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Research Article

Delayed Impulsive Control for Lag Synchronization of Neural Networks with Time-Varying Delays and Partial Unmeasured States

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In this article, a controller with delay impulse is applied to a neural network (NN) with time-varying delays. Firstly, the lag synchronization of the system is discussed. In addition, a sufficient condition for guaranteed lag synchronization requiring all state information is derived by using linear matrix inequality (LMI). In particular, the main results of neural networks (NNs) with time-varying delays and partial unmeasured states are studied by using the delay impulsive control gain matrix derived from dimension expansion. Finally, we will give two examples, both of which can confirm the validity of the results.

1. Introduction

As the most important branch system in modern artificial intelligence, the neural network (NN) [1–3] has always been an important object of scholars' research. In recent decades, NN has made great progress in control systems, signal analysis and processing, traffic safety, pattern recognition, and other fields. However, when using NN to solve practical problems, many scholars have found that the information transmission mode between neurons is not continuous, and the transmission of neurotransmitters may be delayed. Thus, it is necessary to discuss NNs with a time delay.

Time delay is a common phenomenon in many practical NNs and a key factor that directly affects and determines the synchronization of NNs. In [4], the lag synchronization of NNs with time delays is studied by establishing appropriate controllers. In [5], the asymptotic control of nonlinear NNs with time delays is discussed by using adaptive mechanisms and projection operators to estimate unknown time delays. In [6], fixed time synchronization for delayed complex dynamic NNs is studied. At present, the synchronization analysis of linear systems with a constant delay has been widely studied, but there are still many problems in the study of time-varying delay cases. For example, it is difficult to establish sufficient and necessary conditions for the synchronization of time-varying delay NNs, so it is particularly important to seek synchronization conditions with minimum conservatism.

As a branch of time-delay NNs, NNs with time-varying delays have wide application prospects in pattern recognition, optimization calculation, and image processing. Over the past three decades, it has been proved that linear matrix inequality (LMI) methods can be used to obtain more concise ideas and their interrelationships, such as the feasibility of solutions, reversibility, and controller specification form. In [7], some sufficient conditions for the global exponential stability (GES) of high-order Cohen–Grossberg NNs are derived by using induction and the properties of nonsingular M-matrices. In [8], the synchronization problem of complex dynamic neural networks (CDNNs) with time-varying delays is studied by using an impulse distributed control scheme. In [9], exponential adaptive synchronization of time-varying delay NNs is discussed by establishing the Lyapunov functional. It is not hard to find that the synchronization of NNs in material transportation, adaptive control, psychology, and transportation is still worth studying, in addition to the stability problems of a large number of differential NNs with time-varying delays directly proposed in engineering practice. Therefore, the research on synchronization theory of NNs with time-varying delays has an extensive practical engineering background and profound theoretical value.

As one of the most important dynamic behaviors of NNs, synchronization can be divided into quasi-synchronization [10], lag synchronization [11], exponential synchronization [12], and coupling synchronization [13] according to different synchronization behaviors. In [14], a new method of high-order NNs based on time delays and impulses discrete is studied. In [15], an adaptive controller is designed to study the adaptive synchronization problem of NNs with time delays. In [16], the synchronization problem of complex NNs with unknown bounded time-varying delays using LMI is discussed. It is not hard to see that the lag synchronization result of delayed impulsive control has attracted the attention of many scholars. Current research results concentrate on the complete state of information. In other words, when the state of the NN is partial and immeasurable, the above results do not apply to the case of the NN with impulsive action. Therefore, it is a key problem to realize lag synchronization of NNs, which refers to partial unmeasured states under the delayed impulse of NNs with time-varying delays. However, we find that the analysis of lag synchronization of delayed impulses is still in its infancy.

Motivated by the above discussions, the purpose of this article is to discuss the lag synchronization of time-varying NNs with partial unmeasured states under the control of delayed impulse. By increasing the unmeasurable state dimension and extending the measurable state dimension, the unmeasurable state dimension and measurable state dimension can be unified. The results show that some sufficient conditions for lag synchronization of NNs with time-varying delays can be obtained. The main contributions to this article include the following:

(1) A new time-varying delay differential NN is established.

(2) The state transition matrix is used to separate the measurable state and the unmeasurable state, and the dimensions of the two states obtained are consistent through dimension expansion.

(3) The result of lag synchronization is obtained by LMI inequality. At the same time, compared with the general nonlinear system, because of the complexity of time-varying delay NNs, the design of impulsive control gain is more difficult. Appropriate delay impulsive gain can be obtained by using LMI.

The remainder of the article is as follows: Section 2 provides an NN model with a time-varying delay, a definition, a hypothesis, and two useful lemmas. In Section 3, some conditions for satisfying LMI are given, and some theorems satisfying the main results are obtained. In Section 4, two numerical examples are discussed to illustrate the feasibility of the conclusions. Finally, Section 5 is the conclusion of the article.

Notations: in the whole article, R signifies real number set, R^n, and $R^{n\times q}$ stand, respectively, the set of real numbers and all n-dimensional and $n\times q$-dimensional real spaces equipped. $A<0\,(A>0)$ denotes a negative (positive) definite matrix. $\lambda_{\min}(A)\,(\lambda_{\max}(A))$, A^T and A^{-1} denote the minimum (maximum) eigenvalue, the transpose and the inverse of matrix A, respectively. Set $\alpha\vee\beta$ be the maximum value of α and β and $\alpha\wedge\beta$ denotes the minimum value. For any interval $I\subseteq R$, set $S\subseteq R^k\,(1\leq k\leq n)$, $PC(I,S)=\{\theta\colon I\longrightarrow S\}$, where θ is a continuous point except at the finite point t, exist $\theta(t^+)$, $\theta(t^-)$, and $\theta(t^+)=\theta(t)$. $\tau(t)$ denotes time-varying delay, $\tau>0$ is time delay, $PC([-\tau,0],R^n)$ represents the collection of piecewise right-hand functions $h\colon[-\tau,0]\longrightarrow R^n$ with the norm defined by $\|h\|_\tau=\sup_{-\tau\leq s\leq 0}\|h(s)\|$. $\Lambda=\{1,2,\ldots,n\}$ and the symbol $\star$ denote a symmetric block in a symmetric matrix.

2. Preliminaries

2.1. Model. First, time-varying delay NN can be considered as

$$
\begin{cases}\dot{z}(t)=-Az(t)+Bz(t-\tau(t))+Cf(z(t))+Df(z(t-\tau(t)))+I, & t>0,\\ z(s)=\phi(s), & s\in[-\tau,0],\end{cases}\tag{1}
$$

where $z(t)=(z_1,\ldots,z_n)^T\in R^n$ denotes neuron state vector; $A=\operatorname{diag}\{a_1,\ldots,a_n\}$ and $B=\operatorname{diag}\{b_1,\ldots,b_n\}$ are diagonal matrices with $a_i,b_i>0$, $i\in\Lambda$; C and D correspond to the constant connection weight matrix; $f(z(t))=(f_1(z_1(t)),\ldots,f_n(z_n(t)))^T$ denote the neuron activation functions; assume that $0\leq\tau(t)\leq\tau$, $\tau\leq\infty$; I is an external input signal; $\phi(\cdot)\in PC([-\tau,0],R^n)$ represents the initial state.

