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Research Article

Drive-Response Synchronization for Second-Order Memristor-Based Delayed Neural Networks with Settling Time Estimation via Discontinuous Feedback Control

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In this paper, the settling time estimation of synchronization issues for inertial memristive neural networks (IMNNs) with mixed time-varying delays is investigated. First, by using a new reduced order approach and introducing free-weighted coefficients η_i and ξ_i into variable transformation, the original second-order derivative system is transformed into a first-order differential system. Second, appropriate controllers are designed for IMNNs to guarantee the system synchronization in a settling time. In addition, the settling time is explicitly estimated and dependent on time delays and the initial values of the coupled system. Finally, two numerical examples are presented to demonstrate the effectiveness of the theoretical results.

1. Introduction

The circuit memristors (as a contraction of memory and resistor) were first proposed in 1971 [1]. The prototype of the memristor was identified and built [2]. Due to the feature that rapid variation of voltage resulted in irregular change of memristance, memristor behavior was introduced to integrate circuit design [3]. Owing to some successful applications in various areas, more and more people paid attention to the dynamical characteristic analysis of memristor-based neural networks. In [4], passivity analysis of memristor-based neural networks has been considered by constructing appropriate Lyapunov–Krasovskii functionals. However, using the inequality techniques and useful Lyapunov functionals, paper [5] studied global exponential periodicity and stability of a class of memristor-based recurrent neural networks with multiple delays. Meanwhile, finite-time synchronization of memristor-based Cohen–Grossberg neural networks with time-varying delays was investigated in [6]. By designing Lyapunov-like function method and average dwell time technique, some delay-dependent sufficient conditions were given to guarantee the exponential stability of uncertain switched neural networks in [7]. In addition, in [8], prescribed time synchronization of coupled memristive neural networks was considered.

On the other hand, neural networks with inertial items have a strong practical application background in biology and engineering, and then most scholars started to pay more attention to this realm [9–11]. Global exponential stability in Lagrange sense for inertial neural networks was proposed [12]. In [13], matrix measure and Halanay inequality were used in synchronization analysis of IMNNs. Through impulses effect and periodically intermittent control, the global exponential synchronization of coupled IMNNs was investigated [14]. Synchronization of coupled IMNNs with reaction-diffusion terms was discussed by pinning sampled-data control in [15].

In recent years, many scholars have interest in various types of synchronization problems of IMNNs with time delays [16–20], and the solution theory of differential equation in the sense of Filippov has received great attention [21]. Meanwhile, the finite-time synchronization issues of IMNNs with time delays via different control ways were being given increasing amount of attention [22–26]. The finite-time stabilization control problems were discussed in present of discontinuous activations and several delays for

IMNNs [27]. In IMNNs with discrete and distributed delays, global synchronization and passivity analysis were investigated in [28, 29], respectively. In [30, 31], the dynamical analysis employed adaptive control approach and the theory of differential equations with discontinuous right-hand sides. The authors investigated the finite time and fixed time synchronization of IMNNs using the Lyapunov stability theory and Filippov discontinuous theory [32]. In [33], some novel and effective criteria were built to achieve asymptotic synchronization and finite synchronization in the fractional-order model for IMNNs. Novel sufficient conditions were given to guarantee finite-time synchronization with the drive and response delayed IMNNs [34].

As is well known, time delays in particular mixed time-varying delays unavoidably appear during the finite switching speed of neurons. Then, the IMNNs with mixed time-varying delays have become a complicated switched delayed nonlinear system, which means its dynamic analysis tends to be more challenging. During modeling neural networks, mixed time-varying delays are inevitably encountered in the signal transmission among the neurons because of the finite switching speed of the neurons and amplifiers [35]. Meanwhile, owing to the presence of an amount of parallel pathways of kinds of axon sizes and lengths, it is desired that continuously distributed delays are introduced over certain duration of time, such that the distant past has less influence compared to the recent behavior of the state [36]. Hence, it is necessary to pay close attention to discrete and distributed time delays. Taking the influence of actuator failures into account for IMNNs with mixed time-varying delays, improved delay-independent reliable controllers were designed to guarantee finite-time synchronization in [37]. However, to the best of authors' knowledge, there was little work reported on settling time estimation of synchronization issues of IMNNs with mixed time-varying delays.

Motivated by the abovementioned observations, we consider the settling time estimation of synchronization for IMNNs with mixed time-varying delays. The main contributions of this paper include the following aspects:

(1) A novel reduced order approach is proposed, and free-weighted coefficients η_i and ξ_i are introduced into variable transformation, which is less conservative and extend existing results. In previous paper, ξ_i was only introduced.

(2) However, designing new discontinuous controllers, some recent conditions are given to realize the settling time estimation of synchronization for driven-response system for IMNNs with mixed time-varying delays.

(3) Without using existing finite-time stability theorems, synchronization can obtain a settling time for IMNNs via combining new Lyapunov–Krasovskii functionals with recent inequality skills.

This paper is organized as follows: In section 2, model description and preliminaries are presented. In Section 3, we give the main results and its proof. In Section 4, two examples with simulations are presented to illustrate our main results. Finally, the conclusion of this paper is given in Section 5.

2. Preliminaries and Model Description

In this paper, the following inertial neural network is considered:

$$\begin{aligned}\frac{d^2 x_i(t)}{dt^2} &= -a_i\frac{dx_i(t)}{dt} - b_i x_i(t) + \sum_{j=1}^{n} c_{ij}(x_i(t)) f_j\big(x_j(t)\big) + \sum_{j=1}^{n} d_{ij}(x_i(t)) f_j\big(x_j(t-\tau_j(t))\big)\\ &\quad + \sum_{j=1}^{n} h_{ij}(x_i(t)) \int_{t-\sigma(t)}^{t} f_j\big(x_j(s)\big)ds + I_i(t), \quad t \geq 0,\end{aligned} \tag{1}$$

where $i = 1, 2, \ldots, n$, $x_i(t)$ represents the state of the ith neuron at time t, and the second derivative is called an inertial term of (1). $a_i > 0$ and $b_i > 0$ are constant. $c_{ij}(x_i(t))$, $d_{ij}(x_i(t))$, and $h_{ij}(x_i(t))$ are connection weights related to the neurons without delays, with discrete delays and distributed delays, respectively. They are given as follows:

$$\begin{aligned} c_{ij}(x_i(t)) &= \begin{cases} \hat{c}_{ij}, & |x_i(t)| \leq T_i,\\ \check{c}_{ij}, & |x_i(t)| > T_i,\end{cases}\\ d_{ij}(x_i(t)) &= \begin{cases} \hat{d}_{ij}, & |x_i(t)| \leq T_i,\\ \check{d}_{ij}, & |x_i(t)| > T_i,\end{cases}\\ h_{ij}(x_i(t)) &= \begin{cases} \hat{h}_{ij}, & |x_i(t)| \leq T_i,\\ \check{h}_{ij}, & |x_i(t)| > T_i,\end{cases}\end{aligned} \tag{2}$$

in which switching jump $T_i > 0$, and $\widehat{c}_{ij}, c_{ij}, \widehat{d}_{ij}, d_{ij}, \widehat{h}_{ij}$, and h_{ij} are known constants with respect to memristance. $f_j(t)$ stands for neuron activation function of jth neuron at time t with $f_j(0) = 0$. The $\tau_j(t)$ is the time-varying delay which satisfies $0 \le \tau_j(t) \le \tau_j$, $\dot{\tau}_j(t) \le \mu_j < 1$. $\sigma(t)$ is the distributed delay and $0 \le \sigma(t) \le \sigma$, $\dot{\sigma}(t) \le \widetilde{\sigma} < 1$. $I_i(t)$ is the external input on the ith neuron at time t and $|I_i(t)| \le I_i$. I_i is the: constant. The initial conditions of system (1) are as follows:

$$x_i(s) = \phi_i(s), \frac{\mathrm{d}x_i(s)}{\mathrm{d}t} = \varphi_i(s), s \in [-\tau, 0], \tau = \max\left\{\tau_j, \sigma\right\}, j = 1, 2, \ldots, n, \tag{3}$$

where $\phi_i(s)$ and $\varphi_i(s) \in C([-\tau, 0]; R^n)$.

Assumption 1. The activation function $f_j(\cdot)$ meets with the Lipschitz condition, i.e., there exists positive constants $l_i > 0$ such that for any $x, y \in R$, the following inequalities hold:

$$\left|f_j(x) - f_j(y)\right| \le l_j|x - y|, j = 1, 2, \ldots, n, \tag{4}$$

and there exists positive constants F_j such that $|f_j(\cdot)| \le F_j$.

