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Research Article

Optimization of Data Distributed Network System under Uncertainty

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The major network design or data distributed problems may be described as constrained optimization problems. Constrained optimization problems include restrictions imposed by the system designers. These limitations are basically due to the system design's physical limitations or functional requirements of the network system. Constrained optimization is a computationally challenging job whenever the constraints/limitations are nonlinear and nonconvex. Furthermore, nonlinear programming methods can easily deal same optimization problem if somehow the constraints are nonlinear and convex. In this paper, we have addressed a distributed network design problem involving uncertainty that transmits data across a parallel router. This distributed network design problem is a Jackson open-type network design problem that has been formulated based on the M/M/1 queueing system. Because our network design problem is a nonlinear, convex optimization problem, we have employed a well-known Kuhn–Tucker (K-T) optimality algorithm to solve the same. Here, we have used triangular fuzzy numbers to express uncertain traffic rates and data processing rates. Then, by applying α-level interval of fuzzy numbers and their corresponding parametric representation of α-level intervals, the associated network design problem has been transformed to its parametric form and later has been solved. To obtain the optimal data stream rate in terms of interval and to illustrate the applicability of the entire approach, a hypothetical numerical example has been exhibited. Finally, the most important results have been reported.

1. Introduction

A queue is a set of individuals or things that must be handled in a specific way. Erlang [1], a Danish engineer known as the "Father of Queueing Theory," had published articles on the study of telephone traffic congestion in 1909. A queueing network consists of nodes, each of which represents a service facility. In 1957, Jackson [2] found the use of queueing networks. Jackson's network [3] has the most significant contributions to the queueing network service centres, digital communications, communications infrastructure, processing and flexible automation systems, transportation hubs, and healthcare systems which are just a few areas of use for queueing frameworks. Queueing networks are categorized into three types: open, closed, and mixed networks. Users begin receiving from an external device and forwarding to an external destination via open networks. Closed networks have a constant majority of individuals who stream between queues but never leave the same process. Some working phases are created by combining networks in order for them to be open, while others require them to be closed, which are referred to as mixed networks. Retrial queues or queues with frequent orders are queueing concepts that are expected to arrive to the clients who find the host engaged could sometimes retry for which service after quite period of time. Between retrials, the stuck user needs to join a clientele referred as "orbit." Kosten [4] was the first to propose the notion of retrial queues in 1947. Retrial queues have been used as computer simulations in a variety of digital systems, communication networks, telephony, wireless communications, reservation ticket bookings, healthcare organisations, and so on. Cohen [5] explored the fundamental issues of telephone traffic theory. The advanced telecommunication framework comprises personal information, and conventional relay switched networks may not work well with data. To address this issue, packet-switching has been introduced. It is possible that a specific connectivity between routers is busy in a packet-switched system; in that case, the packet should wait until it becomes available.

A router is required to wait to transfer data through an intermediate destination node [6]. Through our conceptual scheme, we have decided to place our queueing network on the Jackson queueing network, for each queue becoming a single server M/M/1 queue. In M/M/1 queues, the exponential distribution is used to distribute both the host time and the processing time. The exponential distribution is also the probability distribution of the time between occurrences in a stochastic process, a process in which events take place continuously and independently at a constant average rate and is therefore more exact for traffic-related cross-functional and cross time and processing time in a real-world data transmission system. Queueing networks are an effective technique for studying the efficiency of ride-sharing or vehicular applications [7–9], transportation networks [10], communication networks [5, 11], and so on. For more details, one may refer to the distinguished works of Kulkarni [12], Rama and Motahar [13], Paschalidis and Tsitsiklis [14], Sommer et al. [15], Cruz and Woensel [16], Roy [17], Kim et al. [18], Melamed [19], Kam et al. [20], Kaul et al. [21], Kam et al. [22], Whitt and You [23], Whitt [24], Xia and Shihada [25], Timmer and Scheinhardt [26], Wang et al. [27], and Alam et al. [28], etc.

2. Motivation of the Study

In conventional network routing problems, it has been presumed that data transmission factors such as network congestion, data transfer rate, and network throughput are clearly recognised. This implies that its system's complete probabilistic information is known advance. In this case, it is implied that the probability value for each probable event is exactly predictable. Nevertheless, susceptible to equipment failures, unstable power supply, unauthorised storage facilities, and certain other matters pertaining to unusual surroundings and all parameters relating to the communication of data network systems may not always be fixed/precise. In this case, it is necessary to retrieve exact system information. As a matter of fact, the performance parameters will be deemed as a reasonable approximation. When dealing with such ambiguous figures, fuzzy set theory helps a lot (Zadeh, 1965). Lotfi A. Zadeh developed fuzzy set theory and logic in 1965 [29]. He extended classical set theory concepts to account for data ambiguity and uncertainty, laying the groundwork for modern fuzzy set theory. Zadeh was personally responsible for the subject's significant and notable advancement. This theory is primarily used to model uncertainty in data across a wide range of research fields.

In this study, we took into account fuzzy values and their corresponding α-level interval numbers while attempting to deal with ambiguous parameters. As a consequence, the fuzzy valued data distributed network system presents a viable conceptual model for addressing the network optimization model of a multifaceted distributed data network with a fuzzy valued parametric arrangement. The parameters associated with this problem have always been considered fuzzy numbers, and the respective problem has been solved using the interval parametric technique. For interval parametric technique, interval arithmetic [30] and a very well parametric representation of α-level interval studied by Sahoo [31] have been used. Throughout this paper, we considered a distributed network design problem with uncertainty which also transmits data over a parallel gateway. This network design problem is a Jackson open-type network problem with a M/M/1 queueing system that has been formed. Even though our network design problem is nonlinear and convex, the authors used well-known Kuhn–Tucker (K-T) optimality techniques [32] to solve it. In this paper, we consider and solve a data-distributed network system under uncertainty. Finally, the most important results were reported.

2.1. Contributions and Objectives. The following are the contributions and objectives of the proposed research work:

(1) We developed and solved data distributed network systems in the presence of uncertainty

(2) Jackson open-type network has been used to formulate the network design optimization problems based on the M/M/1 queueing system

(3) We used fuzzy sets/fuzzy numbers to convey uncertainty

(4) To express all of the parameters, such as data traffic rates and data processing rates to send data in a whole network, we used triangular fuzzy numbers (TFNs)

(5) Interval parametric representation of fuzzy parameters has been accomplished using a parametric representation of interval numbers

(6) For determining the optimal data stream rate over the routers, a well-known Kuhn–Tucker (K-T) optimality technique has been implemented

(7) In an uncertainty context, Kuhn–Tucker (K-T) optimality technique has been described

(8) Using Little's formula and interval arithmetic, we have estimated the data stream rate, expected a number of packets, expected waiting time, mean processing time, and expected number of packets in the router in terms of interval uncertainty

(9) Concluding remarks have been presented, along with the future direction of the proposed research work

Most of the work done in this study can be summarized in the following way:

(i) Jackson open-type network has been used to formulate the network design optimization problems based on the M/M/1 queueing system

(ii) In a data distributed network, all the parameters, such as data traffic rates and data processing rates to send data in a whole network, we have used triangular fuzzy numbers (TFNs)

(iii) Interval parametric representation of fuzzy parameters has been accomplished using a parametric representation of interval numbers

(iv) Kuhn–Tucker (K-T) optimality technique has been described to find optimal data stream rates under uncertain situations

(v) Little's formula has been used to estimate the data stream rate, expected number of packets, expected waiting time, mean processing time, and expected number of packets in the router in terms of interval uncertainty

The rest of the paper is organized as follows: Section 2 gives some basic mathematical foundations to develop the paper. Section 3 presents a problem description in an uncertain environment. Section 4 gives an analytical solution of the problem in terms of interval uncertainty with a numerical illustration. Finally, the concluding remark has been presented in Section 5.

