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Research Article

Stability of a Nonlinear Fractional Langevin System with Nonsingular Exponential Kernel and Delay Control

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Fractional Langevin system has great advantages in describing the random motion of Brownian particles in complex viscous fluid. This manuscript deals with a delayed nonlinear fractional Langevin system with nonsingular exponential kernel. Based on the fixed point theory, some sufficient criteria for the existence and uniqueness of solution are established. We also prove that this system is UH- and UHR-stable attributed to the nonlinear analysis and inequality techniques. As applications, we provide some examples and simulations to illustrate the availability of main findings.

1. Introduction

To expound the random motion of particles in fluid after colliding with each other, Langevin raised the famous Langevin equation in 1908. Afterward, many random phenomena and processes were found to be described by Langevin system [1, 2]. However, the integer-order Langevin equation cannot meet the accuracy requirements in describing complex viscoelasticity. Thereby, the classical Langevin equation has been extended and modified. For example, Kubo [3, 4] put forward a general Langevin equation to simulate the complex viscoelastic anomalous diffusion process. Eab and Lim [5] applied a fractional Langevin equation to describe the single-file diffusion. Sandev and Tomovski [6] established a fractional Langevin equation model to study the motion of free particles driven by power-law noise. Furthermore, the stability of the system with practical application background is the most important dynamic characteristics. Ulam and Hyers [7, 8] proposed a concept of system stability called UH-stability in 1940s. In recent ten years, here have many works (some of them [9–30]) on UH-stability of fractional system.

As far as we know, the papers on fractional Langevin system published at present are all about Caputo and Riemann–Liouville fractional derivatives. However, as the authors [31–33] pointed out, the definitions of Caputo and Riemann–Liouville fractional derivatives have singular kernels. In 2015, Caputo and Fabrizio [34] defined a nonsingular fractional derivative with exponential kernel under a more general framework, which is also called the Caputo–Fabrizio (CF) fractional derivative. The properties and applications of this novel fractional derivative have attracted the attention of many scholars. Losada and Nieto [31] systematically studied the Laplace transform of CF-fractional derivative and its antiderivative, and applied it to study the falling body problem. For more research and application of CF-fractional differential equations, readers refer to references [35–42]. To the best of my knowledge, there are no papers dealing with Ulam–Hyers type stability of CF-fractional Langevin system. Motivated by aforementioned system, this manuscript focuses on the following nonlinear fractional Langevin system with nonsingular exponential kernel and delay control

$$\begin{cases} {}^{C}F\mathscr{D}_{0^+}^{\nu}\left[{}^{CF}\mathscr{D}_{0^+}^{\mu}-\lambda\right]u(t)=f(t,u(t),(\mathscr{C}u)(t)), & t\in(0,T],\\ (\mathscr{C}u)(t)=g(t,u(t-\xi(t))), & t\in(0,T],\\ u(t)=\varpi_1(t),{}^{CF}\mathscr{D}_{0^+}^{\mu}u(t)=\varpi_2(t), & t\in[-\eta,0], \end{cases} \tag{1}$$

where $T>0$, $0<\mu,\nu\leq 1$, and $\lambda>0$ are some constants, ${}^{CF}\mathscr{D}_{0^+}^{*}$ represents the $*$-order fractional derivative with nonsingular exponential kernel, the nonlinear response $f\in C([0,T]\times\mathbb{R}^2,\mathbb{R})$, the control function $g\in C([0,T]\times\mathbb{R},\mathbb{R})$, the delay function $\xi\in C([0,T],\mathbb{R}^+)$ with $\eta=\max\{\xi(t): t\in[0,T]\}$, and the initial functions $\varpi_1,\varpi_2\in C([-\eta,0],\mathbb{R})$.

Compared with previous papers on fractional Langevin equation, the influence of delay control is considered for the first time in our system (1). In fact, it is sometimes necessary and beneficial to manually control and intervene in the random motion of free particles. However, manual control is not instantaneous, but often lagging. Therefore, it is of great practical value to consider the control delay $\mathscr{C}(u)(t)$ in system (1). Meanwhile, for $0<\alpha<1$, the kernel functions of Caputo fractional derivative and CF-fractional derivative with α-order are written by $(t-s)^{-\alpha}$ and $e^{(-\alpha/1-\alpha)(t-s)}$, respectively. Obviously, $(t-s)^{-\alpha}\longrightarrow\infty$ (singular) and $e^{(-\alpha/1-\alpha)(t-s)}\longrightarrow 1$ (nonsingular), as $s\longrightarrow t$. Therefore, it is of great significance to explore the dynamic properties of system (1). The highlights of this paper mainly include two aspects: (a) In the fractional Langevin equation, we consider delay control and nonsingular exponential kernel function, which have not appeared in previous studies. (b) We obtain some new and easily verifiable sufficient criteria for the solvability and stability of system (1).

The structure of the remaining sections of the paper is as follows. Section 2 introduces some fundamental definitions and lemmas of CF-fractional calculus. In Section 3, we obtain some criteria for the existence of solutions to system (1) by utilizing some fixed point theorems. In Section 4, we shall prove that system (1) is UH- and UHR-stable. Section 5 provides some applications to illustrate the correctness of our major outcomes. A brief summary is made in Section 6.

2. Preliminaries

This section gives the concepts of Caputo–Fabrizio fractional derivative and integral as well as some useful results.

Definition 1. (see [31]). For $0\leq\mu\leq 1$, $T>0$ and $u\in H^1(0,T)$, the left-sided μ-order Caputo–Fabrizio fractional integral of function u is defined by

$${}^{CF}\mathscr{I}_0^{\mu}u(t)=\frac{1-\mu}{\mathscr{N}(\mu)}u(t)+\frac{\mu}{\mathscr{N}(\mu)}\int_0^t u(s)\mathrm{d}s, \tag{2}$$

where $\mathscr{N}(\mu)$ represents the normalisation constant with $\mathscr{N}(0)=\mathscr{N}(1)=1$.

Definition 2. (see [34]). For $0\leq\mu\leq 1$, $T>0$ and $u\in H^1(0,T)$, the left-sided μ-order Caputo–Fabrizio fractional derivative of function u is defined by

$${}^{CF}\mathscr{D}_{0^+}^{\mu}u(t)=\frac{\mathscr{N}(\mu)}{1-\mu}\int_0^t e^{(-\mu/1-\mu)(t-s)}u'(s)\mathrm{d}s. \tag{3}$$

Lemma 1 (see [31]). *Let $0\leq\gamma\leq 1$ and $h\in C[0,\infty)$. Consider the below initial value problem*

$$\left\{{}^{C}F\mathscr{D}_{0^+}^{\gamma}w(t)=h(t),\quad t\geq 0, w(0)=w_0.\right. \tag{4}$$

Then, the unique solution of this IVP is read as

$$w(t)=w_0+\frac{1-\gamma}{\mathscr{N}(\gamma)}[h(t)-h(0)]+\frac{\gamma}{\mathscr{N}(\gamma)}\int_0^t h(s)\mathrm{d}s. \tag{5}$$