Consider a time-varying delay NNs (1), its response NN is as follows:

$$
\begin{cases}\dot{w}(t)=-Aw(t)+Bw(t-\tau(t))+Cf(w(t))+Df(w(t-\tau(t)))+I+u(t),\ t>d\\ w(s)=\theta(s), s\in[-\tau+d,d],\end{cases}\tag{2}
$$

where the number of measurable states in the state part of the response NN is q and $\theta(\cdot) \in PC([-\tau + d, d], R^n)$ is the initial state. Let $H = \{r_1, \ldots, r_q\} \subset \{1, \ldots, n\}$ and $G = \{r_{q+1}, \ldots, r_n\} \subset \{1, \ldots, n\}$ represent the collection of measurable and unmeasurable states, respectively. The lag synchronization characteristic of the NN is $w(t) \longrightarrow z(t - d)$ for exist $d > 0$. Assume that the error variable is $e(t) = w(t) - z(t - d)$. Thus, the error dynamics NN of drive and response NN is as follows:

$$\begin{cases} \dot{e}(t) = -Ae(t) + Be(t - \tau(t)) + Cg(e(t)) + Dg(e(t - \tau(t))) + u(t), & t > d \\ e(s) = \theta(s) - \phi(s - d), & s \in [-\tau + d, d], \end{cases} \quad (3)$$

where $g(e) = f(e(\cdot) + z(\cdot)) - f(z(\cdot))$. Let $u(t) \in R^n$ be a controller and

$$u(t) = \sum_{k \in Z_+} (Ke(t - \eta_k) - e(t))\delta(t - t_k), \quad (4)$$

where $\{t_k\}$ denotes the impulse sequence, and it satisfies $\inf\{t_k - t_{k-1}, k \in Z_+\} > 0$. Assume Θ_d is the impulse sequences, and it satisfies $t_k - t_{k-1} \leq \rho$ (ρ is any positive constant), $K \in R^{n\times n}$ denotes the control gain matrix. $\delta(\cdot)$ denotes the Dirac delta function, when $t = t_k$, $\delta(t) = 1$; otherwise, $\delta(t) = 0$. η_k is delayed in the impulse, and it satisfies $0 \leq \eta_k \leq \eta, \eta > 0$.

Define e_i $(i \in H)$ as the measurable state, we shift the e_i in front of e. It is not hard to find a matrix that T satisfies $\vartheta(t) = Te(t)$, where T is a transition matrix; then, we can get

$$\begin{cases} \dot{\vartheta}(t) = -A_\vartheta \vartheta(t) + B_\vartheta \vartheta(t - \tau(t)) + C_\vartheta g(\vartheta(t)) + D_\vartheta g(\vartheta(t - \tau(t))), & t \neq t_k, \\ \vartheta(t) = K_\vartheta \vartheta(t^- - \eta_k), & t = t_k, \\ \vartheta(s) = \theta_\vartheta(s) - \phi_\vartheta(s - d), & s \in [(-\tau + d) \wedge (-\eta + d), d], \end{cases} \quad (5)$$

where $\vartheta = (e_{r_1}, \ldots, e_{r_q}, e_{r_{q+1}}, \ldots, e_{r_n})^T$, $D_\vartheta = \mathrm{TDT}^{-1}$, $C_\vartheta = \mathrm{TCT}^{-1}$, $g(\vartheta) = g(Te)$, $A_\vartheta = \mathrm{TAT}^{-1}$, $B_\vartheta = \mathrm{TBT}^{-1}$, $\phi_\vartheta = T\phi$, $\theta_\vartheta = T\theta$, $K_\vartheta = \mathrm{TKT}^{-1}$. After obtaining the transformed error system, we will discuss two different cases of error state information.

Remark 1. During the past period, lag synchronization of NNs has been investigated in the literature [8, 14, 15]. In [15], lag synchronization of BAM NNs with impulses is discussed. However, few studies consider the existence of unmeasured states of time-varying delay NNs, so the above results for lag synchronization are not applicable. Thus, we improve the known results and study the lag synchronization problem of time-varying delay NNs in both measurable and unmeasurable states. Therefore, we divide impulsive control into two situations: the first is that the number of set H is greater than the number of set G, namely, $q > n - q$; the second is that the number of sets. H is less than or equal to the number of sets G, namely, $q \leq n - q$.

Hypothesis 1. The neuron activation function $f_i(\cdot)$ satisfies

$$|f_i(u) - f_i(v)| \leq l_i |u - v|, \quad (6)$$

where $u, v \in R$, l_i is a positive constant and $i \in \Lambda$.

Definition 1. For any initial conditions $z(s) = \phi(s)$ and $w(s) = \theta(s)$ satisfies

$$|z(t) - w(t - d)| \longrightarrow 0. \quad (7)$$

As $t \longrightarrow +\infty$, then it is said that the drive NN (1) and response NN (2) achieve lag synchronization with the time lag d.

Lemma 1 (see [16]). *Assume that $h(t) \in PC(R, R_+)$ satisfies*

$$\begin{cases} \dfrac{dh(t)}{dt} \leq \lambda h(t) + \delta h(t - \tau(t)), & t \neq t_k, \\ h(t) \leq \theta_k h(t - \eta_k), & t = t_k, k \in Z_+, \end{cases} \quad (8)$$

where $\lambda \in R, \delta \in R_+$, and $\theta_k \in R_+$. If there exist α, β, p are constants, and $\alpha > 0, \beta > 0, p > 0$, such that

$$\begin{aligned} &\alpha + |\lambda| + \beta\delta e^{\alpha\tau} < \ln\beta/p, \\ &\beta e^{\alpha\eta}\theta_k \leq 1, \quad k \in Z_+. \end{aligned} \quad (9)$$

Then the solution of (5) satisfies the following:

$$h(t) \leq \beta\overline{h}(0)e^{-\alpha t}, \quad (10)$$

in Θ_d, where $\overline{h}(0) = sup_{-(\tau\vee\eta)\leq s\leq 0} h(s)$.

Proof. Similar to Lemma 2 proof in [16], setting $T = p, \mu_1 = \exp(\alpha T), \mu_2 = \exp(\alpha\tau), \mu_3 = \exp(\alpha\eta)$, one may derive Lemma 1. □

Lemma 2. *For any real vectors $z, \zeta \in R^n$ and real matrix $P \in R^{n\times n}$, there exists an $n \times n$ real matrix $M > 0$ satisfies the following:*

$$2z^T P\zeta \le z^T PM^{-1}P^T z + \zeta^T M\zeta. \tag{11}$$

3. Main Results

This section presents the main theoretical results of the article; that is, the lag synchronization problem of time-varying delay NNs is proved by delayed impulse. To better demonstrate the comprehensiveness of the results, two cases will be considered, i.e., $q > n - q$ and $q \le n - q$, respectively.

