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Research Article

Pricing American Options by a Fourier Transform Multinomial Tree in a Conic Market

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Based on FFT, a high-order multinomial tree is constructed, and the method to obtain the price of American style options in the Lévy conic market is studied. Firstly, the nature of the Lévy process and the pricing principle of European-style options are introduced. Secondly, the method to construct a high-order multinomial tree based on Fourier transform is presented. It can be proved by theoretical derivation that the multinomial tree can converge to the Lévy process. Thirdly, we introduce the conic market theory based on the concave distortion function and give the discretization method of the concave distortion expectation. Then, the American option pricing method based on reverse iteration is given. Finally, the CGMY process is used to demonstrate how to price the American put option in the Lévy conic market. We can draw conclusions that the Fourier transform multinomial tree can avoid the difficulty of parameter estimation when using traditional moment matching methods to construct multinomial trees. Because the Lévy process has the analytic form characteristic function, this method is a promising method to calculate the prices of options in the Lévy conic market.

1. Introduction

In the traditional market described by the law of one price, buyers and sellers in the market sell goods at the same price at the same moment. However, the law of one price cannot describe a market with insufficient liquidity or a market using the market maker rule, such as Nasdaq or other OTC markets. The conic finance theory developed in recent years can solve this problem well. In the conic theory, the whole market is treated as a virtual counterparty for zero-transaction costs with non-negative cash flows. There are different prices for the same cash flow at the same time. So the conic market is also known as the two-price market [1]. Contrary to the law of one price that risks can be eliminated through hedging, the conic finance theory argues that the risk in the market will not be eliminated completely but only to a certain acceptable extent, which reflects the reality of the market better . The theory of conic finance is the latest development branch of financial economics and financial engineering. It is a hot research topic spot in theory and application in recent years.

A basic model based on a more realistic balance sheet mode was introduced in Refs. [2], [3], where both the assets and liabilities were found to be risky. Thus, bid and ask prices must be treated separately and prudently. According to this model, contingent capital notes are priced based on the conic theory. Using the acceptability and distorted expectations, the author introduced a capital gap-based trigger and summarized the pricing of seven kinds of capital and noncapital bonds. Dilip Madan and Wim Schoutens [3] solved the parameter correction problem of distortion risk measures by using the bid-ask transaction history data in the options market. Fasen and Svejda [4] gave a dynamic consistency version to ensure that acceptability decisions are consistent over time. One of the static risk measures is the so-called distortion measure. Based on the framework of notions of consistency, these static risk measures can be extended to the dynamic setting, and the acceptable or unacceptable risks of bank portfolio are studied based on the dynamic risk measure. Rodriguez [5] studied the theory of no-arbitrage pricing and dynamic cone finance in discrete-time markets where the underlying asset pays dividends and carries transaction costs and pointed out that the no-arbitrage condition in this market also implies the existence of a risk-neutral measure. Madan and Schoutens [6] illustrated a two-price market equilibrium theory that allows investors to trade structured products of their design. Competitive pressures in the market cause the market to reduce asking prices and raise bid prices. Bielecki et al. [7] used the theory of dynamic coherent acceptability indices to derive the theoretical framework of dynamic bid-ask prices for derivatives and amplified the dynamic gain loss ratio to calculate the bid and ask prices of some path-dependent options. Madan et al. [8] constructed a Markov chain to modeling the dynamically consistent sequences of the bid and ask prices. These processes are demonstrated by generating dynamically consistent bid and ask sequences for various structured products. Eberlein et al. [9] clarified that nonlinear discounted martingale is associated with no-arbitrage in a two-price economy because the linear discounted martingale is independent of no-arbitrage in the economy and satisfies the law of one price. In Madan and Wang [10], for the dual problem, the acceptability pricing is attributed to convex programming that can be solved by the CVXOPT. The forward-start option is used to describe the acceptability process which is defined by positive expectations at minima var. Obtaining the boundaries is more restrictive than ignoring the repricing constraints or the acceptability support set. Madan [11] constructed a model combining nonlinear discounting and nonlinear martingale, which illustrates the interaction between the severity of the change and its associated discount rate. Mathematical finance relies extensively on martingale, and in some cases, martingale needs to satisfy constraints. For example, they may need to evolve in compact concentrates; if the interest rate is positive, the discounted zero-coupon bond price is [0, 1] in the risk-neutral martingale. Vrins [12] presented conic martingale to solve this problem, and then the SDE of bounded martingale with separable diffusion coefficients is converted into an SDE with the drift process of the autonomous diffusion coefficient. The method is applied to the modeling of survival probabilities and potential applications include CDS options or CVA of the pricing. Eberlein [13] proposed a two-price market model, which is determined in the absence of market equilibrium so that the risk of loss is acceptable. Acceptability is defined by a series of test measures or scenarios. Therefore, the bid price is the lowest point of the valuation of the test, and the asking price is the highest price of such a valuation. The two prices are related to the nonlinear expectations operator, which discussed liquidity measurement and portfolio theory and other aspects. Junike [14] proposed a new definition of concave conditional performance, proving the duality of the conditional risk measure, and established a new dynamic performance measure. Madan and Schoutens [15] gave a comprehensive introduction to the conic theory which also known as two-price theory. Although the law of one price usually eliminates all risks, the concept of acceptable risk is vital to the two-price theory which considers that it is impossible to eliminate risk in the modern financial economy.