Lemma 2. *Let $T,\lambda>0$, $0<\mu,\nu\leq 1$, $f\in C([0,T]\times\mathbb{R}^2,\mathbb{R})$, $g\in C([0,T]\times\mathbb{R},\mathbb{R})$, $\xi\in C([0,T],\mathbb{R}^+)$ with $\eta=\max\{\xi(t): t\in[0,T]\}$, $\varpi_1,\varpi_2\in C([-\eta,0],\mathbb{R})$. If $\Lambda\triangleq 1-\lambda(1-\mu)/\mathscr{N}(\mu)\neq 0$, then the CF-fractional Langevin equation (1) is equivalent to the following integral equation:*

$$u(t)=\begin{cases} \varpi_1(0)+\dfrac{1}{\Lambda}\left[\dfrac{\mu(\varpi_2(0)-\lambda\varpi_1(0))}{\mathscr{N}(\mu)}t+\dfrac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}[F_u(t)-F_u(0)]\right.\\ +\dfrac{\lambda\mu}{\mathscr{N}(\mu)}\displaystyle\int_0^t u(s)\mathrm{d}s+\dfrac{\mu+\nu-2\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t F_u(s)\mathrm{d}s\\ \left.+\dfrac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\displaystyle\int_0^t (t-s)F_u(s)\mathrm{d}s\right], \quad t\in[0,T]\\ \varpi_1(t),\ t\in[-\eta,0], \end{cases} \tag{6}$$

where $F_u(t) = f(t, u(t), (\mathscr{C}u)(t))$, $(\mathscr{C}u)(t) = g(t, u(t-\xi(t)))$.

Proof. Assume that $u(t) \in C([-\eta, T]$ satisfies system (1). Then, when $t \in [0, T]$, we derive from Lemma 1 that

$$\left[{}^{CF}\mathscr{D}_{0^+}^{\mu} - \lambda\right]u(t) = {}^{CF}\mathscr{D}_{0^+}^{\mu}u(0) - \lambda u(0) + \frac{1-\nu}{\mathscr{N}(\nu)}\left[f(t, u(t), (\mathscr{C}u)(t)) - f(0, u(0), (\mathscr{C}u)(0))\right] + \frac{\nu}{\mathscr{N}(\nu)}\int_0^t f(\tau, u(\tau), (\mathscr{C}u)(\tau))\mathrm{d}\tau. \tag{7}$$

Eqaution (7) gives

$${}^{CF}\mathscr{D}_{0^+}^{\mu}u(t) = \left({}^{CF}\mathscr{D}_{0^+}^{\mu}u(0) - \lambda u(0)\right) + \lambda u(t) + \frac{1-\nu}{\mathscr{N}(\nu)}\left[f(t, u(t), (\mathscr{C}u)(t)) - f(0, u(0), (\mathscr{C}u)(0))\right] + \frac{\nu}{\mathscr{N}(\nu)}\int_0^t f(\tau, u(\tau), (\mathscr{C}u)(\tau))\mathrm{d}\tau. \tag{8}$$

From Lemma 1, $u(0) = \varpi_1(0)$ and (8), we have

$$\begin{aligned}
u(t) &= \varpi_1(0) + \frac{1-\mu}{\mathscr{N}(\mu)}\left[\begin{array}{c}\lambda\left[u(t) - \varpi_1(0)\right] + \frac{1-\nu}{\mathscr{N}(\nu)}\left[f(t, u(t), (\mathscr{C}u)(t)) - f\left(0, \varpi_1(0), (\mathscr{C}u)(0)\right)\right] \\ + \frac{\nu}{\mathscr{N}(\nu)}\int_0^t f(\tau, u(\tau), (\mathscr{C}u)(\tau))\mathrm{d}\tau\end{array}\right] \\
&\quad + \frac{\mu}{\mathscr{N}(\mu)}\int_0^t\left[\begin{array}{c}\left(\varpi_2(0) - \lambda\varpi_1(0)\right) + \lambda u(s) + \frac{1-\nu}{\mathscr{N}(\nu)}f(s, u(s), (\mathscr{C}u)(s)) \\ + \frac{\nu}{\mathscr{N}(\nu)}\int_0^s f(\tau, u(\tau), (\mathscr{C}u)(\tau))\mathrm{d}\tau\end{array}\right]\mathrm{d}s \\
&= \left[1 - \frac{\lambda(1-\mu)}{\mathscr{N}(\mu)}\right]\varpi_1(0) + \frac{\lambda(1-\mu)}{\mathscr{N}(\mu)}u(t) + \frac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\left[f(t, u(t), (\mathscr{C}u)(t)) - f\left(0, \varpi_1(0), (\mathscr{C}u)(0)\right)\right) \\
&\quad + \frac{(1-\mu)\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t f(\tau, u(\tau), (\mathscr{C}u)(\tau))\mathrm{d}\tau \\
&\quad + \frac{\mu\left(\varpi_2(0) - \lambda\varpi_1(0)\right)}{\mathscr{N}(\mu)}t + \frac{\lambda\mu}{\mathscr{N}(\mu)}\int_0^t u(s)\mathrm{d}s \\
&\quad + \frac{\mu(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t f(s, u(s), (\mathscr{C}u)(s))\mathrm{d}s \\
&\quad + \frac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t\left[\int_0^s f(\tau, u(\tau), (\mathscr{C}u)(\tau))\mathrm{d}\tau\right]\mathrm{d}s.
\end{aligned} \tag{9}$$

Exchanging the order of double integrals, the last integral term of (9) is reduced to

$$\begin{aligned}\int_0^t\left[\int_0^s f(\tau,u(\tau),(\mathscr{C}u)(\tau))\mathrm{d}\tau\right]\mathrm{d}s &= \int_0^t f(\tau,u(\tau),(\mathscr{C}u)(\tau))\left[\int_\tau^t \mathrm{d}s\right]\mathrm{d}\tau\\ &= \int_0^t (t-\tau)f(\tau,u(\tau),(\mathscr{C}u)(\tau))\mathrm{d}\tau.\end{aligned}\tag{10}$$

It follows from (9) and (10) that

$$u(t)=\varpi_1(0)+\frac{1}{\Lambda}\left[\begin{array}{c}\dfrac{\mu(\varpi_2(0)-\lambda\varpi_1(0))}{\mathscr{N}(\mu)}t+\dfrac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\left[F_u(t)-F_u(0)\right]\\ +\dfrac{\lambda\mu}{\mathscr{N}(\mu)}\displaystyle\int_0^t u(s)\mathrm{d}s+\dfrac{\mu+\nu-2\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t F_u(s)\mathrm{d}s\\ +\dfrac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\displaystyle\int_0^t(t-s)F_u(s)\mathrm{d}s\end{array}\right].\tag{11}$$

When $t\in[-\eta,0]$, it is clear that $u(t)=\varpi_1(t)$ holds. Thus, system (6) holds, namely, $u(t)\in C([-\eta,T]$ also satisfies system (6). And vice versa, if $u(t)\in C[-\eta,T]$ satisfies integral system (6), then, when $t\in[0,T]$, we know that (8) and (7) hold by finding the fractional derivative ${}^{CF}\mathscr{D}_{0^+}^{\mu}$ at both sides of (6). Next, by finding the fractional derivative ${}^{CF}\mathscr{D}_{0^+}^{\nu}$ at both sides of (7), we easily derive the first fractional equation of (1). When $t\in[-\eta,0]$, let us make a supplementary definition ${}^{CF}\mathscr{D}_{0^+}^{\mu}u(t)=\varpi_2(t)$, then $u(t)=\varpi_1(t)$ and ${}^{CF}\mathscr{D}_{0^+}^{\mu}u(t)=\varpi_2(t)$ satisfy the equation (1). Thus, we verify that $u(t)\in C[-\eta,T]$ also satisfies system (1). The proof is completed.

