Hindawi
Discrete Dynamics in Nature and Society
Volume 2023, Article ID 1153483, 9 pages
https://doi.org/10.1155/2023/1153483

Research Article

Determining the Viability of an Unbounded Polyhedron for a Switched System

Jianfeng Lv and Na Zhao

School of Science, Inner Mongolia University of Science and Technology, Baotou 014010, China

Correspondence should be addressed to Na Zhao; nkdzn@imust.edu.cn

Received 1 February 2023; Revised 8 April 2023; Accepted 18 April 2023; Published 8 May 2023

Academic Editor: Manuel De la Sen

This paper proposes a new method to determine the viability of a switched system on a cone and an unbounded polyhedron. First, we investigate the viability condition on a cone. Then, a sufficient viability criterion for a polyhedron, which is expressed by a convex hull of finite number of extreme points and a nonnegative linear combination of finite extreme directions, is presented by using nonsmooth analysis. Based on this criterion, instead of all boundary points, just several extreme points and extreme directions are needed to be verified whether satisfying some conditions. The advantage of the proposed methods is that determining the viability for a switched system is easy to be implemented. Finally, an example is listed to illustrate the effectiveness of the proposed methods.

1. Introduction

Viability is an important research topic in control theory, and the viability of a dynamic system on a region means that the system states stay inside the region under any initial conditions in the region [1]. The research on viability focuses on continuous evolution of a dynamic system within a constraint region, aiming to maintain the system state within the constraint region by describing the possible trajectories and trying to select appropriate control, which provides essential guarantee for the safe and continuous evolution of the system, making the study of viability theory of great importance. Viability theory has brought a new research approach to the safe evolution of dynamic systems, and it has now been applied to fishery ecosystems [2], renewable energy systems [3, 4], robot control [5, 6], and system fault detection [7].

The research on viability mainly includes two aspects. A determining criterion for dynamic systems regarding whether a given region satisfies viability is established [8–10], and algorithms for computing the viability kernel of dynamical systems are constructed [11–16]. Although the author of [1] has given a sufficient and necessary condition of determining the viability, it is difficult to implement and not feasible in practice because each boundary points of viability constraints have to be checked based on the differential inclusions and the tangent cone. Thus, researchers have considered the viability for some simple systems on special forms of regions. In [8], the viability of a linear system on a region with nonsmooth boundary is studied. In [9], the viability of a class of differential inclusion at a point is verified by determining the consistency of a system of linear inequalities. A viability verification of a polyhedron for a linear control system is researched in [17], and the method of determining the viability for a bounded polyhedron, which is expressed by a convex hull of a finitely many points, can be transformed into verifying the viability condition at vertices. Chen has discussed the viability of a linear system on a bounded polyhedron, and the method of determining viability is transformed into solving a finite number of linear programming problems (see [18–20]) in [21]. Blanchini has characterized the viability condition for a linear system on polyhedral set and ellipsoidal set in [22]. Computation of the viability kernel for dynamical systems is a fundamental problem in the viability theory. It has traditionally been computed using the viability kernel algorithm [12] and level set approach [23]. Mitchell et al. in [23] has presented an algorithm by proofing that the reachable set is the zero sublevel set of the viscosity solution of a particular time-dependent Hamilton–Jacobi–Isaacs partial differential equation. Neznakhin has constructed the viability kernel for a generalized dynamical system by an attainability set in [24] and constructed the viability kernel in the phase constraints for a nonlinear controlled system with a target set in [25]. However, these methods require gridding the state space, and hence, their time and memory complexity grow exponentially with the state dimension. Thus, these methods are feasible only for dynamical systems with low dimension. Deffuant et al. proposed an algorithm for computing the approximation of the viability kernel by support vector machines in [26]. It uses support vector machines as classification techniques and finds a viable control at each time step by gradient optimization techniques. This algorithm allows us to avoid the exponential growth of the computing time with the dimension of the control space.