3. Mathematical Background

This section describes essential terms and concepts related to fuzzy sets, fuzzy numbers, and parametric representation of interval numbers which will be used in this study.

3.1. Expected Number of Packets in $(M/M/1)$: $(\infty/FCFS/\infty)$ Queueing System. Let N be the number of packets in the system. If λ and μ be the arrival rate and processing rate, then according to $(M/M/1)$: $(\infty/FCFS/\infty)$ queueing system, L_s = expected no. of packets in the system, i.e., average no. of the packet in a system$= \sum_{N=0}^{\infty} NP_N = \sum_{N=0}^{\infty} N(1-\rho)\rho^N = \rho/1-\rho = \lambda/\mu-\lambda$, where $\rho = \lambda/\mu$

3.2. KKT Optimality Conditions. Let us consider a minimization problem as follows:

Minimum $f(x) = f(x_1, x_2, \ldots, x_n)$

Subject to

$$g_i(x_1, x_2, \cdots, x_n) \leq 0, \quad i = 1, 2, 3 \ldots, m. \tag{1}$$

The necessary conditions for a solution x^* to be the local optimal for problem (1) are as follows:

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0, \tag{2}$$

$$\lambda_i^* g_i(x^*) = 0, \quad \text{for } i = 1, 2, \ldots, m. \tag{3}$$

$$g_i(x^*) \leq 0, \quad \text{for } i = 1, 2, \ldots, m. \tag{4}$$

$$\lambda^*_i \rangle\, 0, \quad \text{for } i = 1, 2, \ldots, m. \tag{5}$$

where $\nabla f(x) = (\partial f(x)/\partial x_1, \partial f(x)/\partial x_2, \ldots, \partial f(x)/\partial x_n)^T$ and $\nabla g_i(x) = (\partial g_i(x)/\partial x_1, \partial g_i(x)/\partial x_2, \ldots, \partial g_i(x)/\partial x_n)^T$.

Equations (2)–(5) are the Kuhn–Tucker (K-T) conditions and point (x^*, λ^*) is the K-T point. It is to be noted that one of the best approaches to solve a nonlinear programming problem (NLP) is to find (K-T) point (x^*, λ^*) and consider x^* as the optimal solution of problem (1). Again, if the function $f(x)$ is convex and the feasible region is a convex set, then x^* is also the global minimum of problem (1).

3.3. Concept of Fuzzy Set. A membership function $\mu_{\widetilde{A}}(x)$ that maps to each element x in X to a real number in the interval $0 \leq x \leq 1$ forms a fuzzy set. The function $\mu_{\widetilde{A}}(x)$ represents the degree of membership of x in the fuzzy set $\widetilde{A}$.

Definition 1. The α-cut of a fuzzy set $\widetilde{A}$ is a crisp subset of X and is denoted by $A_\alpha = \{x \in X\colon \mu_{\widetilde{A}}(x) \geq \alpha\}$, where $\mu_{\widetilde{A}}(x)$ is the membership function of $\widetilde{A}$ and $\alpha \in [0, 1]$.

Definition 2. A fuzzy set $\widetilde{A}$ is called a normal fuzzy set if there exists at least one $x \in X$ such that $\mu_{\widetilde{A}}(x) = 1$.

Definition 3. A fuzzy set $\widetilde{A}$ is called convex iff for every pair of $x_1, x_2 \in X$, the membership function of $\widetilde{A}$ satisfies $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\}$, where $\lambda \in [0, 1]$.

Definition 4. A fuzzy number $\widetilde{A}$ is a fuzzy set that is both convex and normal.

Definition 5. The triangular fuzzy number (TFN) is a normal fuzzy number denoted as $\widetilde{A} = (a_l, a_m, a_r)$, where $a_l \leq a_m \leq a_r$ and its membership function $\mu_{\widetilde{A}}(x)\colon X \longrightarrow [0, 1]$ is defined by

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - a_l}{a_m - a_l} & \text{if } a_l \leq x \leq a_m, \\ 1 & \text{if } x = a_m, \\ \dfrac{a_r - x}{a_r - a_m} & \text{if } a_m \leq x \leq a_r. \end{cases} \tag{6}$$

3.3.1. α-Level Set of Triangular Fuzzy Numbers. Let $\widetilde{A} = (a_l, a_m, a_r)$ be a triangular fuzzy number, then α-level set of the triangular fuzzy number $\widetilde{A} = (a_l, a_m, a_r)$ is $A_\alpha = \{x \in X\colon \mu_{\widetilde{A}}(x) \geq \alpha\} = [A_\alpha^-, A_\alpha^+]$, where $A_\alpha^- = a_l + (a_m - a_l)\alpha$ and $A_\alpha^+ = a_r - (a_r - a_m)\alpha$, $\alpha \in [0, 1]$.

3.4. Parametric representation of Interval number. Let $A = [a^-, a^+]$ be an interval number. According to Sahoo [22], parametric representation of interval number $A = [a^-, a^+]$ is denoted by $H_A(p)$, which is defined as follows:

$$H_A(p) = \begin{cases} (a^-)^{1-p}(a^+)^p & \text{if } a^- \rangle\, 0 \text{ and } 0 \leq p \leq 1, \\ -(|a^-|)^{1-p}\left(|a^+|\right)^p & \text{if } a^+ \langle\, 0 \text{ and } 0 \leq p \leq 1. \end{cases} \quad (7)$$

If $0 \in [a^-, a^+]$, i.e., if $a^- \langle\, 0$ and $a^+ \rangle\, 0$, then

$$H_A(p) = \begin{cases} -(|a^-|)^{1-p}(|\theta|)^p, \theta \longrightarrow 0^- & \text{if } \left(a^+ - |a^-|\right) \langle\, 0 \text{ and } 0 \leq p \leq 1, \\ (\theta)^{1-p}\left(a^+\right)^p, \theta \longrightarrow 0^+ & \text{if } \left(a^+ - |a^-|\right) \rangle\, 0 \text{ and } 0 \leq p \leq 1. \end{cases} \quad (8)$$

3.4.1. Interval Parametric Representation of Triangular Fuzzy Number. Let $\widetilde{A} = (a_l, a_m, a_r)$ be a triangular fuzzy number and $A_\alpha = [A_\alpha^-, A_\alpha^+]$ be the corresponding α–level interval, where $A_\alpha^- = a_l + (a_m - a_l)\alpha$ and $A_\alpha^+ = a_r - (a_r - a_m)\alpha$, $\alpha \in [0, 1]$, then according to Section 3.4, $H_{A_\alpha}(p)$ be its corresponding parametric representation of $A_\alpha = [A_\alpha^-, A_\alpha^+]$.

4. Problem Formulation

We have considered the following assumptions and notations to develop the entire paper.