3. Existence of Solutions

This section mainly studies the solvability of system (1) by using the below some important fixed point theorems.

Lemma 3 (see [43]). *Let $\mathbb{X}$ be a Banach space and $\phi\neq\mathbb{Y}\subset\mathbb{X}$ be closed convex. Assume that $\mathscr{P}$ and $\mathscr{Q}$ satisfy*

(i) $\mathscr{P}u+\mathscr{Q}v\in\mathbb{Y}$, $\forall u,v\in\mathbb{Y}$.

(ii) $\mathscr{P}$ is contract, and $\mathscr{Q}$ is compact and continuous.

Then, there exists at least an $u^*\in\mathbb{Y}$ satisfying $u^*=\mathscr{P}u^*+\mathscr{Q}u^*$.

Lemma 4 (see [44]). *Let $\mathbb{X}$ be a Banach space and $\phi\neq\mathbb{E}\subset\mathbb{X}$ be closed. If $\mathscr{F}:\mathbb{E}\longrightarrow\mathbb{E}$ is contract, then $\mathscr{F}$ admits a unique fixed point $u^*\in\mathbb{E}$.*

According to Lemma 2, we take $\mathbb{X}=C([-\eta,T],\mathbb{R})$. For all $u\in\mathbb{X}$, we define the norm $\|u\|=\sup\{|u(t)|: -\eta\le t\le T\}$, then $(\mathbb{X},\|\cdot\|)$ is a Banach space. We always argue the existence and stability of solution for system (1) on $(\mathbb{X},\|\cdot\|)$. Throughout the paper, the following fundamental assumptions are needed:

(H1) T,μ,ν, and λ are some constants satisfying $T,\lambda>0$, $0<\mu,\nu\le 1$, and $\Lambda\triangleq 1-\lambda(1-\mu)/\mathscr{N}(\mu)\neq 0$.

(H2) $f\in C([0,T]\times\mathbb{R}^2,\mathbb{R})$, $g\in C([0,T]\times\mathbb{R},\mathbb{R})$, $\xi\in C([0,T],\mathbb{R}^+)$ with $\eta=\max\{\xi(t): t\in[0,T]\}$, $\varpi_1,\varpi_2\in C([-\eta,0],\mathbb{R})$.

Theorem 1. *Assume that (H_1) and (H_2) are true, as well as the following conditions (H_3) and (H_4) also hold:*

(H3) For all $t\in[0,T]$, $u,v\in\mathbb{R}$, there have some continuous functions $\mathscr{M}_1(t)$, $\mathscr{M}_2(t)$, $\mathscr{M}_3(t)\ge 0$ such that

$$\begin{aligned}|f(t,u,v)|&\le\mathscr{M}_1(t)+\mathscr{M}_2(t)|v|,\\ |g(t,v)|&\le\mathscr{M}_3(t).\end{aligned}\tag{12}$$

(H4) $0<\vartheta\triangleq\lambda T/|\mathscr{N}(\mu)-\lambda(1-\mu)|<1$.

Then, system (1) admits at least a solution $u^*(t)\in\mathbb{X}$.

Proof. In the light of Lemma 2, for all $u\in\mathbb{X}$, the operators $\mathscr{P},\mathscr{Q}:\mathbb{X}\longrightarrow\mathbb{X}$ are defined by

$$(\mathscr{P}u)(t) = \begin{cases} \varpi_1(0) + \dfrac{1}{\Lambda}\left\{\dfrac{\mu(\varpi_2(0) - \lambda\varpi_1(0))}{\mathcal{N}(\mu)}t + \dfrac{\lambda\mu}{\mathcal{N}(\mu)}\displaystyle\int_0^t u(s)\mathrm{d}s\right\}, & t \in [0,T], \\ \varpi_1(t),\ t \in [-\eta, 0], \end{cases} \tag{13}$$

$$(\mathscr{Q}u)(t) = \begin{cases} \dfrac{1}{\Lambda}\left[\begin{array}{l} \dfrac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}[F_u(t) - F_u(0)] + \dfrac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\displaystyle\int_0^t F_u(s)\mathrm{d}s \\ + \dfrac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\displaystyle\int_0^t (t-s)F_u(s)\mathrm{d}s \end{array}\right], & t \in [0,T], \\ 0,\ t \in [-\eta, 0], \end{cases} \tag{14}$$

where $F_u(t)$ is defined as (6). It is easy to see from (13) and (14) that $\mathscr{P}u + \mathscr{Q}v \in \mathbb{X}, \forall u, v \in \mathbb{X}$, that is, the condition (i) in Lemma 3 holds. In addition, $\forall t \in [-\eta, T]$, $u, v \in \mathbb{X}$, when $t \in [0,T]$, one has

$$\begin{aligned} |(\mathscr{P}u)(t) - (\mathscr{P}v)(t)| &= \left|\frac{\lambda\mu}{\Lambda\mathcal{N}(\mu)}\int_0^t [u(s) - v(s)]\mathrm{d}s\right| \\ &\leq \frac{\mathcal{N}(\mu)}{|\mathcal{N}(\mu) - \lambda(1-\mu)|} \times \frac{\lambda\mu}{\mathcal{N}(\mu)}\int_0^t |u(s) - v(s)|\mathrm{d}s \\ &\leq \frac{\lambda\mu}{|\mathcal{N}(\mu) - \lambda(1-\mu)|}\int_0^t \mathrm{d}s \cdot \|u - v\| = \frac{\lambda t}{|\mathcal{N}(\mu) - \lambda(1-\mu)|}\|u - v\| \\ &\leq \frac{\lambda T}{|\mathcal{N}(\mu) - \lambda(1-\mu)|}\|u - v\| = \vartheta\|u - v\|. \end{aligned} \tag{15}$$

When $t \in [-\eta, 0]$, one derives from (13) that

$$|(\mathscr{P}u)(t) - (\mathscr{P}v)(t)| = |\varpi_1(t) - \varpi_1(t)| \equiv 0. \tag{16}$$

Equations (15) and (16) mean that

$$\|\mathscr{P}u - \mathscr{P}v\| \leq \frac{\lambda T}{|\mathcal{N}(\mu) - \lambda(1-\mu)|}\|u - v\| = \vartheta\|u - v\|. \tag{17}$$

In view of (H_4) and (17), one concludes that $\mathscr{P}: \mathbb{X} \longrightarrow \mathbb{X}$ is contract.