Resorting to the following variable transformation,

$$y_i(t) = \eta_i \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} + \xi_i x_i(t), \eta_i \ne 0, i = 1, 2, \ldots, n, \tag{5}$$

the inertial neural network (1) can be written as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\dfrac{\xi_i}{\eta_i} x_i(t) + \dfrac{1}{\eta_i} y_i(t), \\ \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -\beta_i y_i(t) + \alpha_i x_i(t) + \eta_i \displaystyle\sum_{j=1}^{n} c_{ij}(x_i(t)) f_j\left(x_j(t)\right) + \eta_i \sum_{j=1}^{n} d_{ij}(x_i(t)) f_j\left(x_j\left(t - \tau_j(t)\right)\right) \\ + \eta_i \displaystyle\sum_{j=1}^{n} h_{ij}(x_i(t)) \int_{t-\sigma(t)}^{t} f_j\left(x_j(s)\right) ds + \eta_i I_i(t), \quad t \ge 0, i = 1, 2, \ldots, n, \end{cases} \tag{6}$$

where $\alpha_i = -(\xi_i^2/\eta_i) + \alpha_i \xi_i - \eta_i b_i, \beta_i = \alpha_i - (\xi_i/\eta_i)$. The initial conditions can be written as follows:

$$\begin{cases} x_i(s) = \phi_i(s), \\ y_i(s) = \eta_i \varphi_i(s) + \xi_i(s)\phi_i(s) \triangleq \psi_i(s), s \in [-\tau, 0], \tau = \max\limits_{j=1,2,\ldots,n}\left\{\tau_j, \sigma\right\}. \end{cases} \tag{7}$$

Remark 2. In [11], only ξ_i was introduced in certain variable transformations, but we introduce free-weighted coefficients η_i and ξ_i into variable transformation (5) in our paper. The different variable transformations can be obtained by selecting η_i and ξ_i with different values, which make our results less conservative.

Remark 3. In systems (1) and (6), since $c_{ij}(t)$, $d_{ij}(t)$, and $h_{ij}(t)$ are discontinuous, the classical definition of the solution for differential equations cannot apply here. To solve the problem, Filippov presented a solution concept for the differential equation with discontinuous right-hand side. Based on the definition, a differential equation with discontinuous right-hand side has the same solution set as certain differential inclusion.

Now, we introduce the concept of Filippov solution [21]. Consider the following differential system:

$$\frac{d^2 x}{dt^2} = f(t, x), \tag{8}$$

where $f(t, x)$ is discontinuous in x.

Definition 4. Consider the set-valued map $F\colon R \times R^n \longrightarrow R^n$ defined as follows:

$$F(t, x) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{\mathrm{co}}\left[f\left(t, \frac{B(x, \delta)}{N}\right)\right], \tag{9}$$

where $B(x, \delta)$ is the ball of center x and radius δ, $\mu(N)$ is the Lebesgue of set N. A vector-value function $x(t)$ defined on a nondegenerate interval $I \subset R$ is called a Filippov solution of system (8), if it is absolutely continuous on any subinterval $[t_1, t_2]$ of I and for a.a. $t \in I$. $x(t)$ satisfies the differential inclusion as follows:

$$\frac{d^2x}{dt^2} = F(t, x). \tag{10}$$

Next, let us consider the differential equation system (6). Let the set-valued maps be as follows:

$$\begin{aligned}
K\left[c_{\mathrm{ij}}\left(x_i(t)\right)\right] &= \begin{cases} \widehat{c}_{\mathrm{ij}}, \left|x_i(t)\right| \le T_i, \\ \check{c}_{\mathrm{ij}}, \left|x_i(t)\right| > T_i, \end{cases} \\
K\left[d_{ij}\left(x_i(t)\right)\right] &= \begin{cases} \widehat{d}_{\mathrm{ij}}, \left|x_i(t)\right| \le T_i, \\ \check{\mathrm{d}}_{\mathrm{ij}}, \left|x_i(t)\right| > T_i, \end{cases} \\
K\left[h_{ij}\left(x_i(t)\right)\right] &= \begin{cases} \widehat{h}_{\mathrm{ij}}, \left|x_i(t)\right| \le T_i, \\ \check{h}_{\mathrm{ij}}, \left|x_i(t)\right| > T_i, \end{cases}
\end{aligned} \tag{11}$$

for $t \in R$ and $i, j = 1, 2, \ldots, n$. It is obvious that $K[c_{\mathrm{ij}}(x_i(t))]$, $K[d_{\mathrm{ij}}(x_i(t))]$, and $K[h_{\mathrm{ij}}(x_i(t))]$ are all closed, convex, and compact.

We define the Filippov solution of system (6) as follows.

Definition 5. (Filippov solution [21]). A function $x(t)$ is said to be a solution of system (6) on $[(0, T)$ with initial condition (7), if $x(t)$ is absolutely continuous on any compact interval of $[(0, T)$ and satisfies differential inclusions

$$\begin{cases}
\dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} \in -\dfrac{\xi_i}{\eta}x_i(t) + \dfrac{1}{\eta_i}y_i(t), \\
\dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} \in -\beta_i y_i(t) + \alpha_i x_i(t) + \eta_i \displaystyle\sum_{j=1}^{n} K\left[c_{\mathrm{ij}}\left(x_i(t)\right)\right] f_j\left(x_j(t)\right) \\
+ \eta_i \displaystyle\sum_{j=1}^{n} K\left[d_{\mathrm{ij}}\left(x_i(t)\right)\right] f_j\left(x_j\left(t - \tau_j(t)\right)\right) \\
+ \eta_i \displaystyle\sum_{j=1}^{n} K\left[h_{\mathrm{ij}}\left(x_i(t)\right)\right] \int_{t-\sigma(t)}^{t} f_j\left(x_j(s)\right) ds + \eta_i I_i(t),
\end{cases} \tag{12}$$

or $\gamma_{\mathrm{ij}} \in K[c_{\mathrm{ij}}(x_i(t))]$, $\zeta_{\mathrm{ij}} \in K[d_{\mathrm{ij}}(x_i(t))]$, and $\nu_{\mathrm{ij}} \in K[h_{\mathrm{ij}}(x_i(t))]$ satisfy

$$\begin{cases}
\dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\dfrac{\xi_i}{\eta_i}x_i(t) + \dfrac{1}{\eta_i}y_i(t), \\
\dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -\beta_i y_i(t) + \alpha_i x_i(t) + \eta_i \displaystyle\sum_{j=1}^{n} \gamma_{\mathrm{ij}} f_j\left(x_j(t)\right) + \eta_i \displaystyle\sum_{j=1}^{n} \zeta_{\mathrm{ij}} f_j\left(x_j\left(t - \tau_j(t)\right)\right) \\
+ \eta_i \displaystyle\sum_{j=1}^{n} \nu_{ij} \int_{t-\sigma(t)}^{t} f_j\left(x_j(s)\right) ds + \eta_i I_i(t), \\
for\ a.a.t \in [0, T), i = 1, 2, \ldots, n.
\end{cases} \tag{13}$$

Based on the concept of drive-response synchronization, the corresponding response system of (13) is given in the following form:

$$\begin{cases} \dfrac{\mathrm{d}m_i(t)}{\mathrm{d}t} = -\dfrac{\xi_i}{\eta_i}m_i(t) + \dfrac{1}{\eta_i}w_i(t) + u_{1i}(t), \\ \dfrac{\mathrm{d}w_i(t)}{\mathrm{d}t} = -\beta_i w_i(t) + \alpha_i m_i(t) + \eta_i \sum_{j=1}^{n} \widetilde{\gamma}_{\mathrm{ij}} f_j\big(m_j(t)\big) + \eta_i \sum_{j=1}^{n} \widetilde{\zeta}_{ij} f_j\big(m_j\big(t-\tau_j(t)\big)\big) \\ + \eta_i \sum_{j=1}^{n} \widetilde{\nu}_{ij} \int_{t-\sigma(t)}^{t} f_j\big(m_j(s)\big)ds + \eta_i I_i(t) + u_{2i}(t), \\ \text{for } a.a.t \in [0,T), i = 1,2,\ldots,n. \end{cases} \tag{14}$$

where $\widetilde{\gamma}_{ij} \in K[c_{\mathrm{ij}}(m_i(t))]$, $\widetilde{\zeta}_{\mathrm{ij}} \in K[d_{\mathrm{ij}}(m_i(t))]$, and $\widetilde{\nu}_{\mathrm{ij}} \in K[h_{\mathrm{ij}}(m_i(t))]$. We define error state $e_{1i}(t) = m_i(t) - x_i(t)$ and $e_{2i}(t) = w_i(t) - y_i(t)$. Then, we can obtain error system from (13) and (14).

