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Research Article

Traveling Wave Solution of the Kaup–Boussinesq System with Beta Derivative Arising from Water Waves

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The main purpose of this paper is to construct the traveling wave solution of the Kaup–Boussinesq system with beta derivative arising from water waves. By using the complete discriminant system method of polynomial, the rational function solution, the trigonometric function solution, the exponential function solution, and the Jacobian function solution of the Kaup–Boussinesq system with beta derivative are obtained. In order to further explain the propagation of the Kaup–Boussinesq system with beta derivative in water waves, we draw its three-dimensional diagram, two-dimensional diagram, density plot, and contour plot by using Maple software.

1. Introduction

The Kaup–Boussinesq system was first proposed by Kaup after adding a first-order nonlinear term to the Boussinesq equation when he studied it in 1975. For more than 40 years, the study of predecessors on the Kaup–Boussinesq system has covered a wide range of fields and made many achievements. Especially, the study of traveling wave solutions of the Kaup–Boussinesq system have become a very important research field [1–6]. Many important methods are also used to construct traveling wave solutions [7–12] and optical soliton solutions [13–17] of the Kaup–Boussinesq system. However, as far as we can tell from the literature, the solutions obtained mainly focus on hyperbolic function solutions, trigonometric function solutions, and rational function solutions. On the contrary, practical problems from the fields of physics, communication, control, and engineering technology are usually simulated by fractional partial differential equations [18–23]. Therefore, the study of traveling wave solution of the fractional Kaup–Boussinesq system is a very important and challenging problem for scientists.

The Kaup–Boussinesq system with beta derivative can be described as follows [24, 25]:

$$\begin{cases} D_t^{\beta}\Omega - \varphi_{xxx} - 2(\Omega\varphi)_x = 0, \\ D_t^{\beta}\varphi - \Omega_x - \left(\varphi^2\right)_x = 0, \end{cases} \tag{1}$$

where $\Omega = \Omega(x,t)$ and $\varphi = \varphi(x,t)$ represent the height of the water surface above the horizontal bottom and the horizontal velocity field, respectively. D_t^{β} stands for the beta fractional derivative. In [24], Wang and his collaborators obtained the traveling wave of system (1) by using the auxiliary equation method. In [25], Kilic and Inc studied the time fractional Kaup–Boussinesq system in the sense of the modified Riemann–Liouville derivative by using first integral method. The abovementioned literature has not obtained the Jacobian function solution, so in this paper, we will use the complete discriminant system to study system (1).

Next, we give the definition of beta fractional derivative.

Definition 1 (see [26]). Let $f\colon [0,\infty \longrightarrow \mathbf{R}$. Then, the beta derivative of $f(t)$ of order β is defined as

$$T^{\beta}(f(t)) = \frac{d^{\beta}f(t)}{dt^{\beta}} = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon(t + (1/\Gamma(\beta)))^{1-\beta}\right) - f(t)}{\varepsilon}, \beta \in (0, 1]. \tag{2}$$

The layout of this article is as follows. In Section 2, we obtain the traveling wave solution of the Kaup–Boussinesq system with beta derivative arising from water waves. In Section 3, we give a summary.

2. Traveling Wave Solution of (1)

In order to obtain the traveling wave solution of (1), we first make the following traveling wave transformation:

$$\begin{aligned} &\Omega(x,t) = \vartheta(\zeta),\\ &\varphi(x,t) = \theta(\zeta),\\ &\zeta = x - \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}. \end{aligned} \tag{3}$$

Substituting (3) into (1), we obtain

$$\begin{cases} -\lambda\vartheta' - \theta''' - 2\vartheta\vartheta' = 0,\\ -\lambda\theta' - \vartheta' - 2\theta\vartheta' = 0. \end{cases} \tag{4}$$

Integrating both sides of the second equation of (4) with respect to ζ at the same time and making the integral constant zero, we get

$$\vartheta = -\lambda\theta - \theta^2. \tag{5}$$

Substituting (5) into the first equation of (4), then it can be rewritten as

$$\left(\lambda^2 + 6\lambda\theta + 6\theta^2\right)\theta' - \theta''' = 0. \tag{6}$$

Integrating equation (6) once with respect to ζ, we get

$$\theta'' - \lambda^2\theta - 3\lambda\theta^2 - 2\theta^3 + b_1 = 0, \tag{7}$$

where b_1 is the integral constant.