Now, we apply Arzelá–Ascoli theorem to prove that $\mathscr{Q}: \mathbb{X} \longrightarrow \mathbb{X}$ is completely continuous. Indeed, for all $t \in [-\eta, T]$, $u \in \mathbb{X}$, when $t \in [0,T]$, it follows from (14) and (H_3) that

$$\begin{aligned}|(\mathcal{Q}u)(t)| &\leq \frac{1}{|\Lambda|}\left[\begin{array}{c}\dfrac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left(|F_u(t)|+|F_u(0)|\right)+\dfrac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\displaystyle\int_0^t|F_u(s)|\mathrm{d}s\\ +\dfrac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\displaystyle\int_0^t(t-s)|F_u(s)|\mathrm{d}s\end{array}\right]\\ &\leq \frac{M}{|\Lambda|}\left[\frac{2(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}+\frac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t ds+\frac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t(t-s)\mathrm{d}s\right]\\ &\leq \frac{M}{|\Lambda|\mathcal{N}(\mu)\mathcal{N}(\nu)}\left[2(1-\mu)(1-\nu)+(\mu+\nu-2\mu\nu)T+\frac{\mu\nu T^2}{2}\right],\end{aligned} \tag{18}$$

where $M=\|\mathcal{M}_1\|_T+\|\mathcal{M}_2\|_T\cdot\|\mathcal{M}_3\|_T$ and $\|\mathcal{M}_i\|_T=\sup\{\mathcal{M}_i(t)\colon 0\leq t\leq T\}$ $(i=1,2,3)$. When $t\in[-\eta,0]$, (14) gives

$$|(\mathcal{Q}u)(t)|\equiv 0, \tag{19}$$

By (18) and (19), we know that $\mathcal{Q}\colon \mathbb{X}\longrightarrow\mathbb{X}$ is uniformly bounded.

In the meantime, for all $u\in\mathbb{X}$, $t_1,t_2\in[-\eta,T]$ with $t_1<t_2$, we verify that the operator $\mathcal{Q}$ is equicontinuous in three cases. □

Case 1. When $0\leq t_1<t_2\leq T$, according to (14) and (H_3), we get

$$\begin{aligned}\left|(\mathcal{Q}u)(t_2)-(\mathcal{Q}u)(t_1)\right| &\leq \frac{1}{|\Lambda|}\left[\begin{array}{c}\dfrac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}|F_u(t_2)-F_u(t_1)|\\ +\dfrac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left|\displaystyle\int_0^{t_2}F_u(s)\mathrm{d}s-\int_0^{t_1}F_u(s)\mathrm{d}s\right|\\ +\dfrac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left|\displaystyle\int_0^{t_2}(t_2-s)F_u(s)\mathrm{d}s-\int_0^{t_1}(t_1-s)F_u(s)\mathrm{d}s\right|\end{array}\right]\\ &\leq \frac{1}{|\Lambda|}\left[\begin{array}{c}\dfrac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}|F_u(t_2)-F_u(t_1)|+\dfrac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\displaystyle\int_{t_1}^{t_2}|F_u(s)|\mathrm{d}s\\ +\dfrac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left(\displaystyle\int_{t_1}^{t_2}(t_2-s)|F_u(s)|\mathrm{d}s+\int_0^{t_1}(t_2-t_1)|F_u(s)|\mathrm{d}s\right)\end{array}\right]\\ &\leq \frac{1}{|\Lambda|}\left[\begin{array}{c}\dfrac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}|F_u(t_2)-F_u(t_1)|+\dfrac{(\mu+\nu-2\mu\nu)M}{\mathcal{N}(\mu)\mathcal{N}(\nu)}(t_2-t_1)\\ +\dfrac{2\mu\nu TM}{\mathcal{N}(\mu)\mathcal{N}(\nu)}(t_2-t_1)\end{array}\right]\\ &\longrightarrow 0,\quad \text{as } t_2\longrightarrow t_1.\end{aligned} \tag{20}$$

Case 2. .When $-\eta\leq t_1<0<t_2\leq T$, then $t_2\longrightarrow t_1$ means that $t_1\longrightarrow 0^-$ and $t_2\longrightarrow 0^+$. From (14), we obtain

$$(\mathcal{Q}u)(t_2)-(\mathcal{Q}u)(t_1)\longrightarrow(\mathcal{Q}u)(0^+)-(\mathcal{Q}u)(0^-)=0,\quad \text{as } t_2\longrightarrow t_1. \tag{21}$$

Case 3. .When $-\eta \le t_1 < t_2 \le 0$, then

$$(\mathcal{Q}u)(t_2) - (\mathcal{Q}u)(t_1) \equiv 0 - 0 = 0 \longrightarrow 0, \quad \text{as}\, t_2 \longrightarrow t_1. \tag{22}$$

From (20)–(22), one concludes that, $\forall \epsilon > 0$, $t_1, t_2 \in [-\eta, T]$ and $u \in \mathbb{X}$, $\exists \delta = \delta(\epsilon) > 0$ such that $|(\mathcal{Q}u)(t_2) - (\mathcal{Q}u)(t_1)| < \epsilon$ provided that $|t_2 - t_1| < \delta$, i.e., $\mathcal{Q}: \mathbb{X} \longrightarrow \mathbb{X}$ is equicontinuous. Thus, the condition (ii) is also true. So, it follows from Lemmas 3 and 2 that there exists at least a fixed point $u^*(t) \in \mathbb{X}$ with $u^*(t) = (\mathcal{P}u^*)(t) + (\mathcal{Q}u^*)(t)$, which satisfies system (1). The proof is completed.

Theorem 2. *Assume that (H_1) and (H_2) are true, as well as the following conditions (H_5) and (H_6) also hold.*

(H5) For all $t \in [0, T]$, $u, \overline{u}, v, \overline{v} \in \mathbb{R}$, there have some continuous functions $\mathcal{H}_1(t)$, $\mathcal{H}_2(t)$, $\mathcal{H}_3(t) \ge 0$ such that

$$\begin{aligned} |f(t,u,v) - f(t,\overline{u},\overline{v})| &\le \mathcal{H}_1(t)|u - \overline{u}| + \mathcal{H}_2(t)|v - \overline{v}|,\\ |g(t,v) - g(t,\overline{v})| &\le \mathcal{H}_3(t)|v - \overline{v}|. \end{aligned} \tag{23}$$

(H6) $0 < \Gamma < 1$, *where* $\Gamma = 1/|\Lambda|[(1-\mu)(1-\nu)\mathcal{H}/\mathcal{N}(\mu)\mathcal{N}(\nu) + \lambda\mu T/\mathcal{N}(\mu) + (\mu + \nu - 2\mu\nu)\mathcal{H}T/\mathcal{N}(\mu)\mathcal{N}(\nu) + \mu\nu\mathcal{H}T^2/2\mathcal{N}(\mu)\mathcal{N}(\nu)]$, $\mathcal{H} = |\mathcal{H}_1|_T + |\mathcal{H}_2|_T \cdot |\mathcal{H}_3|_T$ *and* $\|\mathcal{H}_i\|_T = sup\{\mathcal{H}_i(t)\colon 0 \le t \le T\}$, $i = 1, 2, 3$.

Then, system (1) admits a unique solution $u^*(t) \in \mathbb{X}$.