$$\begin{cases} \dfrac{\mathrm{d}e_{1i}(t)}{\mathrm{d}t} = -\dfrac{\xi_i}{\eta_i}e_{1i}(t) + \dfrac{1}{\eta_i}e_{2i}(t) + u_{1i}(t), \\ \dfrac{\mathrm{d}e_{2i}(t)}{\mathrm{d}t} = -\beta_i e_{2i}(t) + \alpha_i e_{1i}(t) + \eta_i \sum_{j=1}^{n} \widetilde{\gamma}_{\mathrm{ij}} f_j\big(e_{1j}(t)\big) \\ + \eta_i \sum_{j=1}^{n} \big(\widetilde{\gamma}_{\mathrm{ij}} - \gamma_{\mathrm{ij}}\big) f_j\big(x_j(t)\big) + \eta_i \sum_{j=1}^{n} \widetilde{\zeta}_{\mathrm{ij}} f_j\big(e_{1j}\big(t-\tau_j(t)\big)\big) \\ + \eta_i \sum_{j=1}^{n} \big(\widetilde{\zeta}_{\mathrm{ij}} - \zeta_{\mathrm{ij}}\big) f_j\big(x_j\big(t-\tau_j(t)\big)\big) + \eta_i \sum_{j=1}^{n} \widetilde{\nu}_{\mathrm{ij}} \int_{t-\sigma(t)}^{t} f_j\big(e_{1j}(s)\big)\mathrm{d}s \\ \eta_i \sum_{j=1}^{n} \big(\widetilde{\nu}_{\mathrm{ij}} - \nu_{\mathrm{ij}}\big) \int_{t-\sigma(t)}^{t} f_j\big(x_j(s)\big)\mathrm{d}s + u_{2i}(t), \\ \text{for } a.a.t \in [0,T), i = 1,2,\ldots,n, \end{cases} \tag{15}$$

where $f_j(e_{1j}(t)) = f_j(m_j(t)) - f_j(x_j(t))$, $f_j(e_{1j}(t-\tau_j(t))) = f_j(m_j(t-\tau_j(t))) - f_j(x_j(t-\tau_j(t)))$.

Definition 6 (see [18]). The system (13) is said to be synchronized with (14) in a settling time under suitable designed feedback controllers $u_{1i}(t)$ and $u_{2i}(t)$, if there exists a constant $t_1 > 0$ (t_1 depends on the initial state vector error value and time delay) such that $\lim_{t\longrightarrow t_1}(\|e_1(t)\|_1 + \|e_2(t)\|_1) = 0$ and $\|e_1(t)\|_1 + \|e_2(t)\|_1 = 0$ for $\forall t \geq t_1$, where $\|e_1(t)\|_1 + \|e_2(t)\|_1 = \sum_{i=1}^{n}|e_{1i}(t)| + \sum_{i=1}^{n}|e_{2i}(t)|$, $e_1(t) = (e_{11}(t), e_{11}(t), \cdots, e_{1n}(t))^T$, and $e_2(t) = (e_{21}(t), e_{22}(t), \ldots, e_{2n}(t))^T$. t_1 is called the settling time.

3. Results

In this section, suitable controllers are designed for the finite-time synchronization in (13) and (14). By using Lyapunov functionals method and some analytical techniques, several sufficient conditions to ensure synchronization of IMNNs are obtained.

The controllers are designed as follows:

$$\begin{cases} u_{1i}(t) = -K_{1i}e_{1i}(t) - Q_{1i}\mathrm{sgn}(e_{1i}(t)), \\ u_{2i}(t) = -K_{2i}e_{2i}(t) - Q_{2i}\mathrm{sgn}(e_{2i}(t)), \end{cases} \tag{16}$$

where $K_{1i}(t)$, $K_{2i}(t)$, $Q_{1i}(t)$, and $Q_{2i}(t)$ are control gains, and $\mathrm{sgn}(\cdot)$ is the standard sign function.

The following theorems and corollaries are our main results.

Theorem 7. *If $f(\cdot)$ meets Assumption 1, there exists $Q_{1i} > 0$ and K_{1i}, K_{2i}, and Q_{2i} satisfy the following inequalities:*

$$K_{1i} \geq -\frac{\xi_i}{\eta_i} + |\alpha_i| + \sum_{j=1}^{n} |\eta_i| c_{ji}^{+} l_i + \sum_{j=1}^{n} \frac{|\eta_i|}{1-\mu_i} d_{ji}^{+} l_i + \sum_{j=1}^{n} \frac{|\eta_i|}{1-\widetilde{\sigma}} h_{ji}^{+} l_i \sigma, \tag{17}$$

$$K_{2i} \geq \frac{1}{|\eta_i|} - \beta_i, \tag{18}$$

$$Q_{2i} > M_i. \tag{19}$$

Then, the systems (13) and (14) are synchronized in a finite-time under controllers (16). Moreover, the settling time is estimated as follows:

$$\begin{aligned} t_1 \leq \frac{1}{\theta_0} \Bigg[& \sum_{i=1}^{n} |e_{1i}(0)| + \sum_{i=1}^{n} |e_{2i}(0)| + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|\eta_i|}{1-\mu_j} d_{ij}^{+} l_j \int_{-\tau_j}^{0} |e_{1j}(s)| \mathrm{d}s \\ & \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|\eta_i|}{1-\widetilde{\sigma}} h_{ij} l_j \int_{-\sigma}^{0} \int_{s}^{0} |e_{1j}(u)| \mathrm{d}u\mathrm{d}s \Bigg] - \max_{j=1,2,\ldots,n} \{\tau_j, \sigma\}, \end{aligned} \tag{20}$$

where $\widehat{c}_{ij}, c_{ij}, \widehat{d}_{ij}, d_{ij}, \widehat{h}_{ij}, h_{ij}$ are known constants with respect to memristance, $c_{ij}^{+} = \max\{|c_{ij}|, |\widehat{c}_{ij}|\}$, $d_{ij}^{+} = \max\{|d_{ij}|, |\widehat{d}_{ij}|\}$, $h_{ij}^{+} = \max\{|h_{ij}|, |\widehat{h}_{ij}|\}$, $M_i = |\eta_i|\sum_{j=1}^{n}|c_{ij} - \widehat{c}_{ij}|F_j + |\eta_i|\sum_{j=1}^{n}|d_{ij} - \widehat{d}_{ij}| F_j + |\eta_i|\sum_{j=1}^{n}|h_{ij} - \widehat{h}_{ij}|F_j\sigma, \theta_0 = \min_{i=1,2,\ldots,n}\{(Q_{2i} - M_i), Q_{1i}\}$, $\alpha_i = -(\xi_i^2/\eta_i) + \alpha_i\xi_i - \eta_i b_i$, and $\beta_i = \alpha_i - (\xi_i/\eta_i)$, $i = 1, 2, \ldots, n$.

