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Research Article

The Convergence of Attractors for Some Discrete Cahn-Hilliard Systems

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In this article, we use a finite difference scheme to discretize the Cahn-Hilliard equation with the space step size h. We first prove that this semidiscrete system inherits two important properties, called the conservation of mass and the decrease of the total energy, from the original equation. Then, we show that the semidiscrete system has an attractor on a subspace of $\mathbb{R}^{N+1}$. Finally, the convergence of attractors is established as the space step size h of the semidiscrete Cahn-Hilliard equation tends to 0.

1. Introduction

In this paper, we are concerned with some discrete forms of the following Cahn-Hilliard equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\frac{\delta G}{\delta u},\ x \in (0, L), t > 0, \tag{1}$$

$$\frac{\delta G}{\delta u} = -u + u^3 - \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

associated with boundary conditions:

$$\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_{x=0} &= \left.\frac{\partial u}{\partial x}\right|_{x=L} \\ &= 0, \\ \left.\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\right|_{x=0} &= \left.\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\right|_{x=L} \\ &= 0, \end{aligned} \tag{3}$$

where $u(x,t)$ denotes a distribution function of the concentration of one component of the binary mixture, the functional G (called the Ginzburg-Landau free energy) usually denotes a local free energy, and the notation $\delta G/\delta u$ given by (1) is consistent with the variational derivative of

$$G(u(x,t)) = -\frac{1}{2}u^2 + \frac{1}{4}u^4 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2. \tag{4}$$

The Cahn-Hilliard equation was first proposed by Cahn and Hilliard [1] and is usually used to describe the complex phase separation phenomenon which occurs when two miscible substances are rapidly cooled to critical temperatures at high temperatures; see [2–5] for details.

Nowadays, the Cahn-Hilliard equation has attracted much attention from researchers; see [3, 6–13], etc. Especially, Alikakos, Bates, and Fusco [6] established the existence of some extremely slowly evolving solutions of the Cahn-Hilliard equation. Later, Cholewa and Dlotko [7] proved the existence of global attractors for the Cahn-Hilliard equation. Based on this result, Li and Zhong [10] further established the existence of global attractors for the Cahn-Hilliard equation with fast growing nonlinearity. Recently, Furihata [3] proposed a stable and conservative finite difference scheme to solve numerically the Cahn-Hilliard equation. Specifically, they designed a new difference scheme which inherits some characteristic properties from the equation, including the conservation of mass and the decrease of the total energy.

Inspired by these works mentioned above, we use a finite difference scheme (with respect to $x \in (0, L)$) to discretize the Cahn-Hilliard equations (1) and (2) with the space step size h, which can generate $N_h + 1$ nodes. Thus, one can obtain the following equations:

$$\frac{dz_k}{dt} = \delta_{h,k}^{\langle 2\rangle}\left(\frac{\delta G_d}{\delta z_k}\right),\ k = 0, 1, \ldots, N_h, \tag{5}$$

where

$$\begin{aligned} \delta_{h,k}^{\langle 2\rangle} z_k &= \frac{1}{h^2}\left(z_{k-1} - 2z_k + z_{k+1}\right),\\ \left(\frac{\delta G_d}{\delta z_k}\right) &= -z_k + z_k^3 - \delta_{h,k}^{\langle 2\rangle} z_k, \end{aligned} \tag{6}$$

associated with the initial conditions

$$z_k(0) = z_{k,0},\ k = 0, 1, \ldots, N_h. \tag{7}$$

Moreover, the boundary conditions (3) can be discretized as

$$\begin{aligned} z_{-1} &= z_1,\\ z_{-2} &= z_2,\\ z_{N_h-1} &= z_{N_h+1},\\ z_{N_h-2} &= z_{N_h+2}. \end{aligned} \tag{8}$$

We are interested in the existence and the convergence of attractors for this semidiscrete Cahn-Hilliard equation. It is well-known that semidiscretization is a very effective technique in the finite element analysis of solid bodies and computational fluid mechanics; see [14, 15]. The semidiscretization of a partial differential equation means that the space is discretized and the time is continuous, which is widely studied by many researchers. For example, Castro and Micu [16] studied the controllability of a semidiscrete system with a boundary control at one extremity. Mai, Qin, and Zhang [17] studied the Turing instability of a two-dimensional semidiscrete Gierer–Meinhardt system and carried out a series of simulations. Concerning this topic of semidiscrete systems, the interested reader is referred to [18, 19, 20, 21, 22, 23] for some concrete examples.