Multiplying both sides of equation (7) by θ' and integrating it once, then equation (7) can be obtained:

$$\frac{1}{2}(\theta')^2 - \frac{1}{2}\lambda^2\theta^2 - \lambda\theta^3 - \frac{1}{2}\theta^4 + b_1\theta = b_2, \tag{8}$$

where b_2 is the integral constant.

Then, equation (8) can be reduced as

$$(\theta')^2 = \theta^4 + 2\lambda\theta^3 + \lambda^2\theta^2 + a_1\theta + a_0, \tag{9}$$

where $a_1 = -2b_1, a_0 = 2b_2$.

For equation (9), we assume that

$$\begin{cases} \omega = \theta + \dfrac{\lambda}{2},\\ \zeta_1 = \zeta. \end{cases} \tag{10}$$

Substituting (10) into (9), we obtain

$$\omega_{\zeta_1}^2 = F(\omega) = \omega^4 + Q_2\omega^2 + Q_1\omega + Q_0, \tag{11}$$

where $Q_2 = \lambda^2, Q_1 = a_1, Q_0 = (\lambda^4/16) - (a_1\lambda/2) + a_0$.

According to the complete discriminant system of polynomials [27], the integral representation of (11) can be expressed as follows:

$$\pm(\zeta_1 - \zeta_0) = \int \frac{d\omega}{\sqrt{\omega^4 + Q_2\omega^2 + Q_1\omega + Q_0}}, \tag{12}$$

where ζ_0 is an integral constant.

Suppose that $F(\omega) = \omega^4 + Q_2\omega^2 + Q_1\omega + Q_0$, then its complete discrimination system is as follows:

$$\begin{cases} D_1 = 4,\\ D_2 = -Q_2,\\ D_3 = -2Q_2^3 + 8Q_2Q_0 - 9Q_1^2,\\ D_4 = -Q_2^3Q_1^2 + 4Q_2^4Q_0 + 36Q_2Q_1^2Q_0 - 32Q_2^2Q_0^2 - \dfrac{27}{4}Q_1^4 + 64Q_0^3,\\ E_2 = 9Q_2^2 - 32Q_2Q_0. \end{cases} \tag{13}$$

Next, according to the complete discriminant system method of polynomials, we will give all single wave solutions of the Kaup–Boussinesq system with beta derivative.

Case 1. When $D_2 = D_3 = D_4 = 0$, that is, $\omega_{\zeta_1}^2 = \omega^4$. The solutions of (1) can be presented as follows:

$$\begin{cases} \Omega_1(x,t) = -\dfrac{1}{\left[x - (\lambda/\beta)(t + (1/\Gamma(\beta)))^{\beta} - \zeta_0\right]^2} + \dfrac{\lambda^2}{4}, \\ \phi_1(x,t) = -\dfrac{1}{x - (\lambda/\beta)(t + (1/\Gamma(\beta)))^{\beta} - \zeta_0} - \dfrac{\lambda}{2}. \end{cases} \quad (14)$$

Case 2. When $D_2 < 0, D_3 = D_4 = 0$, that is, $\omega_{\zeta_1}^2 = (\omega^2 + u^2)^2$, where u is the positive real number. The solutions of (1) can be presented as follows:

$$\begin{cases} \Omega_2(x,t) = -\dfrac{Q_2}{2}\mathbf{tan}^2\left(\sqrt{\dfrac{Q_2}{2}}\left(x - \dfrac{\lambda}{\beta}\left(t + \dfrac{1}{\Gamma(\beta)}\right)^{\beta} - \zeta_0\right)\right) + \dfrac{\lambda^2}{4}, \\ \phi_2(x,t) = \sqrt{\dfrac{Q_2}{2}}\,\mathbf{tan}\left(\sqrt{\dfrac{Q_2}{2}}\left(x - \dfrac{\lambda}{\beta}\left(t + \dfrac{1}{\Gamma(\beta)}\right)^{\beta} - \zeta_0\right)\right) - \dfrac{\lambda}{2}. \end{cases} \quad (15)$$

By selecting appropriate parameters, we can easily draw three-dimensional, two-dimensional, density plot, and contour plot of the solutions $\phi_2(x,t)$ and $\Omega_2(x,t)$ of system (1) (see Figures 1 and 2).