4.1. Assumptions

(i) The processor can process only one packet at a time

(ii) Packets arrival process is memoryless (M): no matter what, each seemed of a time a new packet arrives with probability λ

(iii) Processing of packets is memoryless (M): if the queue is not empty, in each seemed if a time the server/processor complete processing a packet with probability μ

(iv) Average number of packets P waiting in queue is given by $P = \lambda/\mu - \lambda$

(v) Data rate, additional traffic rate, and processing rate of the traffic network are fuzzy valued

4.2. Notations

$\widetilde{r}_j$: additional traffic rate of j-th server which is fuzzy valued

$\widetilde{C}_j$: processing rate of j-th server which is fuzzy valued

$\widetilde{x}_j$: datastream rate of j-th server which is uncertain

$\widetilde{R}$: data rate of the network which is fuzzy valued

Let us assume that a network sends data of rate $\widetilde{R}$ over a n parallel routers which are n parallel queues. Let the processing rate of j – th router is $\widetilde{C}_j$. Let us suppose that the data be distributed over separate streams of rates $\widetilde{x}_{1,}\widetilde{x}_2, \widetilde{x}_3, \ldots, \widetilde{x}_n$, where data $\widetilde{x}_j$ is sent into j-th queue. Furthermore, assume that the j – th queue serves an additional traffic stream of rate $\widetilde{r}_j$ (see Figure 1). According to $(M/M/1)$: $(\infty/FCFS/\infty)$ queueing model, the average number of data packets in j – th queue is $\widetilde{x}_j + \widetilde{r}_j/\widetilde{C}_j - (\widetilde{x}_j + \widetilde{r}_j)$. Therefore, for all the queues, we want to find $\widetilde{x}_{1,}\widetilde{x}_2, \widetilde{x}_3, \ldots, \widetilde{x}_n$ in such a way that the total average no. of packets in the entire n-queue network is to be minimized. Therefore, the corresponding optimization problem becomes as follows:

$$\begin{aligned} &\text{minimize } \widetilde{P}\left(\widetilde{x}_{1,}\widetilde{x}_2, \widetilde{x}_3, \ldots, \widetilde{x}_n\right) = \sum_{j=1}^{n} \frac{\widetilde{x}_j + \widetilde{r}_j}{\widetilde{C}_j - \left(\widetilde{x}_j + \widetilde{r}_j\right)}, \\ &\text{subject to } \widetilde{g}\left(\widetilde{x}_{1,}\widetilde{x}_2, \widetilde{x}_3, \ldots, \widetilde{x}_n\right) = -\sum_{j=1}^{n} \widetilde{x}_j - \widetilde{R} \leq 0, \\ &\sum_{j=1}^{n} \widetilde{r}_j + \widetilde{R} \leq \sum_{j=1}^{n} \widetilde{C}_j, \\ &\widetilde{x}_j \in \left[\widetilde{0}, \widetilde{C}_j - \widetilde{r}_j\right] \text{and } \widetilde{C}_j \rangle\, \widetilde{1}, \quad \text{for } j = 1, 2, \ldots, n. \end{aligned} \quad (9)$$

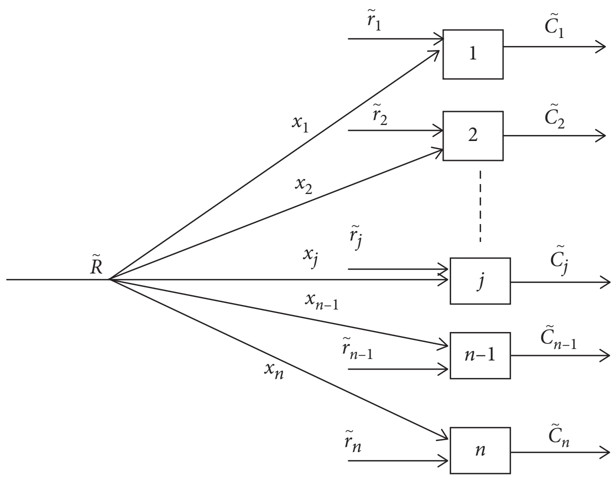

FIGURE 1: Jackson open-type data sending network over n parallel router.

As the optimization problem (9) is related to fuzzy valued objective function as well as fuzzy valued constraints, for solution purposes we have used a parametric representation of the fuzzy number. Now, using Prometric representation, problem (9) reduces to the following:

$$\begin{aligned}
&\text{minimize } H_{\widetilde{P}(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3,\ldots,\widetilde{x}_n)}(p) = \sum_{j=1}^{n} \frac{\widetilde{x}_j + H_{\widetilde{r}_j}(p)}{\left(H_{\widetilde{C}_j}(p) - \left(\widetilde{x}_j + H_{\widetilde{r}_j}(p)\right)\right)},\\
&\text{subject to} - \sum_{j=1}^{n} \widetilde{x}_j - H_{\widetilde{R}}(p) \leq 0,\\
&\sum_{j=1}^{n} H_{\widetilde{r}_j}(p) + H_{\widetilde{R}}(p) \leq \sum_{j=1}^{n} H_{\widetilde{C}_j}(p),\\
&\widetilde{x}_j \in \left[0, H_{\widetilde{C}_j}(p) - H_{\widetilde{r}_j}(p)\right] \text{and } H_{\widetilde{C}_j}(p)\rangle\, 1, \quad \text{for } j = 1, 2, \ldots, n.
\end{aligned} \tag{10}$$

Problem (10) is a nonlinear and convex optimization problem. Therefore, to find out the optimal solution, we have used widely known K-T optimality criteria mentioned in equations (2)–(5).

Deduction 1. Average number of packets in the $j-$th queue is $\widetilde{x}_j + \widetilde{r}_j/\widetilde{C}_j - (x_j + \widetilde{r}_j)$.

Proof. From Figure 1, it has been observed that the arrival rate is $\widetilde{x}_j + \widetilde{r}_j$ at $j-$th server/router and service (processing) rate is $\widetilde{C}_j$ .Therefore, $\widetilde{\rho}_j = (\widetilde{x}_j + \widetilde{r}_j/\widetilde{C}_j)$. Hence, the expected number of packets in the $j-$th server/router, i.e., an average number of packets in a system, is

$$\begin{aligned}
\widetilde{L}_s &= \frac{\widetilde{\rho}_j}{1-\widetilde{\rho}_j}\\
&= \widetilde{x}_j + \widetilde{r}_j/\widetilde{C}_j - \left(\widetilde{x}_j + \widetilde{r}_j\right).
\end{aligned} \tag{11}$$

□

Theorem 1. *The optimization problem (9) is convex.*

Proof. Problem (9) is convex if $\nabla^2 P$ as well as $\nabla^2 g$ is positive definite.

Now,

$$\nabla^2 P = \begin{pmatrix}
\dfrac{\partial^2 P}{\partial x_1^2} & \dfrac{\partial^2 P}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 P}{\partial x_1 \partial x_n}\\
\dfrac{\partial^2 P}{\partial x_2 \partial x_1} & \dfrac{\partial^2 P}{\partial x_2^2} & \cdots & \dfrac{\partial^2 P}{\partial x_2 \partial x_n}\\
\vdots & \vdots & \vdots & \vdots\\
\dfrac{\partial^2 P}{\partial x_n \partial x_1} & \dfrac{\partial^2 P}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 P}{\partial x_n^2}
\end{pmatrix} \tag{12}$$

TABLE 1: Additional traffic ($\tilde{r}_j$) and processing rate ($\tilde{C}_j$) of each server.

Router (j)	Additional traffic ($\tilde{r}_j$) (in kbps)	Processing rate ($\tilde{C}_j$) (in kbps)
1	(1.0, 2.0, 2.5)	(55, 60, 62)
2	(1.5, 3.0, 4.0)	(60, 62, 63)
3	(4.0, 5.0, 6.0)	(59, 63, 64)
4	(1.0, 2.0, 3.0)	(57, 60, 64)
5	(4.0, 6.0, 6.5)	(58, 60, 63)
6	(3.0, 4.0, 5.0)	(61, 64, 66)
7	(6.0, 8.0, 9.0)	(69, 70, 72)
8	(2.0, 3.5, 4.0)	(59, 61, 63)
9	(4.5, 5.0, 6.0)	(63, 65, 66)
10	(0.5,1.0, 2.0)	(57, 60, 63)

TABLE 2: Additional traffic and processing rate of each server in terms of α-level interval.

