and μ is crucial to the synchronization time, and the ideal parameter values can be determined through multiple calculations.

Remark 6. For complex networks with large-scale node books, a controller on each node is unrealistic to apply. Only some network nodes are selected to achieve network synchronization, which is essential to containment control and applicable to practical problems. Therefore, this paper adopts the containment control scheme, which has strong economic benefits. Note that the condition $p+2\gamma c_0\lambda_{\max}(\widehat{A}_l)<0$ in the theorem can only determine the number of pinned nodes and not which node. However, the selection of appropriate nodes is of more significance to the synchronization efficiency. The specific node selection method will be given in Section 3.2.

3.2. Selection Scheme for Pinning Node. Considering the sufficient condition of synchronization in Theorem 1 and giving priority to nodes with high control degrees [23], the procedure for the selection of restraining nodes is as follows.

Step 1. Determine the appropriate parameters to meet the sufficient conditions in Theorem 1.

Step 2. Similar to [23], take the degree of nodes in the network as the reference standard for the selection of pinning nodes. When the network is undirected, prioritize the nodes with high degrees; when the network is directed, calculate the ratio of outgoing and incoming degrees of each point, sort the ratio, and prioritize the nodes with high ratios.

Step 3. Let $l=1$ and check whether the sufficient condition in Theorem 1 is satisfied. If so, the number of pinning nodes is one. If not, increase the number of pinning nodes one by one until the sufficient condition is satisfied.

Remark 7. The pinning node can be selected in two ways. One is to select according to the node degree. For strongly connected networks, select nodes with large degrees as containment nodes [23]. Low-degree nodes should be selected as the pinning node in the weakly connected network, which has better synchronization efficiency [42]. The second approach is to arbitrarily select the pinning node [25]. Although network synchronization is possible as long as the number of pinning nodes meets the sufficient conditions in Theorem 1, in practical applications, the connection mechanism between network nodes affects the transmission efficiency of network information. Therefore, the selection of appropriate nodes can achieve network synchronization faster. After determining the number of pinning nodes, the selection of the control node is critical.

4. Numerical Simulation

A coupled time-delay complex network with 50 identical nodes is considered, and its driving system is as follows:

$$
\begin{aligned}
\dot{x}_i(t)&=f\left(x_i(t)\right)+c_0\sum_{j=1}^{50} a_{ij}\Gamma x_j(t)\\
&\quad +c_1\sum_{j=1}^{50} b_{ij}\Gamma h\left(x_j(t-\tau(t))\right),\quad 1\leq i\leq 50.
\end{aligned}
\tag{21}
$$

The response system is as follows:

$$
\dot{y}_i(t)=\begin{cases}
f\left(y_i(t)\right)+c_0\displaystyle\sum_{j=1}^{50} a_{ij}\Gamma y_j(t)+c_1\sum_{j=1}^{50} b_{ij}\Gamma h\left(y_j(t-\tau(t))\right)+u_i(t), & 1\leq i\leq l,\\
f\left(y_i(t)\right)+c_0\displaystyle\sum_{j=1}^{50} a_{ij}\Gamma y_j(t)+c_1\sum_{j=1}^{50} b_{ij}\Gamma h\left(y_j(t-\tau(t))\right), & l+1\leq i\leq 50,
\end{cases}
\tag{22}
$$