Case 3. When $D_2 D_3 < 0, D_4 = 0$, $F(\omega)$ has a double root and two conjugate complex roots, that is,

$$\omega_{\zeta_1}^2 = (\omega - \alpha)^2\left[(\omega - \kappa)^2 + \gamma^2\right]. \quad (16)$$

The solutions of (1) can be presented as follows:

$$\begin{cases} \Omega_3(x,t) = -\dfrac{\left\{\left[\mathbf{e}^{\pm\sqrt{(\alpha-\kappa)^2+\gamma^2}\,\left(x-(\lambda/\beta)(t+(1/\Gamma(\beta)))^{\beta}-\zeta_0\right)} - \tau\right] + \sqrt{(\alpha-\kappa)^2+\gamma^2}\,(2-\tau)\right\}^2}{\left\{\left[\mathbf{e}^{\pm\sqrt{(\alpha-\kappa)^2+\gamma^2}\,\left(x-(\lambda/\beta)(t+(1/\Gamma(\beta)))^{\beta}-\zeta_0\right)} - \tau\right]^2 - 1\right\}^2} + \dfrac{\lambda^2}{4}, \\ \phi_3(x,t) = \dfrac{\left[\mathbf{e}^{\pm\sqrt{(\alpha-\kappa)^2+\gamma^2}\,\left(x-(\lambda/\beta)(t+(1/\Gamma(\beta)))^{\beta}-\zeta_0\right)} - \tau\right] + \sqrt{(\alpha-\kappa)^2+\gamma^2}\,(2-\tau)}{\left[\mathbf{e}^{\pm\sqrt{(\alpha-\kappa)^2+\gamma^2}\,\left(x-(\lambda/\beta)(t+(1/\Gamma(\beta)))^{\beta}-\zeta_0\right)} - \tau\right]^2 - 1} - \dfrac{\lambda}{2}, \end{cases} \quad (17)$$

where $\tau = (\alpha - 2\kappa/\sqrt{(\alpha-\kappa)^2 + \gamma^2})$; the above solutions are solitary wave solutions.

Case 4. When $D_2 > 0, D_3 > 0, D_4 = 0$, $F(\omega)$ has four real roots, that is,

$$\omega_{\zeta_1}^2 = (\omega - x_1)(\omega - x_2)(\omega - x_3)(\omega - x_4), \quad (18)$$

where x_1, x_2, x_3, x_4 are real numbers satisfying $x_1 > x_2 > x_3 > x_4$.

The solutions of (1) can be presented as follows:

$$\begin{cases} \Omega_{41}(x,t) = -\dfrac{\left\{x_2(x_1-x_4)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x - \lambda/\beta(t+1/\Gamma(\beta))^{\beta} - \zeta_0\right), m\right] - x_1(x_2-x_4)\right\}^2}{\left\{(x_1-x_4)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x - \lambda/\beta(t+1/\Gamma(\beta))^{\beta} - \zeta_0\right), m\right] - (x_2-x_4)\right\}^2} + \lambda^2/4, \\ \phi_{41}(x,t) = \dfrac{x_2(x_1-x_4)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x - \lambda/\beta(t+1/\Gamma(\beta))^{\beta} - \zeta_0\right), m\right] - x_1(x_2-x_4)}{(x_1-x_4)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x - \lambda/\beta(t+1/\Gamma(\beta))^{\beta} - \zeta_0\right), m\right] - (x_2-x_4)} - \dfrac{\lambda}{2}, \end{cases} \quad (19)$$