In this paper, we discuss the attractor of the semidiscrete Cahn-Hilliard system (5)–(8). We first develop some ideas in [3, 7] to give some properties of solutions and prove the existence of attractors. Then, we establish the convergence of the attractor with respect to the space step size h.

This paper is organized as follows. In Section 2, we present two well-known properties of the system (5)–(8) called the conservation of mass and the decrease of the total energy. In Section 3, we prove the existence of attractors for the system (5)–(8) on some affined subspaces. In Section 4, we further establish the convergence of the attractor with respect to the space step size h; that is, the attractor of the system (5)–(8) tends to the attractor of the original system (1)–(3) as $h \longrightarrow 0$.

2. Two Well-Known Properties

In this section, we discuss two well-known properties of the semidiscrete Cahn-Hilliard system (5-8). For the purpose, we let $J_h = \{0, 1, 2, \ldots, N_h\}$ and $Z_h = (z_k(t))_{k\in J_h}$ and define the linear operators by

$$\begin{aligned} \left(\delta_h^{\langle 2\rangle} Z_h\right)_k &= \delta_{h,k}^{\langle 2\rangle} z_k\\ &= \frac{1}{h^2}\left(z_{k-1} - 2z_k + z_{k+1}\right),\\ \left(\delta_h^{\langle 4\rangle} Z_h\right)_k &= \delta_{h,k}^{\langle 2\rangle}\left(\delta_h^{\langle 2\rangle} Z_h\right)_k\\ &= \frac{1}{h^4}\left(z_{k-2} - 4z_{k-1} + 6z_k - 4z_{k+1} + z_{k+2}\right). \end{aligned} \tag{9}$$

Then, one can rewrite the system (5) as

$$\frac{dZ_h}{dt} = -\delta_h^{\langle 4\rangle} Z_h + \delta_h^{\langle 2\rangle}\left(-Z_h + Z_h^3\right),\ t > 0. \tag{10}$$

Without loss of generality, in the following, we first assume that the space step size $h = 1$, hoping that there is no confusion.

Let $J = \{0, 1, 2, \ldots, N\}$ and $U = (U_k(t))_{k\in J}$. Then, the system (5)–(8) can be simply written as the following system

$$\frac{dU}{dt} = -\delta^{\langle 4\rangle} U + \delta^{\langle 2\rangle}\left(-U + U^3\right),\ t > 0, \tag{11}$$

with the initial value

$$\begin{aligned} U(0) &= \left(U_k(0)\right)_{k\in J}\\ &= \left(z_{k,0}\right)_{k\in J}, \end{aligned} \tag{12}$$

and boundary conditions

$$\begin{aligned} U_{-1} &= U_1,\\ U_{-2} &= U_2,\\ U_{N-1} &= U_{N+1},\\ U_{N-2} &= U_{N+2}, \end{aligned} \tag{13}$$

where $\delta^{\langle 2\rangle}$ and $\delta^{\langle 4\rangle}$ are linear operators defined by

$$\begin{aligned} \left(\delta^{\langle 2\rangle} U\right)_k &= \delta_k^{\langle 2\rangle} U_k\\ &= U_{k-1} - 2U_k + U_{k+1}, \end{aligned} \tag{14}$$

$$\begin{aligned} \left(\delta^{\langle 4\rangle} U\right)_k &= \delta_k^{\langle 2\rangle}\left(\delta^{\langle 2\rangle} U\right)_k\\ &= U_{k-2} - 4U_{k-1} + 6U_k - 4U_{k+1} + U_{k+2}. \end{aligned} \tag{15}$$

