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Research Article

Existence and Uniqueness of Positive Solutions for a Coupled System of Fractional Differential Equations

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In this article, we investigate a boundary value problem for a coupled differential system of fractional order that the nonlinear term depends on the unknown functions as well as their lower order fractional derivatives. Firstly, we give Green's functions and prove their properties; secondly, the existence and uniqueness of positive solutions are obtained by using some fixed point theorems. In addition, two examples are presented to demonstrate the application of our main results.

1. Introduction

Fractional differential equations have received considerable attention in recent years due to their wide applications in engineering, physics, economy, and control theory (see the monographs of Das [1], Kilbas et al. [2], and Podlubny [3]). There are a large number of papers dealing with the solvability of nonlinear fractional differential equations, such as [4–8] and references therein. The study of coupled differential systems of fractional order is very significant because this kind of system can often occur in applications. Recently, a series of investigations on boundary value problems for fractional differential equation systems with the nonlinearity depending on the fractional derivative have been presented. Most of them are devoted to the solvability of nonlinear fractional differential equation systems by using techniques of nonlinear analysis [9–15]. It is worth mentioning that the nonlinear terms in these papers are independent of the fractional derivative of the unknown functions. But the opposite case is more difficult and complicated, and this work attempts to deal exactly with this case.

In [16], the authors investigated a three-point boundary value problem for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases} D^{\alpha}u(t) + f\left(t, v(t), D^{p}v(t)\right) = 0, & 0<t<1, \\ D^{\beta}v(t) + g\left(t, u(t), D^{q}u(t)\right) = 0, & 0<t<1, \\ u(0) = 0,\ u(1) = \gamma u(\eta), v(0) = 0, & v(1) = \gamma v(\eta), \end{cases} \tag{1}$$

where $1<\alpha, \beta<2, p, q, \gamma>0, 0<\eta<1, \alpha - q \geq 1, \quad \beta - p \geq 1$, $\gamma\eta^{\alpha-1}<1, \gamma\eta^{\beta-1}<1, f, g\colon\ [0,1]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are given continuous functions. The nonlinear terms in the coupled system involve the fractional derivatives of the unknown functions.

In [12], the author discussed the existence of solutions for a coupled differential system of fractional order:

$$\begin{cases} D^{\alpha}u(t) + f\left(t, v(t), D^{\mu}v(t)\right) = 0, & 0<t<1, \\ D^{\beta}v(t) + g\left(t, u(t), D^{\nu}u(t)\right) = 0, & 0<t<1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases} \tag{2}$$

where $1<\alpha, \beta<2, \mu, \nu>0, \alpha-\nu\geq 1, \beta-\mu\geq 1, f, g\colon\quad (0,1)\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are given functions.

Motivated by the above work, in this paper, we consider the system of nonlinear fractional differential equations:

$$
\begin{cases}
D^{\alpha}u(t) + f\left(t, v(t), D^{\gamma}v(t)\right) = 0, & 0 < t < 1 \\
D^{\beta}v(t) + g\left(t, u(t), D^{\delta}u(t)\right) = 0, & 0 < t < 1, \\
u(0) = u'(0) = 0, & u'(1) = \mu u'(\xi), \\
v(0) = v'(0) = 0, & v'(1) = \mu v'(\xi).
\end{cases} \tag{3}
$$

where $2 < \alpha, \beta \le 3, 1 < \gamma, \delta \le 2$ and $\alpha - \delta \ge 1, \beta - \gamma \ge 1, 0 < \xi < 1, 0 < \mu\xi^{\alpha-2} < 1, 0 < \mu\xi^{\beta-2} < 1$. D^{α} and D^{β} are the standard Riemann–Liouville fractional derivatives and f, g: $[0,1] \times \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+$ are given continuous functions and depend on the unknown functions as well as their lower order fractional derivatives. The properties of Green's function are investigated and the existence and uniqueness results of positive solutions are obtained by applying Schauder fixed point theorem and contraction mapping principle.

2. Preliminaries

For the convenience of the readers, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature [2–5, 7, 17].

Definition 1 (see [2, 3]). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function f: $(0, +\infty) \longrightarrow \mathbb{R}$ is given by

$$
I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \quad t > 0, \tag{4}
$$

provided the integral exists.

Definition 2 (see [2, 3]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function f: $(0, +\infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{aligned}
D^{\alpha}f(t) &= \left(\frac{d}{dt}\right)^n \left(I^{n-\alpha}f\right)(t) \\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, \quad t > 0,
\end{aligned} \tag{5}
$$

where $n = [\alpha] + 1$, in which $[\alpha]$ denotes the integer part of the number α.

Lemma 1 (see [2, 4]). *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation $D^{\alpha}u(t) = 0$ has solutions*

$$
u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_nt^{\alpha-N}, C_i \in \mathbb{R}, i = 1, 2, \ldots, N, \tag{6}
$$

where N is the smallest integer greater than or equal to α.

Lemma 2 (see [2, 4]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha\,(\alpha > 0)$ that belongs to $C(0,1) \cap L(0,1)$, then*

$$
I^{\alpha}D^{\alpha}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_nt^{\alpha-N}, \tag{7}
$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

Remark 1 (see [2, 3]). The following properties are useful for our discussion:

$$
\begin{aligned}
& I^{\alpha}I^{\beta}f(t) = I^{\alpha+\beta}f(t), D^{\alpha}I^{\alpha}f(t) \\
&\qquad = f(t), \alpha, \beta > 0, a.e.\ t \in (0,1), f \in L(0,1); \\
& D^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}, \alpha > 0, \gamma > -1, t > 0.
\end{aligned} \tag{8}
$$

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 3. *Let $h_1 \in C[0,1]$ and $2 < \alpha \le 3$, the unique solution of problem*

$$
D^{\alpha}u(t) + h_1(t) = 0, \ 0 < t < 1, \tag{9}
$$

$$
u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \tag{10}
$$

is $u(t) = \int_0^1 G_1(t,s)h_1(s)ds$, where

$$
G_1(t,s) = \begin{cases}
\dfrac{(1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}, & 0 \le s \le t \le 1, s \le \xi, \\[2ex]
\dfrac{(1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}, & 0 < \xi \le s \le t \le 1, \\[2ex]
\dfrac{(1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}, & 0 \le t \le s \le \xi < 1, \\[2ex]
\dfrac{(1-s)^{\alpha-2}t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}, & 0 \le t \le s \le 1, \xi \le s.
\end{cases} \tag{11}
$$

Here $G_1(t,s)$ is called Green's function of BVP (9) and (10).