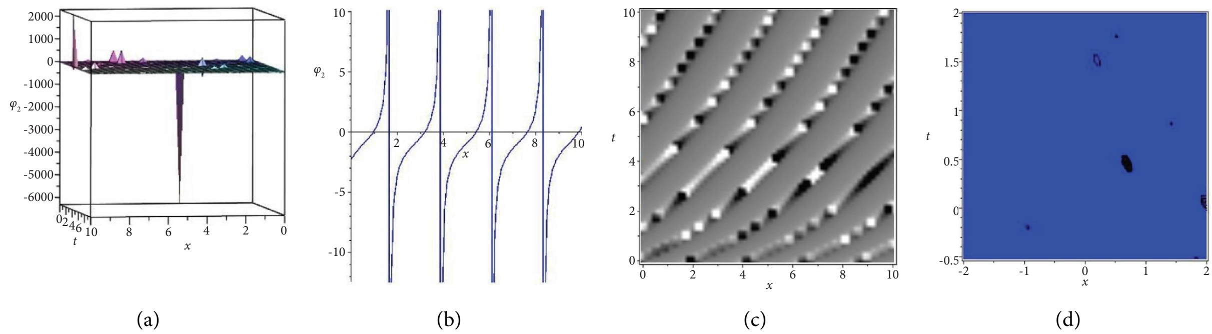

Figure 1: The graphics of $\phi_2(x,t)$ in equation (15) at $a_0 = 3, a_1 = 0, \lambda = 2, b_1 = 0, and\ b_2 = (3/2)$ (a) 3D surface (b) 2D surface (c) Density plot (d) Contour plot.

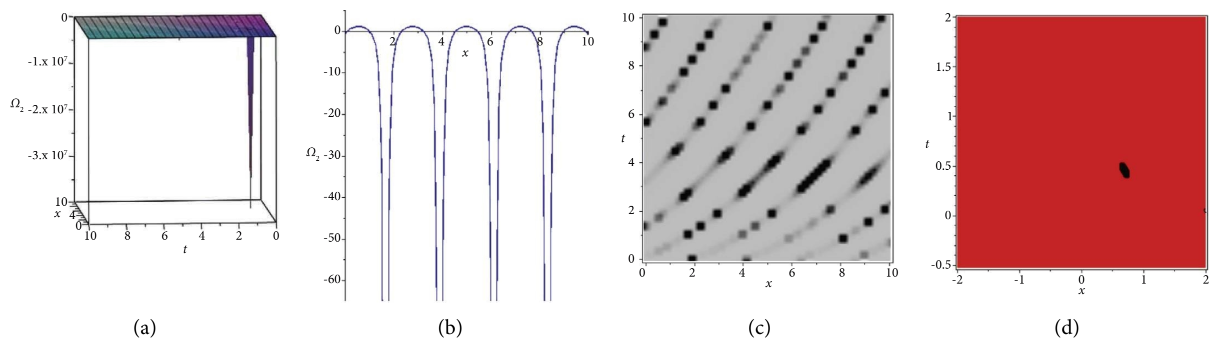

Figure 2: The graphics of $\Omega_2(x,t)$ in equation (15) at $a_0 = 3, a_1 = 0, \lambda = 2, b_1 = 0, and\ b_2 = (3/2)$ (a) 3D surface (b) 2D surface (c) Density plot (d) Contour plot.

$$\begin{cases} \Omega_{42}(x,t) = -\dfrac{\left\{x_4(x_2-x_3)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x-\lambda/\beta(t+1/\Gamma(\beta))^\beta-\zeta_0\right),m\right]-x_3(x_2-x_4)\right\}^2+\lambda^2/4,}{\left\{(x_2-x_3)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x-\lambda/\beta(t+1/\Gamma(\beta))^\beta-\zeta_0\right),m\right]-(x_2-x_4)\right\}^2}, \\ \phi_{42}(x,t) = \dfrac{x_4(x_2-x_3)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x-\lambda/\beta(t+1/\Gamma(\beta))^\beta-\zeta_0\right),m\right]-x_3(x_2-x_4)}{(x_2-x_3)\mathbf{sn}^2\left[\sqrt{(x_1-x_3)(x_2-x_4)}/2\left(x-\lambda/\beta(t+1/\Gamma(\beta))^\beta-\zeta_0\right),m\right]-(x_2-x_4)}-\dfrac{\lambda}{2}, \end{cases} \quad (20)$$

where $m^2 = ((x_1-x_4)(x_2-x_3)/(x_1-x_3)(x_2-x_4))$.

Case 5. When $D_2D_3 \geq 0, D_4 < 0, F(\omega)$ has two different real roots and two conjugate complex roots, that is,

$$\omega_{\zeta_1}^2 = (\omega-y_1)(\omega-y_2)\left[(\omega-y_3)^2+y_4^2\right], \quad (21)$$

where $y_1 > y_2$ and y_3, y_4 are real numbers.