Case 1. The number of set H is greater than the number of set G, namely, $q > n - q$.

First of all, we think about the lag synchronization of time-varying delay NNs via delayed impulsive control, when $q > n - q$. Considering the error states $\vartheta(t)$, $\vartheta = (\vartheta_1^T, \vartheta_2^T)^T$, $\vartheta_1 = (\vartheta_1^{1T}, \vartheta_1^{2T})^T, \vartheta_1^1 = (e_{r_1}, \ldots, e_{r_{n-q}})^T, \vartheta_1^2 = (e_{r_{n-q+1}}, \ldots, e_{r_q})^T$, $\vartheta_2 = (e_{r_{q+1}}, \ldots, e_{r_n})^T$, $g(\vartheta_1^1) = (g_{r_1}(e_{r_1}(\cdot)), \ldots, g_{r_{n-q}}(e_{r_{n-q}}(\cdot)))^T, g(\vartheta_1^2) = (g_{r_{n-q+1}}(e_{r_{n-q+1}}(\cdot)), \ldots, g_{r_q}(e_{r_q}(\cdot)))^T$, and $g(\vartheta_2) = (g_{r_{q+1}}(e_{r_{q+1}}(\cdot)), \ldots, g_{r_n}(e_{r_n}(\cdot)))^T$. After expanding the error state, NN (4) can be rewritten as

$$\begin{cases} \dot{\vartheta}(t) = -\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\begin{pmatrix} \vartheta_1^1(t) \\ \vartheta_1^2(t) \\ \vartheta_2(t) \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}\begin{pmatrix} \vartheta_1^1(t-\tau(t)) \\ \vartheta_1^2(t-\tau(t)) \\ \vartheta_2(t-\tau(t)) \end{pmatrix} \\ +\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}\begin{pmatrix} g\left(\vartheta_1^1(t)\right) \\ g\left(\vartheta_1^2(t)\right) \\ g(\vartheta_2(t)) \end{pmatrix} + \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}\begin{pmatrix} g\left(\vartheta_1^1(t-\tau(t))\right) \\ g\left(\vartheta_1^2(t-\tau(t))\right) \\ g(\vartheta_2 = (t-\tau(t))) \end{pmatrix}, \\ \vartheta(t) = K_\vartheta \vartheta(t^- - \eta_k), \quad t = t_k, \\ \vartheta(s) = \theta_\vartheta(s) - \phi_\vartheta(s-d), \quad s \in [(-\tau+d)\wedge(-\eta+d), d], \end{cases} \tag{12}$$

where $\vartheta_1^1, \vartheta_1^2 \in \vartheta_1$ and ϑ_1 denote measurable state of the neuron and ϑ_2 denote unmeasurable state of the neuron. Compared with the traditional impulsive controller, in this article, unmeasurable and measurable state information are discussed. In other words, state information can be divided into unmeasurable state and measurable state information by transformation matrix. Therefore, we can obtain the control gain matrix as below:

$$K_\varpi = \begin{pmatrix} k_1^1 & 0 & 0 \\ 0 & k_1^2 & 0 \\ k_2 & 0 & 0 \end{pmatrix}, \tag{13}$$

where k_1^1, k_1^2, and k_2 are real matrices. When $q > n - q$, this means that the dimension of the measurable state may be greater than that of the unmeasurable state. When $q > n - q$, extend the dimension of ϑ_1^2 to the dimension of unmeasurable state ϑ_2, then we get $\overline{\vartheta}_2 = (\vartheta_2^T, \vartheta_1^{2T})^T$. Redefine variables after dimension expansion, then one has $\overline{\vartheta} = (\overline{\vartheta}_1^T, \overline{\vartheta}_2^T)^T, \overline{\vartheta}_1 = \vartheta_1, g(\overline{\vartheta}_1(\cdot)) = (g(\vartheta_1^1(\cdot))^T, g(\vartheta_1^2(\cdot))^T) g(\overline{\vartheta}_2(\cdot)) = (g(\vartheta_2(\cdot))^T, g(\vartheta_1^2(\cdot))^T)$, which means that $\overline{\vartheta}_1$, $\overline{\vartheta}_2$ have the same dimension. Hence, NN (6) can be indicated as follows:

$$\begin{cases} \overline{\vartheta} = -\overline{A}\overline{\vartheta}(t) + \overline{B}\overline{\vartheta}(t-\tau(t)) + \overline{C}g(\overline{\vartheta}(t)) + \overline{D}g(\overline{\vartheta}(t-\tau(t))), \quad t \ne t_k, \\ \overline{\vartheta}(t) = \overline{K}\overline{\vartheta}(t^- - \eta_k), \quad t = t_k, \\ \overline{\vartheta}(s) = \overline{\varphi}(s) - \overline{\phi}(s-d), \quad s \in [(-\tau+d)\wedge(-\eta+d), d], \end{cases} \tag{14}$$

where

$$\begin{aligned}
&\overline{A}=\begin{pmatrix}\overline{A}_{11} & \overline{A}_{12}\\ \overline{A}_{21} & \overline{A}_{22}\end{pmatrix},\overline{B}=\begin{pmatrix}\overline{B}_{11} & \overline{B}_{12}\\ \overline{B}_{21} & \overline{B}_{22}\end{pmatrix},\overline{C}=\begin{pmatrix}\overline{C}_{11} & \overline{C}_{12}\\ \overline{C}_{21} & \overline{C}_{22}\end{pmatrix},\overline{D}=\begin{pmatrix}\overline{D}_{11} & \overline{D}_{12}\\ \overline{D}_{21} & \overline{D}_{22}\end{pmatrix},\\
&\overline{A}_{11}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\overline{A}_{12}=\begin{pmatrix}A_{13} & 0\\ A_{23} & 0\end{pmatrix},\overline{A}_{21}=\begin{pmatrix}A_{31} & A_{32}\\ A_{21} & A_{22}\end{pmatrix},\overline{A}_{22}=\begin{pmatrix}A_{33} & 0\\ A_{23} & 0\end{pmatrix},\\
&\overline{B}_{11}=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix},\overline{B}_{12}=\begin{pmatrix}B_{13} & 0\\ B_{23} & 0\end{pmatrix},\overline{B}_{21}=\begin{pmatrix}B_{31} & B_{32}\\ B_{21} & B_{22}\end{pmatrix},\overline{B}_{22}=\begin{pmatrix}B_{33} & 0\\ B_{23} & 0\end{pmatrix},\\
&\overline{C}_{11}=\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix},\overline{C}_{12}=\begin{pmatrix}C_{13} & 0\\ C_{23} & 0\end{pmatrix},\overline{C}_{21}=\begin{pmatrix}C_{31} & C_{32}\\ C_{21} & C_{22}\end{pmatrix},\overline{C}_{22}=\begin{pmatrix}C_{33} & 0\\ C_{23} & 0\end{pmatrix},\\
&\overline{D}_{11}=\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix},\overline{D}_{12}=\begin{pmatrix}D_{13} & 0\\ D_{23} & 0\end{pmatrix},\overline{D}_{21}=\begin{pmatrix}D_{31} & D_{32}\\ D_{21} & D_{22}\end{pmatrix},\overline{D}_{22}=\begin{pmatrix}D_{33} & 0\\ D_{23} & 0\end{pmatrix}.
\end{aligned}\tag{15}$$