In the following, we verify that the solution of the semidiscrete Cahn-Hilliard system (11)–(13) satisfies

$$\frac{d}{dt}\sum_{k=0}^{N}{}'' U_k(t) = 0, \qquad (16)$$

$$\frac{d}{dt}\sum_{k=0}^{N}{}'' G_d(U_k) \le 0, \qquad (17)$$

which corresponds to the following two properties:

$$\int_0^L u(x,t)\mathrm{d}x = \int_0^L u(x,0)\mathrm{d}x, \qquad (18)$$

$$\frac{d}{dt}\int_0^L G(u(x,t))\mathrm{d}x \le 0, \qquad (19)$$

(called the conservation of mass and the decrease of the total energy, respectively.) of the solutions to the Cahn-Hilliard system (1)–(3); see [3, 9] for details.

In (16) and (17), $\sum_{k=0}^{N}{}''$ is a summation operator (see [3]), given by

$$\sum_{k=0}^{N}{}'' U_k = \frac{1}{2}U_0 + \sum_{k=1}^{N-1} U_k + \frac{1}{2}U_N, \qquad (20)$$

$$G_d(U_k) = -\frac{1}{2}U_k^2 + \frac{1}{4}U_k^4 + \frac{1}{2}\,\frac{(\delta_k^+ U_k)^2 + (\delta_k^- U_k)^2}{2}. \qquad (21)$$

is the discrete local free energy, where

$$\begin{aligned}\delta_k^+ U_k &= U_{k+1} - U_k,\\ \delta_k^- U_k &= U_k - U_{k-1}.\end{aligned} \qquad (22)$$

Remark 1. These two properties (18) and (19) play an important role in studying the Cahn-Hilliard equation and are widely investigated by researchers; see e.g., [24, 25]. On the other hand, it is clear to see from (17) that $\int_0^L G(\mathrm{u(x, t)})\,\mathrm{d}x$ can be employed as a Lyapunov function of the Cahn-Hilliard system (see e.g., [26]). Similarly, one can regard $\sum_{k=0}^{N}{}'' G_d(U_k)$ as a Lyapunov function of (11) and (12).

Proposition 1. *The solution of the semidiscrete Cahn-Hilliard system (11)–(13) satisfies the properties (16) and (17).*

Proof. First, we show that the solution of (11)–(13) satisfies the conservation of mass (16). Indeed, by the definition of the summation operator and (5), we easily see that

$$\frac{d}{dt}\sum_{k=0}^{N}{}'' U_k(t) = \sum_{k=0}^{N}{}'' \frac{d}{dt}U_k(t) = \sum_{k=0}^{N}{}'' \delta_k^{\langle 2\rangle}\left(\frac{\delta G_d}{\delta U}\right)_k = 0, \qquad (23)$$

where $(\delta G_d/\delta U)_k = -U_k + U_k^3 - \delta_k^{\langle 2\rangle} U_k$.

Now, let us prove that the solution of (11)–(13) satisfies (17). By the definition of $G_d(U_k)$, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{N}{}'' G_d(U_k) &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{N}{}''\left(-\frac{1}{2}U_k^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{N}{}''\left(\frac{1}{4}U_k^4\right)\\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{N}{}''\frac{1}{4}\left((U_{k+1}-U_k)^2 + (U_k - U_{k-1})^2\right)\\
&= \sum_{k=0}^{N}{}''(-\mathrm{U_k})\frac{\mathrm{d}U_k}{\mathrm{d}t} + \sum_{k=0}^{N}{}''\left(U_k^3\right)\frac{\mathrm{d}U_k}{\mathrm{d}t}\\
&\quad - \sum_{k=0}^{N}{}''(U_{k-1} - 2U_k + U_{k+1})\frac{\mathrm{d}U_k}{\mathrm{d}t}\\
&= \sum_{k=0}^{N}{}''\left(\frac{\delta G_d}{\delta U}\right)_k\frac{\mathrm{d}U_k}{\mathrm{d}t}\\
&= \sum_{k=0}^{N}{}''\left(\frac{\delta G_d}{\delta U}\right)_k\left(\delta_k^{\langle 2\rangle}\left(\frac{\delta G_d}{\delta U}\right)_k\right).\end{aligned} \qquad (24)$$