Proof. Define the following Lyapunov–Krasovskii functional:

$$V(t) = \sum_{i=1}^{3} V_i(t), \tag{21}$$

where

$$\begin{aligned} V_1(t) &= \sum_{i=}^{n} |e_{1i}(t)| + \sum_{i=}^{n} |e_{2i}(t)|, \\ V_2(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|\eta_i|}{1-\mu_j} d_{ij}^{+} l_j \int_{t-\tau_j(t)}^{t} |e_{1j}(s)| \mathrm{d}s, \\ V_3(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|\eta_i|}{1-\widetilde{\sigma}} h_{ij}^{+} l_j \int_{-\sigma(t)}^{0} \int_{t+s}^{t} |e_{1j}(u)| \mathrm{d}u\mathrm{d}s. \end{aligned} \tag{22}$$

Considering the upper right-hand derivative of $V_i(t)$ $(i = 1, 2, 3)$ along the trajectory of the error system (15), the proof of process will be given as follows:

$$\dot{V}_1(t) = \sum_{i=1}^{n} \mathrm{sgn}(e_{1i}(t))\dot{e}_{1i}(t) + \sum_{i=1}^{n} \mathrm{sgn}(e_{2i}(t))\dot{e}_{2i}(t). \tag{23}$$

Through combining (15) with (23), we can obtain

$$\begin{aligned}\dot{V}_1(t) = {} & \sum_{i=1}^{n} \operatorname{sgn}(e_{1i}(t))\left[-\frac{\xi_i}{\eta_i}e_{1i}(t) + \frac{1}{\eta_i}e_{2i}(t) - K_{1i}e_{1i}(t) - Q_{1i}\operatorname{sgn}(e_{1i}(t))\right] \\ & + \left[\sum_{i=1}^{n} \operatorname{sgn}(e_{2i}(t)) - \beta_i e_{2i}(t) + \alpha_i e_{1i}(t) + \eta_i \sum_{j=1}^{n} \widetilde{\gamma}_{ij} f_j(e_{1j}(t)) + \eta_i \sum_{j=1}^{n} (\widetilde{\gamma}_{ij} - \gamma_{ij}) f_j(x_j(t))\right. \\ & + \eta_i \sum_{j=1}^{n} \widetilde{\zeta}_{ij} f_j(e_{1j}(t-\tau_j(t))) + \eta_i \sum_{j=1}^{n} (\widetilde{\zeta}_{ij} - \zeta_{ij}) f_j(x_j(t-\tau_j(t))) + \eta_i \sum_{j=1}^{n} \widetilde{\nu}_{ij} \int_{t-\sigma(t)}^{t} f_j(e_{1j}(s))\mathrm{d}s \\ & \left. + \eta_i \sum_{j=1}^{n} (\widetilde{\nu}_{ij} - \nu_{ij}) \int_{t-\sigma(t)}^{t} f_j(x_j(s))\mathrm{d}s - K_{2i}e_{2i}(t) - Q_{2i}\operatorname{sgn}(e_{2i}(t))\right].\end{aligned} \tag{24}$$

By Assumption 1, and let $c_{ij}^{+} = \max\left\{|c_{ij}|, |\widehat{c}_{ij}|\right\}, d_{ij}^{+} = \max\left\{|d_{ij}|, |\widehat{d}_{ij}|\right\}, h_{ij}^{+} = \max\left\{|h_{ij}|, |\widehat{h}_{ij}|\right\}$, we can have

$$\begin{aligned}& \operatorname{sgn}(e_{2i}(t))\left[\eta_i \sum_{j=1}^{n} \widetilde{\gamma}_{ij} f_j(e_{1j}(t)) + \eta_i \sum_{j=1}^{n} \widetilde{\zeta}_{ij} f_j(e_{1j}(t-\tau_j(t))) + \eta_i \sum_{j=1}^{n} \widetilde{\nu}_{ij} \int_{t-\sigma(t)}^{t} f_j(e_{1j}(s))\right]\mathrm{d}s \\ & \leq |\eta_i| \sum_{j=1}^{n} c_{ij}^{+} l_j |e_{1j}(t)| + |\eta_i| \sum_{j=1}^{n} d_{ij}^{+} l_j |e_{1j}(t-\tau_j(t))| + |\eta_i| \sum_{j=1}^{n} h_{ij}^{+} l_j \int_{t-\sigma(t)}^{t} |e_{1j}(s)|\mathrm{d}s,\end{aligned} \tag{25}$$

and

$$\begin{aligned}& \operatorname{sgn}(e_{2i}(t))\left\{\eta_i \sum_{j=1}^{n} (\widetilde{\gamma}_{ij} - \gamma_{ij}) f_j(x_j(t)) + \eta_i \sum_{j=1}^{n} (\widetilde{\zeta}_{ij} - \zeta_{ij}) f_j(x_j(t-\tau_j(t))) + \eta_i \sum_{j=1}^{n} (\widetilde{\nu}_{ij} - \nu_{ij}) \int_{t-\sigma(t)}^{t} f_j(x_j(s))\mathrm{d}s\right\} \\ & \leq \left\{|\eta_i| \sum_{j=1}^{n} |\check{c}_{ij} - \widehat{c}_{ij}| F_j + |\eta_i| \sum_{j=1}^{n} |\check{d}_{ij} - \widehat{d}_{ij}| F_j + |\eta_i| \sum_{j=1}^{n} |\check{h}_{ij} - \widehat{h}_{ij}| F_j \sigma\right\} |\operatorname{sgn}(e_{2i}(t))| = M_i |\operatorname{sgn}(e_{2i}(t))|.\end{aligned} \tag{26}$$

When $e_{1i}(t) \neq 0$, we will find that $-\operatorname{sgn}(e_{1i}(t))Q_{1i}\operatorname{sgn}(e_{1i}(t)) = -Q_{1i}$, otherwise $-\operatorname{sgn}(e_{1i}(t))Q_{1i}\operatorname{sgn}(e_{1i}(t)) = 0$. Similarly, $-\operatorname{sgn}(e_{2i}(t))Q_{2i}\operatorname{sgn}(e_{2i}(t)) = -Q_{2i}$ for $e_{2i}(t) \neq 0$ and $-\operatorname{sgn}(e_{2i}(t))Q_{2i}\operatorname{sgn}(e_{2i}(t)) = 0$ for $e_{2i}(t) = 0$. Therefore,

$$-\operatorname{sgn}(e_{1i}(t))Q_{1i}\operatorname{sgn}(e_{1i}(t)) = -Q_{1i}\lambda_{1i}, \tag{27}$$

$$-\operatorname{sgn}(e_{2i}(t))Q_{2i}\operatorname{sgn}(e_{2i}(t)) = -Q_{2i}\lambda_{2i}, \tag{28}$$

where $\lambda_{1i} = 1$ if $e_{1i}(t) \neq 0$, otherwise $\lambda_{1i} = 0$; $\lambda_{2i} = 1$ if $e_{2i}(t) \neq 0$, otherwise $\lambda_{2i} = 0$.

Substituting the systems (25–28) into (24), it is derived that

$$\begin{aligned}\dot{V}_1(t) \leq {} & \sum_{i=1}^{n}\left\{\left(-\frac{\xi_i}{\eta_i} - K_{1i} + |\alpha_i|\right)|e_{1i}(t)| + |\eta_i| \sum_{j=1}^{n} c_{ij}^{+} l_j |e_{1j}(t)| + |\eta_i| \sum_{j=1}^{n} d_{ij}^{+} l_j |e_{1j}(t-\tau_j(t))|\right. \\ & \left. + |\eta_i| \sum_{j=1}^{n} h_{ij}^{+} l_j \int_{t-\sigma(t)}^{t} |e_{1j}(s)|\mathrm{d}s + \left(\frac{1}{|\eta_i|} - \beta_i - K_{2i}\right)|e_{2i}(t)| - Q_{1i}\lambda_{1i} + (M_i - Q_{2i})\lambda_{2i}\right\}.\end{aligned} \tag{29}$$

It is obtained from $V_2(t)$, $V_3(t)$ that

$$\begin{aligned}\dot{V}_2(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|}{1-\mu_j}d_{ij}^{+}l_j\left|e_{1j}(t)\right| - \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|\left(1-\dot{\tau}_j(t)\right)}{1-\mu_j}d_{ij}^{+}l_j\left|e_{1j}\left(t-\tau_j(t)\right)\right| \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|}{1-\mu_j}d_{ij}^{+}l_j\left|e_{1j}(t)\right| - \sum_{i=1}^{n}\sum_{j=1}^{n}|\eta_i|d_{ij}^{+}l_j\left|e_{1j}\left(t-\tau_j(t)\right)\right|,\end{aligned} \tag{30}$$

and

$$\begin{aligned}\dot{V}_3(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|}{1-\widetilde{\sigma}}h_{ij}^{+}l_j\sigma(t)\left|e_{1j}(t)\right| - \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|\left(1-\dot{\sigma}_j(t)\right)}{1-\widetilde{\sigma}}h_{ij}^{+}l_j\int_{t-\sigma(t)}^{t}\left|e_{1j}(s)\right|ds \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|}{1-\widetilde{\sigma}}h_{ij}^{+}l_j\sigma(t)\left|e_{1j}(t)\right| - \sum_{i=1}^{n}\sum_{j=1}^{n}|\eta_i|h_{ij}^{+}l_j\int_{t-\sigma(t)}^{t}\left|e_{1j}(s)\right|\mathrm{d}s.\end{aligned} \tag{31}$$