It can be seen that measurable state $\overline{\vartheta}_1$ and unmeasurable state $\overline{\vartheta}_2$ of NN (12) have the same dimension. Therefore, the lag synchronization problem of partial unmeasured time-varying delay NNs can be studied under delayed impulsive control, and control gain matrix $\overline{K}\in R^{2q\times 2q}$ is shown below:

$$\overline{K}=\begin{pmatrix}K_1 & 0\\ K_2 & 0\end{pmatrix},\tag{16}$$

where $K_1=\begin{pmatrix}k_1^1 & 0\\ 0 & k_1^2\end{pmatrix}, K_2=\begin{pmatrix}k_2 & 0\\ 0 & k_1^2\end{pmatrix}$. Then, the error dynamical system is given as

$$\begin{cases}\overline{\vartheta}=-\overline{A}\overline{\vartheta}(t)+\overline{B}\overline{\vartheta}(t-\tau(t))+\overline{C}g(\overline{\vartheta}(t))+\overline{D}g(\overline{\vartheta}(t-\tau(t))), & t\neq t_k,\\ \overline{\vartheta}_1(t)=K_1\overline{\vartheta}_1(t^- -\eta_k), & t=t_k,\\ \overline{\vartheta}_2(t)=K_2\overline{\vartheta}_1(t^- -\eta_k), & t=t_k,\\ \overline{\vartheta}(s)=\overline{\theta}(s)-\overline{\phi}(s-d), & s\in[(-\tau+d)\wedge(-\eta+d),d].\end{cases}\tag{17}$$

Theorem 1. *Assume that Hypothesis 1 is satisfied, there exist* $(n-q)\times(n-q)$ *real matrices* $N_1>0,\Phi_1>0,\Psi_{11},\Psi_{21}$, $(2q-n)\times(2q-n)$ *real matrices* $N_2>0,\Phi_2>0,\Psi_{12},\Psi_{22}, 2q\times 2q$ *diagonal matrices* $U_1>0,U_2>0$, *and* $\gamma>0,\delta>0,\beta>1,p>0,\alpha>0$ *are constants sufficing*

$$\alpha+|\lambda|+\beta\delta e^{\alpha\tau}<\frac{\ln\beta}{p},$$

$$\begin{pmatrix}\Omega & \Upsilon\overline{C} & \Upsilon\overline{D}\\ \star & -U_1 & 0\\ \star & \star & -U_2\end{pmatrix}\leq 0,$$

$$\overline{B}^T\Upsilon+\Upsilon\overline{B}+LU_2L-\delta\Upsilon\leq 0,$$

$$\begin{pmatrix}-\overline{\beta}N & \Psi_1 & \Psi_2\\ \star & -N & 0\\ \star & \star & -\Phi\end{pmatrix}\leq 0,\tag{18}$$

where $\overline{\beta}=\beta^{-1}e^{-\alpha\eta}$ *and*

$$\begin{aligned}&N=\operatorname{diag}\{N_1,N_2\},\Omega\\ &L=\operatorname{diag}\left\{l_{r_1},\ldots,l_{r_q},l_{r_{q+1}},\ldots,l_{r_n},l_{r_{n-q+1}},\ldots,l_{r_q}\right\},\\ &\Phi=\operatorname{diag}\{\Phi_1,\Phi_2\},\Upsilon\\ &\Psi_1=\operatorname{diag}\{\Psi_{11},\Psi_{12}\},\Psi_2\end{aligned}\tag{19}$$

This signifies that NN (2) is globally lag synchronized with the drive NN (1) for

$$K=T^{-1}\begin{pmatrix}N_1^{-1}\Psi_{11}^T & 0 & 0\\ 0 & N_2^{-1}\Psi_{12}^T & 0\\ \Phi_1^{-1}\Psi_{21}^T+N_1^{-1}\Psi_{11}^T & 0 & 0\end{pmatrix}T,\tag{20}$$

in Θ_d, *where real matrix* T *is* $n\times n$.

Proof. Consider Lyapunov functions

$$\begin{aligned}V(t)&=\overline{\vartheta}_1^T(t)N\overline{\vartheta}_1(t)+\left(\overline{\vartheta}_2(t)-\overline{\vartheta}_1(t)\right)^T\Phi\left(\overline{\vartheta}_2(t)-\overline{\vartheta}_1(t)\right)\\ &=\overline{\vartheta}(t)Y\overline{\vartheta}(t).\end{aligned}\tag{21}$$

When $t\in[t_{k-1},t_k)$, taking the derivative of $V(t)$ in NN (14), we can get

$$\begin{aligned}\dot{V}(t) &= \dot{\overline{\vartheta}}(t)^T\Upsilon\overline{\vartheta}(t) + \overline{\vartheta}(t)^T\Upsilon\dot{\overline{\vartheta}}(t)\\ &= \overline{\vartheta}(t)^T\left(-\overline{A}^T\Upsilon - \Upsilon\overline{A}\right)\overline{\vartheta}(t) + \overline{\vartheta}(t-\tau(t))^T\\ &\quad\left(\overline{B}^T\Upsilon + \Upsilon\overline{B}\right)\overline{\vartheta}(t-\tau(t))\\ &\quad + 2\overline{\vartheta}(t)^T\Upsilon\overline{C}g(\overline{\vartheta}(t)) + 2\overline{\vartheta}(t)^T\Upsilon\overline{D}g(\overline{\vartheta}(t-\tau(t))).\end{aligned} \tag{22}$$