Proof. The function $u(t)$ is said be a solution of BVP (9) and (10) if it satisfies the factional differential equation (9) and boundary conditions (10) in the classical sense. We may apply Lemma 2 to reduce equation (9) to an equivalent integral equation

$$u(t) = -I^{\alpha}h_1(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + C_3t^{\alpha-3}, \tag{12}$$

for some $C_1, C_2, C_3 \in \mathbb{R}$. Consequently, the general solution of equation (9) is

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h_1(s)ds + C_1t^{\alpha-1} + C_2t^{\alpha-2} + C_3t^{\alpha-3}. \tag{13}$$

By $u(0) = u'(0) = 0$, there are $C_2 = C_3 = 0$.

On the other hand, $u'(1) = \mu u'(\xi)$ combining with

$$\begin{aligned} u'(1) &= -\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}h_1(s)ds + (\alpha-1)C_1, \\ u'(\xi) &= -\int_0^{\xi} \frac{(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)}h_1(s)ds + (\alpha-1)C_1\xi^{\alpha-2}, \end{aligned} \tag{14}$$

one has $C_1 = (1/(1-\mu\xi^{\alpha-2}))\int_0^1 (1-s)^{\alpha-2}h_1(s)/\Gamma(\alpha)ds - (\mu/(1-\mu\xi^{\alpha-2}))\int_0^{\xi}(\xi-s)^{\alpha-2}h_1(s)/\Gamma(\alpha)ds$.

Therefore, the unique solution of problem (9) and (10) is

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds + \frac{1}{1-\mu\xi^{\alpha-2}}\int_0^1 \frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds \\ &- \frac{\mu}{1-\mu\xi^{\alpha-2}}\int_0^{\xi}\frac{(\xi-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds. \end{aligned} \tag{15}$$

For $t \le \xi < 1$, one has

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds + \frac{1}{1-\mu\xi^{\alpha-2}}\left[\int_0^t + \int_t^{\xi} + \int_{\xi}^1\right]\frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds \\ &- \frac{\mu}{1-\mu\xi^{\alpha-2}}\left[\int_0^t + \int_t^{\xi}\right]\frac{(\xi-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds \\ = &\int_0^t \frac{(1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}h_1(s)ds \\ &+ \int_t^{\xi}\frac{(1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}h_1(s)ds + \int_{\xi}^1 \frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}h_1(s)ds \\ = &\int_0^1 G_1(t,s)h_1(s)ds. \end{aligned} \tag{16}$$

For $t \ge \xi$, one has

$$\begin{aligned} u(t) = &-\left[\int_0^{\xi} + \int_{\xi}^t\right]\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds + \frac{1}{1-\mu\xi^{\alpha-2}}\left[\int_0^{\xi} + \int_{\xi}^t + \int_t^1\right]\frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds \\ &- \frac{\mu}{1-\mu\xi^{\alpha-2}}\int_0^{\xi}\frac{(\xi-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\xi} \frac{(1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)} h_1(s)\mathrm{d}s \\
&\quad + \int_{\xi}^{t} \frac{(1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)} h_1(s)\mathrm{d}s + \int_t^1 \frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)} h_1(s)\mathrm{d}s \\
&= \int_0^1 G_1(t,s)h_1(s)\mathrm{d}s.
\end{aligned} \tag{17}
$$

The proof is finished. □

Lemma 4. *The function $G_1(t,s)$ defined by equation (11) possesses the following properties:*

(1) $G_1(t,s) > 0$, for $t, s \in (0,1)$;

(2) $t^{\alpha-1}G_1(1,s) \le G_1(t,s) < G_1(1,s)$, for $t, s \in (0,1)$.

Proof. Let

$$
\begin{aligned}
g_1(t,s) &= (1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right), \\
g_2(t,s) &= (1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right), \\
g_3(t,s) &= (1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1}, \\
g_4(t,s) &= (1-s)^{\alpha-2}t^{\alpha-1}.
\end{aligned} \tag{18}
$$

(1) For $s \le t, s \le \xi$, let $\Delta = (1-s/\xi)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}$, If $\Delta \ge 0$, by $2 < \alpha \le 3, 0 < \mu\xi^{\alpha-2} < 1, 0 < \xi < 1$, there is

$$
\begin{aligned}
g_1(t,s) &= (1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right) \\
&= \left[(1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\right] - \mu\xi^{\alpha-2}\left[\left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\right] \\
&> (1-s)^{\alpha-2}t^{\alpha-1} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-1} \\
&= t^{\alpha-1}\left[(1-s)^{\alpha-2} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}\right] \ge 0,
\end{aligned} \tag{19}
$$

If $\Delta < 0$, by $2 < \alpha \le 3, 0 < \mu\xi^{\alpha-2} < 1, 0 < \xi < 1$, there is

$$
\begin{aligned}
g_1(t,s) &= \left[(1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\right] - \mu\xi^{\alpha-2}\left[\left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\right] \\
&= t^{\alpha-1}\left\{\left[(1-s)^{\alpha-2} - \left(1-\frac{s}{t}\right)^{\alpha-1}\right] - \mu\xi^{\alpha-2}\left[\left(1-\frac{s}{\xi}\right)^{\alpha-2} - \left(1-\frac{s}{t}\right)^{\alpha-1}\right]\right\} > 0.
\end{aligned} \tag{20}
$$