Making the following transformation:

$$\omega = \frac{\mu_1 \mathbf{cos}\,\psi + \mu_2}{\mu_3 \mathbf{cos}\,\psi + \mu_4}, \quad (22)$$

where $\mu_1 = (1/2)(y_1+y_2)\mu_3 - (1/2)\ (y_1+y_2)\mu_4, \mu_2 = (1/2)\ (y_1+y_2)\mu_4 - (1/2)(y_1+y_2)\mu_3, \mu_3 = y_1 - y_3 - (y_4/\ m_1)$, $\mu_4 = y_1 - y_3 - y_4 m_1, E = y_4^2 + (y_1-y_3)(y_2-y_3)/\ y_4(y_1+y_2)$, and $m_1 = E \pm \sqrt{E^2+1}$.

The solutions of (1) can be presented as follows:

$$\begin{cases} \Omega_5(x,t) = -\dfrac{\left\{\mu_1 \mathbf{cn}\left[\sqrt{-2y_4m_1(y_1-y_2)}/2mm_1\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\mu_2\right\}^2}{\left\{\mu_3 \mathbf{cn}\left[\sqrt{-2y_4m_1(y_1-y_2)}/2mm_1\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\mu_4\right\}^2}+\dfrac{\lambda^2}{4}, \\ \phi_5(x,t) = \dfrac{\mu_1 \mathbf{cn}\left[\sqrt{-2y_4m_1(y_1-y_2)}/2mm_1\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\mu_2}{\mu_3 \mathbf{cn}\left[\sqrt{-2y_4m_1(y_1-y_2)}/2mm_1\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\mu_4}-\dfrac{\lambda}{2}, \end{cases} \tag{23}$$

where the solutions are the doubly-periodic solutions of elliptic function.

Case 6. When $D_2D_3 \leq 0, D_4 > 0$, $F(\omega)$ has two pairs of conjugate complex roots, that is,

$$\omega_{\zeta_1}^2 = \left[(\omega-z_1)^2+s_1^2\right]\left[(\omega-z_2)^2+s_2^2\right], \tag{24}$$

where $z_1, z_2, s_1, \textit{and}\ s_2$ are real numbers and $s_1 \geq s_2 > 0$.

Making the following transformation:

$$\omega = \frac{\sigma_1 \tan\psi + \sigma_2}{\sigma_3 \tan\psi + \sigma_4}, \tag{25}$$

where $\sigma_1 = z_1\sigma_3 + s_1\sigma_4, \sigma_2 = z_1\sigma_4 - s_1\sigma_3, \sigma_3 = -s_1 - (s_2/m_1), \sigma_4 = z_1 - z_2, E = ((z_1-z_2)^2 + s_1^2 + s_2^2/2s_1s_2), m_1 = E + \sqrt{E^2-1}$, and $m^2 = (m_1^2 - 1/m_1^2)$.

The solutions of (1) can be presented as follows:

$$\begin{cases} \Omega_6(x,t) = -\dfrac{\left\{\sigma_1 \mathbf{sn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\sigma_2 \mathbf{cn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]\right\}^2}{\left\{\sigma_3 \mathbf{sn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\sigma_4 \mathbf{cn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]\right\}^2}+\dfrac{\lambda^2}{4}, \\ \phi_6(x,t) = \dfrac{\sigma_1 \mathbf{sn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\sigma_2 \mathbf{cn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]}{\sigma_3 \mathbf{sn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]+\sigma_4 \mathbf{cn}\left[v\left(x-\lambda/\beta(t+1/\Gamma(\beta))^{\beta}-\zeta_0\right),m\right]}-\dfrac{\lambda}{2}, \end{cases} \tag{26}$$

where $v = (s_2\sqrt{(\sigma_3^2+\sigma_4^2)(m_1^2\sigma_3^2+\sigma_4^2)}/\sigma_3^2+\sigma_4^2)$.

3. Conclusion

In the paper, we have obtained the traveling wave solution of the Kaup–Boussinesq system with beta derivative arising from water waves by using the complete discriminant system method of polynomial. The rational function solution, the trigonometric function solution, the exponential function solution, and the Jacobian function solution are obtained. Compared with the existing literature [1–6, 24, 25], the paper not only considers the fractional version of the Kaup–Boussinesq system, but also obtains more abundant solutions. It is worth noting that this paper draws three-dimensional and two-dimensional diagrams to explain the propagation of the Kaup–Boussinesq system with beta derivatives in water waves. The Kaup–Boussinesq system with beta derivative arising from water waves is a very important water wave equation. In future research, our work will focus on the dynamic behavior and traveling wave solution of the Kaup–Boussinesq system.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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