According to Hypothesis 1 and Lemma 2, we can get

$$\begin{aligned}&2\overline{\vartheta}(t)^T\Upsilon\overline{C}g(\overline{\vartheta}(t))\\ &\leq \overline{\vartheta}(t)^T\Upsilon\overline{C}U_1^{-1}\overline{C}^T\Upsilon\overline{\vartheta}(t) + g^T(\overline{\vartheta}(t))U_1g(\overline{\vartheta}(t))\\ &\leq \overline{\vartheta}(t)^T\Upsilon\overline{C}U_1^{-1}\overline{C}^T\Upsilon\overline{\vartheta}(t) + \overline{\vartheta}^T(t)LU_1L\overline{\vartheta}(t),\\ &2\overline{\vartheta}(t)^T\Upsilon\overline{D}g(\overline{\vartheta}(t-\tau(t)))\\ &\leq \overline{\vartheta}^T(t)\Upsilon\overline{D}U_2^{-1}\overline{D}^T\Upsilon\overline{\vartheta}(t) + g^T(\overline{\vartheta}(t-\tau(t)))U_2g(\overline{\vartheta}(t-\tau(t)))\\ &\leq \overline{\vartheta}(t)^T\Upsilon\overline{D}U_2^{-1}\overline{D}^T\Upsilon g(\overline{\vartheta}(t)) + \overline{\vartheta}^T(t-\tau(t))LU_2L\overline{\vartheta}(t-\tau(t)).\end{aligned} \tag{23}$$

Substituting the above inequality into (19), considering inequalities (16) and (17), we obtain

$$\begin{aligned}\dot{V}(t) &\leq \dot{\overline{\vartheta}}(t)^T\left(-A^T\Upsilon - \Upsilon A + LU_1L + \Upsilon\overline{C}U_1^{-1}\overline{C}^T\Upsilon + \Upsilon\overline{D}U_2^{-1}\overline{D}^T\Upsilon\right)\overline{\vartheta}(t)\\ &\quad + \overline{\vartheta}^T(t-\tau(t))\left(\overline{B}^T\Upsilon + \Upsilon\overline{B} + LU_2L\right)\overline{\vartheta}(t-\tau(t))\\ &\leq \lambda\overline{\vartheta}^T(t)\Upsilon\overline{\omega}(t) + \delta\overline{\vartheta}^T(t-\tau(t))\Upsilon\overline{\vartheta}(t-\tau(t)).\end{aligned} \tag{24}$$

When $t = t_k$, by (11), we can get

$$\begin{aligned}V(t_k^+) &= \overline{\vartheta}_1(t_k-\eta_k)^T K_1^T N K_1\overline{\vartheta}_1(t_k-\eta_k)\\ &\quad + \overline{\vartheta}_1(t_k-\eta_k)^T(K_2-K_1)^T\Phi(K_2-K_1)\overline{\vartheta}_1(t_k-\eta_k)\\ &\leq (\beta e^{\alpha\eta})^{-1}V(t_k^- - \eta_k).\end{aligned} \tag{25}$$

Considering the inequality in the condition, and using Lemma 1, we can get

$$V(t) \leq \beta\sup_{\overline{s}\leq s\leq d}V(s)e^{-\alpha t}, \tag{26}$$

which implies that

$$|\vartheta(t)| \leq \sqrt{\frac{\beta\lambda_{\max}(Y)}{\lambda_{\min}(Y)}}\vartheta(d)\exp\left(-\frac{\alpha}{2}t\right), \quad t \geq t_0, \tag{27}$$

where $\vartheta(d) = \sup_{\overline{s}\leq s\leq d}\vartheta(s), \overline{s} = (-\tau+d)\wedge(-\eta+d)$. From the above proof, it follows that error NN (17) converges exponentially to zero. That is, the coupled NN with time-varying delay achieves lag synchronization. The proof has been completed. □

Case 2. The number of set H is less than or equal the number of set G; i.e., $q \leq n-q$.

Then we consider the lag synchronization of NNs with time-varying delayed impulsive control, under the $q \leq n-q$. Considering error sates ϑ, set $\vartheta = (\vartheta_1^T, \vartheta_2^T)^T$, $\vartheta_1 = (e_{r_1}, \ldots, e_{r_q})^T, \vartheta_2 = (\vartheta_2^{1T}, \vartheta_2^{2T})^T, \vartheta_2^1 = (e_{r_{n-q+1}}, \ldots, e_{r_{2q}})^T$, $\vartheta_2^2 = (e_{r_{2q+1}}, \ldots, e_{r_n})^T$, $g(\vartheta_1(\cdot)) = (g_{r_1}(e_{r_1}(\cdot)), \ldots, g_{r_q}(e_{r_q}(\cdot)))^T$, $g(\vartheta_2^1) = (g_{r_{q+1}}(e_{r_{q+1}}(\cdot)), \ldots, g_{r_{2q}}(e_{r_{2q}}(\cdot)))^T$, and $g(\vartheta_2^2) = (g_{r_{2q+1}}(e_{r_{2q+1}}(\cdot)), \ldots, g_{r_n}(e_{r_n}(\cdot)))^T ss$. After expanding the error state, NN (4) can be rewritten as

$$\begin{cases}\dot{\vartheta}(t) = -\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}\begin{pmatrix}\vartheta_1(t)\\ \vartheta_2^1(t)\\ \vartheta_2^2(t)\end{pmatrix} + \begin{pmatrix}B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\end{pmatrix}\begin{pmatrix}\vartheta_1(t-\tau(t))\\ \vartheta_2^1(t-\tau(t))\\ \vartheta_2^2(t-\tau(t))\end{pmatrix}\\ +\begin{pmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{pmatrix}\begin{pmatrix}g(\vartheta_1(t))\\ g(\vartheta_2^1(t))\\ g(\vartheta_2^2(t))\end{pmatrix} + \begin{pmatrix}D_{11} & D_{12} & D_{13}\\ D_{21} & D_{22} & D_{23}\\ D_{31} & D_{32} & D_{33}\end{pmatrix}\begin{pmatrix}g(\vartheta_1(t-\tau(t)))\\ g(\vartheta_2^1(t-\tau(t)))\\ g(\vartheta_2^2(t-\tau(t)))\end{pmatrix},\\ \vartheta(t) = K_\vartheta\vartheta(t^- - \eta_k), \quad t = t_k,\\ \vartheta(s) = \theta_\vartheta(s) - \phi_\vartheta(s-d), \quad s \in [(-\tau+d)\wedge(-\eta+d), d],\end{cases} \tag{28}$$

where $\vartheta_2^1, \vartheta_2^2 \in \vartheta_2$, since $q \le n-q$, and the control gain matrix is shown below:

$$K_\vartheta = \begin{pmatrix} k_1 & 0 & 0 \\ k_2^1 & 0 & 0 \\ 0 & k_2^2 & 0 \end{pmatrix}, \quad (29)$$