Proof. According to Lemma 2, an operator $\mathcal{F}: \mathbb{X} \longrightarrow \mathbb{X}$ is defined by

$$(\mathcal{F}u)(t) = \begin{cases} \varpi_1(0) + \dfrac{1}{\Lambda}\left[\begin{array}{c} \dfrac{\mu(\varpi_2(0) - \lambda\varpi_1(0))}{\mathcal{N}(\mu)}t + \dfrac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}[F_u(t) - F_u(0)] \\ + \dfrac{\lambda\mu}{\mathcal{N}(\mu)}\displaystyle\int_0^t u(s)\mathrm{d}s + \dfrac{\mu + \nu - 2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\displaystyle\int_0^t F_u(s)ds \\ + \dfrac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\displaystyle\int_0^t (t-s)F_u(s)\mathrm{d}s \end{array}\right], & t \in [0, T] \\ \varpi_1(t),\ t \in [-\eta, 0], \end{cases} \tag{24}$$

where $F_u(t)$ is the same as (6). Then, for all $u, v \in \mathbb{X}$, when $t \in [0, T]$, it follows from (H_5) that

$$\begin{aligned}
|(\mathcal{T}u)(t)-(\mathcal{T}v)(t)| &\le \frac{1}{|\Lambda|}\left[\begin{array}{c}\frac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left|F_u(t)-F_v(t)\right|+\frac{\lambda\mu}{\mathcal{N}(\mu)}\int_0^t|u(s)-v(s)|\mathrm{d}s\\ +\frac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t\left|F_u(s)-F_v(s)\right|\mathrm{d}s\\ +\frac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t(t-s)\left|F_u(s)-F_v(s)\right|\mathrm{d}s\end{array}\right]\\
&\le \frac{1}{|\Lambda|}\left[\begin{array}{c}\frac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left[\mathcal{H}_1(t)|u(t)-v(t)|+\mathcal{H}_2(t)|(\mathcal{C}u)(t)-(\mathcal{C}v)(t)|\right]\\ +\frac{\lambda\mu}{\mathcal{N}(\mu)}\int_0^t|u(s)-v(s)|\mathrm{d}s\\ +\frac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t\left[\mathcal{H}_1(s)|u(s)-v(s)|+\mathcal{H}_2(s)|(\mathcal{C}u)(s)-(\mathcal{C}v)(s)|\right]\mathrm{d}s\\ +\frac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t(t-s)\left[\begin{array}{c}\mathcal{H}_1(s)|u(s)-v(s)|\\ +\mathcal{H}_2(s)|(\mathcal{C}u)(s)-(\mathcal{C}v)(s)|\end{array}\right]\mathrm{d}s\end{array}\right]\\
&\le \frac{1}{|\Lambda|}\left[\begin{array}{c}\frac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left[\begin{array}{c}\mathcal{H}_1(t)|u(t)-v(t)|\\ +\mathcal{H}_2(t)\mathcal{H}_3(t)|u(s-\xi(s))-v(s-\xi(s))|\end{array}\right]\\ +\frac{\lambda\mu}{\mathcal{N}(\mu)}\int_0^t|u(s)-v(s)|\mathrm{d}s\\ +\frac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t\left[\begin{array}{c}\mathcal{H}_1(s)|u(s)-v(s)|\\ +\mathcal{H}_2(s)\mathcal{H}_3(s)|u(t-\xi(t))-v(t-\xi(t))|\end{array}\right]\mathrm{d}s\\ +\frac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t(t-s)\left[\begin{array}{c}\mathcal{H}_1(s)|u(s)-v(s)|\\ +\mathcal{H}_2(s)\mathcal{H}_3(s)|u(s-\xi(s))-v(s-\xi(s))|\end{array}\right]ds\end{array}\right]\\
&\le \frac{1}{|\Lambda|}\left[\begin{array}{c}\frac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left[\left\|\mathcal{H}_1\right\|_T\cdot\|u-v\|+\left\|\mathcal{H}_2\right\|_T\cdot\left\|\mathcal{H}_3\right\|_T\cdot\|u-v\|\right]+\frac{\lambda\mu}{\mathcal{N}(\mu)}\int_0^t\|u-v\|\mathrm{d}s\\ +\frac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t\left[\left\|\mathcal{H}_1\right\|_T\cdot\|u-v\|+\left\|\mathcal{H}_2\right\|_T\cdot\left\|\mathcal{H}_3\right\|_T\cdot\|u-v\|\right]\mathrm{d}s\\ +\frac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t(t-s)\left[\left\|\mathcal{H}_1\right\|_T\cdot\|u-v\|+\left\|\mathcal{H}_2\right\|_T\cdot\left\|\mathcal{H}_3\right\|_T\cdot\|u-v\|\right]\mathrm{d}s\end{array}\right]\\
&\le \frac{1}{|\Lambda|}\left[\frac{(1-\mu)(1-\nu)\mathcal{H}}{\mathcal{N}(\mu)\mathcal{N}(\nu)}+\frac{\lambda\mu T}{\mathcal{N}(\mu)}+\frac{(\mu+\nu-2\mu\nu)\mathcal{H}T}{\mathcal{N}(\mu)\mathcal{N}(\nu)}+\frac{\mu\nu\mathcal{H}T^2}{2\mathcal{N}(\mu)\mathcal{N}(\nu)}\right]\|u-v\|\\
&=\Gamma\|u-v\|,
\end{aligned}\tag{25}$$

where Γ is the same as the condition (H_6). When $t \in [-\eta, 0]$, (24) leads to

$$|(\mathscr{F}u)(t) - (\mathscr{F}v)(t)| = |\varpi_1(t) - \varpi_1(t)| \equiv 0. \tag{26}$$

From (25) and (26), one has

$$\|\mathscr{F}u - \mathscr{F}v\| \leq \Gamma\|u - v\|. \tag{27}$$

By (27) and (H_6), one knows that $\mathscr{F}: \mathbb{X} \longrightarrow \mathbb{X}$ is a contraction mapping. Thus, it follows from Lemmas 3 and 2 that a mapping $\mathscr{F}$ exists a unique fixed point $u^*(t) \in \mathbb{X}$, which satisfies system (1). The proof is completed. □

4. UH- and UHR-Stability

This section mainly establishes the UH- and UHR-stability of system (1). Let $z \in \mathbb{X}$, $\varepsilon > 0$, $0 < \mu, \nu \leq 1$, and $\varphi \in C([0, T], \mathbb{R}^+)$ be nondecreasing. Consider the following two inequalities:

$$\begin{cases} |^{CF}\mathscr{D}_{0^+}^{\nu}\left[{}^{CF}\mathscr{D}_{0^+}^{u} - \lambda\right]\mathscr{W}(t) - f(t, \mathscr{W}(t), (\mathrm{AE})(t)) \leq \varepsilon, & t \in (0, T], \\ (\mathrm{AE})(t) = g(t, \mathscr{W}(t - \xi(t))), & t \in (0, T], \\ \mathscr{W}(t) = \varpi_1(t), {}^{CF}\mathscr{D}_{0^+}^{u}\mathscr{W}(t) = \varpi_2(t), t \in [-\eta, 0], \end{cases} \tag{28}$$

$$\begin{cases} |^{CF}\mathscr{D}_{0^+}^{\nu}\left[{}^{CF}\mathscr{D}_{0^+}^{u} - \lambda\right]\mathscr{W}(t) - f(t, \mathscr{W}(t), (\mathrm{AE})(t)) \leq \varphi(t)\varepsilon, & t \in (0, T], \\ (\mathrm{AE})(t) = g(t, \mathscr{W}(t - \xi(t))), & t \in (0, T], \\ \mathscr{W}(t) = \varpi_1(t), {}^{CF}\mathscr{D}_{0^+}^{u}\mathscr{W}(t) = \varpi_2(t), t \in [-\eta, 0]. \end{cases} \tag{29}$$