From (29–31), one has

$$\begin{aligned}\dot{V}(t) &\leq \sum_{i=1}^{n}\left(-\frac{\xi_i}{\eta_i}-K_{1i}+|\alpha_i|+\sum_{j=1}^{n}|\eta_i|c_{ji}^{+}l_i+\sum_{j=1}^{n}\frac{|\eta_i|}{1-\mu_j}d_{ji}^{+}l_i+\sum_{j=1}^{n}\frac{|\eta_i|}{1-\sigma}h_{ji}^{+}l_i\sigma\right)\left|e_{1i}(t)\right| \\ &\quad +\sum_{i=1}^{n}\left(\frac{1}{|\eta_i|}-\beta_i-K_{2i}\right)\left|e_{2i}(t)\right|+\sum_{i=1}^{n}\left[\left(M_i-Q_{2i}\right)\lambda_{2i}-Q_{1i}\lambda_{1i}\right].\end{aligned} \tag{32}$$

When $\|e_1(t)\|_1+\|e_2(t)\|_1\neq 0$. From (17–19) and (32), we can obtain that

$$\dot{V}(t)\leq\sum_{i=1}^{n}\left[\left(Q_{2i}-M_i\right)\lambda_{2i}-Q_{1i}\lambda_{1i}\right]\leq-\theta_0<0, \tag{33}$$

where $\theta_0=\min_{j=1,2,\ldots,n}\{(Q_{2i}-M_i),Q_{1i}\}$.

Integrating both sides of the inequality (33) from 0 to t, we can get the following inequality:

$$V(t)-V(0)\leq-\theta_0 t. \tag{34}$$

There exists $t_1\in(0,+\infty)$ such that

$$\begin{aligned}&\lim_{t\longrightarrow t_1}\left(\left\|e_1(t)\right\|_1+\left\|e_2(t)\right\|_1\right) \\ &\quad=0 \text{ and } \left\|e_1(t)\right\|_1+\left\|e_2(t)\right\|_1\equiv 0, \forall t\geq t_1.\end{aligned} \tag{35}$$

By virtue of (34) and (35), there obviously exists $t_2=t_1+\max_{j=1,2,\ldots,n}\left\{\pi_j,\sigma\right\}$ such that

$$\lim_{t\longrightarrow t_2}V(t)=0\,and\,V(t)\equiv 0,\quad\forall t\geq t_2. \tag{36}$$

From (33) and (36), we can get that $\dot{V}(t)\leq-\theta_0$ for $t<t_2$. Integrating both sides of the inequality from 0 to t_2 obtains that

$$t_2\leq\frac{V(0)}{\theta_0}=\frac{1}{\theta_0}\left\{\sum_{i=1}^{n}\left|e_{1i}(0)\right|+\sum_{i=1}^{n}\left|e_{2i}(0)\right|+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|}{1-\mu_j}d_{ij}^{+}l_j\int_{-\tau_j}^{0}\left|e_{1j}(s)\right|\mathrm{d}s+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|\eta_i|}{1-\widetilde{\sigma}}h_{ij}^{+}l_j\int_{-\sigma}^{0}\int_{s}^{0}\left|e_{1j}(u)\right|\mathrm{d}u\mathrm{d}s\right\}. \tag{37}$$

According to $t_1=t_2-\max_{j=1,2,\ldots,n}\left\{\tau_j,\sigma\right\}$, the inequality (20) can easily be derived. This completes the proof. □

Remark 8. In [6, 22, 25, 26], the authors considered finite-time synchronization by finite-time stability theorem based on the inequality $\dot{V}(x) \le -\alpha V^{\eta}(x)$, where $\alpha > 0$, $0 < \eta < 1$ are constants, $\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$ with $\kappa-$class function $\alpha_1(\bullet)$ and $\alpha_2(\bullet)$. $V(x)$ is a continuous and positive definite function. In this paper, some recent inequality techniques are used to guarantee the system synchronizing in a settling time, which is explicitly estimated and dependent on time delays and initial values of the coupled system. On the other hand, the discontinue controllers are designed. Then, the error system is not discussed with using existing finite-time stability theorems.

Remark 9. Theorem 1 is achieved on the basis of 1-norm and the inequality (33) is the key step. The inequality (33) cannot be obtained if we use the 2-norm-based Lyapunov functions as those in [6, 22, 25, 26], and we draw inspiration from the ideas of [18, 27]. So, we get some new results about finite-time synchronization for IMNNs with time-varying delays.

When $\eta_i = 1\,(i = 1, 2, \ldots, n)$, we will get the following master system:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\xi_i x_i + y_i(t), \\ \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -\widehat{\beta}_i y_i(t) + \widehat{\alpha}_i x_i(t) + \displaystyle\sum_{j=1}^{n} \gamma_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} \zeta_{ij} f_j\big(x_j\big(t-\tau_j(t)\big)\big) \\ + \displaystyle\sum_{j=1}^{n} \nu_{ij} \int_{t-\sigma(t)}^{t} f_j\big(x_j(s)\big)\mathrm{d}s + I_i, \\ \text{for } a.a.t \in [0, T), i = 1, 2, \ldots, n, \end{cases} \tag{38}$$

where $\widehat{\alpha}_i = -\xi_i^2 + \alpha_i\xi_i - b_i$, $\widehat{\beta}_i = \alpha_i - \xi_i$, $i = 1, 2, \ldots, n$. By the system (38), we can get slave system and give Corollary 10.

Corollary 10. *If $f(\cdot)$ meets Assumption 1, there exists $Q_{1i} > 0$, and K_{1i}, K_{2i}, and Q_{2i} satisfy the following inequalities:*

$$\begin{aligned} &K_{1i} \ge -\xi_i + \left|\widehat{\alpha}_i\right| + \sum_{j=1}^{n} c_{ji}^{+} l_i + \sum_{j=1}^{n} \frac{1}{1-\mu_i} d_{ji}^{+} l_i + \sum_{j=1}^{n} \frac{1}{1-\widetilde{\sigma}} h_{ji}^{+} l_i \sigma, \\ &K_{2i} \ge 1 - \widehat{\beta}_i, \\ &Q_{2i} > \widehat{M}_i. \end{aligned} \tag{39}$$

Then, the master-slave-based systems are synchronized in a finite-time under controllers (16). Moreover, the settling time is estimated as follows:

$$\begin{aligned} t_1 \le \frac{1}{\widehat{\theta}_0} &\left[\sum_{i=1}^{n} \left|e_{1i}(0)\right| + \sum_{i=1}^{n} \left|e_{2i}(0)\right| + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{1-\mu_j} d_{ij}^{+} l_j \int_{-\tau_j}^{0} \left|e_{1j}(s)\right|\mathrm{d}s \right. \\ &\left. + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{1-\widetilde{\sigma}} h_{ij}^{+} l_j \int_{-\sigma}^{0}\int_{s}^{0} \left|e_{1j}(u)\right|\mathrm{d}u\mathrm{d}s \right] - \max_{j=1,2,\ldots,n}\left\{\tau_j, \sigma\right\}, \end{aligned} \tag{40}$$

where $\widehat{M}_i = \sum_{j=1}^{n} |c_{ij} - \widehat{c}_{ij}|F_j + \sum_{j=1}^{n} |d_{ij} - \widehat{d}_{ij}|F_j + \sum_{j=1}^{n} |h_{ij} - \widehat{h}_{ij}|F_j\sigma$, $i, j = 1, 2, \ldots, n$, $\widehat{\alpha}_i = -\xi_i^2 + \alpha_i\xi_i - b_i$, $\widehat{\beta}_i = \alpha_i - \xi_i$, and $\theta_0 = \min_{i=1,2,\ldots,n}\{(Q_{2i} - M_i), Q_{1i}\}$.

Remark 11. When $\eta_i = 1$ $(i = 1, 2, \ldots n)$, the system (38) becomes the system in [27, 38]. Corollary 10 considers the external input and expands the existing results.

When $h_{\mathrm{ij}}(x_i(t)) = 0$ $(i, j = 1, 2, \ldots, n)$, we will get the following master system:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\dfrac{\xi_i}{\eta_{ii}}x_i + \dfrac{1}{\eta_i}y_i(t), \\ \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -\widehat{\beta}_i y_i(t) + \widehat{\alpha}_i x_i(t) + \eta_i \sum\limits_{j=1}^{n} \gamma_{ij} f_j\left(x_j(t)\right) + \eta_i \sum\limits_{j=1}^{n} \zeta_{\mathrm{ij}} f_j\left(x_j\left(t - \tau_j(t)\right)\right) + \eta_i I_i(t), \\ \text{for } a.a.t \in [0, T), i = 1, 2, \ldots, n, \end{cases} \tag{41}$$

where α_i, β_i $(i, j = 1, 2, \ldots, n)$ are the same as the system (13); we can get slave system and we will give Corollary 12.