In the conic market, the bid-ask spread reflects the market liquidity, and the transaction direction determines the transaction price. Therefore, risk measurement and management and hedging strategies in the conic market are completely different from those in one price market. Corcuera et al. [16] introduced the concept of the implicit liquidity of a market. Implicid liquidity is the ability to describe the liquidity risk embedded in the prices of financial derivatives. Albrecher et al. [17] used the bid and ask model prices to fit their market quotes, and then obtain the implied liquidity of financial instruments which is called Lambda: The lower the lambda, the higher the mobility. Estimating parameters before the credit crisis and after the credit crisis showed that long-term options tend to reduce liquidity, and short-term options enhanced liquidity during troubled times. Masimba et al. [18] presented a method to estimate the bid-ask price based on the LIBOR option, which is used for the determination of the caps and floors premium. In the framework of conic finance, Leippold and Schaerer [19] developed a stochastic fluidity model that extends Madan's (2010) discrete-time constant mobility model. With this extension, it is possible to better describe the term structure such as skewness and kurtosis of bid-ask spreads that are typically observed in the options market. Masimba et al. [20] examined the quantification of the risk of incomplete market trading strategies based on the reasonable price of conic finance, which can be determined only by the probability distributions of cash flows. Madan et al. [21] conducted a study based on the conic finance theory to analyze assets demand, optimal debt level, and the value of return of the loss to the tax payer's option for a firm with a lognormal distribution and assets and liabilities, debt and equity costs, as well as the level of the securities ultimately reported in the balance sheet to analyze and report in detail the relationship between these entities and the risk characteristics of the firm. Madan [2] used all the underlying option surfaces to hedge complex positions on multiple underlies. The hedging goal is to minimize the asking price and the remaining risk after hedging is acceptable at a predetermined level. It indicates that such hedging requires the use of risk-neutral principles for potential risks. The comparison of neutral risk, risk-neutral method, and statistical method to estimate the joint risk of multiple underlying assets from multiple option surfaces shows that risk-neutral is superior to its statistical counterparty, and hedging can significantly reduce the asking price. Based on the conic finance theory, Madan [22] studied the evolution of credit assessment adjustment (CVA) from the counterparty's credit risk, including the influence of its default (DVA), the recognition of risk, and the possibility of joint default. Madan [23] introduced the definition of acceptable risk of loss when supply and demand are defined in different event spaces. The risk of acceptable loss is modeled by the convex cone containing all non-negative variables, and financial balance is usually defined as a two-price economy. Acceptable risk of loss has an impact on accounting and risk management since debt is usually valued by ask price and the asset is valued at the bid price. In two-price economy, the best of marking to market is marking two prices. Madan et al. [24] used dynamic concave bid prices and convex ask price functions to define a new hedging strategy called dynamic-conic hedging. The main points of these strategies are to maximize the expected position of the nonlinear condition. Guillaume and Schoutens [25] used the calibrated risk for bid-ask pricing and market-oriented cash flow valuation based on the conic theory. Calibration of the different asset pricing models for liquid trading derivatives leads to different risk-neutral measures by using a variety of reasonable calibration methods, which can be viewed as a test measure for evaluating the (not) acceptability of risk. Madan et al. [26] used nonlinear conditional expectations of nonadditive probabilities in the discrete-time Markovian environment to estimate trading strategies by exaggerating the upward caudal loss event and exaggerating the downward tail gain event to obtain conservatism and the steady-state fixed point of the value and strategy. Madan [27] constructed the best portfolio for full-size positions, long-short mix positions, and volatility-constrained portfolios, compared to the mean-variance portfolio, reflecting a lower degree of concentration in the conic portfolio. It has a comparable sample upside performance and higher downside results. Madan [27] examined portfolio selection issues within the framework of conic finance obtaining the level of risk from the market, selecting weights for each asset to maximize the bid price, comparing different distortion measures to achieve the expected cash flows, solving the problem under different constraints of size and weight, and using a variety of algorithms to calibrate the model. In the conic market, the economic equilibrium allows for different trading prices for nontraded positions. Marking the market turns to marking two prices. Madan [28] applied CoCoCoA to continuously contemporary conservative accounting and used these two mark-to-market systems for 77 companies from November 2005 to July 2015 and the average return of trading performance indicators. It showed that the two accounting systems have substantial differences in business rankings. Hellmers et al. [29] reduced the imbalance in order to improve the total profitability of the asset portfolio by increasing the implicit penalty. A comprehensive mathematical model was proposed to study business strategy in two-price economy. It was found that due to the two-price structure of the equilibrium market, the combined strategy is the most profitable. Li [30, 31] studied the European option pricing problem in the conic market. Van [32], Vazifedan [33], and Vega [34] investigated the application of risk measurement, no-arbitrage, and mechanism selection in the pricing of derivatives in conic financial markets and the impact of credit and default on asset values. Michielon [35] proposed a conic theory-based approach that is able to obtain risk-neutral implied volatilities by use of bid-ask quotes and does not require any restrictive assumptions. Madan [36] provided a systematic and in-depth introduction to the pricing of derivatives in conic financial markets.