For $\xi \leq s \leq t$, by $2 < \alpha \leq 3, 0 < \mu\xi^{\alpha-2} < 1, 0 < 1 - \mu\xi^{\alpha-2} < 1$, there is

$$\begin{aligned} g_2(t,s) &= (1-s)^{\alpha-2}t^{\alpha-1} - (t-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right) > (1-s)^{\alpha-1}t^{\alpha-1} - (t-s)^{\alpha-1} \\ &= (t-ts)^{\alpha-1} - (t-s)^{\alpha-1} \geq 0. \end{aligned} \tag{21}$$

For $t \leq s \leq \xi$, by $0 < \mu\xi^{\alpha-2} < 1, 0 < \xi < 1$, there is

$$\begin{aligned} g_3(t,s) &= (1-s)^{\alpha-2}t^{\alpha-1} - \mu(\xi-s)^{\alpha-2}t^{\alpha-1} = (1-s)^{\alpha-2}t^{\alpha-1} - \mu\xi^{\alpha-2}\left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-1} \\ &> (1-s)^{\alpha-2}t^{\alpha-1} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-1} = t^{\alpha-1}\left[(1-s)^{\alpha-2} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}\right] \geq 0. \end{aligned} \tag{22}$$

For $t \leq s, \xi \leq s$, $g_4(t,s) > 0$ holds clearly. Therefore, $G_1(t,s) > 0$, for $t, s \in (0,1)$.

(2) In the following, we prove $g_i(t,s)\,(i = 1,2,3,4)$ are monotone increasing functions for t. For $s \leq t, s \leq \xi$, if $t \leq \xi$, by $0 < \mu\xi^{\alpha-2} < 1, 0 < \xi < 1$, there is

$$\begin{aligned} \frac{\partial g_1(t,s)}{\partial t} &= (\alpha-1)(1-s)^{\alpha-2}t^{\alpha-2} - \mu(\alpha-1)(\xi-s)^{\alpha-2}t^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}\left(1-\mu\xi^{\alpha-2}\right) \\ &= (\alpha-1)\left\{\left[(1-s)^{\alpha-2}t^{\alpha-2} - (t-s)^{\alpha-2}\right] - \mu\xi^{\alpha-2}\left[\left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-2} - (t-s)^{\alpha-2}\right]\right\} \\ &> (\alpha-1)\left[(1-s)^{\alpha-2}t^{\alpha-2} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-2}\right] \\ &= (\alpha-1)t^{\alpha-2}\left[(1-s)^{\alpha-2} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}\right] \geq 0, \end{aligned} \tag{23}$$

if $t > \xi$, by $0 < \mu\xi^{\alpha-2} < 1, 0 < \xi < 1$, there is

$$\begin{aligned} \frac{\partial g_1(t,s)}{\partial t} &= (\alpha-1)(1-s)^{\alpha-2}t^{\alpha-2} - \mu(\alpha-1)(\xi-s)^{\alpha-2}t^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}\left(1-\mu\xi^{\alpha-2}\right) \\ &= (\alpha-1)\left\{\left[(1-s)^{\alpha-2}t^{\alpha-2} - (t-s)^{\alpha-2}\right] - \mu\xi^{\alpha-2}\left[\left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-2} - (t-s)^{\alpha-2}\right]\right\} \\ &= (\alpha-1)t^{\alpha-2}\left\{\left[(1-s)^{\alpha-2} - \left(1-\frac{s}{t}\right)^{\alpha-2}\right] + \mu\xi^{\alpha-2}\left[\left(1-\frac{s}{t}\right)^{\alpha-2} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}\right]\right\} > 0. \end{aligned} \tag{24}$$

For $\xi \le s \le t$, by $2 < \alpha \le 3, 0 < \mu\xi^{\alpha-2} < 1, 0 < 1 - \mu\xi^{\alpha-2} < 1$, there is

$$\begin{aligned}\frac{\partial g_2(t,s)}{\partial t} &= (\alpha-1)(1-s)^{\alpha-2}t^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}\left(1-\mu\xi^{\alpha-2}\right) \\ &> (\alpha-1)\left[(1-s)^{\alpha-2}t^{\alpha-2} - (t-s)^{\alpha-2}\right] > (\alpha-1)\left[(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}\right] \ge 0.\end{aligned} \tag{25}$$

For $t \le s \le \xi$, by $0 < \mu\xi^{\alpha-2} < 1, 0 < \xi < 1$, there is

$$\begin{aligned}\frac{\partial g_3(t,s)}{\partial t} &= (\alpha-1)(1-s)^{\alpha-2}t^{\alpha-2} - \mu(\alpha-1)(\xi-s)^{\alpha-2}t^{\alpha-2} \\ &= (\alpha-1)\left[(1-s)^{\alpha-2}t^{\alpha-2} - \mu\xi^{\alpha-2}\left(1-\frac{s}{\xi}\right)^{\alpha-2}t^{\alpha-2}\right] \\ &> (\alpha-1)t^{\alpha-2}\left[(1-s)^{\alpha-2} - \left(1-\frac{s}{\xi}\right)^{\alpha-2}\right] \ge 0.\end{aligned} \tag{26}$$

For $t \le s, \xi \le s$, $\partial g_4(t,s)/\partial t = (\alpha-1)(1-s)^{\alpha-2} t^{\alpha-2} > 0$. So $G_1(t,s)$ is monotone increasing function for t. Therefore, $G_1(t,s) < G_1(1,s)$, for $t, s \in (0,1)$.