Thanks to the general identity (summation by parts), we have

$$\begin{aligned}\sum_{k=0}^{N}{}''\frac{(\delta_k^+ U_k)^2 + (\delta_k^- U_k)^2}{2} &= \left(\left(s_k^{\langle 1\rangle}U_k\right)\delta_k^{\langle 1\rangle}U_k\right)\Big|_{k=0}^{N}\\
&\quad - \sum_{k=0}^{N}{}'' U_k\left(\delta_k^{\langle 2\rangle}U_k\right),\end{aligned} \qquad (25)$$

where

$$\begin{aligned}s_k^{\langle 1\rangle}U_k &= \frac{U_{k+1} + U_{k-1}}{2},\\ \delta_k^{\langle 1\rangle}U_k &= \frac{U_{k+1} - U_{k-1}}{2}.\end{aligned} \qquad (26)$$

Thus, it follows from (24) and (25) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{N}{}'' G_d(U_k) &= \sum_{k=0}^{N}{}''\left(\frac{\delta G_d}{\delta U}\right)_k\left(\delta_k^{\langle 2\rangle}\left(\frac{\delta G_d}{\delta U}\right)_k\right)\\
&= \frac{1}{4}\left(\left(\frac{\delta G_d}{\delta U}\right)_{k+1}^2 - \left(\frac{\delta G_d}{\delta U}\right)_{k-1}^2\right)\Bigg|_{k=0}^{N}\\
&\quad - \frac{1}{2}\sum_{k=0}^{N}{}''\left(\left(\delta_k^+\left(\frac{\delta G_d}{\delta U}\right)_k\right)^2 + \left(\delta_k^-\left(\frac{\delta G_d}{\delta U}\right)_k\right)^2\right)\\
&= -\frac{1}{2}\sum_{k=0}^{N}{}''\left(\left(\delta_k^+\left(\frac{\delta G_d}{\delta U}\right)_k\right)^2 + \left(\delta_k^-\left(\frac{\delta G_d}{\delta U}\right)_k\right)^2\right) \le 0,\end{aligned} \qquad (27)$$

which completes the proof of the proposition. □

3. Attractors of the Semidiscrete System

In this section, we aim to prove the existence of attractors for the semidiscrete Cahn-Hilliard system. We first give some properties of the solutions to (11)–(13). To this end, we use $\mathbb{R}^{N+1}$ to denote the $N+1$ Euclid space and equip it with the norm $\|\cdot\|$ defined by

$$\|U\| = \left(\sum_{k=0}^{N} |U_k|^2 \right)^{(1/2)}, U \in \mathbb{R}^{N+1}. \quad (28)$$

We denote by $C(s)$ a positive constant depending on s, whose value may change from line to line.

Note that the equation (11) is finite-dimensional. By the basic knowledge on ODEs, one can easily see that the Cauchy problem of (11)–(13) is well-posed in $\mathbb{R}^{N+1}$. Specifically, for each $U_0 \in \mathbb{R}^{N+1}$, the system (11)–(13) has a unique solution $U(t)$ on $[0, T)$ for some $T > 0$ with $U(0) = U_0$.

Let Φ denote the local semiflow generated by (11)–(13). Then, we have the following fundamental facts on the solutions.

3.1. The Boundedness of the Solution

Theorem 1. *Let $U(t)$ be a solution of (11)–(13) with the initial data $U(0)$. Then,*

$$\|U(t)\| \le C(\|U(0)\|),\ t > 0. \quad (29)$$

Proof. We define a norm $\|\cdot\|_d$ as follows:

$$\|U\|_d = \left| \sum_{k=0}^{N-1} (\delta_k^+ U_k)^2 + \sum_{k=0}^{N}{}'' U_k^2 \right|^{(1/2)}. \quad (30)$$