In recent years, the Fourier transform has become one of the most frequently used tools in derivatives pricing. One reason is that using fast Fourier transform can reduce the amount of computation, which significantly saves the use of computing resources. The other reason is that the Lévy process is considered to be a better description of stochastic movement of risky asset prices. Mordecki [37], Sheu and Tsai [38], and Yamazaki [39] studied optimal stopping time and permanent American option pricing problems based on Fourier transforms. Zhylyevskyy [40] and Gyulov and Valkov [41] focused on the stochastic volatility and American options pricing on finite intervals based on Fourier transform. Pellegrino and Sabino [42], Chan [43], Ruijter and Oosterlee [44] used the multidimensional Fourier transform. Chan [43]studied the derivative pricing by the complex Fourier transform. Wong and Guan [45] obtained a Markov chain for option pricing by the use of FFT. Madan and Yor [46]came up with a type of time-varied Brownian representation theorem about the Lévy process. Asiimwe et al. [47] gave a mechanism to select risk pricing in the Lévy market. Kulczycki and Ryznar [48] studied the problem of estimating the probability transfer function of the Lévy stochastic process. Neufeld and Nutz [49] and Vrins [50] focused on the characteristic functions of Lévy processes. Jovan and Ahčan [51] researched on the Lévy model using structural methods to predict the default. Gong and Zhuang [52] and Lian et al. [53], respectively, studied the Lévy process under the American options and discrete barrier option pricing methods.

Most Lévy processes do not have analytic forms of probability density functions but definitely have analytic forms of characteristic functions. Hu et al. [54] presented an algorithm to create a multinomial tree based on saddle-point approximation for pricing options in the Lévy market. Based on FFT, authors proposed a new method to construct a high-order recombination multinomial tree, which can be applied to the American option pricing in Lévy model conic markets. The algorithm has low computational resource requirements and is very easy to program, which solves the difficulty of constructing a risk-neutral multinomial tree by traditional moment matching methods.

When the market is incomplete, the equivalence martingale measure of pricing is not unique, and thus the traditional pricing theory is also able to obtain the bid-ask spread of prices of the same financial derivatives. In a noncomplete market, the bid-ask price spread determined by the equivalence martingale measure is too broad compared with the actual bid-ask spread in the market. In contrast, the parameters of the distortion measure in conic market theory can be obtained by use of the market data, and the obtained bid-ask spread better reflects the actual market situation. The remainder of this article is organized as follows: Section 2 describes the Lévy process and European option pricing. In Section 3, we present an algorithm to create a multinomial tree based on FFT. Section 4 introduces the concave distortion expectation and pricing in the conic market and gives the discretization method of the concave distortion function. Section 5 gives an American option pricing method based on inverse iteration. Section 6 demonstrates the application of the algorithm to pricing

American put option through the CGMY process. In Section 7, we draw conclusions and provide an outlook.