On the other hand, when $0 < s \le t < 1, s \le \xi$, we have

$$\begin{aligned}G_1(t,s) &= \frac{t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}\left[(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - \left(1-\frac{s}{t}\right)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)\right] \\ &\ge \frac{t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}\left[(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)\right] \\ &= t^{\alpha-1}G_1(1,s).\end{aligned} \tag{27}$$

When $0 < \xi \le s \le t < 1$, we obtain

$$\begin{aligned}G_1(t,s) &= \frac{t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}\left[(1-s)^{\alpha-2} - \left(1-\frac{s}{t}\right)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)\right] \\ &\ge \frac{t^{\alpha-1}}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}\left[(1-s)^{\alpha-2} - (1-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)\right] \\ &= t^{\alpha-1}G_1(1,s).\end{aligned} \tag{28}$$

In the similar discussion, we can deduce

$$G_1(t,s) \geq t^{\alpha-1} G_1(1,s), \tag{29}$$

where

$$G_1(1,s) = \begin{cases} \dfrac{(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}, & 0 \leq s \leq \xi, \\[2ex] \dfrac{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}\left(1-\mu\xi^{\alpha-2}\right)}{\left(1-\mu\xi^{\alpha-2}\right)\Gamma(\alpha)}, & \xi \leq s \leq 1. \end{cases} \tag{30}$$

The proof is finished.

Similarly, we can obtain $G_2(t,s)$ if α is replaced by β,

$$G_2(t,s) = \begin{cases} \dfrac{(1-s)^{\beta-2}t^{\beta-1} - \mu(\xi-s)^{\beta-2}t^{\beta-1} - (t-s)^{\beta-1}\left(1-\mu\xi^{\beta-2}\right)}{\left(1-\mu\xi^{\beta-2}\right)\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, s \leq \xi, \\[2ex] \dfrac{(1-s)^{\beta-2}t^{\beta-1} - (t-s)^{\beta-1}\left(1-\mu\xi^{\beta-2}\right)}{\left(1-\mu\xi^{\beta-2}\right)\Gamma(\beta)}, & 0 < \xi \leq s \leq t \leq 1, \\[2ex] \dfrac{(1-s)^{\beta-2}t^{\beta-1} - \mu(\xi-s)^{\beta-2}t^{\beta-1}}{\left(1-\mu\xi^{\beta-2}\right)\Gamma(\beta)}, & 0 \leq t \leq s \leq \xi < 1, \\[2ex] \dfrac{(1-s)^{\beta-2}t^{\beta-1}}{\left(1-\mu\xi^{\beta-2}\right)\Gamma(\beta)}, & 0 \leq t \leq s \leq 1, \xi \leq s. \end{cases} \tag{31}$$

The function $G_2(t,s)$ defined by (31) has the same properties as $G_1(t,s)$. □

Lemma 5 (see [17]). *If S is a closed, bounded and convex subset of a Banach space X and $T\colon S \longrightarrow S$ is completely continuous, then T has a fixed point in S.*

3. Main Results and Proof

In this section, we establish the existence and uniqueness of positive solutions for (3).

The vector $(u,v) \in X \times Y$ is said to be a positive solution of BVP (3) if (u,v) satisfies (3) and $u(t) > 0, v(t) > 0$ for $t \in (0,1)$, where $X = \{u(t) | u(t) \in C^1[0,1] \cap D^\delta u(t) \in C[0,1], 1 < \delta \leq 2\}$ and $Y = \{v(t) | v(t) \in C^1[0,1] \cap D^\gamma v(t) \in C[0,1], 1 < \gamma \leq 2\}$.

By using Green's functions $G_i(t,s)\,(i=1,2)$ from Section 2, problem (3) can be written equivalently as the following nonlinear system of integral equations:

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s) f\left(s, v(s), D^\gamma v(s)\right) ds, \; 0 \leq t \leq 1, \\[2ex] v(t) = \displaystyle\int_0^1 G_2(t,s) g\left(s, u(s), D^\delta u(s)\right) ds, \; 0 \leq t \leq 1. \end{cases} \tag{32}$$

We define the space $X = \{u(t) | \in C^1[0,1]$ and $D^\delta u(t) \in C[0,1], 1 < \delta \leq 2\}$ and the norm

$$\|u\|_X = \|u\| + \left\|D^\delta u\right\| = \max_{0 \leq t \leq 1} |u(t)| + \max_{0 \leq t \leq 1} \left|D^\delta u(t)\right|. \tag{33}$$

Lemma 6. *$(X, \|\cdot\|_X)$ is a Banach space.*

Proof. Let $\{u(t)\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(X, \|\cdot\|_X)$; then clearly $\{u_n\}_{n=1}^{\infty}$ and $\{D^\delta u_n\}_{n=1}^{\infty}$ are Cauchy sequences in the space $C[0,1]$. Therefore, $\{u_n\}_{n=1}^{\infty}$ and $\{D^\delta u_n\}_{n=1}^{\infty}$ converge to some v and w on $[0,1]$ uniformly and $v, w \in C[0,1]$. We need to prove that $w = D^\delta v$.

Note that

$$\left|I^{\delta}D^{\delta}u_n(t)-I^{\delta}w(t)\right|\le\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\left|D^{\delta}u_n(s)-w(s)\right|ds$$
$$\le\frac{1}{\Gamma(\delta+1)}\max_{t\in[0,1]}\left|D^{\delta}u_n(t)-w(t)\right|. \tag{34}$$

By the convergence of $\{D^{\delta}u_n\}_{n=1}^{\infty}$, we have $\lim_{n\longrightarrow\infty} I^{\delta}D^{\delta}u_n(t)=I^{\delta}w(t)$ uniformly for $t\in[0,1]$. On the other hand, by Remark 1 one has $I^{\delta}D^{\delta}u_n(t)=u_n(t)$. Hence, $v(t)=I^{\delta}w(t)$, Remark 1 implies that it is equivalent to the relation $w=D^{\delta}v$. This completes the proof.

Also we define the space $Y=\{v(t)|v(t)\in C^1[0,1]$ and $D^{\gamma}v(t)\in C[0,1], 1<\gamma\le 2\}$ and the norm

$$\|v\|_Y=\|v\|+\left\|D^{\gamma}v\right\|=\max_{0\le t\le1}|v(t)|+\max_{0\le t\le1}\left|D^{\gamma}v(t)\right|. \tag{35}$$

Again $(Y,\|\cdot\|_Y)$ is a Banach space.