Switched systems, which consist of two or more subsystems and a switching rule orchestrating switching between these subsystems, have attracted extensive attention in recent years. To the best of our knowledge, the viability of switched systems has received little attention. Gao has characterized the viability for a hybrid system in [27] and an uncertain impulse system in [28]. Haimovich has developed the problem of invariant set computation for a switched linear system in [29]. Lv and Gao have proposed a method of computing the viability kernel for a switched system in [14]. Lv et al. have studied the viability problem for switched nonlinear systems in [30]. A determining approach of a viable set and an attraction region for switched systems in which Lyapunov functions are piecewise smooth has been proposed. However, these results have not given a specific method for determining viability on an unbounded region. Although method of determining the viability for switched systems has been proposed in [10], it should be noted that this work only considers a bounded polyhedron. In fact, an unbounded region can also be regarded as the security region for a switched system. Determining the viability of an unbounded region makes the viability criterion more complex. However, this determining criterion plays an important role in security evolution of systems. As we know, any unbounded region can be approximated by some unbounded polyhedrons. Thus, considering the viability for a switched system on an unbounded polyhedron is meaningful and important. We study this problem in the paper based on the results of [10]. Our contribution is extending the results of [10] to a cone and an unbounded polyhedron. It is not a natural extension due to the complex features of a switched system. We have constructed the viability criterion for a switched system on a cone and an unbounded polyhedron by means of nonsmooth analysis theory.

The rest of the paper is organized as follows: Section 2 provides some necessary preliminaries. Sections 3 and 4 are presented the viability of a cone and an unbounded polyhedron, respectively. In Section 5, we give an example to illustrate the effectiveness of the given methods. Section 6 is the conclusion.

2. Preliminaries

Consider the following switched system

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t), \tag{1}$$

where $x \in \mathrm{R}^n$ is the system state variable, the switching rule $\sigma(t)$: $[t_0, +\infty) \longrightarrow \Lambda$ is a segmented constant-valued function of time t, the indicator set is $\Lambda = \{1, 2, \cdots, N\}$, and $\sigma(t) = i (i = 1, 2, \cdots, N)$ indicates that the i-th subsystem $\dot{x}(t) = A_i x$ comes into play. u is a control variable. The system jumps at the moment of switching, and its solution is continuous everywhere and nonsmooth.

Definition 1 (see [1]). Let $W \subset \mathrm{R}^n$ be nonempty. If for any initial state $x_0 \in W$, there exists a solution $x(t) = x(t, x_0)$ of system (1), such that $x(t) \in W$ for all $t \geq 0$, then the set W is called viable under system (1). The solution $x(t)$ is said to be a viable solution.

The tangent cone of the set is required in the viability criterion, and it is defined as follows.

Definition 2 (see [1]). Let $W \subset \mathrm{R}^n$ be nonempty and the tangent cone of the set W at $x \in W$ is defined as

$$T_W(x) = \left\{ v \in \mathrm{R}^n \,\middle|\, \liminf_{t \longrightarrow 0^+} \frac{1}{t} d_W(x + tv) = 0 \right\}, \tag{2}$$

where $d_W(x)$ represents the distance from x to W, i.e., $d_W(x) = \inf_{y \in W} |x - y|$.

It is convenient to have characterization of the tangent cone in terms of sequences: $v \in T_W(x)$ if and only if there exist a sequence of $h_k > 0$ converging to 0^+ and a sequence of $v_k \in \mathrm{R}^n$ converging to v such that

$$x + h_k v_k \in W, \ \forall k > 0. \tag{3}$$

Tangent cone is the generalization of tangent plane from smooth case to nonsmooth case. We give some tangent cones at some boundary points in Figure 1. With this notion, a viability condition is given by the following lemma.