where k_1, k_2^1, and k_2^2 are real matrices. When $q \le n-q$, this means that the dimension of measurable state may be smaller than the value of unmeasurable state. For the sake of easy, the dimension of measurable state is extended to ϑ_1 part of measurable state e_{r_q}, and then we get $\tilde{\vartheta}_1 = (\vartheta_1^T, \hat{\vartheta}_1^T)^T$, where $\hat{\vartheta}_1 = (e_{r_q}, e_{r_q}, \ldots, e_{r_q})^T$. So, we can get $\tilde{\vartheta} = (\tilde{\vartheta}_1^T, \tilde{\vartheta}_2^T)^T, \tilde{\vartheta}_2 = \vartheta_2, g(\tilde{\vartheta}_1(\cdot)) = s(g(\vartheta_1(\cdot))^T, g\ (\hat{\vartheta}_1(\cdot))^T)^T$, $g(\tilde{\vartheta}_2(\cdot)) = g(\vartheta_2(\cdot))$, which means that $\tilde{\vartheta}_1$ and $\tilde{\vartheta}_2$ have the same dimension. When $q = n-q$, that is, H has the same dimension as G, $\tilde{\vartheta}_1 = \vartheta_1$ and $\tilde{\vartheta}_2 = \vartheta_2$. Hence, NN (26) can be indicated as follows:

$$\begin{cases} \tilde{\vartheta} = -\tilde{A}\tilde{\vartheta}(t) + \tilde{B}\tilde{\vartheta}(t-\tau(t)) + \tilde{C}g(\tilde{\vartheta}(t)) + \tilde{D}g(\tilde{\vartheta}(t-\tau(t))), & t \ne t_k, \\ \tilde{\vartheta}(t) = \tilde{K}\tilde{\vartheta}(t^- - \eta_k), & t = t_k, \\ \tilde{\vartheta}(s) = \tilde{\theta}(s) - \tilde{\phi}(s-d), & s \in = [(-\tau+d)\wedge(-\eta+d), d], \end{cases} \quad (30)$$

where

$$\begin{gathered} \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \tilde{B} = \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{pmatrix}, \tilde{C} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{pmatrix}, \tilde{D} = \begin{pmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{pmatrix}, \\ \tilde{A}_{11} = \begin{pmatrix} A_{11} & 0 \\ \hat{A}_1 & 0 \end{pmatrix}, \tilde{A}_{12} = \begin{pmatrix} A_{12} & A_{13} \\ \hat{A}_2 & \hat{A}_3 \end{pmatrix}, \tilde{A}_{21} = \begin{pmatrix} A_{21} & 0 \\ A_{31} & 0 \end{pmatrix}, \tilde{A}_{22} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, \\ \tilde{B}_{11} = \begin{pmatrix} B_{11} & 0 \\ \hat{B}_1 & 0 \end{pmatrix}, \tilde{B}_{12} = \begin{pmatrix} B_{12} & B_{13} \\ \hat{B}_2 & \hat{B}_3 \end{pmatrix}, \tilde{B}_{21} = \begin{pmatrix} B_{21} & 0 \\ B_{31} & 0 \end{pmatrix}, \tilde{B}_{22} = \begin{pmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{pmatrix}, \\ \tilde{C}_{11} = \begin{pmatrix} C_{11} & 0 \\ \hat{C}_1 & 0 \end{pmatrix}, \tilde{C}_{12} = \begin{pmatrix} C_{12} & C_{13} \\ \hat{C}_2 & \hat{C}_3 \end{pmatrix}, \tilde{C}_{21} = \begin{pmatrix} C_{21} & 0 \\ C_{31} & 0 \end{pmatrix}, \tilde{C}_{22} = \begin{pmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{pmatrix}, \\ \tilde{D}_{11} = \begin{pmatrix} D_{11} & 0 \\ \hat{D}_1 & 0 \end{pmatrix}, \tilde{D}_{12} = \begin{pmatrix} D_{12} & D_{13} \\ \hat{D}_2 & \hat{D}_3 \end{pmatrix}, \tilde{D}_{21} = \begin{pmatrix} D_{21} & 0 \\ D_{31} & 0 \end{pmatrix}, \tilde{D}_{22} = \begin{pmatrix} D_{22} & D_{23} \\ D_{32} & D_{33} \end{pmatrix}, \\ \hat{A}_1 = \begin{pmatrix} a_{r_{q,1}} & \cdots & a_{r_{q,q}} \\ \vdots & \vdots & \vdots \\ a_{r_{q,1}} & \cdots & a_{r_{q,q}} \end{pmatrix}, \hat{A}_2 = \begin{pmatrix} a_{r_{q,q+1}} & \cdots & a_{r_{q,2q}} \\ \vdots & \vdots & \vdots \\ a_{r_{q,q+1}} & \cdots & a_{r_{q,2q}} \end{pmatrix}, \hat{A}_3 = \begin{pmatrix} a_{r_{q,2q+1}} & \cdots & a_{r_{q,n}} \\ \vdots & \vdots & \vdots \\ a_{r_{q,2q+1}} & \cdots & a_{r_{q,n}} \end{pmatrix}, \end{gathered} \quad (31)$$

and $\hat{B}_i, \hat{C}_i, \hat{D}_i, i = 1, 2, 3$, similar to $\hat{A}_1, \hat{A}_2, \hat{A}_3$.

The control gain matrix $\tilde{K}$ is shown below:

$$\tilde{K} = \begin{pmatrix} K_1 & 0 \\ K_2 & 0 \end{pmatrix}, \quad (32)$$

where $\hat{k}_1$ is a real matrix, $K_1 = \begin{pmatrix} k_1 & 0 \\ 0 & \hat{k}_1 \end{pmatrix}$, and $K_2 = \begin{pmatrix} k_2^1 & 0 \\ 0 & k_2^2 \end{pmatrix}$.

Then, the error dynamical system is given as

$$\begin{cases} \tilde{\vartheta} = -\tilde{A}\tilde{\vartheta}(t) + \tilde{B}\tilde{\vartheta}(t-\tau(t)) + \tilde{C}g(\tilde{\vartheta}(t)) + \tilde{D}g(\tilde{\vartheta}(t-\tau(t))), & t \ne t_k, \\ \tilde{\vartheta}_1(t) = K_1\tilde{\vartheta}_1(t^- - \eta_k), & t = t_k, \\ \tilde{\vartheta}_2(t) = K_2\tilde{\vartheta}_1(t^- - \eta_k), & t = t_k, \\ \tilde{\vartheta}(s) = \tilde{\varphi}(s) - \tilde{\phi}(s-d), & s \in [(-\tau+d)\wedge(-\eta+d), d]. \end{cases} \quad (33)$$