2. The Lévy Process and European Option Pricing

Given a Lévy process X_t, based on the Lévy–Khintchine formula, the characteristic exponent of X_1 can be presented as follows:

$$\Psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{(-1,1)}\right)\nu(dx). \tag{1}$$

The characteristic function of X_t is

$$\begin{aligned}\Phi(u) &= E_{X_t}\left[e^{iux}\right]\\ &= e^{t\Psi(u)}.\end{aligned} \tag{2}$$

Denote the probability density function (PDF) by $f(x)$, and denote cumulative distribution function (CDF) by $F(x)$. The characteristic function is the Fourier transform of the PDF is given by

$$\Phi(u) = \int_{-\infty}^{+\infty} e^{iux} f(x)dx. \tag{3}$$

The PDF and the CDF can be obtained from the characteristic function as

$$\begin{aligned}f(x) &= \frac{1}{\pi}\int_0^{+\infty} \Re\left(e^{-ixu}\Phi(u)\right)du,\\ F(x) &= \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} \Im\left(\frac{e^{-ixu}\Phi(u)}{u}\right)du,\end{aligned} \tag{4}$$

$\Re(\cdot)$ is the real part of the complex number, and $\Im(.)$ refers to the imaginary component of the complex number. The mature period is T. When $t = 0$, the price of a European option with the payoff function $g(.)$ is

$$\begin{aligned}C &= e^{-rT}E^{Q}[g(x)]\\ &= e^{-rT}\int_{-\infty}^{+\infty} g(x)f(x)dx.\end{aligned} \tag{5}$$

But for most Lévy processes, there is no analytic form of the PDF, and you need the help of numerical calculation methods, but the efficiency of doing so is extremely low. Carr and Madan [55], Lewis [56], and Bates [57] used different techniques and all obtained analytical formulas for European option pricing in the Fourier transform space.

For exotic options with path dependence and American options that can be terminated before the mature time, there is no analytic form of pricing formula. In the use of numerical methods to calculate Fourier transform or inverse Fourier transform, FFT (IFFT) reduced to $N\log_2^N$ times to calculate the discrete Fourier transform or inverse Fourier transform which required N^2 times calculation, which greatly improves the computational efficiency. This means that in the pricing of derivatives if the analytic pricing formula of Fourier space can be obtained, it is also called the analytic pricing formula.

3. FFT-Based Multinomial Trees

Compared with the Markov chain, which holds fixed nodes during the whole computation process, the multinomial tree has fewer nodes at the time when price fluctuation is relatively small and has more nodes at a relatively large price fluctuation time and also has higher computational efficiency when calculating the price of derivatives. In the present literature, the moment matching method is applicated to making the moments of the multinomial tree as close as possible to the moments of continuous distribution. Determining the multinomial tree to meet the requirements of the parameters is very difficult, and we often need some extra constraints, such as Yamada and Primbs [58]; the formulation of parameter estimation about a quintuple tree. However, this requires that higher order moments of the continuous distribution must exist and satisfy certain relational constraints; otherwise, the existence of equivalent probability measures for the obtained multinomial trees cannot be guaranteed.

We subject the price of the risky asset to the exponential Lévy process as

$$S_t = S_0 e^{\left(rt - \mu t + X_t\right)}, \tag{6}$$

where S_0 is the stock price when $t = 0$, and $r \rangle 0$ is the risk-free interest rate, X_t is a Lévy process, μ is the constant that makes the expected return of risky asset to be the same as the risk-free interest rate. $T \rangle 0$ is the maturity of the derivatives, and we make $\delta = T/N$ by using the independent-stationary-increments property of the Lévy process, and we have

$$\log\left(S_{n\delta}\right) - \log\left(S_{(n-1)\delta}\right) = (r - \mu)\delta + X_\delta. \tag{7}$$

Let $S_n := S_{n\delta}, X_n := X_{n\delta}$. The multinomial tree with $L = 2^k$ is the state which is similar to the Lévy distribution in $[0, \delta]$. The truncation interval X_δ is $[-a, a]$. Let $\Delta = 2a/(L-1)$ and then, we have

$$x_{1,j} = -a + \Delta j, \quad j = 0, 1, 2, \ldots, L - 1. \tag{8}$$

The L state of the reorganized multinomial tree, in the first $n\,(n \leq N)$ total of $(L-1)n + 1$ nodes, and the j node has

$$x_{n,j} = -na + \Delta j, \quad j = 0, 1, 2, \ldots, (L-1)n. \tag{9}$$

In period n, the truncation interval X_n is $[-na, na]$ because

$$\begin{aligned}X_{n+1,j+k} &= -(n+1)a + (j+k)\Delta\\ &= -na + j\Delta - a + k\Delta\\ &= X_{n,j} + X_{1,k},\end{aligned} \tag{10}$$

where $k = 0, 1, 2, \ldots, L - 1$, so the node j in the period n has L subnodes of the period $(n+1)$: $X_{n+1,j}, \ldots, X_{n+1,j+L-1}$, and the price of the underlying asset at the node j in the period n is