Lemma 1 (see [1]). *The nonempty closed set $W \subset \mathrm{R}^n$ is viable under the system $\dot{x} = f(x)$ if and only if*

$$T_W(x) \cap f(x) \neq \varnothing, \forall x \in W, \tag{4}$$

where $\varnothing$ is an empty set.

Applying Lemma 1 to the switched system, the following conclusion is reached.

Theorem 1. *The nonempty closed set $W \subset \mathrm{R}^n$ is viable under system (1), if and only if*

$$T_W(x) \cap \left(\bigcup_{i=1}^{m} (A_i x + B_i u) \right) \neq \varnothing \quad \forall x \in W. \tag{5}$$

According to the definition of the tangent cone, when x is an inner point of W, $T_W(x) = \mathrm{R}^n$, and then, equation (5) always holds. Therefore, to determine whether the equation (5) holds, it is only required to consider the boundary points of W.

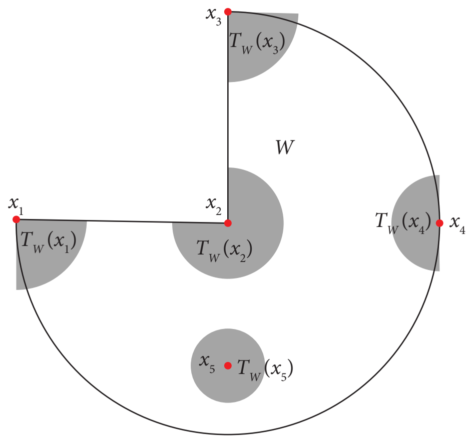

Figure 1: Tangent cones at points on set W.

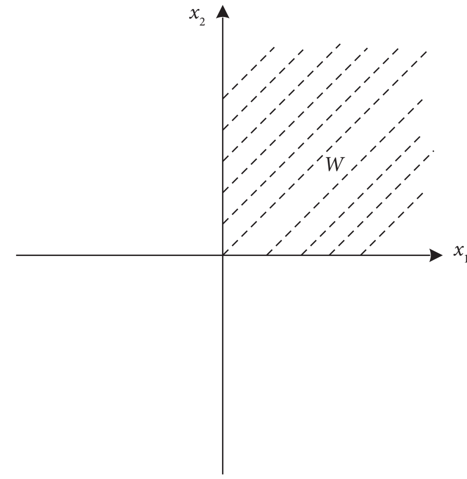

Figure 2: A cone generated by $d_1 = [1\ \ 0]^{\mathrm{T}}$ and $d_2 = [0\ \ 1]^{\mathrm{T}}$.

3. Viability Determining on a Cone

We will discuss the viability on a cone for switched system (1) in this section. Let

$$W = \operatorname{cone}\{d_1, \cdots, d_m\}, \tag{6}$$

be a cone, where $d_i \in \mathrm{R}^n, 1 \le i \le m$ denote the extreme directions. Figure 2 gives a cone represented by $d_1 = [1\ \ 0]^{\mathrm{T}}$ and $d_2 = [0\ \ 1]^{\mathrm{T}}$.

We consider the viability for the switched system (1) on the cone represented by (6). Let the control input set be a cone. Based on the nonsmooth analysis, an approach for determining the viability on a cone has proposed.

Theorem 2. *Let the nonempty cone W be given by (6). If there exists a subsystem $A_k\,(k \in \Lambda)$ of (1) such that A_k satisfies the viability condition at any direction on each facet of W, then switched system (1) is viable on W.*

Proof. By the literature [31], the cone W can be expressed as

$$W = \left\{x \middle| c_i^{\mathrm{T}} x \le 0, i = 1, \cdots, p\right\}. \tag{7}$$

Define the following index set:

$$I(x) = \left\{i \middle| c_i^{\mathrm{T}} x = 0, i \in \{1, \cdots, p\}\right\}. \tag{8}$$

According to constraint qualifications shown in [10], the tangent cone can be expressed as

$$T_W(x) = \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}} y \le 0, i \in I(x)\right\}. \tag{9}$$