Obviously the product space $(X\times Y,\|\cdot\|_{X\times Y})$ is a Banach space with norm

$$\|(u,v)\|_{X\times Y}=\max\{\|u\|_X,\|v\|_Y\}, \tag{36}$$

for $(u,v)\in X\times Y$.

We define the operators $T_1\colon Y\longrightarrow X, T_2\colon X\longrightarrow Y$ and $T\colon X\times Y\longrightarrow X\times Y$ by

$$\begin{cases} T_1v(t)=\displaystyle\int_0^1 G_1(t,s)f\left(s,v(s),D^{\gamma}v(s)\right)ds, 0\le t\le1,\\ T_2u(t)=\displaystyle\int_0^1 G_2(t,s)g\left(s,u(s),D^{\delta}u(s)\right)ds, 0\le t\le 1,\end{cases} \tag{37}$$

and $T(u,v)=(T_1v,T_2u)$, $(u,v)\in X\times Y$. Thus, the solutions of our problem (3) are the fixed points of the operator T.

Now, we give the main result of this work. For convenience, we introduce the following notations:

$$p=\left(1-\mu\xi^{\alpha-2}\right)^{-1},$$
$$q=\left(1-\mu\xi^{\beta-2}\right)^{-1},$$
$$A=\frac{p}{(\alpha-1)\Gamma(\alpha)}+\frac{1+p+\mu p}{\Gamma(\alpha-\delta+1)}, \tag{38}$$
$$B=\frac{q}{(\beta-1)\Gamma(\beta)}+\frac{1+q+\mu q}{\Gamma(\beta-\gamma+1)}.$$

□

Theorem 1. *If $f,g\colon [0,1]\times\mathbb{R}_+\times\mathbb{R}\longrightarrow\mathbb{R}_+$ are continuous functions and $f(t,0,0), g(t,0,0)$ are not identically zero. Suppose that one of the following conditions is satisfied.*

(H_1) There are two nonnegative functions $a(t), b(t)\in C[0,1]$ such that. $f(t,x,y)\le a(t)+c_1|x|^{\rho_1}+c_2|y|^{\rho_2}$ and $g(t,x,y)\le b(t)+d_1|x|^{\theta_1}+d_2|y|^{\theta_2}$, where $c_i,d_i\ge0$, $0<\rho_i,\theta_i<1$ for $i=1,2$

(H_2) $f(t,x,y)\le c_1|x|^{\rho_1}+c_2|y|^{\rho_2}$ and $g(t,x,y)\le d_1|x|^{\theta_1}+d_2|y|^{\theta_2}$, where $c_i,d_i\ge0,\rho_i,\theta_i>1$ for $i=1,2$

Then boundary value problem (3) has a positive solution $(u(t),v(t)), t\in[0,1]$.

Proof. First, let condition (H_1) be valid. Define

$$S=\{(u,v)|(u,v)\in X\times Y, u\ge0, v\ge0, \|(u,v)\|_{X\times Y}\le R, t\in[0,1]\}, \tag{39}$$

where $R\ge\max\left\{(3Ac_1)^{(1/1-\rho_1)}, (3Ac_2)^{(1/1-\rho_2)}, (3Bd_1)^{(1/1-\theta_1)}, (3Bd_2)^{(1/1-\theta_2)}, 3k, 3l\right\}$., and

$$k=\int_0^1 G_1(1,s)a(s)ds+\frac{1+p}{\Gamma(\alpha-\delta)}\int_0^1(1-s)^{\alpha-\delta-1}a(1)ds+\frac{\mu p}{\Gamma(\alpha-\delta)}\int_0^{\xi}(\xi-s)^{\alpha-2}a(\xi)ds,$$
$$l=\int_0^1 G_2(1,s)b(s)ds+\frac{1+q}{\Gamma(\beta-\gamma)}\int_0^1(1-s)^{\beta-\gamma-1}b(1)ds+\frac{\mu q}{\Gamma(\beta-\gamma)}\int_0^{\xi}(\xi-s)^{\beta-2}b(\xi)ds. \tag{40}$$

Observe that S is a closed convex set of the Banach space $X\times Y$.

Now we prove that $T\colon S\longrightarrow S$. For any $(u,v)\in S$, from the nonnegativeness of f,g and Lemma 4, it is easy to know that $T_1v(t)\ge0, T_2u(t)\ge0$.

Again by Remark 1 and Lemma 4, we have

$$\left|T_1v(t)\right|=\left|\int_0^1 G_1(t,s)f\left(s,v(s),D^{\gamma}v(s)\right)ds\right|\le\int_0^1\left|G_1(t,s)a(s)\right|ds+\left(c_1R^{\rho_1}+c_2R^{\rho_2}\right)\int_0^1 G_1(t,s)ds$$
$$\le\int_0^1 G_1(1,s)a(s)ds+\left(c_1R^{\rho_1}+c_2R^{\rho_2}\right)\int_0^1 G_1(1,s)ds$$
$$<\int_0^1 G_1(1,s)a(s)ds+\left(c_1R^{\rho_1}+c_2R^{\rho_2}\right)\frac{1}{(\alpha-1)\Gamma(\alpha)\left(1-\mu\xi^{\alpha-2}\right)}, \tag{41}$$