Then, by (20) and a simple calculation, we have

$$\begin{aligned} \|U\|_d^2 &= \left| \sum_{k=0}^{N}{}'' \frac{(\delta_k^+ U_k)^2 + (\delta_k^- U_k)^2}{2} + \sum_{k=0}^{N}{}'' U_k^2 \right| \\ &\le 2 \left| \sum_{k=0}^{N}{}'' \left(\frac{1}{2} \frac{(\delta_k^+ U_k)^2 + (\delta_k^- U_k)^2}{2} + U_k^2 \right) \right| \\ &\le 2 \left| \sum_{k=0}^{N}{}'' \left(\frac{1}{2} \frac{(\delta_k^+ U_k)^2 + (\delta_k^- U_k)^2}{2} - \frac{1}{2} U_k^2 + \frac{1}{4} U_k^4 + \frac{9}{4} \right) \right| \\ &\le 2 \left| \sum_{k=0}^{N}{}'' \left(\frac{1}{2} \frac{(\delta_k^+ U_k)^2 + (\delta_k^- U_k)^2}{2} - \frac{1}{2} U_k^2 + \frac{1}{4} U_k^4 \right) \right| + \frac{9}{2} N \\ &= 2 \left| \sum_{k=0}^{N}{}'' G_d(U_k) \right| + \frac{9}{2} N. \end{aligned} \quad (31)$$

Here, we used the inequality $x^2 \le -(1/2)x^2 + (1/4)x^4 + (9/4)$ (see [3]). Thereby, we deduce from (17) that

$$\|U\|_d^2 \le 2 \left| \sum_{k=0}^{N}{}'' G_d(U_k(0)) \right| + \frac{9}{2} N. \quad (32)$$

On the other hand, it is trivial to show that

$$\begin{aligned} \left| \sum_{k=0}^{N}{}'' G_d(U_k) \right| &= \left| -\frac{1}{2} \sum_{k=0}^{N}{}'' U_k^2 + \frac{1}{4} \sum_{k=0}^{N}{}'' U_k^4 + \frac{1}{2} \sum_{k=0}^{N-1} (\delta_k^+ U_k)^2 \right| \\ &\le \frac{1}{4} \sum_{k=0}^{N} |U_k|^2 + \frac{1}{4} \sum_{k=0}^{N} |U_k|^4 + 2 \sum_{k=0}^{N} |U_k|^2 \\ &\le C\left(\|U\|^2\right). \end{aligned} \quad (33)$$

Thus, by the equivalence of norms, (32) and (33), we can conclude that (29) holds true. □

3.2. Properties of the Stationary Solution. As noted in the Remark 1, one can see $\sum_{k=0}^{N}{}'' G_d(U_k)$ is a Lyapunov function. Next, we present some basic results on $\sum_{k=0}^{N}{}'' G_d(U_k)$.

Lemma 1. *If $U = (U_k)_{k \in J}$ is a solution of the system (11)–(13) satisfying*

$$\sum_{k=0}^{N}{}'' G_d(U_k) = \text{constant},\ t > 0, \quad (34)$$

then U is a stationary (time independent) solution.

Proof. By (27), we have

$$\begin{aligned} \frac{d}{dt} \sum_{k=0}^{N}{}'' G_d(U_k) &= \sum_{k=0}^{N}{}'' \left(\left(\delta_k^+ \left(\frac{\delta G_d}{\delta U} \right)_k \right)^2 + \left(\delta_k^- \left(\frac{\delta G_d}{\delta U} \right)_k \right)^2 \right) \\ &= 0, \end{aligned} \quad (35)$$

from which it can be seen that

$$\left(\frac{\delta G_d}{\delta U} \right)_{k-1} = \left(\frac{\delta G_d}{\delta U} \right)_k = \left(\frac{\delta G_d}{\delta U} \right)_{k+1},\ \forall k \in J. \quad (36)$$

Thereby,

$$\delta_k^{\langle 2 \rangle} \left(\frac{\delta G_d}{\delta U} \right)_k = \left(\frac{\delta G_d}{\delta U} \right)_{k-1} - 2 \left(\frac{\delta G_d}{\delta U} \right)_k + \left(\frac{\delta G_d}{\delta U} \right)_{k+1} = 0, \quad (37)$$

which together with (11) gives that

$$\frac{dU_k}{dt} = 0,\ \forall k \in J. \quad (38)$$

Hence, the result of Lemma 1 holds true. □

Lemma 2. *Let U be a solution of (11)–(13). Then, U is a stationary solution if and only if there is a constant γ such that*

$$\left(\frac{\delta G_d}{\delta U} \right)_k = \gamma,\ \forall k \in J, \quad (39)$$

where $(\delta G_d/\delta U)_k = -U_k + U_k^3 - \delta_k^{\langle 2\rangle} U_k$.