We only need to consider the viability for boundary points of W. In other words, we need to consider the points which make the index set to be nonempty. If the index set $I(x) \ne \varnothing$, then x is on the one of the facet of W. Assuming that x is on the facet H,

$$H = \operatorname{cone}\left\{d_{i_1}, \cdots, d_{i_q}\right\}, i_1, \cdots, i_q \in \{1, \cdots, m\}, \tag{10}$$

then, there exist $\mu_i(x) > 0, i = 1, \cdots, q$, such that

$$x = \mu_1(x) d_{i_1} + \cdots + \mu_q(x) d_{i_q}. \tag{11}$$

By known condition, there exist $u_i \in U, i = 1, \cdots, q$, such that the following formulas hold:

$$\begin{aligned} T_W\left(d_{i_1}\right) \cap \left(A_k d_{i_1} + B_k u_1\right) \ne \varnothing, \cdots, T_W\left(d_{i_q}\right) \\ \cap \left(A_k d_{i_q} + B_k u_q\right) \ne \varnothing. \end{aligned} \tag{12}$$

In nonsmooth optimization, two frequently used constraint qualifications are shown as follows. Let the function $g(x) \le 0$ be the boundary of the region W. Then, we have

Constraint Qualification 1 There exists $y_0 \in \mathrm{R}^n$, such that $g'(x; y_0) < 0$.

Constraint Qualification 2 $\mathrm{cl}\gamma(x) = \Gamma(x)$, where

$$\begin{aligned} \gamma(x) &= \left\{y \in \mathrm{R}^n \middle| g'(x; y) < 0\right\}, \\ \Gamma(x) &= \left\{y \in \mathrm{R}^n \middle| g'(x; y) \le 0\right\}. \end{aligned} \tag{13}$$

In fact, the set W satisfies Constraint Qualification 1 or 2 at $x \in \mathrm{R}^n$, and then, $T_W(x) = \Gamma(x)$. On the other hand, the cone can be expressed as $W = \{x \mid c_i^{\mathrm{T}} x \le 0, i = 1, \cdots, p\}$, and then,

$$T_W(x) = \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}} y \le 0, i \in I(x)\right\}. \tag{14}$$

Substituting tangent cone given by (14) into (12), we will get

$$\begin{cases} \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}} y \le 0, i \in I\left(d_{i_1}\right)\right\} \cap \left(A_k d_{i_1} + B_k u_1\right) \ne \varnothing, \\ \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}} y \le 0, i \in I\left(d_{i_2}\right)\right\} \cap \left(A_k d_{i_2} + B_k u_2\right) \ne \varnothing, \\ \cdots\cdots\cdots, \\ \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}} y \le 0, i \in I\left(d_{i_q}\right)\right\} \cap \left(A_k d_{i_q} + B_k u_q\right) \ne \varnothing. \end{cases} \tag{15}$$

Thus,

$$\begin{cases} \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}}\left(A_k d_{i_1} + B_k u_1\right) \le 0, i \in I\left(d_{i_1}\right)\right\} \ne \varnothing, \\ \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}}\left(A_k d_{i_2} + B_k u_2\right) \le 0, i \in I\left(d_{i_2}\right)\right\} \ne \varnothing, \\ \cdots\cdots\cdots, \\ \left\{y \in \mathrm{R}^n \middle| c_i^{\mathrm{T}}\left(A_k d_{i_q} + B_k u_q\right) \le 0, i \in I\left(d_{i_q}\right)\right\} \ne \varnothing. \end{cases} \quad (16)$$

They are equivalent to the consistency of some linear inequalities as follows:

$$\begin{cases} c_i^{\mathrm{T}} A_k d_{i_1} + c_i^{\mathrm{T}} B_k u_1 \le 0, i \in I\left(d_{i_1}\right), \\ c_i^{\mathrm{T}} A_k d_{i_2} + c_i^{\mathrm{T}} B_k u_2 \le 0, i \in I\left(d_{i_2}\right), \\ \vdots \\ c_i^{\mathrm{T}} A_k d_{i_q} + c_i^{\mathrm{T}} B_k u_q \le 0, i \in I\left(d_{i_q}\right). \end{cases} \quad (17)$$