$$\begin{aligned}
\left|D^{\delta}T_1 v(t)\right| &= \Big| -I^{\alpha-\delta} f\left(t, v(t), D^{\gamma} v(t)\right) + \frac{\Gamma(\alpha-1)t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)\left(1-\mu\xi^{\alpha-2}\right)}\left[I^{\alpha-1} f\left(\cdot, v(\cdot), D^{\gamma} v(\cdot)\right)(1)\right.\\
&\quad \left.-\mu I^{\alpha-1} f\left(\cdot, v(\cdot), D^{\gamma} v(\cdot)\right)(\xi)\right]\Big|\\
&\leq I^{\alpha-\delta} a(t) + \frac{\Gamma(\alpha-1)p}{\Gamma(\alpha-\delta)} I^{\alpha-1} a(1) + \frac{\Gamma(\alpha-1)\mu p}{\Gamma(\alpha-\delta)} I^{\alpha-1} a(\xi)\\
&\quad + \frac{c_1 R^{\rho_1} + c_2 R^{\rho_2}}{\Gamma(\alpha-\delta)}\left[\int_0^t (t-s)^{\alpha-\delta-1}\mathrm{d}s + p\int_0^1 (1-s)^{\alpha-2}\mathrm{d}s + \mu p\int_0^{\xi} (\xi-s)^{\alpha-2}\mathrm{d}s\right]\\
&\leq \frac{1+p}{\Gamma(\alpha-\delta)}\int_0^1 (1-s)^{\alpha-\delta-1} a(1)\mathrm{d}s + \frac{\mu p}{\Gamma(\alpha-\delta)}\int_0^{\xi} (\xi-s)^{\alpha-2} a(\xi)\mathrm{d}s\\
&\quad + \left(c_1 R^{\rho_1} + c_2 R^{\rho_2}\right)\frac{1+p+\mu p}{\Gamma(\alpha-\delta+1)}.
\end{aligned} \tag{42}$$

Hence,

$$\left\|T_1 v\right\|_X \leq k + \left(c_1 R^{\rho_1} + c_2 R^{\rho_2}\right)A \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R. \tag{43}$$

Similarly, one has

$$\left\|T_2 u\right\|_Y \leq l + \left(d_1 R^{\theta_1} + d_2 R^{\theta_2}\right)B \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R. \tag{44}$$

Therefore, we get $\|T(u,v)\|_{X\times Y} \leq R$. Notice that $T_1 v(t), T_2 u(t), D^{\delta}T_1 v(t)$ and $D^{\gamma}T_2 u(t)$ are continuous on $[0,1]$. Thus, we have $T\colon S \longrightarrow S$.

Now we are in the position to let (H_2) be satisfied. Choose

$$0 < R \leq \min\left\{(2Ac_1)^{\frac{1}{1-\rho_1}}, (2Ac_2)^{\frac{1}{1-\rho_2}}, (2Bd_1)^{\frac{1}{1-\theta_1}}, (2Bd_2)^{\frac{1}{1-\theta_2}}\right\}. \tag{45}$$

Repeating arguments similar to that above we can obtain

$$\left\|T_1 v\right\|_X \leq \left(c_1 R^{\rho_1} + c_2 R^{\rho_2}\right)A = Ac_1 R^{\rho_1-1}R + Ac_2 R^{\rho_2-1}R \leq \frac{Ac_1}{2\,Ac_1}R + \frac{Ac_2}{2\,Ac_2}R = \frac{R}{2} + \frac{R}{2} = R, \tag{46}$$

$$\left\|T_2 u\right\|_Y \leq \left(d_1 R^{\theta_1} + d_2 R^{\theta_2}\right)B \leq \frac{R}{2} + \frac{R}{2} = R. \tag{47}$$

Consequently, we have $T\colon S \longrightarrow S$.

In view of the continuity of G_1, G_2, f, and g, it is easy to see that the operator T is continuous.

Next, we proof TS is equicontinuous. We take $M = \max_{0\leq t\leq 1} f(t, v(t), D^{\gamma} v(t))$, $N = \max_{0\leq t\leq 1} g(t, u(t), D^{\delta} u(t))$, for any $(u,v) \in S$, let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, then we have

$$\begin{aligned}
\left|T_1 v(t_2) - T_1 v(t_1)\right| &= \left|\int_0^1 \left(G_1(t_2,s) - G_1(t_1,s)\right) f\left(s, v(s), D^{\gamma} v(s)\right)\mathrm{d}s\right|\\
&\leq \frac{M}{\Gamma(\alpha)}\left|\int_0^{t_2} (t_2-s)^{\alpha-1}\mathrm{d}s - \int_0^{t_1} (t_1-s)^{\alpha-1}\mathrm{d}s\right|\\
&\quad + \frac{M}{\Gamma(\alpha)\left(1-\mu\xi^{\alpha-2}\right)}\int_0^1 (1-s)^{\alpha-2}\mathrm{d}s\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu M}{\Gamma(\alpha)\left(1-\mu\xi^{\alpha-2}\right)}\int_0^{\xi}(\xi-s)^{\alpha-2}\mathrm{d}s\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|\\
&\le\frac{M}{\Gamma(\alpha+1)}\left|t_2^{\alpha}-t_1^{\alpha}\right|+\frac{Mp(1+\mu)}{(\alpha-1)\Gamma(\alpha)}\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|,
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
\left|D^{\delta}T_1v(t_2)-D^{\delta}T_1v(t_1)\right|&\le\left|I^{\alpha-\delta}f(t_2,v(t_2),D^{\gamma}v(t_2))-I^{\alpha-\delta}f(t_1,v(t_1),D^{\gamma}v(t_1))\right|\\
&\quad+\frac{\Gamma(\alpha-1)p}{\Gamma(\alpha-\delta)}I^{\alpha-1}f(1,v(1),D^{\gamma}v(1))\left|t_2^{\alpha-\delta-1}-t_1^{\alpha-\delta-1}\right|\\
&\quad+\frac{\Gamma(\alpha-1)\mu p}{\Gamma(\alpha-\delta)}I^{\alpha-1}f(\xi,v(\xi),D^{\gamma}v(\xi))\left|t_2^{\alpha-\delta-1}-t_1^{\alpha-\delta-1}\right|\\
&\le\frac{M}{\Gamma(\alpha-\delta)}\left|\int_0^{t_2}(t_2-s)^{\alpha-\delta-1}\mathrm{d}s-\int_0^{t_1}(t_1-s)^{\alpha-\delta-1}\mathrm{d}s\right|\\
&\quad+\frac{Mp}{\Gamma(\alpha-\delta)}\int_0^1(1-s)^{\alpha-2}\mathrm{d}s\left|t_2^{\alpha-\delta-1}-t_1^{\alpha-\delta-1}\right|\\
&\quad+\frac{\mu Mp}{\Gamma(\alpha-\delta)}\int_0^{\xi}(\xi-s)^{\alpha-2}\mathrm{d}s\left|t_2^{\alpha-\delta-1}-t_1^{\alpha-\delta-1}\right|\\
&\le\frac{M}{\Gamma(\alpha-\delta+1)}\left|t_2^{\alpha-\delta}-t_1^{\alpha-\delta}\right|+\frac{Mp(1+\mu)}{(\alpha-1)\Gamma(\alpha-\delta)}\left|t_2^{\alpha-\delta-1}-t_1^{\alpha-\delta-1}\right|.
\end{aligned}
\tag{49}
$$