Definition 3. Problem (1) is Ulam–Hyers (UH) stable, if for all $\epsilon > 0$ and any solution $\mathscr{W} \in \mathbb{X}$ of (28), there have a unique solution $u^* \in \mathbb{X}$ of (1) and a constant $C_1 > 0$ satisfying

$$|\mathscr{W}(t) - u^*(t)| \leq C_1\epsilon. \tag{30}$$

Definition 4. Problem (1) is generalised Ulam–Hyers (GUH) stable, if for all $\epsilon > 0$ and any solution $\mathscr{W} \in \mathbb{X}$ of (28), there have a unique solution $u^* \in \mathbb{X}$ of (1) and function $\vartheta(\cdot) \in C(\mathbb{R}, \mathbb{R}^+)$ with $\vartheta(0) = 0$ satisfying

$$|\mathscr{W}(t) - u^*(t)| \leq \vartheta(\epsilon). \tag{31}$$

Definition 5. Problem (1) is Ulam-–Hyers–Rassias (UHR) stable, if for all $\epsilon > 0$ and any solution $\mathscr{W} \in \mathbb{X}$ of (29), there have a unique solution $u^* \in \mathbb{X}$ of (1) and constant $C_2 > 0$ satisfying

$$|\mathscr{W}(t) - u^*(t)| \leq C_2\varphi(t)\epsilon, \quad t \in [-\eta, T]. \tag{32}$$

Definition 6. Problem (1) is generalised Ulam-Hyers-Rassias (GUHR) stable, if for any solution $\mathscr{W} \in \mathbb{X}$ of (29), there have a unique solution $u^* \in \mathbb{X}$ of (1) and constant $C_3 > 0$ satisfying

$$|\mathscr{W}(t) - u^*(t)| \leq C_3\varphi(t), \quad t \in [-\eta, T]. \tag{33}$$

Obviously, UH-stable ⇒ GUH-stable, and UHR-stable ⇒ GUHR-stable.

Remark 1. $\mathscr{W} \in \mathbb{X}$ satisfies the inequality (28) iff there has $\phi \in \mathbb{X}$ such that

(1) $|\phi(t)| \leq \epsilon$, $0 < t \leq T$.

(2) ${}^{CF}\mathscr{D}_{0^+}^{\nu}[{}^{ML}\mathscr{D}_{0^+}^{\mu} - \lambda]\mathscr{W}(t) = f(t, \mathscr{W}(t), (\mathrm{AE})(t)) + \phi(t)$, $0 < t \leq T$.

(3) $\mathscr{W}(t) = \varpi_1(t), {}^{CF}\mathscr{D}_{0^+}^{\mu}\mathscr{W}(t) = \varpi_2(t)$, $t \in [-\eta, 0]$.

Remark 2. $\mathscr{W} \in \mathbb{X}$ satisfies the inequality (29) iff there has $\psi \in \mathbb{X}$ such that

(1) $|\psi(t)| \leq \varphi(t)\epsilon$, $0 < t \leq T$.

(2) ${}^{CF}\mathscr{D}_{0^+}^{\nu}[{}^{ML}\mathscr{D}_{0^+}^{\mu} - \lambda]\mathscr{W}(t) = f(t, \mathscr{W}(t), (\mathscr{C})\mathscr{W}(t)) + \psi(t)$, $0 < t \leq T$.

(3) $\mathscr{W}(t) = \varpi_1(t), {}^{CF}\mathscr{D}_{0^+}^{\mu}\mathscr{W}(t) = \varpi_2(t)$, $t \in [-\eta, 0]$.

Theorem 3. *If all the conditions of Theorem 2 are true, then system (1) is UH-stable and also GUH-stable.*

Proof. By Lemma 2 and Remark 1, the solution $\mathscr{W}(t)$ of inequality (28) is formulated as

$$\mathscr{W}(t) = \begin{cases} \varpi_1(0) + \dfrac{1}{\Lambda}\left[\begin{array}{c} \dfrac{\mu(\varpi_2(0) - \lambda\varpi_1(0))}{\mathscr{N}(\mu)}t + \dfrac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\left[(F_{\mathscr{W}}(t) - F_{\mathscr{W}}(0)) + (\phi(t) - \phi(0))\right] \\ + \dfrac{\lambda\mu}{\mathscr{N}(\mu)}\displaystyle\int_0^t \mathscr{W}(s)ds + \dfrac{\mu+\nu-2\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t [F_{\mathscr{W}}(s) + \phi(s)]ds \\ + \dfrac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\displaystyle\int_0^t (t-s)[F_{\mathscr{W}}(s) + \phi(s)]ds \end{array}\right], & t \in [0, T] \\ \varpi_1(t), \; t \in [-\eta, 0], \end{cases} \tag{34}$$

By Theorem 2 and Lemma 2, the unique solution $u^*(t)$ of (1) satisfies

$$u^*(t) = \begin{cases} \varpi_1(0) + \dfrac{1}{\Lambda}\left[\begin{array}{c} \dfrac{\mu(\varpi_2(0) - \lambda\varpi_1(0))}{\mathscr{N}(\mu)}t + \dfrac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\left[F_{u^*}(t) - F_{u^*}(0)\right] \\ + \dfrac{\lambda\mu}{\mathscr{N}(\mu)}\displaystyle\int_0^t u^*(s)ds + \dfrac{\mu+\nu-2\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t F_{u^*}(s)ds \\ + \dfrac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\displaystyle\int_0^t (t-s)F_{u^*}(s)ds \end{array}\right], & t \in [0, T] \\ \varpi_1(t), \; t \in [-\eta, 0], \end{cases} \tag{35}$$

Similar to (25) and (26), we derive from (34) and (35) that

$$\begin{aligned}\left|\mathscr{W}(t)-u^{*}(t)\right| \leq & \frac{1}{|\Lambda|}\left\{\frac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\left[\left|F_{\mathscr{W}}(t)-F_{u^{*}}(t)\right|+|\phi(t)|+|\phi(0)|\right]\right. \\ & +\frac{\lambda\mu}{\mathscr{N}(\mu)}\int_0^t\left|\mathscr{W}(s)-u^{*}(s)\right|\mathrm{d}s+\frac{(\mu+\nu-2\mu\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t\left[\left|F_{\mathscr{W}}(s)-F_{u^{*}}(s)\right|+|\phi(s)|\right]\mathrm{d}s \\ & \left.+\frac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t(t-s)\left[\left|F_{\mathscr{W}}(s)-F_{u^{*}}(s)\right|+|\phi(s)|\right]\mathrm{d}s\right\} \\ \leq & \frac{1}{|\Lambda|}\left\{\frac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\left[\mathscr{H}\left\|\mathscr{W}(t)-u^{*}(t)\right\|+\varepsilon+\varepsilon\right]+\frac{\lambda\mu}{\mathscr{N}(\mu)}\int_0^t\left\|\mathscr{W}(s)-u^{*}(s)\right\|\mathrm{d}s\right. \\ & +\frac{\mu+\nu-2\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t\left[\mathscr{H}\left\|\mathscr{W}(s)-u^{*}(s)\right\|+\varepsilon\right]ds \\ & \left.+\frac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t(t-s)\left[\mathscr{H}\left\|\mathscr{W}(s)-u^{*}(s)\right\|+\varepsilon\right]\mathrm{d}s\right\} \\ \leq & \Gamma\left\|\mathscr{W}(t)-u^{*}(t)\right\|+\Upsilon\varepsilon, \quad t\in[0,T],\end{aligned} \tag{36}$$

where $\Upsilon=(1/|\Lambda|)[2(1-\mu)(1-\nu)\mathscr{H}/\mathscr{N}(\mu)\mathscr{N}(\nu)+\lambda\mu T/\mathscr{N}(\mu)+(\mu+\nu-2\mu\nu)\mathscr{H}T/\mathscr{N}(\mu)\mathscr{N}(\nu)+\mu\nu\mathscr{H}T^2/2\mathscr{N}(\mu)\mathscr{N}(\nu)]$, and

$$\left|\mathscr{W}(t)-u^{*}(t)\right|=\left|\varpi_1(t)-\varpi_1(t)\right|\equiv 0, \quad t\in[-\eta,0]. \tag{37}$$