Since the extreme directions $d_{i_1}, \cdots, d_{i_q}$ on the same facet of W, then $I(d_{i_1}) \cap \cdots \cap I(d_{i_q}) \ne \varnothing$ and (17) is meaningful. Multiplying $\mu_1(x), \cdots, \mu_q(x)$ on each inequality of (17), respectively, and adding up, we can obtain

$$\begin{aligned} &c_i^{\mathrm{T}} A_k\left(\mu_1(x) d_{i_1} + \cdots + \mu_q(x) d_{i_q}\right) \\ &\quad + c_i^{\mathrm{T}} B_k\left(\mu_1(x) u_1 + \cdots + \mu_q(x) u_q\right) \le 0. \end{aligned} \quad (18)$$

Letting $\overline{u} = \mu_1(x) u_1 + \cdots + \mu_q(x) u_q$. Since the set U is a cone and $u_i \in U (i = 1, \cdots, q)$, $\mu_i > 0 (i = 1, \cdots, q)$, according to the definition of the cone, then $\overline{u} \in U$. Thus, (18) can be rewritten as

$$c_i^{\mathrm{T}}\left(A_k x + B_k \overline{u}\right) \le 0, i \in I\left(d_{i_1}\right) \cap \cdots \cap I\left(d_{i_q}\right). \quad (19)$$

It implies that

$$T_W(x) \cap \left(A_k x + B_k \overline{u}\right) \ne \varnothing. \quad (20)$$

This concludes the proof of the theorem.

Theorem 2 has presented a method of determining the viability of a cone for the switched system. For each facet of the cone, we can find all the extreme directions contained in this facet. We next determine whether the viability condition on these directions is satisfied for each subsystem. If there exists a subsystem satisfying the viability condition at any direction on a facet of the cone, then the points on the facet are viable. If each facet of the cone satisfies the viability condition, then the cone is a viable region.

4. Viability Determining on an Unbounded Polyhedron

We restrict our attention to determining the viability on an unbounded polyhedron in this section.

The representation of an unbounded polyhedron is presented below. Let $a_1, \cdots, a_m \in \mathrm{R}^n$, $\lambda_i \ge 0, i = 1, \cdots, m$, and $\sum_{i=1}^{m} \lambda_i = 1$, $a = \sum_{i=1}^{m} \lambda_i a_i$ is called a convex combination of $a_1, \cdots, a_m$. The convex hull of the set S, denoted co$S$, is a set formed by all convex combinations in S. In other words, $a \in \mathrm{co}S$, if and only if a can be expressed as $a = \sum_{i=1}^{k} \lambda_i a_i$, where k is a positive integer, $\sum_{i=1}^{k} \lambda_i = 1$ and $a_i \in S, \lambda_i \ge 0, i = 1, \cdots, k$. For the set $\{a_1, \cdots, a_m\}$, where $a_i \in \mathrm{R}^n, i = 1, \cdots, m$, its convex hull $\mathrm{co}\{a_1, \cdots, a_m\}$ can be expressed as

$$\mathrm{co}\{a_1, \cdots, a_m\} = \left\{\sum_{j=1}^{m} \lambda_i a_i \middle| \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \ge 0, i = 1, \cdots, m\right\}. \quad (21)$$