Proof. Let U be a solution of (11)–(13). Then, if U is a stationary solution, we see from (6) and (11) that

$$\delta_k^{\langle 2\rangle}\left(\frac{\delta G_d}{\delta U}\right)_k = 0,\ \forall k \in J. \quad (40)$$

By the boundary conditions (13), one has

$$\left(\frac{\delta G_d}{\delta U}\right)_0 = \left(\frac{\delta G_d}{\delta U}\right)_1 = \left(\frac{\delta G_d}{\delta U}\right)_2 = \cdots = \left(\frac{\delta G_d}{\delta U}\right)_N := \text{constant}. \quad (41)$$

Conversely, if there is a constant γ such that $(\delta G_d/\delta U)_k = \gamma$, then

$$\delta_k^{\langle 2\rangle}\left(\frac{\delta G_d}{\delta U}\right)_k = \frac{dU_k}{dt} = 0, \quad (42)$$

which completes the proof of what we desired.

Let γ be the constant given by Lemma 2. Assume U is a stationary solution of (11)–(13) such that (39) holds. Then,

$$\sum_{k=0}^{N}{}''\left(\frac{\delta G_d}{\delta U}\right)_k = \sum_{k=0}^{N}{}''\left(-U_k + U_k^3\right) + \sum_{k=0}^{N}{}''\left(-\delta_k^{\langle 2\rangle} U_k\right) = \sum_{k=0}^{N}{}''\left(-U_k + U_k^3\right) = N\gamma, \quad (43)$$

which shows that the γ can be expressed as

$$\gamma = \frac{1}{N}\sum_{k=0}^{N}{}''\left(-U_k + U_k^3\right). \quad (44)$$

Let $\overline{U}$ denote the spatial average of U, i.e.,

$$\overline{U}(t) = \frac{1}{N}\sum_{k=0}^{N}{}'' U_k(t). \quad (45)$$

Then, one can see from (44) that $\gamma = \overline{-U + U^3}$. Write

$$\mathscr{D} = \left\{U \in \mathbb{R}^{N+1}: \left(\frac{\delta G_d}{\delta U}\right)_k = \overline{-U + U^3}\right\} \text{ and } \mathscr{D}_\alpha = \left\{U \in \mathscr{D}: |\overline{U}| \le \alpha\right\}, \quad (46)$$

where $\alpha > 0$. Then, one can deduce that $\mathscr{D}$ is actually the set of all stationary solutions of the system (11)–(13). In the following, we show that $\mathscr{D}_\alpha$ is bounded for each $\alpha > 0$, which plays an important part in constructing the attractor. □

Theorem 2. *For each fixed* $\alpha > 0$, $\mathscr{D}_\alpha$ *is a bounded set of* $\mathbb{R}^{N+1}$.

Proof. Let $U \in \mathscr{D}_\alpha$. We infer from Lemma 2 that

$$\gamma\sum_{k=0}^{N}{}'' U_k = \sum_{k=0}^{N}{}'' U_k\left(\frac{\delta G_d}{\delta U}\right)_k = \sum_{k=0}^{N}{}''\left(-U_k^2 + U_k^4 - U_k\left(\delta_k^{\langle 2\rangle} U_k\right)\right), \quad (47)$$

where γ is given by (44).