Router (j)	Additional traffic (in kbps)	Processing rate (in kbps)
1	$[1.0 + \alpha, 2.5 - 0.5\alpha]$	$[55 + 5\alpha, 62 - 2\alpha]$
2	$[1.5 + 1.5\alpha, 4.0 - \alpha]$	$[60 + 2\alpha, 63 - \alpha]$
3	$[4.0 + \alpha, 6.0 - \alpha]$	$[59 + 4\alpha, 64 - \alpha]$
4	$[1.0 + \alpha, 3.0 - \alpha]$	$[57 + 3\alpha, 64 - 4\alpha]$
5	$[4.0 + 2\alpha, 6.5 - 0.5\alpha]$	$[58 + 2\alpha, 63 - 3\alpha]$
6	$[3.0 + \alpha, 5.0 - \alpha]$	$[61 + 3\alpha, 66 - 2\alpha]$
7	$[6.0 + 2\alpha, 9.0 - \alpha]$	$[69 + \alpha, 72 - 2\alpha]$
8	$[2.0 + 1.5\alpha, 4.0 - 0.5\alpha]$	$[59 + 2\alpha, 63 - 2\alpha]$
9	$[4.5 + 0.5\alpha, 6.0 - \alpha]$	$[63 + 2\alpha, 66 - \alpha]$
10	$[0.5 + 0.5\alpha, 2.0 - \alpha]$	$[57 + 3\alpha, 63 - 3\alpha]$

TABLE 3: Additional traffic and processing rate of each server in terms of parametric representation.

Router (j)	Additional traffic (in kbps)	Processing rate (in kbps)
1	$(1.0 + \alpha)^{1-p}(2.5 - 0.5\alpha)^p$	$(55 + 5\alpha)^{1-p}(62 - 2\alpha)^p$
2	$(1.5 + 1.5\alpha)^{1-p}(4.0 - \alpha)^p$	$(60 + 2\alpha)^{1-p}(63 - \alpha)^p$
3	$(4.0 + \alpha)^{1-p}(6.0 - \alpha)^p$	$(59 + 4\alpha)^{1-p}(64 - \alpha)^p$
4	$(1.0 + \alpha)^{1-p}(3.0 - \alpha)^p$	$(57 + 3\alpha)^{1-p}(64 - 4\alpha)^p$
5	$(4.0 + 2\alpha)^{1-p}(6.5 - 0.5\alpha)^p$	$(58 + 2\alpha)^{1-p}(63 - 3\alpha)^p$
6	$(3.0 + \alpha)^{1-p}(5.0 - \alpha)^p$	$(61 + 3\alpha)^{1-p}(66 - 2\alpha)^p$
7	$(6.0 + 2\alpha)^{1-p}(9.0 - \alpha)^p$	$(69 + \alpha)^{1-p}(72 - 2\alpha)^p$
8	$(2.0 + 1.5\alpha)^{1-p}(4.0 - 0.5\alpha)^p$	$(59 + 2\alpha)^{1-p}(63 - 2\alpha)^p$
9	$(4.5 + 0.5\alpha)^{1-p}(6.0 - \alpha)^p$	$(63 + 2\alpha)^{1-p}(66 - \alpha)^p$
10	$(0.5 + 0.5\alpha)^{1-p}(2.0 - \alpha)^p$	$(57 + 3\alpha)^{1-p}(63 - 3\alpha)^p$

where $\partial^2 P/\partial x_i \partial x_j = \begin{cases} 2C_j/\{C_j - (x_j + r_j)\}^3 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Here, $2C_j/\{C_j - (x_j + r_j)\}^3$ is always positive and hence $\nabla^2 P$ is positive definite. Similarly, $\nabla^2 g$ is also positive definite, where $g = x_1 + x_2 + x_3 + \cdots + x_n - R \geq 0$. □

Theorem 2. *If x^* is a local optimizer of (9), then x^* is also a global optimizer of (9).*

Proof. As the optimization problem (9) is convex, also if x^* is a local optimizer of (9) then x^* is also a global optimizer of (9). □

5. Analytical Solution of Problem with Numerical Illustration

Using KKT conditions (2)–(5), analytical solution of problem (10) is as follows:

$$\lambda = \left(\frac{\sum_{j=1}^{n} \sqrt{H_{\tilde{C}_j}(p)}}{\sum_{j=1}^{n} H_{\tilde{C}_j}(p) - \sum_{j=1}^{n} H_{\tilde{r}_j}(p) - H_{\tilde{R}}(p)} \right)^2, \quad (13)$$

$$\tilde{x}_j = \left(H_{\tilde{C}_j}(p) - H_{\tilde{r}_j}(p) - \sqrt{\frac{H_{\tilde{C}_j}(p)}{\lambda}} \right), \quad (14)$$

Table 4: Data stream rate for different values of p and $\alpha = 0$.

Router (j)	Data stream rate when $\alpha = 0$										
	$p = 0$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 1$
1	45.32	46.02	46.73	47.45	48.19	48.93	49.69	50.46	51.24	52.03	52.84
2	49.43	49.71	49.99	50.27	50.55	50.84	51.12	51.41	51.70	51.99	52.28
3	46.01	46.50	47.01	47.52	48.03	48.55	49.07	49.60	50.14	50.68	51.23
4	47.16	47.82	48.50	49.18	49.87	50.57	51.28	52.00	52.74	53.48	54.23
5	45.08	45.53	45.99	46.44	46.91	47.37	47.85	48.32	48.81	49.29	49.78
6	48.86	49.36	49.87	50.38	50.90	51.42	51.95	52.49	53.03	53.57	54.13
7	53.27	53.52	53.77	54.03	54.28	54.53	54.79	55.04	55.30	55.56	55.82
8	48.01	48.42	48.83	49.25	49.67	50.10	50.53	50.96	51.40	51.84	52.28
9	49.21	49.59	49.97	50.35	50.74	51.13	51.52	51.92	52.32	52.72	53.13
10	47.66	48.28	48.92	49.56	50.21	50.86	51.53	52.21	52.89	53.58	54.28

Table 5: Data stream rate for different values of p and $\alpha = 0.2$.

Router (j)	Data stream rate when $\alpha = 0.2$										
	$p = 0$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 1$
1	46.17	46.74	47.32	47.90	48.49	49.09	49.70	50.31	50.93	51.56	52.19
2	49.64	49.87	50.09	50.32	50.54	50.77	51.00	51.23	51.46	51.69	51.93
3	46.69	47.09	47.49	47.90	48.32	48.73	49.15	49.58	50.00	50.43	50.87
4	47.65	48.19	48.73	49.28	49.84	50.40	50.97	51.54	52.12	52.71	53.30
5	45.19	45.55	45.92	46.29	46.66	47.03	47.41	47.79	48.17	48.56	48.95
6	49.35	49.76	50.17	50.58	51.00	51.42	51.84	52.27	52.70	53.13	53.57
7	53.21	53.41	53.61	53.81	54.02	54.22	54.42	54.63	54.83	55.04	55.25
8	48.22	48.55	48.88	49.22	49.56	49.90	50.24	50.59	50.93	51.28	51.64
9	49.62	49.93	50.23	50.54	50.85	51.17	51.48	51.80	52.12	52.44	52.76
10	48.25	48.76	49.27	49.78	50.30	50.83	51.36	51.90	52.44	52.99	53.55

Table 6: Data stream rate for different values of p and $\alpha = 0.4$.