Corollary 12. *If $f(\cdot)$ meets Assumption 1, the $Q_{1i} > 0$, K_{1i}, K_{2i}, and Q_{2i} satisfy the following inequalities:*

$$\begin{aligned} K_{1i} &\geq -\frac{\xi_i}{\eta_i} + \left|\breve{\alpha}_i\right| + \sum_{j=1}^{n}\left|\eta_i\right|c_{ji}^{+} l_i + \sum_{j=1}^{n}\frac{\left|\eta_i\right|}{1-\sigma} d_{\mathrm{ji}}^{+} l_i, \\ K_{2i} &\geq \frac{1}{\left|\eta_i\right|} - \breve{\beta}_{i,} \\ Q_{2i} &> \breve{M}_i. \end{aligned} \tag{42}$$

Then, the master-slave-based systems are synchronized in a finite time under controllers (16). Moreover, the settling time is estimated as follows:

$$t_1 \leq \frac{1}{\breve{\theta}_0}\left[\sum_{i=1}^{n}\left|e_{1i}(0)\right| + \sum_{i=1}^{n}\left|e_{2i}(0)\right| + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left|\eta_i\right|}{1-\mu_j} d_{ij}^{+} l_j \int_{-\tau_j}^{0}\left|e_{1j}(s)\right|\mathrm{d}s\right] - \max_{j=1,2,\ldots,n}\left\{\tau_j\right\}, \tag{43}$$

where $M_i = |\eta_i|\sum_{j=1}^{n}|c_{ij} - \widehat{c}_{ij}|F_j + |\eta_i|\sum_{j=1}^{n}|d_{ij} - \widehat{d}_{\mathrm{ij}}|F_j$, $\min_{i=1,2,\ldots,n}\{(Q_{2i} - M_i), Q_{1i}\}$, $\alpha_{\mathrm{i}} = -(\xi_{\mathrm{i}}^2/\eta_{\mathrm{i}}) + \alpha_{\mathrm{i}}\xi_{\mathrm{i}} - \eta_{\mathrm{i}}\mathrm{b}_{\mathrm{i}}$, $-\beta_i = \alpha_i - (\xi_i/\eta_i), i, j = 1, 2, \ldots, n$, and $\theta_0 = \min_{i=1,2,\ldots,n}\{(Q_{2i} - M_i), Q_{1i}\}$.

Define the following Lyapunov–Krasovskii functional:

$$\breve{V}(t) = \sum_{i=1}^{2} V_i(t), \tag{44}$$

where $V_1(t)$ and $V_2(t)$ are the same as definition of (21). The means of the proof are similar to Theorem 7.

Remark 13. When $\eta_i = 1$ and $h_{\mathrm{ij}}(x_i(t)) = 0$ $(i, j = 1, 2, \ldots, \mathrm{n})$, the system (1) is same to the system in [11, 24]. Corollary 12 obtains the condition of finite-time synchronization for the system (41) with η_i (η_i is not a constant). Therefore, our paper is more general.

Remark 14. When $\eta_i = 1$ $(i, j = 1, 2, \ldots n)$, $h_{\mathrm{ij}}(x_i(t)) = h$ (h is a constant, $i, j = 1, 2, \ldots, n$), we can gain the result which is similar to Theorem 2 in [17].

Remark 15. In the above theorems and corollaries, since $c_{\mathrm{ij}}(t)$, $d_{\mathrm{ij}}(t)$, and $h_{\mathrm{ij}}(t)$ are discontinuous, the solution way for differential equations in [24] cannot apply here. To solve the problem, Filippov et al. have presented a solution concept for the differential equations with discontinuous right-hand side [23, 27]. Based on the definition, a differential equation with discontinuous right-hand side has the same solution set as certain differential inclusion.

4. Illustrative Examples

In this section, two examples are given to present the effectiveness of our results achieved in this paper.

Example 1. Consider the following IMNNs:

$$\begin{cases} \dfrac{d^2x_1(t)}{dt^2} = -\dfrac{dx_1(t)}{dt} - 2x_1(t) + \displaystyle\sum_{j=1}^{2} c_{1j}(x_1(t)) f_j(x_j(t)) \\ + \displaystyle\sum_{j=1}^{2} d_{1j}(x_1(t)) f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{2} h_{1j}(x_1(t)) \int_{t-\sigma(t)}^{t} f_j(x_j(s)) ds + I_1(t), \\ \dfrac{d^2x_2(t)}{dt^2} = -2\dfrac{dx_2(t)}{dt} - 4x_i(t) + \displaystyle\sum_{j=1}^{2} c_{2j}(x_2(t)) f_j(x_j(t)) \\ + \displaystyle\sum_{j=1}^{2} d_{2j}(x_2(t)) f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{2} h_{2j}(x_2(t)) \int_{t-\sigma(t)}^{t} f_j(x_j(s)) ds + I_2(t), \end{cases} \tag{45}$$

with $f_i(x_i) = \cos(x_i), \tau_1(t) = \tau_2(t) = 0.5\sin(t) + 0.5$, $\sigma(t) = 0.5\cos(t) + 0.5, I_1(t) = 100 + 5\sin(t), I_2(t) = 100 + 5\cos(t)$, where

$$\begin{aligned} c_{11}(x_1(t)) &= \begin{cases} -0.2, & |x_1| \le 0.5, \\ 0.2, & |x_1| > 0.5, \end{cases} \quad c_{12}(x_1(t)) = \begin{cases} 2.0, & |x_1| \le 0.5, \\ 3.0, & |x_1| > 0.5, \end{cases} \\ c_{21}(x_2(t)) &= \begin{cases} 3.0, & |x_2| \le 0.5, \\ 2.0, & |x_2| > 0.5, \end{cases} \quad c_{22}(x_2(t)) = \begin{cases} -0.1, & |x_2| \le 0.5, \\ 0.1, & |x_2| > 0.5, \end{cases} \\ d_{11}(x_1(t)) &= \begin{cases} -0.1, & |x_1| \le 0.5, \\ 0.2, & |x_1| > 0.5, \end{cases} \quad d_{12}(x_1(t)) = \begin{cases} 2.0, & |x_1| \le 0.5, \\ 2.5, & |x_1| > 0.5, \end{cases} \\ d_{21}(x_2(t)) &= \begin{cases} 3.0, & |x_2| \le 0.5, \\ 2.5, & |x_2| > 0.5, \end{cases} \quad d_{22}(x_2(t)) = \begin{cases} -0.1, & |x_2| \le 0.5, \\ -0.2, & |x_2| > 0.5, \end{cases} \\ h_{11}(x_1(t)) &= \begin{cases} -0.2, & |x_1| \le 0.5, \\ 0.1, & |x_1| > 0.5, \end{cases} \quad h_{12}(x_1(t)) = \begin{cases} 2.0, & |x_1| \le 0.5, \\ 3.0, & |x_1| > 0.5, \end{cases} \\ h_{21}(x_2(t)) &= \begin{cases} 3.0, & |x_2| \le 0.5, \\ 2.5, & |x_2| > 0.5, \end{cases} \quad h_{22}(x_2(t)) = \begin{cases} -0.2, & |x_2| \le 0.5, \\ 0.2, & |x_2| > 0.5. \end{cases} \end{aligned} \tag{46}$$

Due to $\tau_1(t) = \tau_2(t) = 0.5\sin(t) + 0.5$ and $\sigma(t) = 0.5\cos(t) + 0.5$, we can let $\tau_1 = \tau_2 = 1, \mu_1 = \mu_2 = 0.5, \sigma = 1, \tilde{\sigma} = 0.5$. According to Assumption 1, we let $l_1 = l_2 = 1, F_1 = F_2 = 1, \xi_1 = 2, \xi_2 = 4, \eta_1 = 1, \eta_2 = 2$. Through simple computing, we get $\alpha_1 = -4, \alpha_2 = -8, \beta_1 = -1, \beta_2 = 0, K_{11} \ge 33, K_{12} \ge 21.8, K_{21} \ge 2, K_{22} \ge 0.5, M_1 = 3.5, M_2 = 5.4, Q_{21} > 3.5, Q_{22} > 5.4$.