Theorem 2. *Assume that Hypothesis 1 is satisfied, there exist* $q\times q$ *real matrices* $N_1>0,\Phi_1>0,\Psi_{11},\Psi_{21}$, $(n-2q)\times(n-2q)$ *real matrices* $N_2>0,\Phi_2>0,\Psi_{12},\Psi_{22}$, $2(n-q)\times 2(n-q)$ *diagonal matrices* $U_1>0,U_2>0$, *and* $\gamma>0,\delta>0,\beta>1,p>0,\alpha>0$ *are constants sufficing*

$$\alpha+|\lambda|+\beta\delta e^{\alpha\tau}<\frac{\ln\beta}{p},$$

$$\begin{pmatrix}\Omega & \Upsilon\overline{C} & \Upsilon\overline{D}\\ \star & -U_1 & 0\\ \star & \star & -U_2\end{pmatrix}\le 0,$$

$$\overline{B}^T\Upsilon+\Upsilon\overline{B}+LU_2L-\delta\Upsilon\le 0,$$

$$\begin{pmatrix}-\overline{\beta}N & \Psi_1 & \Psi_2\\ \star & -N & 0\\ \star & \star & -\Phi\end{pmatrix}\le 0, \tag{34}$$

where $\overline{\beta}=\beta^{-1}e^{-\alpha\eta}$, *and*

$$\begin{aligned}&N=\operatorname{diag}\{N_1,N_2\},\Omega=-A^T\Upsilon-\Upsilon A-\lambda\Upsilon+LU_1L,\\ &L=\operatorname{diag}\left\{1_{r_1},\ldots,1_{r_q},1_{r_q},\ldots,1_{r_q},1_{r_{n-q+1}},\ldots,1_{r_n}\right\},\\ &\Phi=\operatorname{diag}\{\Phi_1,\Phi_2\},\Upsilon=\begin{pmatrix}N+\Phi & -\Phi\\ -\Phi & \Phi\end{pmatrix},\\ &\Psi_1=\operatorname{diag}\{\Psi_{11},\Psi_{12}\},\Psi_2=\operatorname{diag}\{\Psi_{21},\Psi_{22}\}.\end{aligned} \tag{35}$$

Then, it signifies that NN (2) is globally lag synchronized with the drive NN (1) for

$$K=T^{-1}\begin{pmatrix}N_1^{-1}\Psi_{11}^T & 0 & 0\\ \Phi_1^{-1}\Psi_{21}^T+N_1^{-1}\Psi_{11}^T & 0 & 0\\ 0 & \Phi_2^{-1}\Psi_{22}^T+N_2^{-1}\Psi_{12}^T & 0\end{pmatrix}T, \tag{36}$$

over the class Θ_d, *where real matrix* T *is* $n\times n$.

Proof. Similarly, the proof of Theorem 2 is similar to Theorem 1. □

Remark 2. Compared with the general time delay NN, due to the complex structure of NN, it is particularly difficult to design delayed impulsive control for time-varying delay NNs, so in order to overcome this problem, this article designs two controllers, which make it easier to monitor the measurable status of the impulsive time information. In other words, when the state of the NN is not measurable, the information of the instantaneous measurable state can also be adopted through impulsive control.

Remark 3. Throughout the article, we solve the global lag synchronization problem on different dimensions of measurable and unmeasurable states using Theorems 1 and 2. It can be seen from this article that, in Theorem 1, when the dimension of the measurable state is greater than that of the unmeasurable state, we extend the dimension of the unmeasurable state and get lag synchronization; in Theorem 2, the dimension of the measurable state is less than the unmeasurable state, and we obtain the lag synchronization result by extending the dimension of the measurable state.

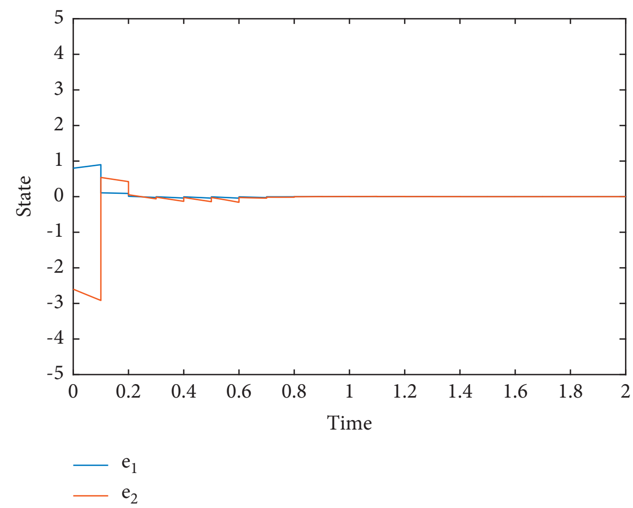

Figure 1: The trajectory of an error NN (12) with control inputs.

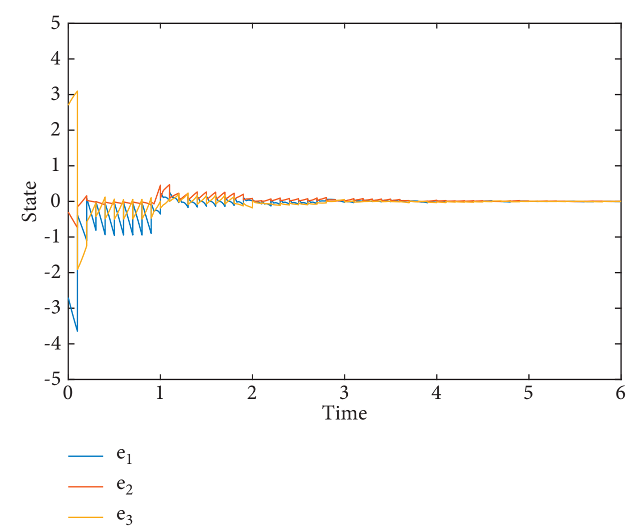

Figure 2: The trajectory of an error NN (26) with control inputs.

Remark 4. In [11], LMI is used to derive some sufficient conditions for lag synchronization of NNs with time-delayed. However, studies on NNs with time-varying delays are excluded. Our results not only study the lag synchronization of NNs with time-varying delays but also relax the restrictions on upper and lower bounds, which greatly reduces the time to reach the lag synchronization.

4. Examples

At last, a 2D and a 3D example are used to verify the main results of this article.

Example 1. Consider the 2D autonomous time-varying delay NNs

$$\begin{cases} \dot{z}(t) = -Az(t) + Bz(t-\tau(t)) + Cf(z(t)) + Df(z(t-\tau(t))) + J, & t > 0 \\ z(s) = \phi(s), & s \in [-\tau, 0], \end{cases} \quad (37)$$

where $A = B = I, f_1 = f_2 = \tanh(s), \tau(t) = 1/2|\cos(t)|$ and

$$C = \begin{pmatrix} 3.0 & -0.1 \\ -4.0 & 2.0 \end{pmatrix}, \quad D = \begin{pmatrix} -2.0 & -0.2 \\ -0.1 & 1.5 \end{pmatrix}. \quad (38)$$

In the simulation, the initial values of NN (36) are set as $\phi = (0.5, 0.8)^T$. The response NN is as follows:

$$\begin{cases} \dot{w}(t) = -Aw(t) + Bw(t-\tau(t)) + Cf(w(t)) \\ +Df(w(t-\tau(t))) + I + u(t), & t > d, \\ w(s) = \theta(s), & s \in [-\tau + d, d], \end{cases} \quad (39)$$

where the initial conditions of NN (37) are set as $\theta(s) = (1.3, -1.8)^T, d = 1$. The error dynamical system is given as

$$\begin{cases} \dot{e}(t) = -Ae(t) + Be(t-\tau(t)) + Cg(e(t)) + Dg(e(t-\tau(t))) + u(t), & t > d \\ e(t) = Ke(t^- - \eta_k), & t = t_k, \\ e(s) = \theta(s) - \phi(s-d), & s \in [(-\tau + d) \wedge (-\eta + d), d], \end{cases} \quad (40)$$

where $K = \begin{pmatrix} 0.1286 & 0 \\ 0.678 & 0 \end{pmatrix}$, $\sup_{k \in Z_+}\{t_k - t_{k-1}\} = 0.1, \eta_k = 0.03$.