Router (j)	Data stream rate when $\alpha = 0.4$										
	$p = 0$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 1$
1	47.03	47.47	47.90	48.35	48.79	49.24	49.69	50.15	50.61	51.08	51.55
2	49.85	50.02	50.19	50.36	50.53	50.70	50.87	51.05	51.22	51.39	51.57
3	47.37	47.67	47.98	48.29	48.60	48.91	49.23	49.54	49.86	50.18	50.51
4	48.14	48.55	48.96	49.38	49.79	50.22	50.64	51.07	51.50	51.94	52.38
5	45.30	45.57	45.85	46.13	46.40	46.69	46.97	47.25	47.54	47.83	48.11
6	49.85	50.16	50.47	50.78	51.09	51.41	51.73	52.04	52.37	52.69	53.01
7	53.15	53.30	53.45	53.60	53.75	53.90	54.06	54.21	54.36	54.52	54.67
8	48.43	48.68	48.93	49.18	49.44	49.69	49.95	50.21	50.47	50.73	50.99
9	50.04	50.27	50.50	50.73	50.97	51.20	51.44	51.67	51.91	52.15	52.39
10	48.84	49.23	49.61	50.00	50.39	50.79	51.19	51.59	52.00	52.40	52.81

$$\sum_{j=1}^{n} \tilde{x}_j = H_{\tilde{R}}(p). \tag{15}$$

Using (13)–(15), we obtain

$$\tilde{x}_j = H_{\tilde{C}_j}(p) - H_{\tilde{r}_j}(p) - \frac{\sqrt{H_{\tilde{C}_j}(p)}\left(\sum_{j=1}^{n} H_{\tilde{C}_j}(p) - \sum_{j=1}^{n} H_{\tilde{r}_j}(p) - H_{\tilde{R}}(p)\right)}{\sum_{j=1}^{n}\sqrt{H_{\tilde{C}_j}(p)}}, \quad j = 1, 2, \ldots, n. \tag{16}$$

Case 1. If $H_{\tilde{r}_1}(p) = H_{\tilde{r}_2}(p) = \ldots\ldots = H_{\tilde{r}_n}(p) = H_{\tilde{r}}(p)$ and $H_{\tilde{C}_1}(p) = H_{\tilde{C}_2}(p) = \ldots\ldots = H_{\tilde{C}_n}(p) = H_{\tilde{C}}(p)$, then $\tilde{x}_j = H_{\tilde{R}}(p)/n$.

Case 2. If $H_{\tilde{r}_1}(p) = H_{\tilde{r}_2}(p) = \ldots\ldots = H_{\tilde{r}_n}(p) = 0$ and $H_{\tilde{C}_1}(p) = H_{\tilde{C}_2}(p) = \ldots\ldots = H_{\tilde{C}_n}(p) = H_{\tilde{C}}(p)$, then $\tilde{x}_j = H_{\tilde{R}}(p)$

Case 3. If $H_{\tilde{r}_1}(p) = H_{\tilde{r}_2}(p) = \ldots\ldots = H_{\tilde{r}_n}(p) = H_{\tilde{r}}(p)$ and $H_{\tilde{C}_1}(p) = H_{\tilde{C}_2}(p) = \ldots\ldots = H_{\tilde{C}_n}(p) = 0$, then $\tilde{x}_j = H_{\tilde{R}}(p)/n$.

Case 4. If $H_{\tilde{r}_1}(p) = H_{\tilde{r}_2}(p) = \ldots\ldots = H_{\tilde{r}_n}(p) = 0$ and $H_{\tilde{C}_1}(p) = H_{\tilde{C}_2}(p) = \ldots\ldots = H_{\tilde{C}_n}(p) = 0$, then $x_j = H_{\tilde{R}}(p)$.

Table 7: Data stream rate for different values of p and $\alpha = 0.6$.

Router (j)	Data stream rate when $\alpha = 0.6$										
	$p = 0$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 1$
1	47.89	48.19	48.48	48.78	49.08	49.38	49.68	49.98	50.29	50.60	50.91
2	50.06	50.18	50.29	50.40	50.52	50.63	50.75	50.86	50.98	51.09	51.21
3	48.05	48.25	48.46	48.67	48.88	49.09	49.30	49.51	49.72	49.93	50.14
4	48.64	48.91	49.19	49.47	49.75	50.03	50.31	50.59	50.88	51.17	51.46
5	45.41	45.59	45.78	45.96	46.15	46.33	46.52	46.71	46.90	47.09	47.28
6	50.35	50.56	50.76	50.97	51.18	51.39	51.60	51.82	52.03	52.24	52.46
7	53.08	53.18	53.28	53.39	53.49	53.59	53.69	53.79	53.89	54.00	54.10
8	48.63	48.80	48.97	49.14	49.31	49.48	49.65	49.82	50.00	50.17	50.34
9	50.45	50.61	50.76	50.92	51.07	51.23	51.39	51.55	51.70	51.86	52.02
10	49.44	49.69	49.95	50.22	50.48	50.74	51.01	51.27	51.54	51.81	52.08

Table 8: Data stream rate for different values of p and $\alpha = 0.8$.

Router (j)	Data stream rate when $\alpha = 0.8$										
	$p = 0$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 1$
1	48.76	48.91	49.05	49.20	49.35	49.50	49.66	49.81	49.96	50.11	50.26
2	50.27	50.33	50.39	50.44	50.50	50.56	50.62	50.67	50.73	50.79	50.84
3	48.73	48.83	48.94	49.04	49.15	49.25	49.36	49.46	49.57	49.67	49.78
4	49.13	49.27	49.41	49.55	49.69	49.83	49.97	50.11	50.25	50.39	50.54
5	45.51	45.61	45.70	45.79	45.89	45.98	46.07	46.17	46.26	46.36	46.45
6	50.85	50.95	51.06	51.16	51.27	51.37	51.48	51.58	51.69	51.80	51.90
7	53.02	53.07	53.12	53.17	53.22	53.27	53.32	53.37	53.42	53.47	53.52
8	48.84	48.93	49.01	49.10	49.18	49.27	49.35	49.44	49.53	49.61	49.70
9	50.86	50.94	51.02	51.10	51.18	51.26	51.33	51.41	51.49	51.57	51.65
10	50.03	50.16	50.29	50.42	50.55	50.68	50.82	50.95	51.08	51.22	51.35

Table 9: Data stream rate for different values of p and $\alpha = 1.0$.