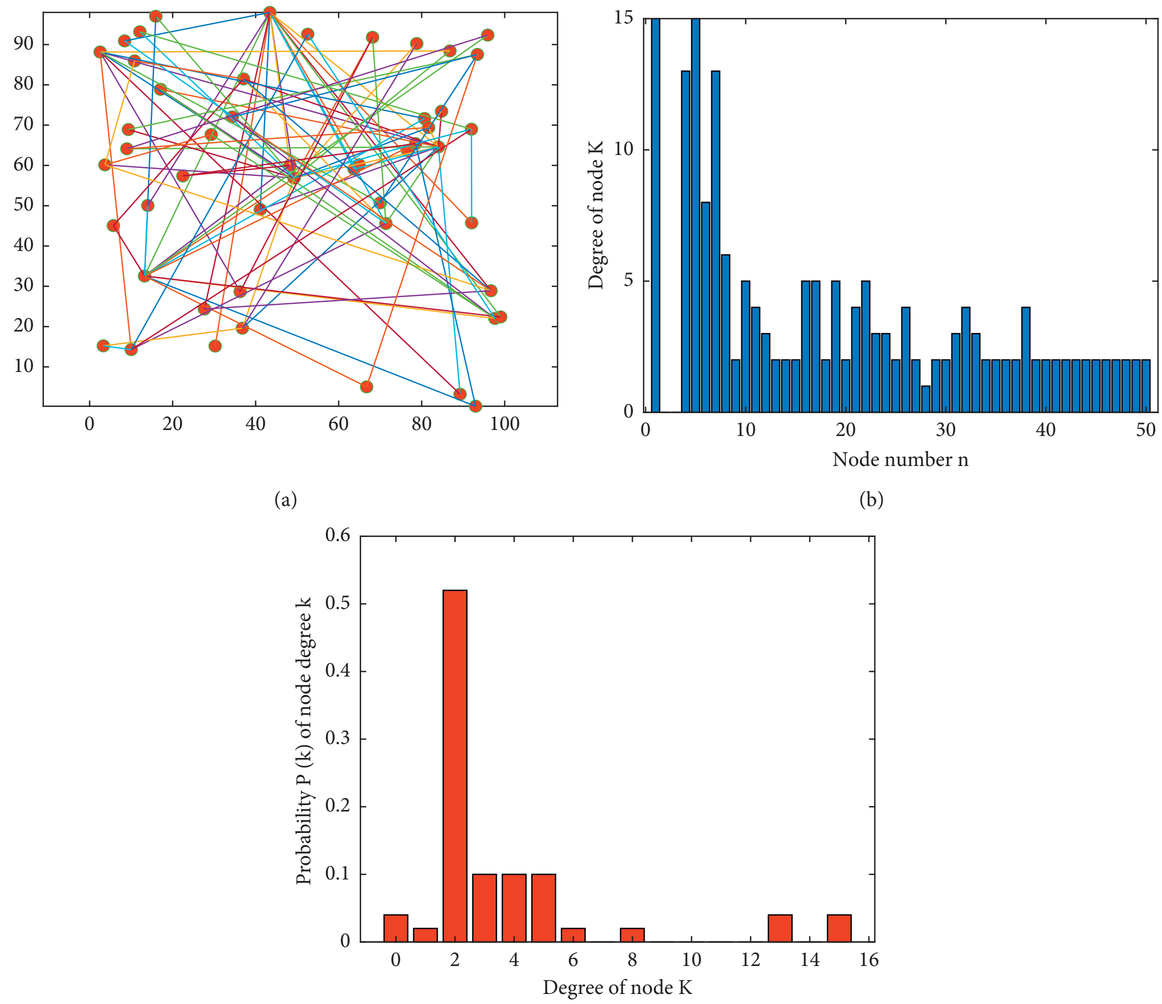


FIGURE 1: Scale-free network and related statistics. (a) Scale-free network topology. (b) Distribution diagram of node degree. (c) Probability distribution of node degree in the network.

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T$, $y_i(t) = (y_{i1}(t), y_{i2}(t), y_{i3}(t))^T$ and $\Gamma = \mathrm{diag}\{1, 1.2, 1\}$. The coupling configuration $A = (a_{ij})_{50\times 50}$ is a network that conforms to scale-free characteristics. The number of nodes in the initial state of the network is set to $m_0 = 3$. Each time a new node is introduced, the number of newly generated edges is $m = 2$. Using the roulette algorithm, m nodes are selected from the existing nodes to connect with the newly added nodes to obtain a scale-free network and further generate matrix **A**, as shown in Figure 1(a). The sufficient conditions of Theorem 1 are obtained as $\lambda_{\max}(AA^T) = 31.4537$; let $B = 0.1A$, coupling strength of edges $c_0 = 0.2$, $c_1 = 0.1$, and the nonlinear term

$$\begin{aligned} h\Big(x_j(t-\tau(t))\Big) = \Big(&x_{j1}(t-\tau(t)) \\ &+ \sin\Big(x_{j1}(t-\tau(t))\Big), x_{j2}(t-\tau(t)) \\ &+ \sin\Big(x_{j2}(t-\tau(t))\Big), x_{j3}(t-\tau(t)) \\ &+ \sin\Big(x_{j3}(t-\tau(t))\Big)\Big), \end{aligned} \tag{23}$$

where $\tau(t) = e^t/1 + e^t$.