By (14), one has

$$\begin{aligned}\sum_{k=0}^{N}{}''\left(-U_k\left(\delta_k^{\langle 2\rangle} U_k\right)\right) &= \sum_{k=0}^{N}{}''\,(U_{k+1} - U_k)^2 + \frac{1}{2}(U_1 - U_0)^2 - \frac{1}{2}(U_{N-1} - U_N)^2 \\ &\ge \sum_{k=0}^{N}{}''\,(U_{k+1} - U_k)^2 - \frac{1}{2}\sum_{k=0}^{N}{}''\,(U_{k+1} - U_k)^2 \\ &= \frac{1}{2}\sum_{k=0}^{N}{}''\,(U_{k+1} - U_k)^2.\end{aligned} \quad (48)$$

Thus, it follows from (47) and (48) that

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{N}{}''\,(U_{k+1} - U_k)^2 &\le \gamma\sum_{k=0}^{N}{}'' U_k - \sum_{k=0}^{N}{}''\left(-U_k^2 + U_k^4\right) \\ &= \overline{U}\sum_{k=0}^{N}{}''\left(-U_k + U_k^3\right) - \sum_{k=0}^{N}{}''\left(-U_k^2 + U_k^4\right).\end{aligned} \quad (49)$$

Here, we have used the definition of γ (see (44)).

Now, by Young inequality, one can deduce that for every $r > 0$, there exists $C(r)$ such that

$$\left|-U_k + U_k^3\right| \le r\left|U_k\right|^4 + C(r). \quad (50)$$

It is trivial to pick positive constants k_0 and k_1 satisfying

$$U_k^2 - U_k^4 \le -k_0\left|U_k\right|^4 + k_1. \quad (51)$$

Substituting (50) and (51) into (49), we obtain that

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{N}{}''\,(U_{k+1} - U_k)^2 &\le |\overline{U}|\sum_{k=0}^{N}{}''\left(r|U_k^4| + C(r)\right) + \sum_{k=0}^{N}{}''\left(-k_0|U_k|^4 + k_1\right) \\ &= (|\overline{U}|r - k_0)\sum_{k=0}^{N}{}''\,|U_k|^4 + N(|\overline{U}|C(r) + k_1).\end{aligned} \quad (52)$$

Take $r = k_0/(|\overline{U}| + 1)$ in the above estimate, then

$$\left|\sum_{k=0}^{N}{}''\,(U_{k+1} - U_k)^2\right| \le 2N(|\overline{U}|C(r) + k_1). \quad (53)$$

Now, we define a new norm $\|\cdot\|_H$ as follows:

$$\|U\|_H = \left(\left| \sum_{k=0}^{N} {}'' (U_{k+1} - U_k)^2 \right| + \left| \frac{1}{N} \sum_{k=0}^{N} {}'' U_k \right|^2 \right)^{(1/2)}. \quad (54)$$

Then, by (53), we find $\|U\|_H \le C(|\overline{U}|)$. Therefore, by the equivalence of the norms, one can immediately conclude that

$$\|U\| \le C(|\overline{U}|) \le C(\alpha), \quad (55)$$

which completes the proof of the theorem. □

3.3. The Existence of the Attractors. Inspired by much literature (see e.g., [7, 10]) on the global attractor for the Cahn-Hilliard dynamical system (1)–(3), we consider the attractor on the space $\mathscr{H}_\alpha = \{U \in \mathbb{R}^{N+1} : |\overline{U}| \le \alpha\}$, where $\alpha > 0$.

Lemma 3. *There exists a bounded subset $\mathscr{B}$ of $\mathscr{H}_\alpha$ attracting each point of $\mathscr{H}_\alpha$.*

Proof. Define the set $\mathscr{B}$ by

$$\mathscr{B} = \bigcup_{U \in \mathscr{H}_\alpha} \omega(U), \quad (56)$$

where $\omega(U)$ denotes the *ω-limit set* of U.