Router (j)	Data stream rate when $\alpha = 1.0$										
	$p = 0$	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 1$
1	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62
2	50.48	50.48	50.48	50.48	50.48	50.48	50.48	50.48	50.48	50.48	50.48
3	49.41	49.41	49.41	49.41	49.41	49.41	49.41	49.41	49.41	49.41	49.41
4	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62	49.62
5	45.62	45.62	45.62	45.62	45.62	45.62	45.62	45.62	45.62	45.62	45.62
6	51.35	51.35	51.35	51.35	51.35	51.35	51.35	51.35	51.35	51.35	51.35
7	52.95	52.95	52.95	52.95	52.95	52.95	52.95	52.95	52.95	52.95	52.95
8	49.05	49.05	49.05	49.05	49.05	49.05	49.05	49.05	49.05	49.05	49.05
9	51.28	51.28	51.28	51.28	51.28	51.28	51.28	51.28	51.28	51.28	51.28
10	50.62	50.62	50.62	50.62	50.62	50.62	50.62	50.62	50.62	50.62	50.62

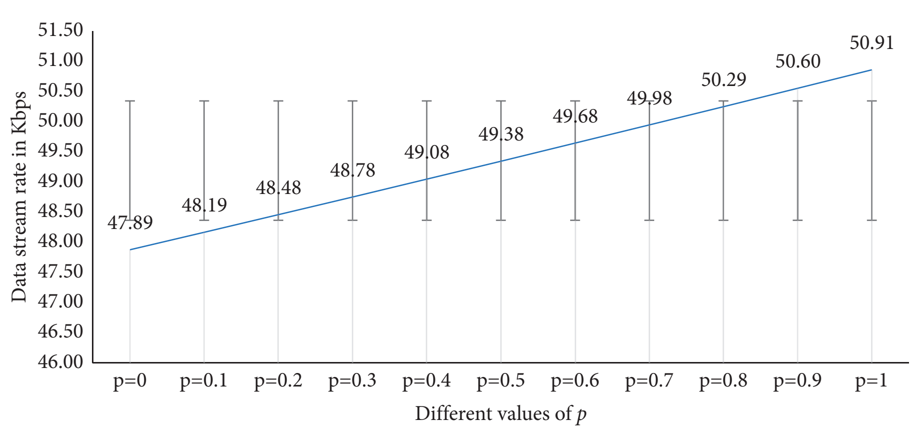

Figure 2: Graphical representation of change of data stream ($\tilde{x}_1$) for server/router 1 when $\alpha = 0.6$.

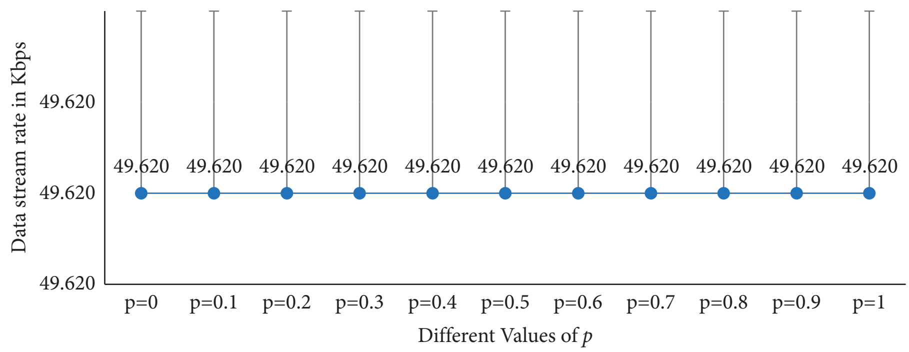

FIGURE 3: Graphical representation of change of data stream ($\tilde{x}_1$) for server/router 1 when $\alpha = 1$.

TABLE 10: Data stream rate in terms of interval for different values of α.

Router (j)	$\alpha = 0$ $x_j = [x_j^-, x_j^+]$ (in kbps)	$\alpha = 0.2$ $x_j = [x_j^-, x_j^+]$ (in kbps)	$\alpha = 0.4$ $x_j = [x_j^-, x_j^+]$ (in kbps)	$\alpha = 0.6$ $x_j = [x_j^-, x_j^+]$ (in kbps)	$\alpha = 0.8$ $x_j = [x_j^-, x_j^+]$ (in kbps)	$\alpha = 1.0$ $x_j = [x_j^-, x_j^+]$ (in kbps)
1	[45.32, 52.84]	[46.17, 52.19]	[47.03, 51.55]	[47.89, 50.91]	[48.76, 50.26]	[49.62, 49.62]
2	[49.43, 52.28]	[49.64, 51.93]	[49.85, 51.57]	[50.06, 51.21]	[50.27, 50.84]	[50.48, 50.48]
3	[46.01, 51.23]	[46.69, 50.87]	[47.37, 50.51]	[48.05, 50.14]	[48.73, 49.78]	[49.41, 49.41]
4	[47.16, 54.23]	[47.65, 53.30]	[48.14, 52.38]	[48.64, 51.46]	[49.13, 50.54]	[49.62, 49.62]
5	[45.08, 49.78]	[45.19, 48.95]	[45.30, 48.11]	[45.41, 47.28]	[45.51,46.45]	[45.62, 45.62]
6	[48.86, 54.13]	[49.35, 53.57]	[49.85, 53.01]	[50.35, 52.46]	[50.85, 51.90]	[51.35, 51.35]
7	[53.27, 55.82]	[53.21, 55.25]	[53.15, 54.67]	[53.08, 54.10]	[53.02, 53.52]	[52.95, 52.95]
8	[48.01, 52.28]	[48.22, 51.64]	[48.43, 50.99]	[48.63, 50.34]	[48.84, 49.70]	[49.05, 49.05]
9	[49.21, 53.13]	[49.62, 52.76]	[50.04, 52.39]	[50.45, 52.02]	[50.86, 51.65]	[51.28, 51.28]
10	[47.66, 54.28]	[48.25, 53.55]	[48.84, 52.81]	[49.44, 52.08]	[50.03, 51.35]	[50.62, 50.62]

TABLE 11: Different characteristics of the proposed data network for $\alpha = 0.8$.

Router (j)	Traffic intensity	Expected number of packets in the queue	Expected waiting time	Expected number of packets in the system	Mean processing time	Expected number of packets in the router that from time to time
1	[0.84, 0.89]	[4.30, 7.00]	[0.08, 0.14]	[5.14, 7.89]	[0.10, 0.16]	[6.14, 8.89]
2	[0.85, 0.88]	[4.89, 6.28]	[0.09, 0.12]	[5.74, 7.15]	[0.11, 0.13]	[6.74, 8.15]
3	[0.85, 0.88]	[4.69, 6.73]	[0.09, 0.13]	[5.54, 7.61]	[0.10, 0.14]	[6.54, 8.61]
4	[0.84, 0.89]	[4.32, 7.03]	[0.08, 0.14]	[5.16, 7.92]	[0.10, 0.15]	[6.16, 8.92]
5	[0.84, 0.88]	[4.54, 6.57]	[0.09, 0.13]	[5.39, 7.45]	[0.10, 0.15]	[6.39, 8.45]
6	[0.85, 0.88]	[4.75, 6.80]	[0.08, 0.12]	[5.60, 7.69]	[0.10, 0.14]	[6.60, 8.69]
7	[0.86, 0.88]	[5.33, 6.76]	[0.09, 0.11]	[6.20, 7.64]	[0.10, 0.13]	[7.20, 8.64]
8	[0.85, 0.88]	[4.71, 6.42]	[0.09, 0.12]	[5.56, 7.30]	[0.10, 0.14]	[6.56, 8.30]
9	[0.86, 0.88]	[5.06, 6.46]	[0.09, 0.12]	[5.91, 7.34]	[0.10, 0.13]	[6.91, 8.34]
10	[0/84, 0.88]	[4.43, 6.79]	[0.08, 0.13]	[5.27, 7.67]	[0.10, 0.15]	[6.27, 8.67]

From Case 1, it is observed that $\tilde{R}$ amount of data packets are equally distributed over n parallel queues. If the processing rate of each server is equal and there is no traffic rate, then $\tilde{R}$ amount of data packets remain stored which have been observed from Case 2. If there is no processing rate, then $\tilde{R}$ amount of data packets are equally distributed over n parallel queue and they are waiting for processing which has been depicted in Case 3. From Case 4, it has also been observed that $\tilde{R}$ amount of data packets are always stored without processing. For the numerical purpose, we have considered the following example.

Let us assume there is a data network with 10 parallel routers and (480, 500, and 530) kbps data has been sent over the 10 routers. Additional traffic ($\tilde{r}_j$) and processing rate ($\tilde{C}_j$) of each server are given in Table 1.