Similarly,

$$
\begin{aligned}
\left|T_2u(t_2)-T_2u(t_1)\right|&\le\frac{N}{\Gamma(\beta+1)}\left|t_2^{\beta}-t_1^{\beta}\right|+\frac{Nq(1+\mu)}{(\beta-1)\Gamma(\beta)}\left|t_2^{\beta-1}-t_1^{\beta-1}\right|,\\
\left|D^{\gamma}T_2u(t_2)-D^{\gamma}T_2u(t_1)\right|&\le\frac{N}{\Gamma(\beta-\gamma+1)}\left|t_2^{\beta-\gamma}-t_1^{\beta-\gamma}\right|+\frac{Nq(1+\mu)}{(\beta-1)\Gamma(\beta-\gamma)}\left|t_2^{\beta-\gamma-1}-t_1^{\beta-\gamma-1}\right|.
\end{aligned}
\tag{50}
$$

Now, using the fact that the functions $t^{\alpha}, t^{\alpha-1}, t^{\alpha-\delta}, t^{\alpha-\delta-1}, t^{\beta}, t^{\beta-1}, t^{\beta-\gamma}, t^{\beta-\gamma-1}$ are uniformly continuous on the interval $[0,1]$, we conclude that TS is equicontinuous. Obviously, it is uniformly bounded since $TS\subseteq S$. Thus, T is completely continuous. By using Lemma 5, we conclude that the boundary value problem (3) has at least one positive solution. □

Theorem 2. *Assume that f, g satisfy*

$$
\begin{aligned}
|f(t,x,y)-f(t,\overline{x},\overline{y})|&<m(|x-\overline{x}|+|y-\overline{y}|),\\
|g(t,x,y)-g(t,\overline{x},\overline{y})|&<n(|x-\overline{x}|+|y-\overline{y}|),
\end{aligned}
\tag{51}
$$

for $t\in[0,1], x,\overline{x}\in\mathbb{R}_+, y,\overline{y}\in\mathbb{R}$. Then, the boundary value problem (3) has a unique positive solution if

$$
\begin{aligned}
m&<\left(\frac{p}{(\alpha-1)\Gamma(\alpha)}+\frac{2p}{\Gamma(\alpha-\delta+1)}\right)^{-1},\\
n&<\left(\frac{q}{(\beta-1)\Gamma(\beta)}+\frac{2q}{\Gamma(\beta-\gamma+1)}\right)^{-1}.
\end{aligned}
\tag{52}
$$

Proof. We consider the operator $T: S\longrightarrow S$, by the definition of $G_i(t,s)(i=1,2)$, for $(u_i,v_i)\in S(i=1,2)$, we have the estimate

$$
\begin{aligned}
\left|T_1v_1(t)-T_1v_2(t)\right|&=\int_0^1G_1(t,s)\left|f(s,v_1(s),D^{\gamma}v_1(s))-f(s,v_2(s),D^{\gamma}v_2(s))\right|\mathrm{d}s\\
&<\int_0^1G_1(1,s)m\left[\left|v_1(s)-v_2(s)\right|+\left|D^{\gamma}v_1(s)-D^{\gamma}v_2(s)\right|\right]\mathrm{d}s
\end{aligned}
$$

$$\le m\int_0^1 G_1(1,s)\mathrm{d}s\|v_1-v_2\| < \frac{mp}{(\alpha-1)\Gamma(\alpha)}\|v_1-v_2\|, s$$

$$\begin{aligned}
\left|D^{\delta}T_1v_1(t)-D^{\delta}T_1v_2(t)\right| &\le I^{\alpha-\delta}\left|f\left(t,v_1(t),D^{\gamma}v_1(t)\right)-f\left(t,v_2(t),D^{\gamma}v_2(t)\right)\right| \\
&\quad+\frac{\Gamma(\alpha-1)t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)\left(1-\mu\xi^{\alpha-2}\right)}I_{0+}^{\alpha-1}\left|f\left(1,v_1(1),D^{\gamma}v_1(1)\right)-f\left(1,v_2(1),D^{\gamma}v_2(1)\right)\right| \\
&\quad+\frac{\mu\Gamma(\alpha-1)t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)\left(1-\mu\xi^{\alpha-2}\right)}I_{0+}^{\alpha-1}\left|f\left(\xi,v_1(\xi),D^{\gamma}v_1(\xi)\right)-f\left(\xi,v_2(\xi),D^{\gamma}v_2(\xi)\right)\right| \\
&< \frac{m\|v_1-v_2\|}{\Gamma(\alpha-\delta)}\left[\int_0^t (t-s)^{\alpha-\delta-1}\mathrm{d}s+\frac{1}{1-\mu\xi^{\alpha-2}}\int_0^1(1-s)^{\alpha-2}\mathrm{d}s\right. \\
&\quad\left.+\frac{\mu}{1-\mu\xi^{\alpha-2}}\int_0^{\xi}(\xi-s)^{\alpha-2}\mathrm{d}\right] \\
&= \frac{m\|v_1-v_2\|}{\Gamma(\alpha-\delta)}\left[\frac{t^{\alpha-\delta}}{\alpha-\delta}+\frac{1}{(\alpha-1)\left(1-\mu\xi^{\alpha-2}\right)}+\frac{\mu\xi^{\alpha-1}}{(\alpha-1)\left(1-\mu\xi^{\alpha-2}\right)}\right] \\
&\le \frac{m\|v_1-v_2\|}{\Gamma(\alpha-\delta)}\frac{1-\mu\xi^{\alpha-2}+1+\mu\xi^{\alpha-1}}{(\alpha-\delta)\left(1-\mu\xi^{\alpha-2}\right)} \\
&\le \frac{2mp}{\Gamma(\alpha-\delta+1)}\|v_1-v_2\|.
\end{aligned} \tag{53}$$