$$\begin{aligned}S_{n,j} &= S_0 e^{\left((r-\mu)\delta n + X_{n,j}\right)}\\ &= S_0 e^{((r-\mu)\delta n - na + j\Delta)}, \quad j = 0, 1, 2, \ldots, (L-1)n,\end{aligned} \tag{11}$$

where $\Phi(u)$ is the characteristic function of the Lévy process in $[0, \delta]$, and the PDF is discretized as

$$f(x) = \frac{1}{\pi}\int_0^{+\infty} \Re\left(e^{-ixu}\Phi(u)\right)du \approx \frac{1}{\pi}\sum_{j=0}^{L-1} \Re\left(e^{-ixu_j}\Phi\left(u_j\right)\right)\eta. \tag{12}$$

The upper bound of the integral in (11) is $b = L\eta$ and $\Delta\eta = 2\pi/L$. Thus, the FFT calculation formula can be used. FFT returns L and the value x, and then we get

$$x_j = -a + \Delta j, \quad j = 0, 1, 2, \ldots, L-1. \tag{13}$$

When$x_j \in [-a, a]$, further, let $u_j = \eta j$, then the (12) can be expressed as

$$f\left(x_k\right) \approx \frac{1}{\pi}\sum_{j=0}^{L-1} \Re\left(e^{-i\Delta\eta k j}\Phi\left(u_j\right)\eta w_j\right), \tag{14}$$

where $k = 0, 1, 2, \ldots, L-1$, and w_j is based on different numerical integration methods to determine the weighting coefficients such as Trapezoidal rule weight, and its weight is $w_j = 1/2$. When $j = 0, L-1$; otherwise, $w_{j=1}$, and then the weighting coefficient of Simpson's rule is

$$w_j = \begin{cases} \frac{1}{3}, j = 0, L-1, \\ \frac{4}{3}, j = 1, 3, 5, \ldots, L-3, \\ \frac{2}{3}, j = 2, 4, 6, \ldots, L-2. \end{cases} \tag{15}$$

Based on FFT, we can obtain the approximate formula of the PDF in $[-a, a]$ an equidistant grid point.

In the L state of the multinomial tree, the probability of occurrence of the j state is

$$P_j = \frac{f\left(x_j\right)}{\sum_{k=0}^{L-1} f\left(x_k\right)}. \tag{16}$$

The denominator in (16) acts as a normalization to ensure that all states in the multinomial tree can form a complete probability space.

Theorem 1. *When $a \longrightarrow +\infty, L \longrightarrow +\infty$, then the distribution of $X_{1,j}$ converges to the continuous distribution X_δ.*

Proof 1. The PDF of X_δ is $f(x), x \in (-\infty, +\infty)$, $X_{1,j}$ is the L aliquot of $[-a, a]$, then the discrete random variable $X_{1,j}$ can be extended to the continuous distribution within $[-a, a]$, and the CDF after the extension is

$$\widetilde{F}(x) = \begin{cases} P_0, & x \leq -a, \\ \sum_{k=0}^{j} P_k, & -a + \Delta j < x \leq -a + \Delta(j+1), \\ 1, & x \rangle a. \end{cases} \tag{17}$$

(I) When $x \leq -a$, by $F(x)$ monotonically increasing, there is

$$\begin{aligned} |F(x) - \widetilde{F}(x)| &\leq F(x) + \widetilde{F}(x) \\ &\leq F(-a) + \widetilde{F}(-a) \\ &\leq F(-a) + f(-a). \end{aligned} \tag{18}$$

$$\because \lim_{-a \longrightarrow -\infty} F(-a) = \lim_{-a \longrightarrow -\infty} f(-a) = 0.$$
$$\lim_{\substack{-a \longrightarrow -\infty \\ x \leq -a}} |F(x) - \widetilde{F}(x)| = 0. \tag{19}$$

(II) When $x \leq -a$, then

$$|F(x) - \widetilde{F}(x)| = |F(x) - 1|. \tag{20}$$

$$\because \lim_{a \longrightarrow +\infty} F(a) = 1.$$
$$\lim_{\substack{a \longrightarrow +\infty \\ x \geq a}} |F(x) - \widetilde{F}(x)| = 0. \tag{21}$$

(III) When $-a + \Delta j \langle x \leq -a + \Delta(j+1)$, $j = 1, 2, \ldots, L-1$, then

$$\widetilde{F}(x) = \sum_{k=0}^{j} P_k = \frac{\sum_{k=0}^{j} f\left(x_k\right)}{\sum_{i=0}^{L-1} f\left(x_i\right)} = \frac{\sum_{k=0}^{j} f\left(x_k\right)\Delta}{\sum_{i=0}^{L-1} f\left(x_i\right)\Delta}. \tag{22}$$