When $Q_{11} > 0, Q_{12} > 0$, we let $K_{11} = 33, K_{12} = 22, K_{21} = 2, K_{22} = 1, Q_{21} = 6, Q_{22} = 8, Q_{11} = 2, Q_{12} = 2$. Then, we can obtain $\theta_0 = 2$. In the case that initial conditions are chosen as $x = (0.5, 1)^T, y = (-3, 3)^T$, trajectories of drive system (45) are presented in Figure 1. We choose the initial value of the response system as $m = (-0.5, 1)^T, w = (-2, 2)^T, t \in [-1, 0]$. So, $|e_{11}(0)| = |e_{12}(0)| = |e_{21}(0)| = |e_{22}(0)| = 1$ and $t_1 < 14.9$.

Moreover, we know that the synchronization can be realized before 14.9 by Figure 2. The theoretical analysis of Theorem 7 is verified.

Example 2. Consider the following IMNNs:

$$\begin{cases} \dfrac{d^2x_1(t)}{dt^2} = -4\dfrac{dx_1(t)}{dt} - 8x_1(t) + \sum_{j=1}^{2} c_{1j}(x_1(t)) f_j(x_j(t)) \\ + \sum_{j=1}^{2} d_{1j}(x_1(t)) f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{2} h_{1j}(x_1(t)) \int_{t-\sigma(t)}^{t} f_j(x_j(s))ds + I_1(t), \\ \dfrac{d^2x_2(t)}{dt^2} = -6\dfrac{dx_2(t)}{dt} - 12x_i(t) + \sum_{j=1}^{2} c_{2j}(x_2(t)) f_j(x_j(t)) \\ + \sum_{j=1}^{2} d_{2j}(x_2(t)) f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{2} h_{2j}(x_2(t)) \int_{t-\sigma(t)}^{t} f_j(x_j(s))ds + I_2(t), \end{cases} \tag{47}$$

with $f_i(x_i) = \tan h(x_i), \tau_1(t) = \tau_2(t) = 0.5\sin(t) + 1.5, \sigma(t) = 0.5\cos(t) + 0.5, I_1(t) = 200 + 10\sin(t),\ I_2(t) = 200 + 10\cos(t)$, where

$$\begin{aligned} c_{11}(x_1(t)) &= \begin{cases} 1.0, & |x_1| \le 1, \\ 2.0, & |x_1| > 1, \end{cases} \quad c_{12}(x_1(t)) = \begin{cases} 2.0, & |x_1| \le 1, \\ 3.0, & |x_1| > 1, \end{cases} \\ c_{21}(x_2(t)) &= \begin{cases} 0.8, & |x_2| \le 1, \\ 1.0, & |x_2| > 1, \end{cases} \quad c_{22}(x_2(t)) = \begin{cases} 1.8, & |x_2| \le 0.5, \\ 0.8, & |x_2| > 1, \end{cases} \\ d_{11}(x_1(t)) &= \begin{cases} -1.5, & |x_1| \le 1, \\ -1.2, & |x_1| > 1, \end{cases} \quad d_{12}(x_1(t)) = \begin{cases} 1.0, & |x_1| \le 1, \\ 0.8, & |x_1| > 1, \end{cases} \\ d_{21}(x_2(t)) &= \begin{cases} 0.8, & |x_2| \le 1, \\ 1.0, & |x_2| > 1, \end{cases} \quad d_{22}(x_2(t)) = \begin{cases} -1.4, & |x_2| \le 0.5, \\ -1.6, & |x_2| > 1, \end{cases} \\ h_{11}(x_1(t)) &= \begin{cases} -0.5, & |x_1| \le 1, \\ -1.0, & |x_1| > 1, \end{cases} \quad h_{12}(x_1(t)) = \begin{cases} 2.0, & |x_1| \le 1, \\ 3.0, & |x_1| > 1, \end{cases} \\ h_{21}(x_2(t)) &= \begin{cases} 1.0, & |x_2| \le 1, \\ 2.0, & |x_2| > 1, \end{cases} \quad h_{22}(x_2(t)) = \begin{cases} -1.6, & |x_2| \le 1, \\ -1.2, & |x_2| > 1. \end{cases} \end{aligned} \tag{48}$$

Due to $\tau_1(t) = \tau_2(t) = 0.5\sin(t) + 1.5$ and $\sigma(t) = 0.5\cos(t) + 0.5$, we can let $\tau_1 = \tau_2 = 2, \mu_1 = \mu_2 = 0.5, \sigma = 1, \tilde{\sigma} = 0.5$. By Assumption 1, let $l_1 = l_2 = 1, F_1 = F_2 = 1, \xi_1 = 4, \xi_2 = 6, \eta_1 = 2, \eta_2 = 2$. After simple computing, we get $\alpha_1 = -8, \alpha_2 = -6, \beta_1 = 2, \beta_2 = 3, K_{11} \ge 34, K_{12} \ge 41.4, K_{21} \ge -1.5, K_{22} \ge -2.5, M_1 = 8, M_2 = 6, Q_{21} > 8, Q_{22} > 6$.

When $Q_{11} > 0, Q_{12} > 0$, let $K_{11} = 34, K_{12} = 42, K_{21} = 2, K_{22} = 1, Q_{21} = 10, Q_{22} = 8, Q_{11} = 2, Q_{12} = 2$. Then, we can obtain $\theta_0 = 2$. In the case that initial conditions are chosen as $x = (0.5, 1)^T, y = (-3, 3)^T$, trajectories of drive system (47) are presented in Figure 3. We choose the initial value of the response system as $m = (0.7, 0.5)^T, w = (-2, 2.2)^T, t \in [-2, 0]$. So, $|e_{11}(0)| = 0.2, |e_{12}(0)| = 0.5, |e_{21}(0)| = 1, |e_{22}(0)| = 0.8$ and $t_1 < 16.05$. Moreover, we know that the synchronization can be realized before 16.05 by Figure 4. The theoretical analysis of Theorem 7 is verified.

Remark 16. When η_1 is the selected different values and other coefficients are invariant in the abovementioned examples, we get different K_{11} and K_{12} thought simple computing. Then, the settling time is explicitly estimated, which depends on time delays and initial values of the coupled system. Moreover, we can find the appropriate value of η_i, which makes the system synchronize more quickly. The system dynamical characteristic with η_i for IMNNs would have specific physical meanings and biological backgrounds.

Remark 17. In Example 1, if η_i is selected different values, we can obtain different $K_{11}, K_{12}, K_{21}, K_{22}, M_1, M_2, Q_{21}$, and Q_{22}. Accordingly, the settling time t_1 is estimated when other parameters remain unchanged. Changed parameters under different η_i are shown in Table 1.

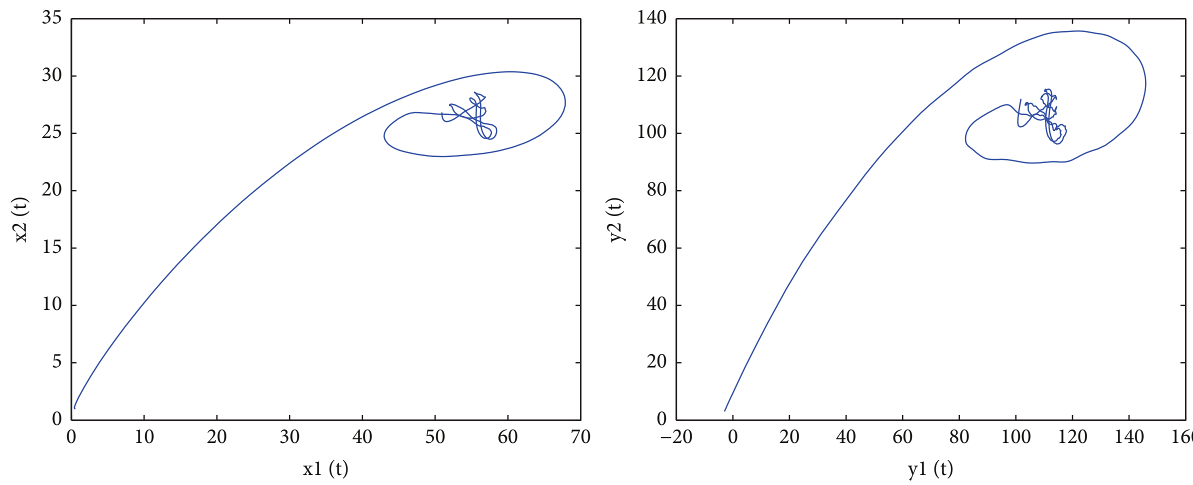

Figure 1: Trajectories of $x(t)$ and $y(t)$ of (45) with initial conditions $x = (0.5, 1)^T$, $y = (-3, 3)^T$.