So, $\|T_1v_1(t)-T_1v_2(t)\| < \eta_1\|v_1-v_2\|$, where

$$\eta_1 = m\left(\frac{p}{(\alpha-1)\Gamma(\alpha)}+\frac{2p}{\Gamma(\alpha-\delta+1)}\right) < 1. \tag{54}$$

Similarly, $\|T_2u_1(t)-T_2u_2(t)\| < \eta_2\|u_1-u_2\|$, where

$$\eta_2 = n\left(\frac{q}{(\beta-1)\Gamma(\beta)}+\frac{2q}{\Gamma(\beta-\gamma+1)}\right) < 1. \tag{55}$$

Therefore,

$$\% \left\|T\left(u_1,v_1\right)-T\left(u_2,v_2\right)\right\| < \max\{\eta_1,\eta_2\}\left\|\left(u_1-u_2,v_1-v_2\right)\right\|. \tag{56}$$

By using contraction mapping principle, we conclude that problem (3) has a unique positive solution. □

4. Examples

Now, we present two examples to illustrate the main results.

Example 1. Consider the following system of fractional differential equations:

$$\begin{cases}
D^{5/2}u(t)+\dfrac{e^t}{1+e^t}+\dfrac{\sin^2 t}{2}\left(|v(t)|^{\rho_1}+|D^{7/6}v(t)|^{\rho_2}\right)=0, & 0<t<1, \\
D^{7/3}v(t)+\dfrac{t}{1+t}+\dfrac{\cos^2 t}{4}\left(|u(t)|^{\theta_1}+|D^{5/4}u(t)|^{\theta_2}\right)=0, & 0<t<1, \\
u(0)=u'(0)=0, \quad u'(1)=\mu u'(\xi), \\
v(0)=v'(0)=0, \quad v'(1)=\mu v'(\xi).
\end{cases} \tag{57}$$

In system (57), $\alpha=5/2, \beta=7/3, \gamma=7/6, \delta=5/4, \mu=3/4, \xi=1/4$. Let

$$\begin{aligned}
f\left(t,v,D^{7/6}v(t)\right) &= \frac{e^t}{1+e^t}+\frac{\sin^2 t}{2}\left(|v(t)|^{\rho_1}+|D^{5/4}v(t)|^{\rho_2}\right), \\
g\left(t,u,D^{5/4}u(t)\right) &= \frac{t}{1+t}+\frac{\cos^2 t}{4}\left(|u(t)|^{\theta_1}+|D^{7/6}u(t)|^{\theta_2}\right),
\end{aligned} \tag{58}$$

for $(t,u,v)\in[0,1]\times\mathbb{R}_+\times\mathbb{R}$.

We deduce

$$\begin{aligned} &f(t,x,y) \leq a(t) + c_1|x|^{\rho_1} + c_2|y|^{\rho_2}, 0<\rho_1,\rho_2<1,\\ &g(t,x,y) \leq b(t) + d_1|x|^{\theta_1} + d_2|y|^{\theta_2}, 0<\theta_1,\theta_2<1. \end{aligned} \tag{59}$$

where $a(t) = e^t/(1+e^t), b(t) = t/(1+t), c_1 = c_2 = 1/2, d_1 = d_2 = 1/4$.

Therefore, by Theorem 1, there exists one positive solution.

Example 2. Consider the following system of fractional differential equations:

$$\begin{cases} D^{5/2}u(t) + \dfrac{e^{-t}}{6}\left(t^2 + |v(t)| + |D^{7/6}v(t)|\right) = 0, \quad 0<t<1,\\ D^{7/3}v(t) + \dfrac{\sin^2 t}{8}\left(t + |u(t)| + |D^{5/4}u(t)|\right) = 0, \quad 0<t<1,\\ u(0) = u'(0) = 0, u'(1) = \mu u'(\xi),\\ v(0) = v'(0) = 0, v'(1) = \mu v'(\xi). \end{cases} \tag{60}$$

In system (60), $\alpha = 5/2, \beta = 7/3, \gamma = 7/6, \delta = 5/4, \mu = 3/4, \xi = 1/4$. We can easily get

$$\begin{aligned} &\left(\frac{p}{(\alpha-1)\Gamma(\alpha)} + \frac{2p}{\Gamma(\alpha-\delta+1)}\right)^{-1} \approx 0.2741,\\ &\left(\frac{q}{(\beta-1)\Gamma(\beta)} + \frac{2q}{\Gamma(\beta-\gamma+1)}\right)^{-1} \approx 0.2522. \end{aligned} \tag{61}$$

We deduce

$$\begin{aligned} &|f(t,x,y) - f(t,\overline{x},\overline{y})| < \frac{1}{6}(|x-\overline{x}| + |y-\overline{y}|),\\ &|g(t,x,y) - g(t,\overline{x},\overline{y})| < \frac{1}{8}(|x-\overline{x}| + |y-\overline{y}|). \end{aligned} \tag{62}$$

where $m = 1/6 < 0.2741, n = 1/8 < 0.2522$. Therefore, by Theorem 2, there exists a unique positive solution.

5. Conclusion

The existence and uniqueness of positive solutions for a coupled differential system of fractional order with three-point boundary conditions are obtained. The main tools are some fixed point theorems in cones. It is worth mentioning that the nonlinear terms in the system are dependent on the fractional derivative of the unknown functions.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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