Table 2 provides the additional traffic and processing rate of each server in terms of α-level interval. Additional traffic and processing rate of each server in terms of parametric representation have been provided in Table 3.

Using (16), we have obtained data stream rate for different values of α and p (see Table 4–9). For illustration purpose, graphical representation of the change of data

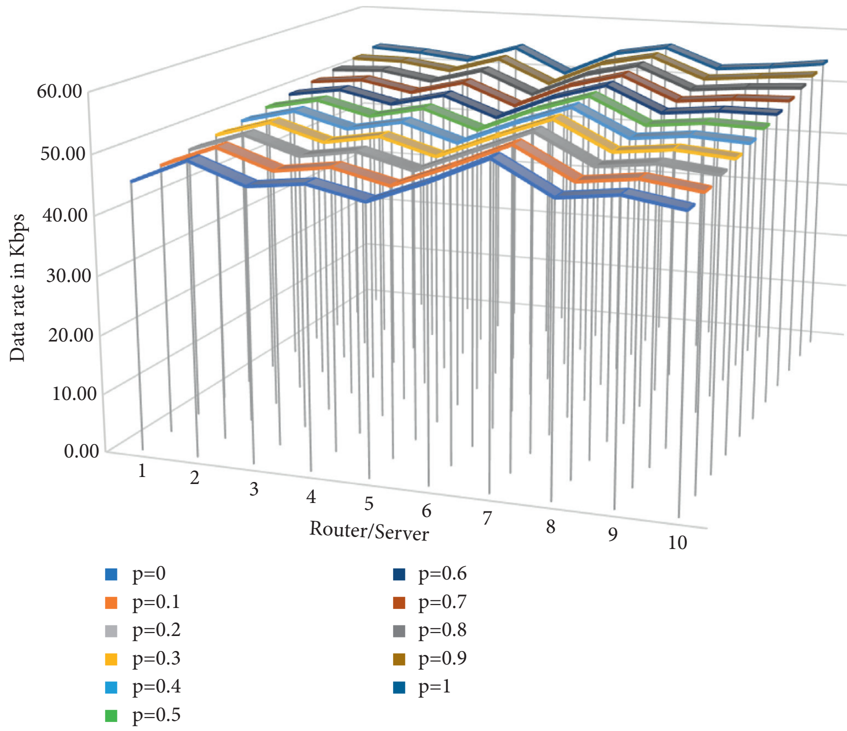

Figure 4: Graphical representation of data stream rate when $\alpha = 0$.

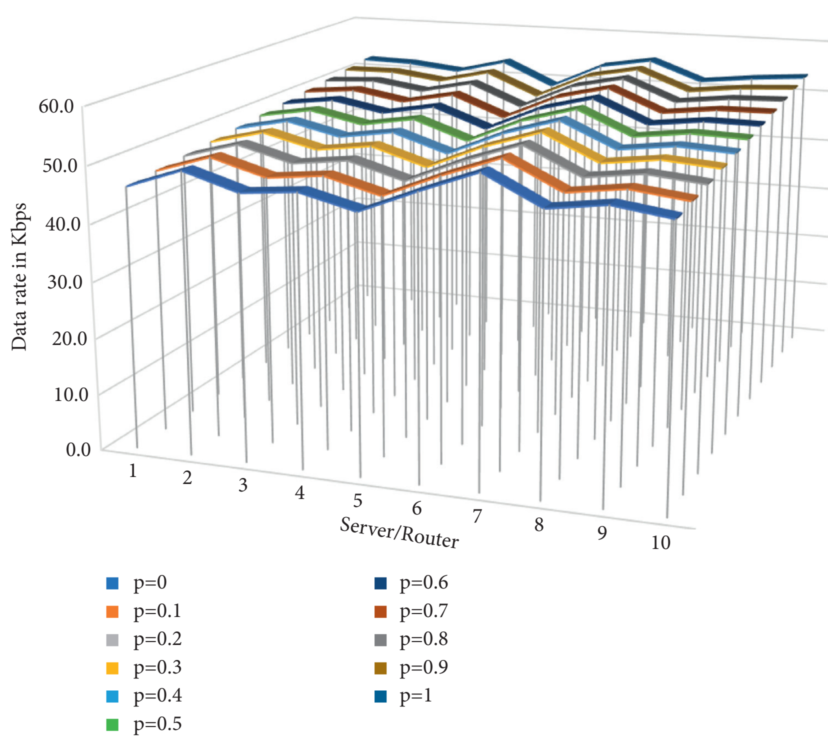

Figure 5: Graphical representation of data stream rate when $\alpha = 0.2$.

stream $(\tilde{x}_1)$ for server/router 1 when $\alpha = 0.6$ and $\alpha = 1$ have been depicted in Figures 2 and 3. From Figure 2, it has been observed that the data stream rate varies from 47.89 to 50.91. Hence, we may conclude that $\tilde{x}_1 \in [47.89, 50.91]$. Again, from Figure 2, it is seen that the data stream rate varies from 49.62 to 49.62. So, we may claim that, for $\alpha = 1$, the entire setup is precise valued but in terms of interval uncertainty we write $\tilde{x}_1 \in [49.62, 49.62]$. Employing these two

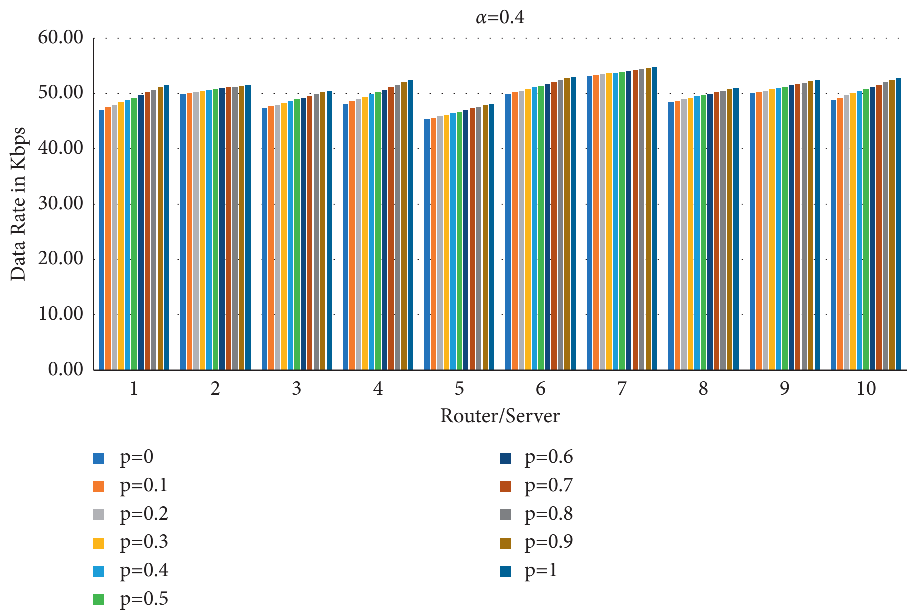

Figure 6: Graphical representation of data stream rate when $\alpha = 0.4$.

arguments, we have obtained $\tilde{x}_j$, $j = 1, 2, \ldots, 10$, in terms of interval uncertainty for different values of p and α. Table 10 provides the data stream rate in terms of interval for different values of α. Different numerical characteristics of the proposed data network in terms of interval uncertainty have been presented in Table 11 when $\alpha = 0.8$. Graphical representation of data stream for $\alpha = 0$, $\alpha = 0.2$, and $\alpha = 0.4$ has been depicted in Figures 4, 5 and 6 respectively.