From (36) and (37), we have

$$\left\|\mathscr{W}(t)-u^{*}(t)\right\|\leq\frac{\Upsilon}{1-\Gamma}\varepsilon. \tag{38}$$

Thus, it follows from (36) and Definitions 3 and 4 that system (1) is UH-stable and also GUH-stable. The proof is completed.

Theorem 4. *If all the conditions of Theorem 2 are true, then system (1) is UHR-stable and also GUHR-stable.*

Proof. According to Lemma 2 and Remark 2, the solution $\mathscr{W}(t)$ of inequality (29) is formulated as

$$\mathscr{W}(t)=\begin{cases}\varpi_1(0)+\dfrac{1}{\Lambda}\left[\begin{array}{c}\dfrac{\mu(\varpi_2(0)-\lambda\varpi_1(0))}{\mathscr{N}(\mu)}t+\dfrac{(1-\mu)(1-\nu)}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\left[(F_{\mathscr{W}}(t)-F_{\mathscr{W}}(0))+(\psi(t)-\psi(0))\right] \\ +\dfrac{\lambda\mu}{\mathscr{N}(\mu)}\displaystyle\int_0^t\mathscr{W}(s)\mathrm{d}s+\dfrac{\mu+\nu-2\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\int_0^t\left[F_{\mathscr{W}}(s)+\psi(s)\right]\mathrm{d}s \\ +\dfrac{\mu\nu}{\mathscr{N}(\mu)\mathscr{N}(\nu)}\displaystyle\int_0^t(t-s)\left[F_{\mathscr{W}}(s)+\psi(s)\right]\mathrm{d}s\end{array}\right], & t\in[0,T] \\ \varpi_1(t),\ t\in[-\eta,0],\end{cases} \tag{39}$$

Since $\varphi\geq 0$ is nondecreasing, one derives from (35) and (39) that

$$
\begin{aligned}
\left|\mathcal{W}(t)-u^*(t)\right| &\leq \frac{1}{|\Lambda|}\left\{\frac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left[\left|F_{\mathcal{W}}(t)-F_{u^*}(t)\right|+|\psi(t)|+|\psi(0)|\right],\right.\\
&+\frac{\lambda\mu}{\mathcal{N}(\mu)}\int_0^t\left|\mathcal{W}(s)-u^*(s)\right|ds+\frac{(\mu+\nu-2\mu\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t\left[\left|F_{\mathcal{W}}(s)-F_{u^*}(s)\right|+|\psi(s)|\right]ds\,,\\
&\left.+\frac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t(t-s)\left[\left|F_{\mathcal{W}}(s)-F_{u^*}(s)\right|+|\psi(s)|\right]ds\right\},\\
&\leq\frac{1}{|\Lambda|}\left\{\frac{(1-\mu)(1-\nu)}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\left[\mathcal{H}\left\|\mathcal{W}(t)-u^*(t)\right\|+\varphi(t)\varepsilon+\varphi(t)\varepsilon\right],\right.\\
&+\frac{\lambda\mu}{\mathcal{N}(\mu)}\int_0^t\left\|\mathcal{W}(s)-u^*(s)\right\|ds+\frac{\mu+\nu-2\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t\left[\mathcal{H}\left\|\mathcal{W}(s)-u^*(s)\right\|+\varphi(t)\varepsilon\right]ds\,,\\
&\left.+\frac{\mu\nu}{\mathcal{N}(\mu)\mathcal{N}(\nu)}\int_0^t(t-s)\left[\mathcal{H}\left\|\mathcal{W}(s)-u^*(s)\right\|+\varphi(t)\varepsilon\right]ds\right\},\\
&\leq\Gamma\left\|\mathcal{W}(t)-u^*(t)\right\|+\Upsilon\varepsilon, t\in[0,T],
\end{aligned}
\tag{40}
$$

and

$$\left|\mathcal{W}(t)-u^*(t)\right|=\left|\varpi_1(t)-\varpi_1(t)\right|\equiv 0, t\in[-\eta,0], \tag{41}$$

By (40) and (41), one has

$$\left\|\mathcal{W}(t)-u^*(t)\right\|\leq\frac{\Upsilon}{1-\Gamma}\varphi(t)\varepsilon, t\in[0,T]. \tag{42}$$

Thereby, from (42) and Definitions 5 and 6, one concludes that system (1) is UHR-stable and also GUHR-stable. The proof is completed.

5. Illustrative Examples

In this section, we shall apply our findings to solve the existence and stability of solutions for two specific systems.

5.1. Theoretical Analysis

Example 1. In (1), take $T=1$, $\mu=0.7$, $\nu=0.4$, $\lambda=1/2$, $f(t,u,w)=t^2+\sin(2u)+\sin(t)w/1+w^2$, $w=g(t,z)=\cos(t)\arctan(z)$, $z=u(t-\xi(t))$, $\xi(t)=2+\sin(2t)/6$, $\varpi_1(t)=2\cos(t)$, $\varpi_2(t)=t$, $\mathcal{N}(x)=1-x+x/\Gamma(x)$, $0<x\leq 1$. A simple calculation gives $\eta=0.5$, $\mathcal{M}_1(t)=1+t^2$, $\mathcal{M}_2(t)=|\sin(t)|$, $\mathcal{M}_3(t)=\pi/2\ (\cos(t))$, $\mathcal{N}(0)=\mathcal{N}(1)=1$ and

$$\Lambda=1-\frac{\lambda(1-\mu)}{\mathcal{N}(\mu)}\approx 0.8213>0, \vartheta=\frac{\lambda T}{|\mathcal{N}(\mu)-\lambda(1-\mu)|}\approx 0.7254<1. \tag{43}$$

Thus, the conditions (H_1)–(H_4) all fulfil. It follows from Theorem 1 that the system of Example 1 exists at least a solution $u^*(t)\in C([-0.5,1],\mathbb{R})$.