All bounded convex polyhedron in space R^n can be expressed in the form of the above equation, where $a_i (i = 1, \cdots, m)$ is geometrically seen as the vertices of the corresponding polyhedron. An unbounded polyhedron can be expressed in the following form:

$$W = \mathrm{co}\{w_1, \cdots, w_m\} + \mathrm{cone}\{d_1, \cdots, d_n\}, \quad (22)$$

where $w_1, \cdots, w_m$ represent the extreme points, and $d_1, \cdots, d_n$ represent the extreme directions of the polyhedron. According to (22), for any $x \in W$, there exists $\mu_i \ge 0, \eta_j \ge 0, i = 1, \cdots, m; j = 1, \cdots, n, \sum_{i=1}^{m} \mu_i = 1$ such that

$$x = \sum_{i=1}^{m} \mu_i w_i + \sum_{j=1}^{n} \eta_j d_j, \quad (23)$$

holds. Figure 3 presents an unbounded polyhedron represented by the convex hull of w_1, w_2 and the nonnegative linear combination of d_1, d_2, where $w_1 = [\,0 \;\; 0\,]^{\mathrm{T}}$, $w_2 = [\,0 \;\; 1\,]^{\mathrm{T}}$, $d_1 = [\,1 \;\; 0\,]^{\mathrm{T}}$, and $d_2 = [\,1 \;\; 1\,]^{\mathrm{T}}$.

Viability of the switched system (1) on a region depends on whether the boundary points of the region satisfy the viability condition, that is, for each boundary point, whether there exists a subsystem A_k, where $k \in \Lambda$, such that the viability condition holds. However, this method is not feasible in practice as the region has infinite number of boundary points. In what follows, the viability of the switched system on an unbounded polyhedron is studied, and a sufficient viability criterion has proposed based on nonsmooth analysis.

Let the control input set of the switched system (1) be a convex set. The viability condition of the unbounded polyhedron W is presented as follows.

Theorem 3. *Let the nonempty unbounded polyhedron W be given by equation (22), H be any facet of W, and $H = co\left\{w_1, \cdots, w_p\right\} + cone\left\{d_1, \cdots, d_q\right\}$, if there exists a subsystem $A_k (k \in \Lambda)$ of (1) such that A_k satisfies the viability condition at extreme points $w_1, \cdots, w_p$, and for any extreme directions $d_1, \cdots, d_q$, there exist $\lambda_r \ge 0 (r = 1, \cdots, q)$ such that $A_k d_j = \sum_{r=1}^{q} \lambda_r d_r$ holds, then the system (1) is viable on W.*

Proof. It is sufficient to prove that the boundary points of W satisfy the viability condition. Let x be any boundary point of W, then x must be located on a facet of W. Let x be on the facet H of W, then we have

$$x = \sum_{i=1}^{p} \mu_i w_i + \sum_{j=1}^{q} \eta_j d_j, \quad (24)$$

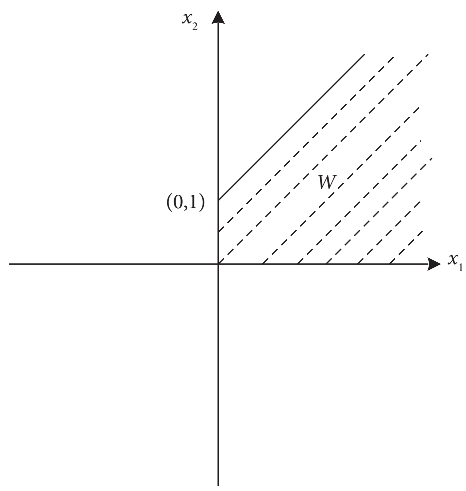

Figure 3: An unbounded polyhedron.

where $\mu_i \geq 0, \sum_{i=1}^{p}\mu_i = 1, \eta_j \geq 0, j = 1, \cdots, q$. Let us prove that the system (1) satisfies the viability condition at x.