6. Concluding remarks

In this paper, first time a data distributed network optimal design problem under uncertainty that sends data over a sequential access point has been solved. This data network design optimization problem is a Jackson open-type network problem based on the M/M/1 queueing system. Because of the uncertainty, the performance parameters involved with the same have also been considered imprecise valued. This impreciseness can be represented in several ways. This imprecision is represented here by a triangular fuzzy number (TFN), and the fuzzy number has been converted into a crisp number using interval parametric technique. The problem was then converted into a crisp nonlinear programming problem and solved by well-known Kuhn–Tucker (K-T) optimality techniques. As the network design optimization problem is nonlinear and convex, Kuhn–Tucker (K-T) optimality techniques provide the best optimal solutions. Using the interval parametric technique, we have obtained an optimal data stream in terms of interval uncertainty. Again, using Little's formula and interval arithmetic, we have estimated the data stream rate, expected number of packets, expected waiting time, mean processing time, and expected number of packets in the router in terms of interval uncertainty that is used from time to time in the distributed data network system. For further findings, numerous different optimization techniques, like meta-heuristic algorithms and evolutionary computation, could be used to solve the problem presented in this paper.

6.1. Future Scope of Research. The solution methodology described here is simple and easy to implement. From a computational perspective, we can surmise that the entire procedure used here will be helpful to address other network design optimization problems with uncertain parameters in the near future, such as analysis and design of service processes, manufacturing assembly lines, wireless communication networks, multitasking computers, transportation/traffic maintenance systems, and routing in wireless sensor networks.

Data Availability

This study has no data associated with it.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. K. Erlang, "The theory of probabilities and telephone conversations," *Nyt Tidsskrift Math, B*, vol. 20, pp. 33–39, 1909.

[2] J. R. Jackson, "Networks of waiting lines," *Operations Research*, vol. 5, no. 4, pp. 518–521, 1957.

[3] J. R. Jackson, "Jobshop-like queueing systems," *Management Science*, vol. 10, no. 1, pp. 131–142, 1963.

[4] L. Kosten, "On the influence of repeated calls in the theory of probabilities of blocking," *Ingenieur*, vol. 59, pp. 1–25, 1947.

[5] J. W. Cohen, "Basic problems of telephone traffic theory and the influence of repeated calls," *Philips Telecommunication Review*, vol. 18, pp. 49–100, 1957.

[6] M. W. Edwin, *Optimization of Simulation of Overflows in Markovian Queueing Models*, University of Twente, Netherlands, 2006.

[7] S. Banerjee, R. Johari, and C. Riquelme, "Pricing in ride-sharing platforms: a queueing-theoretic approach. Proceedings of the sixteenth acm conference on economics and computation," *ACM*, vol. 639–639, 2015.

[8] S. Banerjee, D. Freund, and T. Lykouris, "Pricing and optimization in shared vehicle systems: an approximation framework. Proceedings of the 2017 ACM Conference on Economics and Computation," *ACM*, vol. 517–517, 2017.

[9] A. Waserhole and V. Jost, "Pricing in vehicle sharing systems: optimization in queuing networks with product forms," *EURO Journal on Transportation and Logistics*, vol. 5, no. 3, pp. 293–320, 2016.

[10] D. Adelman, "Price-directed control of a closed logistics queueing network," *Operations Research*, vol. 55, no. 6, pp. 1022–1038, 2007.

[11] Y. Masuda and S. Whang, "Dynamic pricing for network service: equilibrium and stability," *Management Science*, vol. 45, no. 6, pp. 857–869, 1999.

[12] V. G. Kulkarni, "On queueing systems by retrials," *Journal of Applied Probability*, vol. 20, no. 02, pp. 380–389, 1983.

[13] R. P. Rama and R. Motahar, "Analysis and Estimation of Overflow Probability in Jackson Queueing Networks," in *Proceedings of the IEEE conference C2SPCA*, Bangalore, India, October 2013.

[14] I. C. Paschalidis and J. N. Tsitsiklis, "Congestion-dependent pricing of network services," *IEEE/ACM Transactions on Networking*, vol. 8, no. 2, pp. 171–184, 2000.

[15] J. Sommer, J. Berkhout, H. Daduna, and B. Heidergott, "Analysis of Jackson networks with infinite supply and unreliable nodes," *Queueing Systems*, vol. 87, no. 1-2, pp. 181–207, 2017.

[16] F. R. B. Cruz and T. Woensel, "Finite Queuing Modeling Optimization. A selected Review," *Mathematical Modeling and Optimization of Industrial Problems*, vol. 2014, Article ID 374962, 2014.

[17] D. Roy, "Semi-open queuing networks: a review of stochastic models, solution methods and new research areas," *International Journal of Production Research*, vol. 54, no. 6, pp. 1735–1752, 2015.

[18] J. Kim, A. Dudin, S. Dudin, and C. Kim, "Analysis of a semi-open queueing network with Markovian arrival process," *Performance Evaluation*, vol. 120, pp. 1–19, 2018.

[19] B. Melamed, "On Poisson traffic processes in discrete-state Markovian systems by applications to queueing theory," *Advances in Applied Probability*, vol. 11, no. 1, pp. 218–239, 1979.

[20] C. Kam, S. Kompella, G. D. Nguyen, and A. Ephremides, "Effect of message transmission path diversity on status age," *IEEE Transactions on Information Theory*, vol. 62, no. 3, pp. 1360–1374, 2015.

[21] S. K. Kaul and R. D. Yates, "Timely updates by multiple sources: the m/m/1 queue revisited," in *Proceedings of the IEEE 54th Annual Conference on Information Sciences and Systems (CISS)*, vol. 1–6, Princeton, NJ, USA, March 2020.

[22] C. Kam, J. P. Molnar, and S. Kompella, "Age of information for queues in tandem," in *Proceedings of the MILCOM 2018 - 2018 IEEE Military Communications Conference (MILCOM)*, vol. 1–6, Los Angeles, CA, USA, October 2018.

[23] W. Whitt and W. You, "Heavy-traffic limits for stationary network flows," *Queueing Systems*, vol. 95, pp. 53–68, 2020.

[24] W. Whitt, "Time-varying queues," *Queueing Models and Service Management*, vol. 1, no. 2, pp. 79–164, 2018.

[25] L. Xia and B. Shihada, "Max-Min optimality of service rate control in closed queueing networks," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 1051–1056, 2013.

[26] J. Timmer and W. R. W. Scheinhardt, "Cost sharing of cooperating queues in a Jackson network," *Queueing Syst. Theory Appl.*vol. 75, no. 1–17, 2013.

[27] Y. Wang, D. Kim, S.-H. Kim, and H. Lee, "Designing highway access control system using multi-class M/G/C/C state dependent queueing model and cross-entropy method," in *Proceedings of the. Winter Simulation Conf. (WSC)*, pp. 1443–1454, Las Vegas, NV, USA, December 2017.

[28] M. G. R. Alam, T. M. Suma, S. M. Uddin et al., "Queueing theory based vehicular traffic management system through Jackson network model and optimization," *IEEE Access*, vol. 9, Article ID 136018, 2021.

[29] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–352, 1965.

[30] R. E. Moore, R. B. Kearfott and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, PA, 2009.

[31] L. Sahoo, "An interval parametric technique for solving fuzzy matrix games," *Elixir Applied Mathematics*, vol. 93, Article ID 39392, 2016.

[32] S. Dempe and S. Franke, "Solution of Bilevel Optimization Problems Using the KKT Approach," *Optimization*, vol. 68, no. 8, 2019.