For each $U_0 \in \mathscr{B}$, there is $W_0 \in \mathscr{H}_\alpha$ such that $U_0 \in \omega(W_0)$. Let $\{T(t)\}_{t\ge 0}$ be the family of operators which form a continuous semigroup satisfying $V(t) = T(t)V_0$ on the space $\mathscr{H}_\alpha$, where $V(t)$ is the solution of (11) with the initial value V_0. By (17), we know that $\sum_{k=0}^{N} {}'' G_d((T(t)W_0)_k)$ is decreasing along the solution $T(t)W_0 = ((T(t)W_0)_k)_{k\in J}$ and bounded below. Thus, there exists $\beta \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} \sum_{k=0}^{N} {}'' G_d((T(t)W_0)_k) = \beta. \quad (57)$$

Then, one can deduce that

$$\sum_{k=0}^{N} {}'' G_d(U_k(t)) = \beta, \ \forall t \ge 0, \quad (58)$$

where $(U_k(t))_{k\in J} = T(t)U_0$. Hence, by (16) and Lemma 1, we see that U_0 is a stationary solution of the system (11)–(13) belonging to $\mathscr{D}_\alpha$. Consequently, one can conclude from Theorem 2 that $\mathscr{B}$ is bounded.

On the other hand, by Theorem 1, one finds that the semiflow Φ generated by the system (11)–(13) takes bounded sets into bounded sets. Moreover, by the definition of $\mathscr{H}_\alpha$, we see that $\{T(t)\}_{t\ge 0}$ is compact. Hence, $\mathscr{B}$ attracts each point of $\mathscr{H}_\alpha$.

According to [14], Theorem 4.2.4, we can obtain the existence of the attractor for the semifolw Φ on $\mathscr{H}_\alpha$. □

Theorem 4. *The semiflow Φ generated by the system (11)–(13) possesses a connected global attractor on a metric space $\mathscr{H}_\alpha$.*

4. The Convergence of Attractors

Notice that we considered a special case where the space step size h equals 1 in the above arguments. By using the same arguments as Theorem 3.6, one can easily verify that the system (11)–(13) has an attractor $\mathscr{A}_h$ with respect to the general space step size h.

Thanks to [11, Theorem 3], we deduce that the solution $U(t)$ of (11)–(13) converges to the solution $u(x,t)$ of (1)–(3) as the space step size $h \longrightarrow 0$. Based on this fact, in what follows we further show that the attractor $\mathscr{A}_h$ in $\mathscr{H}_\alpha$ tends to the attractor $\mathscr{A}$ of the original system (1-3) as $h \longrightarrow 0$.

Theorem 3. *Let $\mathscr{A}$ be the attractor of the original system (1)–(3) given by [7]. Then,*

$$\operatorname{dist}_{\mathbb{R}^{N_h+1}}(\mathscr{A}, \mathscr{A}_h) \longrightarrow 0 \text{ as } h \longrightarrow 0^+, \quad (59)$$

where $\operatorname{dist}_{\mathbb{R}^{N_h+1}}(\cdot,\cdot)$ denotes the Hausdorff semidistance in $\mathbb{R}^{N_h+1}$.

Proof. We argue by contradiction. If the conclusion is not true, then there exists $\varepsilon_0 > 0$ and a sequence $x_n \in \mathscr{A}_{h_n}$ with $h_n \longrightarrow 0$ (as $n \longrightarrow 0$) such that

$$\operatorname{dist}_{\mathbb{R}^{N_{h_n}+1}}(\mathscr{A}, x_n) \ge \varepsilon_0. \quad (60)$$

On the other hand, we infer from the invariance of $\mathscr{A}_{h_n}$ that there exists a bounded complete solution γ_n of Φ in $\mathscr{A}_{h_n}$ such that $\gamma_n(0) = x_n$. By a standard argument, it can be easily shown that γ_n has a subsequence converging some bounded full trajectory γ_0 of Φ in $\mathscr{A}$ uniformly on any compact interval of $\mathbb{R}$. Hence,

$$\operatorname{dist}_{\mathbb{R}^{N_{h_n}+1}}(\gamma_0, \gamma_n) \longrightarrow 0 \text{ as } h_n \longrightarrow 0. \quad (61)$$

Especially, $\operatorname{dist}_{\mathbb{R}^{N_{h_n}+1}}(\gamma_0(0), x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, which contradicts (60) and completes the proof of the theorem. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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