$\because$ When a is fixed, $L \longrightarrow +\infty$, $\Delta = 2a/L - 1 \longrightarrow 0$, $x_j \longrightarrow x \therefore \lim_{L \longrightarrow +\infty} \widetilde{F}(x) = F(x) - F(-a)/F(a) - F(-a) \because$ 当. $a \longrightarrow +\infty, F(-a) \longrightarrow 0, F(a) \longrightarrow 1$. $\therefore \lim_{a \longrightarrow +\infty} \lim_{L \longrightarrow +\infty} \widetilde{F}(x) = F(x)$.

Combining (i), (ii), and (iii), $\forall x \in (-\infty, +\infty)$, and we conclude that

$$\lim_{\substack{L \longrightarrow +\infty \\ a \longrightarrow +\infty}} \widetilde{F}(x) = F(x). \tag{23}$$

□

4. Distort Expectations, Pricing, and Discretization

Cherny and Madan [59] defined the concave distortion function. The so-called concave distortion function $\Psi(\cdot)$ is in

[0, 1] itself a monotone concave function, and there are $\Psi(0) = 0$ and $\Psi(1) = 1$.

The commonly used CDF has CVaR, and the function form is as follows:

$$\Psi^{\lambda}(u) = \min\left(\frac{u}{\lambda}, 1\right). \quad (24)$$

Minmax var and its function form is as follows:

$$\Psi^{\gamma}(u) = 1 - \left(1 - u^{1/1+\gamma}\right)^{1+\gamma}. \quad (25)$$

Wang transform [60] and the function form is as follows:

$$\Psi^{\alpha}(u) = N\left(N^{-1}(u) + \alpha\right). \quad (26)$$

In Figure 1 shows CVaR parameters for the $\lambda = 0.05$, minmax var parameters for the $\gamma = 0.5$, and Wang transformation parameters for the $\alpha = 0.75$.

As can be seen in Figure 1, the concave distortion function amplifies the downside (corresponding loss) of the distribution, while the top of the distribution (upside, corresponding yields) is narrow. This reflects the real decision-making psychology of investors, which can be explained by the prospect theory of behavioral economics [61]. We use the bid and ask price history data of the new financial products to estimate the parameters according to the method given by Bannör and Scherer. Based on the concave distortion function, Madan and Schoutens defined the concave distortion measure, the market receivable cash flow X, the CDF $F(x)$, and the PDF $f(x)$. Using the concave distortion measure, the bid-ask prices of cash flow X are given as

$$\text{bid}(X) = e^{-rT} \int_{-\infty}^{+\infty} x \, d\psi(F(x)). \quad (27)$$

Then,

$$\text{ask}(X) = -e^{-rT} \int_{-\infty}^{+\infty} x \, d\Psi\left(F_{-X}(x)\right). \quad (28)$$

The risk-neutral price is

$$\text{neutral}(X) = e^{-rT} \int_{-\infty}^{+\infty} x \, dF(x). \quad (29)$$

From the nature of the distortion function, we get

$$\text{bid}(X) \le \text{neutral}(X) \le \text{ask}(X), \quad (30)$$

T denotes the mature time, and r denotes the risk-free rate.

To price American options with early execution characteristics, a multinomial, dynamic concave distortion measure must be used. Taking into account the temporal consistency of prices, the multiperiod dynamic uniform concave distortion measure given by Fasen and Svejda [4] is used to obtain the price of American options.

When the cash flow increases with the price of the risky asset, such as the payoff function of call options, the bid price is calculated using the following method of discretization of the concave distortion measure:

$$\begin{aligned} P_{L-l}^{\text{bid}} &= \Psi\left(\sum_{i=0}^{l} P_{L-l+i}\right) - \Psi\left(\sum_{i=1}^{l} P_{L-l+i}\right), \quad l = 1, 2, 3, \ldots, L-1, \\ P_L^{\text{bid}} &= \Psi\left(P_L\right). \end{aligned} \quad (31)$$

When calculating the asking price, the ask cash flow can be seen as the negative of negative cash flow expectations of the bid cash flow. So the cash flow increases with the underlying asset price rise. The asking price is calculated using the following concave distortion measure discretization:

$$\begin{aligned} P_l^{\text{ask}} &= \Psi\left(\sum_{i=1}^{l} P_i\right) - \Psi\left(\sum_{i=1}^{l-1} P_i\right), \quad l = 2, 3, \ldots, L, \\ P_1^{\text{ask}} &= \Psi\left(P_1\right). \end{aligned} \quad (32)$$

When the cash flow decreases with the price of the risky asset, such as the put option payoff function, to calculate the bid price, we use the concave distortion measure discretization method given by (31) and use the concave distortion measure discretization method given in (32) to calculate the asking price.