Since the subsystem A_k satisfies the viability condition at the extreme points $w_1, \cdots, w_p$, i.e., there exist $u_i \in U, i = 1, \cdots, p$ such that

$$(A_k w_i + B_k u_i) \in T_W(w_i), i = 1, \cdots, p. \tag{25}$$

By the definition of tangent cone, if $y \in T_W(x)$, there exist $s > 0, \xi \in W$ such that $y = s(\xi - x)$ holds. Therefore, for each w_i in (25), there exist $s_i > 0$, $\xi_i \in W$ such that

$$\begin{cases} s_1(\xi_1 - w_1) = A_k w_1 + B_k u_1, \\ s_2(\xi_2 - w_2) = A_k w_2 + B_k u_2, \\ \cdots \\ s_p(\xi_p - w_p) = A_k w_p + B_k u_p, \end{cases} \tag{26}$$

where

$$\begin{gathered} \xi_i = \sum_{k=1}^{m} a_{ik} w_k + \sum_{j=1}^{n} b_{ij} d_j, i = 1, \cdots, p, \\ \sum_{k=1}^{m} a_{ik} = 1, a_{ik} \geq 0, b_{ij} \geq 0, j = 1, \cdots, n. \end{gathered} \tag{27}$$

Take $s = \max_{1\leq i\leq p}\{s_i\}$, and let

$$\gamma_i = w_i + \frac{s_i}{s}(\xi_i - w_i), \quad i = 1, \cdots, p. \tag{28}$$

Substituting ξ_i into the above equation, we will get

$$\begin{gathered} \gamma_i = w_i + \frac{s_i}{s}\left(\sum_{k=1}^{m} a_{ik} w_k + \sum_{j=1}^{n} b_{ij} d_j - w_i\right), \\ \gamma_i = \left(1 - \frac{s_i}{s}\right) w_i + \sum_{k=1}^{m} \frac{s_i}{s} a_{ik} w_k + \sum_{j=1}^{n} \frac{s_i}{s} b_{ij} d_j, i = 1, \cdots, p. \end{gathered} \tag{29}$$

Since the coefficient $(1 - s_i/s) + s_i/s\sum_{k=1}^{m} a_{ik} = 1$, and $1 - s_i/s \geq 0$, $(s_i/s)a_{ik} \geq 0$, $(s_i/s)b_{ij} \geq 0$, that is, γ_i can be expressed as a convex combination of extreme points of W and a nonnegative linear combination of extreme directions. Therefore, $\gamma_i \in W$, and

$$\gamma_i = w_i + \frac{s_i}{s}(\xi_i - w_i) \Longrightarrow s(\gamma_i - w_i) = s_i(\xi_i - w_i). \tag{30}$$

According to equations (26) and (30), we get

$$s(\gamma_i - w_i) = A_k w_i + B_k u_i, i = 1, \cdots, p, \tag{31}$$

where $\gamma_i \in W$. Let

$$\begin{gathered} \gamma_i = \sum_{k=1}^{m} \mu_{ik} w_k + \sum_{j=1}^{n} \eta_{ij} d_j, i = 1, \cdots, p, \\ \sum_{k=1}^{m} \mu_{ik} = 1, \mu_{ik} \geq 0, \eta_{ij} \geq 0, j = 1, \cdots, n. \end{gathered} \tag{32}$$

Substituting γ_i into equation (31), we have

$$\begin{gathered} s\left(\sum_{k=1}^{m} \mu_{ik} w_k + \sum_{j=1}^{n} \eta_{ij} d_j - w_i\right) = A_k w_i + B_k u_i, i = 1, \cdots, p, \\ s\left(\sum_{k=1}^{m} \mu_{ik} w_k + \sum_{j=1}^{n} \eta_{ij} d_j\right) - s w_i = A_k w_i + B_k u_i, i = 1, \cdots, p, \\ s\left(\sum_{k=1}^{m} \mu_{ik} w_k + \sum_{j=1}^{n} \eta_{ij} d_j\right) = (A_k + sI) w_i + B_k u_i, \\ i = 1, \cdots, p. \end{gathered} \tag{33}$$