Example 2. In (1), take $T=1$, $\mu=0.6$, $\nu=0.8$, $\lambda=1/5$, $f(t,u,w)=\cos(t)+t^2/20\log(1+u^2+w^2)$, $w=g(t,z)=\sin(t)+2+\sin(3t)/30\arctan(z)$, $z=u(t-\xi(t))$, $\xi(t)=2+\sin(2t)/6$, $\varpi_1(t)=2\cos(t)$, $\varpi_2(t)=t$, $\mathcal{N}(x)=1-x+x/\Gamma(x), 0<x\leq 1$. By a direct calculation, one has $\eta=0.5$, $\mathcal{N}(0)=\mathcal{N}(1)=1$, $\mathcal{H}_1(t)=\mathcal{H}_2(t)=t^2/20$, $\mathcal{H}_3(t)=2+\sin(3t)/30$, $\|\mathcal{H}_1\|_T=\|\mathcal{H}_2\|_T=1/20$, $\|\mathcal{H}_3\|_T=1/10$ and

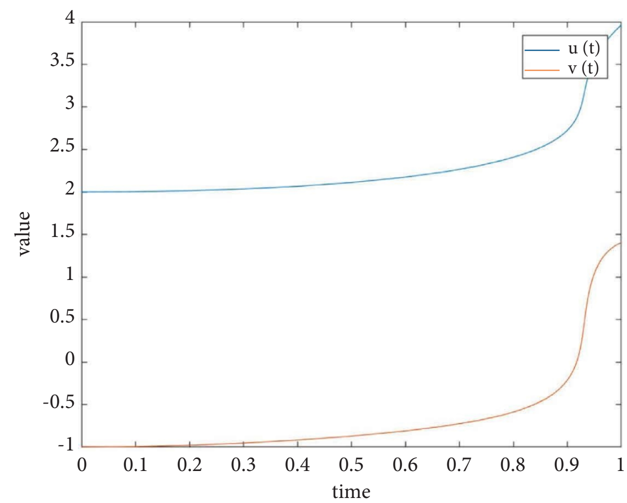

Figure 1: The solution of Example 1.

$$\Lambda = 1 - \frac{\lambda(1-\mu)}{\mathcal{N}(\mu)} \approx 0.9004 > 0, \mathcal{H} = \|\mathcal{H}_1\|_T + \|\mathcal{H}_2\|_T \cdot \|\mathcal{H}_3\|_T = \frac{11}{200},$$
$$\Gamma = \frac{1}{|\Lambda|}\left[\frac{(1-\mu)(1-\nu)\mathcal{H}}{\mathcal{N}(\mu)\mathcal{N}(\nu)} + \frac{\lambda\mu T}{\mathcal{N}(\mu)} + \frac{(\mu+\nu-2\mu\nu)\mathcal{H}T}{\mathcal{N}(\mu)\mathcal{N}(\nu)} + \frac{\mu\nu\mathcal{H}T^2}{2\mathcal{N}(\mu)\mathcal{N}(\nu)}\right] \approx 0.2312 < 1. \tag{44}$$

Thus, we verify that the conditions (H_1), (H_2), (H_5), and (H_6) hold. From Theorem 2 and Theorems 3 and 4, we conclude that the system of Example 2 has a unique solution $u^*(t) \in C([-0.5, 1], \mathbb{R})$, which is UH-, GUH-, UHR-, and GUHR-stable.

5.2. Numerical Simulation. Let $v(t) = ({}^{CF}\mathcal{D}^{\mu}_{0^+} - \lambda)u(t)$, then system (1) is transformed into a system of equations as follows:

$$\begin{cases} {}^{CF}\mathcal{D}^{\mu}_{0^+} u(t) = \lambda u(t) + v(t), t \in (0,T], {}^{CF}\mathcal{D}^{\nu}_{0^+} v(t) = f(t, u(t), (\mathcal{C}u)(t)), t \in (0,T], \\ (\mathcal{C}u)(t) = g(t, u(t-\xi(t))), t \in (0,T], \\ u(t) = \varpi_1(t), v(t) = \varpi_2(t) - \lambda\varpi_1(t), t \in [-\eta, 0]. \end{cases} \tag{45}$$

The simulations of solution of Examples 1 and 2 are shown as Figures 1 and 2, respectively. $u(t)$ is the solution of Langevin system in the figures. The simulation of UH-stability of Example 2 is shown as Figure 3. It follows from the images of $\varepsilon = 0.1$ and $\varepsilon = 0.01$ that, when $0 < \varepsilon \ll 1$, the solution curve of the inequality (28) almost coincides with that of system (45), which shows that system (45) is UH-stable.

6. Summaries

It is well known that the Langevin equation is a powerful tool in describing the random motion of particles in fluid. In a particularly complex viscous liquid, the integer-order Langevin equation describing the motion of particles is no longer accurate. The manuscript focuses on the existence, uniqueness, and UH-like stability of a nonlinear fractional order Langevin equation with nonsingular exponential kernel and delay control. Theoretical analysis and numerical simulation of two examples verify the correctness and effectiveness of our main conclusions. Our findings provide mathematical theoretical support for some physical problems. Furthermore, the mathematical theories and methods used in this paper can be made use of a reference for the study of other fractional differential system. In addition, we can further study the existence, multiplicity, and stability of solutions of some impulsive nonsingular fractional Langevin equations with nonlocal or integral boundary value conditions in the future.

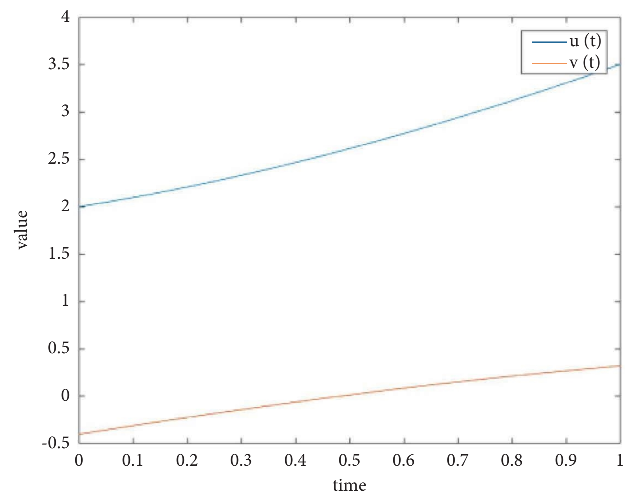

Figure 2: The solution of Example 2.

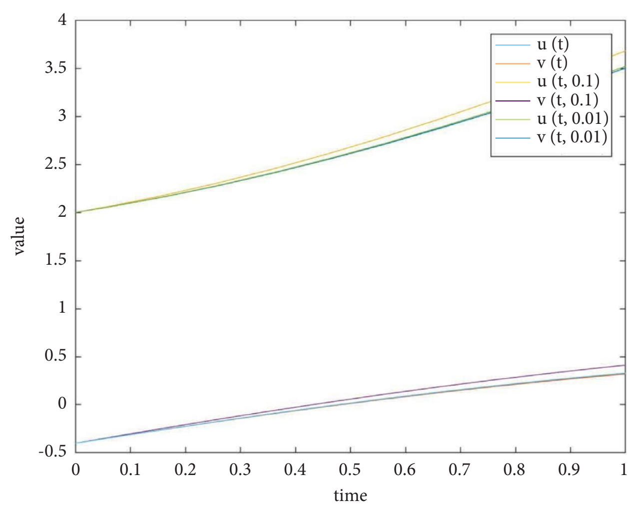

Figure 3: The UH-stability of Example 2 with $\varepsilon = 0.1, 0.01$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no competing interests.

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