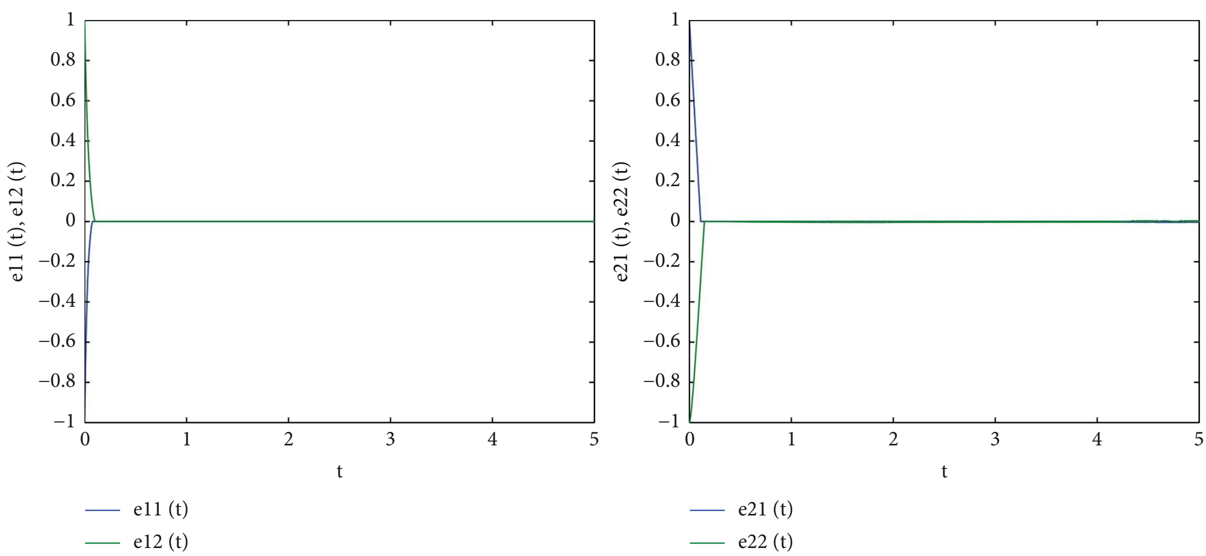

Figure 2: State responses of error system under the feedback controllers (16).

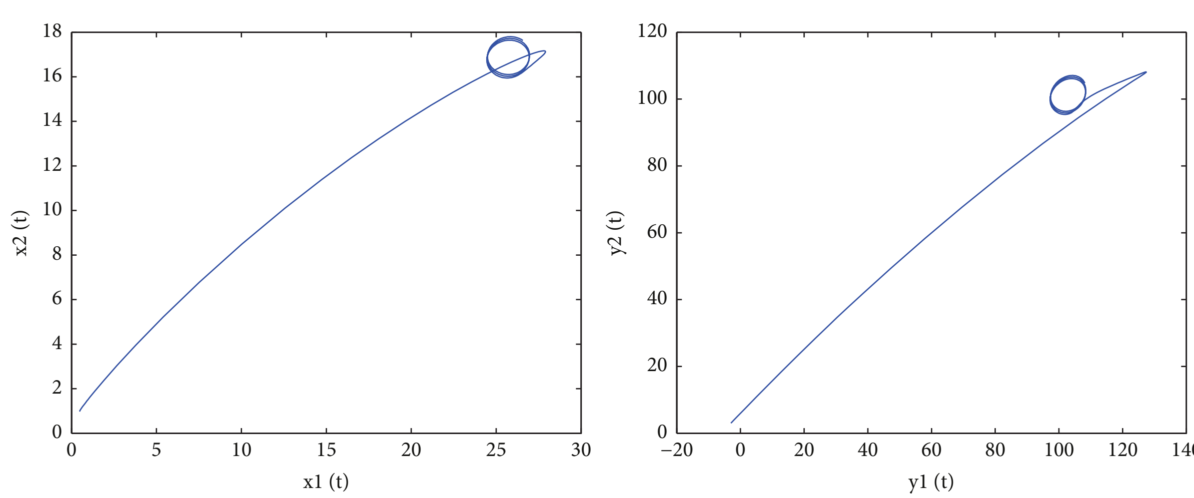

Figure 3: Trajectories of $x(t)$ and $y(t)$ of (47) with initial conditions $x = (0.5, 1)^T$, $y = (-3, 3)^T$.

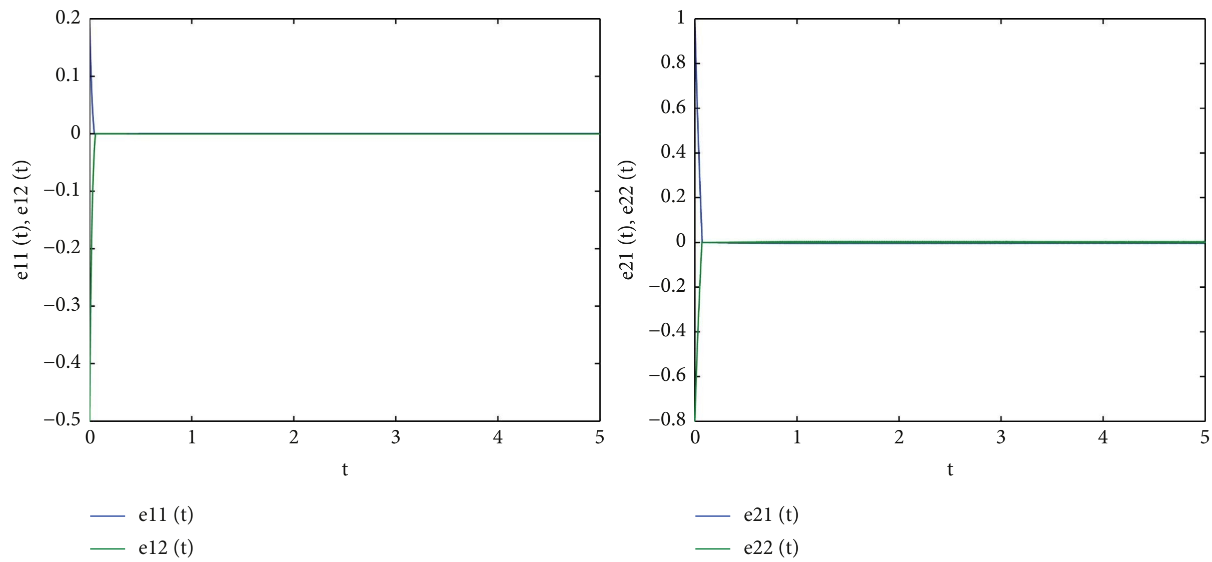

Figure 4: State responses of error system under the feedback controllers (16).

Table 1: Changed parameters with different η_i.

η_1	η_2	M_1	M_2	K_{11}	K_{12}	K_{21}	K_{22}	t_1
1	2	3.5	5.4	33	21.8	2	0.5	14.9
1	1	3.5	2.7	18	22.9	2	2	10.1
2	2	7	5.4	35	35.8	0.5	0.5	19.2
0.5	0.5	1.75	1.35	11	25.45	5	5	5.55

5. Conclusion

In this paper, the settling time estimation of synchronization issues for IMNNs with mixed time-varying delays is discussed. By designing appropriate controllers and Lyapunov–Krasovskii functionals, sufficient conditions are obtained to guarantee synchronization in a settling time without using existing finite-time stability theorem. Due to taking free-weight coefficients into account, the conservatism is reduced and synchronization between master and slave system can be quicker. Meanwhile, the obtained conditions are more general and expend some existing results in [17, 27, 37]. As we all know, various disturbances and uncertainties are unavoidable in lots of practical systems for IMNNs. Future work would concern the finite-time synchronization of IMNN with stochastic disturbances and parameter mismatch. On the other hand, pining impulsive control approaches are a hot topic of concern [38, 39]. Then, we would also pay close attention to them for IMNNs.

Data Availability

Two group numerical simulation data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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