Assuming that the nodes of the driving network and the response network are Lorenz systems, we write the node dynamic equations to unify the representation of network nodes as follows:

$$\begin{cases} \dot{x}_{i1} = 10\left(x_{i2} - x_{i1}\right), \\ \dot{x}_{i2} = 28x_{i1} - x_{i2} - x_{i1}x_{i3}, \\ \dot{x}_{i3} = -\dfrac{8}{3}x_{i3} + x_{i1}x_{i2}. \end{cases} \tag{24}$$

Set the initial conditions of each node of the drive and response system, respectively: $x_i(0) = (7.5 + 0.6i, 0.2 + 0.1i, 3 + 0.1i)^T$, $y_i(0) = (0.5 + 0.6i, 3.2 + 0.1i, 7 + 0.2i)^T$, $i = 1, 2, \ldots, 50$. Figures 2 and 3 show the attractor and time series of a Lorenz system of the first node of the driving system and the response system, respectively.

Here, we only show the trajectory of one node for the driving network and the response network, and the other trajectories will not be repeated.

Based on the parameter design of Li et al. [43], set $\tau = 1/4$, $\beta = 1$, $\alpha = 8$, $\lambda = 2$, and $\mu = 1/2$ in controller $u_i(t)$.

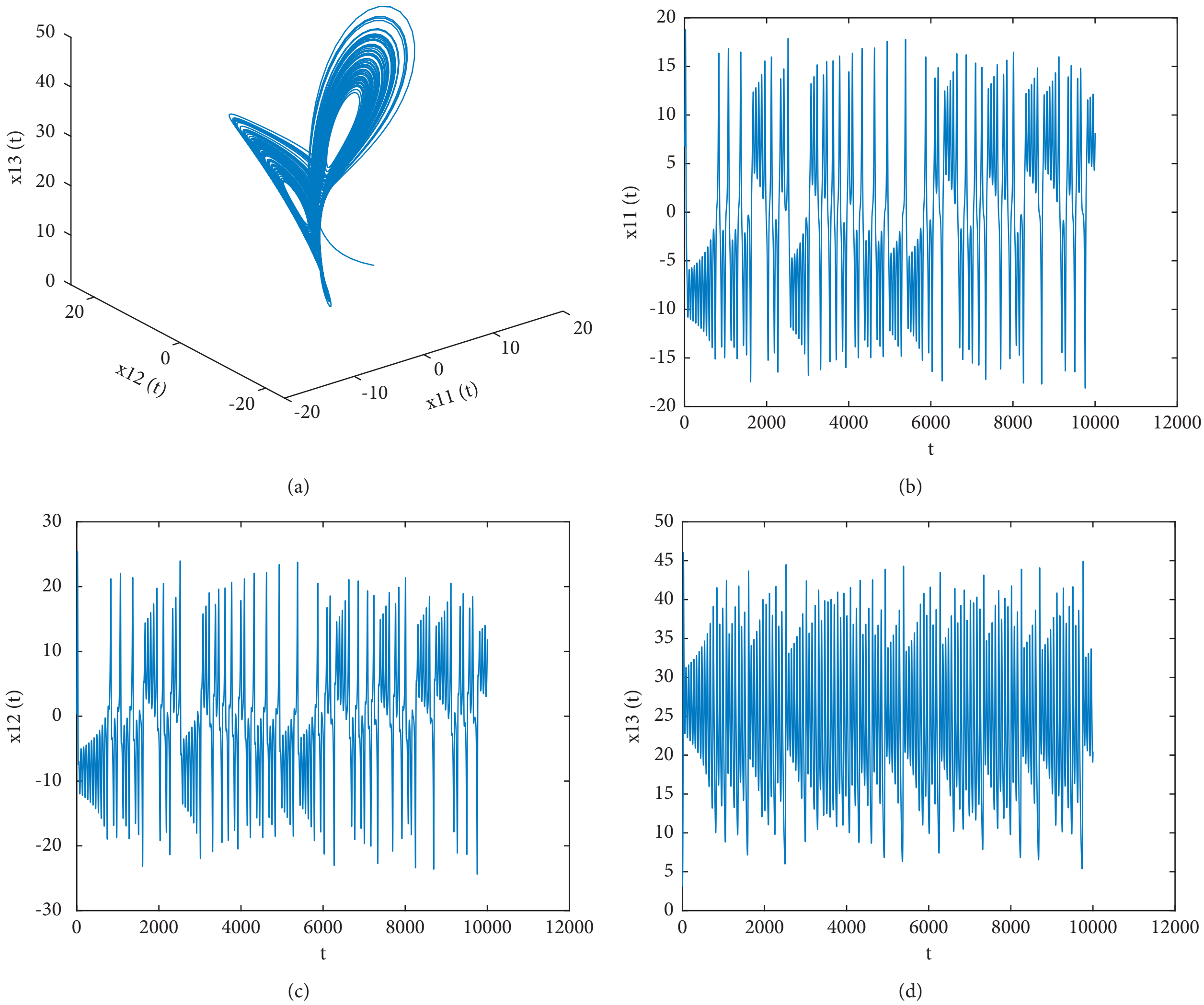


Figure 2: Attractor and time series of the first node of the driving system. (a) Lorenz attractor. (b) x time series. (c) y time series. (d) z time series.

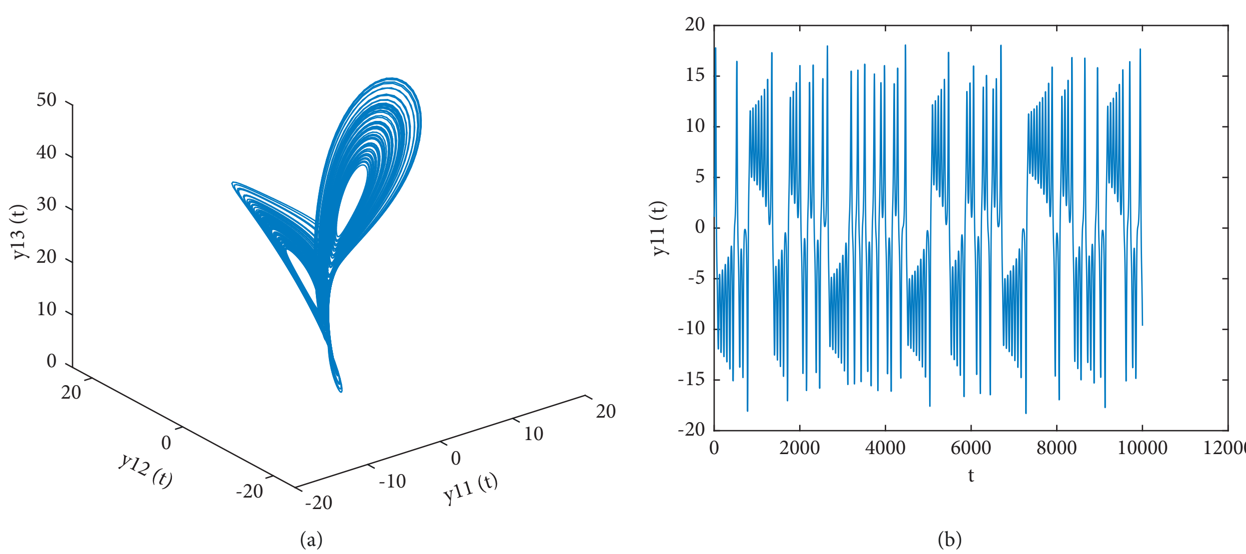
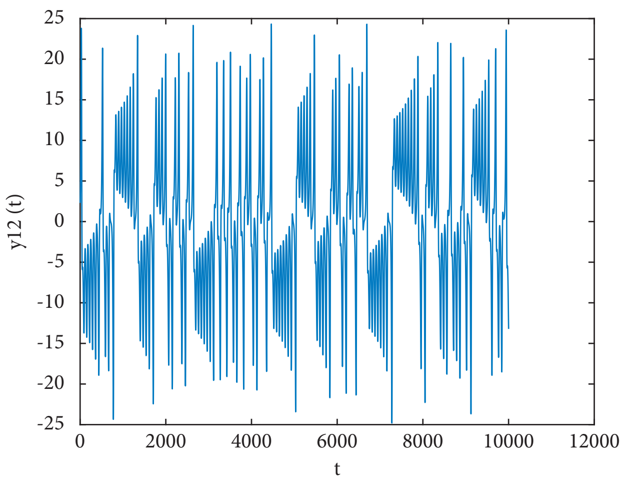
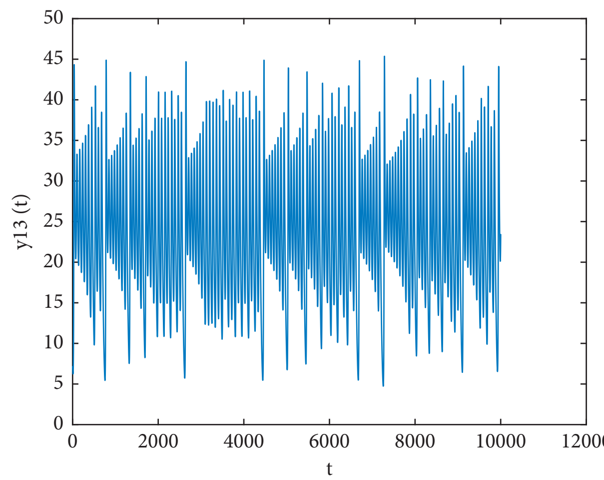


Figure 3: Continued.