5. American Option Pricing Based on Reverse Iteration

Using the dynamic consistency method given by Fasen and Svejda [4] to price American options, reverse iteration based on dynamic programming can be used. Assume that the American option has a payoff function of $M(S)$, the probability of L nodes corresponding to each node in pricing is $P_1, P_2, \ldots, P_L$, at the period i node j, then the continued holding value of American options remains $C_{(i,j)}$. The American option has the value $H_{(i,j)}$ at the node. So the conic L state reorganization of multinomial tree pricing American options of the specific process is as follows:

Step 1: When $i = N$, then $H_{(N,j)} = M(S_{(N,j)})$, $C_{(N,j)} = 0$, $j = 1, 2, \ldots, (L-1)N + 1$.

Step 2: At the period i and node j, the value of holding the American option is

$$C_{(i,j)} = e^{-r\Delta T} \sum_{k=1}^{L} P_k H_{(i+1,j+k-1)}, \quad j = 1, 2, \ldots, (L-1)i + 1. \quad (33)$$

The value of American options is

$$H_{(i,j)} = \max\left(M\left(S_{(i,j)}\right), C_{(i,j)}\right), \quad (34)$$

if there is

$$M\left(S_{(i,j)}\right) \ge C_{(i,j)}. \quad (35)$$

(i) The American option is executed ahead of time at this node.

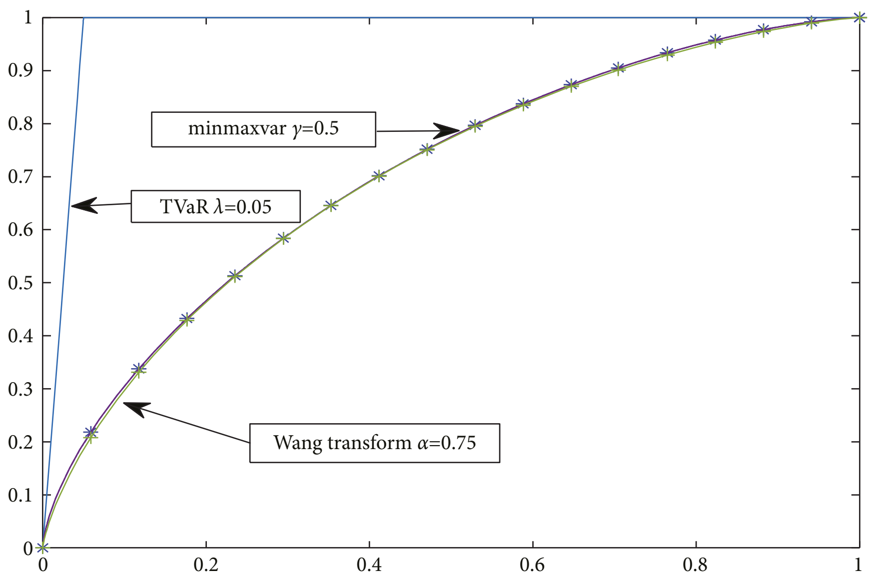

FIGURE 1: Comparisons of different distortion functions.

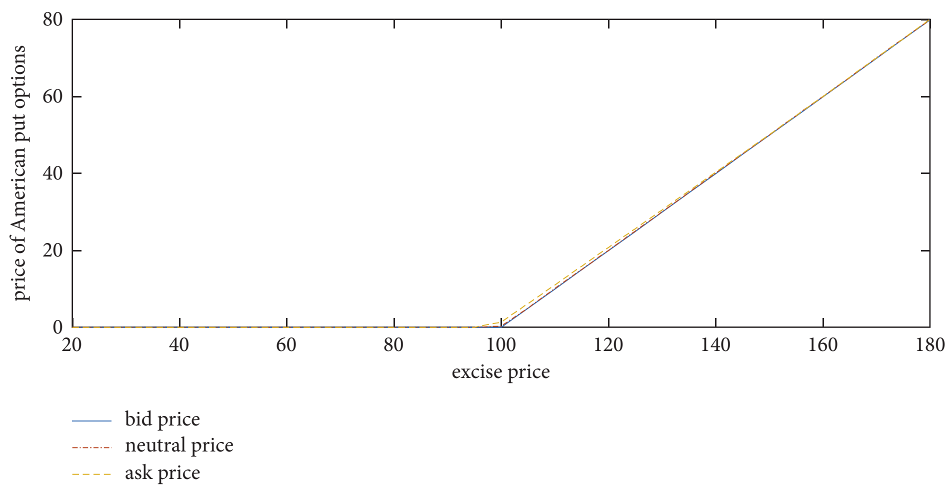

FIGURE 2: Prices of American put options with different excise prices.