Let $\alpha = 0.02, \lambda = 11, \delta = 1, \beta = 15$. According to Theorem 3.1, via solving the LMI matrix, we can get

$$\begin{aligned} \Upsilon &= \begin{pmatrix} 7.7063 & -0.0799 \\ -0.0799 & 0.0799 \end{pmatrix}, \\ U_1 &= \begin{pmatrix} 39.2851 & 0 \\ 0 & 39.2851 \end{pmatrix} \\ U_2 &= \begin{pmatrix} 21.3584 & 0 \\ 0 & 21.3584 \end{pmatrix}, \\ \Psi_1 &= 1.0714, \\ \Psi_2 &= 0.0954. \end{aligned} \quad (41)$$

In the example, the condition in Theorem 1 holds, so on the basis of Theorem 1, NNs with time-varying delays are lag synchronization.

Then the numerical simulation is shown in Figure 1.

Example 2. Consider the 3D autonomous time-varying delayed NNs involving unmeasurable states w_1 and w_3 with

$$C = \begin{pmatrix} 1.3 & -3 & -3 \\ -3 & 1.2 & -4.5 \\ -3 & 4.5 & 1.5 \end{pmatrix}, \quad D = \begin{pmatrix} 6.5 & -8 & -3 \\ -2.8 & 1.4 & -5 \\ -3 & -4.6 & -2.5 \end{pmatrix}. \quad (42)$$

According to Theorem 2, it can be known that the number of measurable states is less than or equal to that of unmeasurable states. In order to get same dimensions, the impulsive control gain matrix is designed by

$$K = \begin{pmatrix} 0 & 0.5236 & 0 \\ 0 & 0.1997 & 0 \\ 0.5236 & 0 & 0 \end{pmatrix}, \quad (43)$$

where $A = B = I, \tau(t) = 1/2|\sin t| + 1/3, f_1 = f_2 = f_3 = \tanh(s), \phi = (0.2, 0.2, -0.4)^T, \theta = (-0.5, 0.5, 0.3)^T$, and Θ_d: $t_{2n-1} = 0.04n + 0.4, t_{2n} = 0.04n + 0.5$. So, we can easily get

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (44)$$

The error system is transformed to

$$\begin{cases} \dot{\vartheta}(t) = -A_\vartheta \vartheta(t) + B_\vartheta \vartheta(t-\tau(t)) + C_\vartheta g(\vartheta(t)) + D_\vartheta g(\vartheta(t-\tau(t))), & t \neq t_k, \\ \vartheta(t) = K_\vartheta \vartheta(t^- - \eta_k), & t = t_k, \\ \vartheta(s) = \theta_\vartheta(s) - \phi_\vartheta(s-d), & s \in [(-\tau + d) \wedge (-\eta + d), d], \end{cases} \quad (45)$$

where $\vartheta = (\vartheta_1, \vartheta_2)^T, \vartheta_1 = x_2,$ and $\vartheta_2 = (\vartheta_2^1, \vartheta_2^1)^T = (x_1, x_3)^T,$

$$C_\vartheta = \begin{pmatrix} 1.2 & -3.0 & -4.5 \\ -3.0 & 1.3 & -3.0 \\ 4.5 & -3.0 & 1.5 \end{pmatrix}, D_\vartheta = \begin{pmatrix} 1.4 & -2.8 & -5 \\ -8 & 6.5 & -3 \\ 4.6 & -3 & -2.5 \end{pmatrix}, K_\vartheta = \begin{pmatrix} 0.1997 & 0 & 0 \\ 0.5236 & 0 & 0 \\ 0 & 0.5236 & 0 \end{pmatrix}. \quad (46)$$

Therefore, a new error system can be obtained

$$\begin{cases} \overline{\vartheta} = -\overline{A}\overline{\vartheta}(t) + \overline{B}\overline{\vartheta}(t-\tau(t)) + \overline{C}g(\overline{\vartheta}(t)) + \overline{D}g(\overline{\vartheta}(t-\tau(t))), & t \neq t_k, \\ \overline{\vartheta}_1(t) = K_1\overline{\vartheta}_1(t^- - \eta_k), & t = t_k, \\ \overline{\vartheta}_2(t) = K_2\overline{\vartheta}_1(t^- - \eta_k), & t = t_k, \\ \overline{\vartheta}(s) = \overline{\theta}(s) - \overline{\phi}(s-d), & s \in [(-\tau+d)\wedge(-\eta+d), d], \end{cases} \quad (47)$$

where

$$K_1 = \begin{pmatrix} 0.1997 & 0 \\ 0 & 0.1997 \end{pmatrix}, K_2 = \begin{pmatrix} 0.5236 & 0 \\ 0 & 0.5236 \end{pmatrix},$$
$$\widetilde{C} = \begin{pmatrix} 1.2 & 0 & -3.0 & -4.5 \\ 1.2 & 0 & -3.0 & -4.5 \\ -3 & 0 & 1.3 & -3.0 \\ 4.5 & 0 & -3.0 & 1.5 \end{pmatrix}, \widetilde{D} = \begin{pmatrix} 1.4 & 0 & -2.8 & -5 \\ 1.4 & 0 & -2.8 & -5 \\ -8 & 0 & 6.5 & -3 \\ 4.6 & 0 & -3 & -2.5 \end{pmatrix}. \quad (48)$$

Set $\alpha = 0.01, \lambda = 82, \delta = 7,$ and $\beta = 8$. From Example 1, similarly, we can see that it is not hard to check that LMI in Theorem 3 holds.

In the example, the condition in Theorem 2 holds, so on the basis of Theorem 2, NNs with time-varying delays are lag synchronization.

Then the numerical simulation is shown in Figure 2.

5. Concluding Remarks

Looking through the article, we study the synchronization problem of delay impulsive control for time-varying delay NNs. In this article, the measurable state and the unmeasurable state are separated by a transformation matrix. In Theorems 1 and 2, we use different methods to extend the dimension to obtain sufficient conditions for lag synchronization of NNs with time-varying delays derived from the control gain matrix. Finally, we also give two examples to confirm the validity of the theoretical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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