(c)



(d)

Figure 3: Attractor and time series of the first node of the response system. (a) Lorenz attractor. (b) x time series. (c) y time series. (d) z time series.

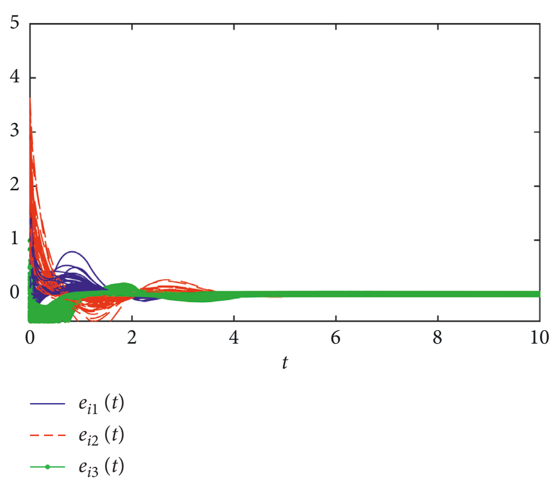


Figure 4: Synchronization error of 13 node networks controlled according to the node selection rules in this paper.

From Assumption 1, we can verify that $L_1 = 1.7342$ and $L_2 = 1.0225$. For the sufficient condition $\lambda > L_2^2/2\varepsilon$ of Theorem 1, set $\varepsilon = 0.3$, and $\gamma = 1.2$ can be obtained from $\|\Gamma\|_2 = \gamma$. Input parameters into sufficient conditions $p + 2\gamma c_0 \lambda_{\max}(\widehat{A}_l) < 0$ to obtain $\lambda_{\max}(\widehat{A}_l) < -0.8077$. Arrange the nodes according to the degree and the selection procedure of the pinning nodes: $\lambda_{\max}(\widehat{A}_{13}) = -0.8176$, $\lambda_{\max}(\widehat{A}_{12}) = -0.7869$. The first 13 nodes can synchronize the drive network and the response network, as shown in Figure 4.

Remark 8. The coupling strength of the network is closely related to the network synchronization ability. In this paper, the value $\lambda_{\max}(\widehat{A})$ also depends on the coupling strengths c_0 and c_1 of the network. In other words, the coupling strength of the network determines the number of pinning nodes. Therefore, in some studies of random pinning node selections (e.g., [25]), the effect of synchronous control can continue to be optimized.

From the above calculation and simulation results, we conclude that the finite-time pinning synchronization control of time-delay coupled complex networks is realized.

5. Conclusions

The effectiveness of this method, which focuses on the selection of pinning nodes, is verified by numerical simulation. This paper attempts to select pinning nodes based on the degree to which sufficient conditions are met. Research results on the identification of key nodes and edges in complex networks are available [44]. Following this, we aim to control the synchronization of complex networks, defining key nodes as pinning nodes.

The relevant research results can be applied to the synchronous control of a supply chain network. For example, the change of products promotes the evolution of a supply chain network and affects the coupling relationship between companies within the network. In particular, the coupling time delay problem greatly impacts the efficiency of the supply chain. Currently, we choose "degree" when considering the time-delay coupling problem among companies in a supply chain network, connect larger enterprises as the nodes of containment control, and finally realize a smooth synchronous evolution path of the supply chain. Our approach can also be applied to UAV clusters to achieve synergy, providing the ability to perform complex, changeable, and dangerous tasks; improve the overall load capacity, information perception, and processing ability; and avoid attacks or low efficiency when a single UAV performs tasks. However, it is precise because we consider the containment control through some nodes, which is conservative compared with controlling all nodes to achieve synchronization. Therefore, reducing conservatism is also an important direction for future research.

In practical applications, the number of pinning nodes and time limitations are significant issues. Therefore, the selection of fewer pinning nodes in a strict, limited time is a necessary research direction for the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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