Step 3: If $i = 0$, end; otherwise, $i := i - 1$, turn to step 2.

The American option value at the initial period is $C_{(0,0)}$.

When using conic multinomial tree pricing American options, the early exercise boundaries for American style options are not the same with the continuous-time and not the same with determining the early exercise boundary of the traditional market. In a discrete multinomial tree market, there may not be a node where the immediate execution value is exactly equal to the value of the American option. At the same node, the same American options, the direction of the transaction on the value of American options, and early implementation also have an impact and decisive role. In period i, the long-position owner of an American option decides whether to execute early or not. For American call options, the early execution boundary is determined as follows:

$$\begin{aligned} &\min_{j=1,\dots,(L-1)i} j \\ &\text{s.t.} \\ &M\left(S_{(i,j)}\right) \geq C_{(i,j)}. \end{aligned} \tag{36}$$

For American put options, the early execution boundary is determined as follows:

$$\begin{aligned} &\max_{j=1,\dots,(L-1)i} j \\ &s.t. \\ &M\left(S_{(i,j)}\right) \geq C_{(i,j)}. \end{aligned} \tag{37}$$

6. Illustration

The CGMY process proposed by Carr et al. [62] belong to a purely jumping Lévy stochastic process without continuous motion. Within an appropriate range of parameters, there can be infinite jumps in any given time interval. Infinitely small jumps and a few big jumps constitute the price movement of the risk assets, which is closer to the financial market reality. The characteristic function of CGMY is

$$\Phi(\mu) = e^{C\Gamma(-Y)\left((G+i\mu)^Y - G^Y + (M-i\mu)^Y - M^Y\right)}, \tag{38}$$

where $\Gamma(.)$ is the gamma function, and C, G, M, Y are the parameters of the CGMY distribution $Y\langle 2$, but when $1 \leq Y\langle 2$, there is no finite second-moment matrix. When $0 \leq Y\langle 1$, then CGMY has infinite jumps. German [63] used PJM's data to obtain the parameters of the CGMY: $C = 0.279627$, $G = 1.497869$, $M = 1.97856$, and $Y = 0.257689$. When $T = 0.5$, the year risk-free interest rate is $r = 5\%$, the beginning of the share price is $S_0 = 100$, and the exercise prices of American options, respectively, are: $K = 95, 100, 105$. The TVaR with parameter $\lambda = 0.95$ is selected as the concave distortion function, and the state $L = 9$, using the method described above to calculate the price of American options in Figure 2.

It can be seen from Figure 2, the bid price is lower than the neutral price, and the neutral price is always lower than the asking price. The imaginary value of the American put option quickly becomes zero because the underlying asset price of the multinomial tree is moving in a limited range. As can be seen from Figure 2, the bid-ask spread obtained by the conic method is close to the actual bid-ask spread in the market. In traditional risk-neutral priced financial derivatives, the bid-ask spreads obtained based on the risk-neutral set are too broad to be of practical value because the risk-neutral measure is not unique.

7. Conclusion

By using FFT to construct high-order restructuring multinomial trees, the pricing of American options in the Lévy model conic market is studied. It proved that the multinomial tree converges in distribution to the Lévy process. It introduces the conic market based on the concave distortions function and pricing and gives the discretization method of the concave distortions function. It gives the American option pricing method based on reverse iteration and demonstrates the American put option pricing in the Lévy-conic market by the CGMY process. It is found that, for the deep-out-of-money American option, the option price obtained by a multinomial tree quickly becomes zero due to the restriction of the price range of the risky asset. The bid price is always less than the risk-neutral price, while the risk-neutral price is always less than the asking price. So with the random concave distortions function parameter changing, bid-ask spreads can be arbitrarily small.

When constructing a high-order restructured multinomial tree using FFT, for the Lévy model in some specific parameter combination, due to the existence of the truncation error, the value of the PDF is sometimes less than zero when calculating the PDF from the characteristic function using IFFT. We must be careful to observe that once the probability density function has a value less than zero, we can consider using the threshold method to make the density function value less than zero to become zero. In the future, we will further study high-precision algorithms, or other alternative methods, and strive to avoid the value of PDF less than zero.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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