Both ends of the above equation are multiplied by μ_1 when $i = 1$, and multiplied by μ_p when $i = p$, we get

$$\begin{cases} s\left(\sum_{k=1}^{m} \mu_1 \mu_{1k} w_k + \sum_{j=1}^{n} \mu_1 \eta_{1j} d_j\right) = (A_k + sI)\mu_1 w_1 + B_k \mu_1 u_1, \\ \cdots\cdots\cdots \\ s\left(\sum_{k=1}^{m} \mu_p \mu_{pk} w_k + \sum_{j=1}^{n} \mu_p \eta_{pj} d_j\right) = (A_k + sI)\mu_p w_p + B_k \mu_p u_p. \end{cases} \tag{34}$$

Adding up the above p equations, we obtain

$$\begin{gathered} s\left(\sum_{i=1}^{p} \mu_i \left(\sum_{k=1}^{m} \mu_{ik} w_k\right) + \sum_{j=1}^{n}\left(\sum_{i=1}^{p} \mu_i \eta_{ij}\right) d_j\right) \\ = (A_k + sI)\sum_{i=1}^{p}(\mu_i w_i) + B_k \sum_{i=1}^{p}(\mu_i u_i). \end{gathered} \tag{35}$$

Add $A_k(\sum_{j=1}^{q}\eta_j d_j)$ to both ends of the above equation, then

$$
\begin{aligned}
&s\left(\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j\right)+A_k\left(\sum_{j=1}^{q}\eta_j d_j\right)\\
&=\left(A_k+sI\right)\sum_{i=1}^{p}\left(\mu_i w_i\right)+B_k\sum_{i=1}^{p}\left(\mu_i u_i\right)+A_k\left(\sum_{j=1}^{q}\eta_j d_j\right),\\
&s\left(\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j\right)+s\frac{1}{s}A_k\left(\sum_{j=1}^{q}\eta_j d_j\right)\\
&=A_k\left(\sum_{i=1}^{p}\left(\mu_i w_i\right)+\sum_{j=1}^{q}\eta_j d_j\right)+B_k\sum_{i=1}^{p}\left(\mu_i u_i\right)+s\sum_{i=1}^{p}\left(\mu_i w_i\right),\\
&s\left(\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j-\sum_{i=1}^{p}\mu_i w_i+\frac{1}{s}\sum_{j=1}^{q}\eta_j A_k d_j\right)\\
&=A_k\left(\sum_{i=1}^{p}\mu_i w_i+\sum_{j=1}^{q}\eta_j d_j\right)+B_k\sum_{i=1}^{p}\mu_i u_i.
\end{aligned}
\tag{36}
$$

Substitute the given $A_k d_j = \sum_{r=1}^{q}\lambda_r d_r$ into the left end of equation (36), then

$$
\begin{aligned}
&s\left(\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j+\frac{1}{s}\sum_{j=1}^{q}\eta_j\sum_{r=1}^{q}\lambda_r d_r-\sum_{i=1}^{p}\mu_i w_i\right)\\
&=A_k\left(\sum_{i=1}^{p}\mu_i w_i+\sum_{j=1}^{q}\eta_j d_j\right)+B_k\sum_{i=1}^{p}\mu_i u_i,\\
&s\left(\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j+\frac{1}{s}\sum_{j=1}^{q}\eta_j\sum_{r=1}^{q}\lambda_r d_r+\sum_{j=1}^{q}\eta_j d_j-\left(\sum_{i=1}^{p}\mu_i w_i+\sum_{j=1}^{q}\eta_j d_j\right)\right)\\
&=A_k\left(\sum_{i=1}^{p}\mu_i w_i+\sum_{j=1}^{q}\eta_j d_j\right)+B_k\sum_{i=1}^{p}\mu_i u_i.
\end{aligned}
\tag{37}
$$

Since U is a convex set, where $u_i\in U, \mu_i\geq 0, i=1,\cdots,p, \sum_{i=1}^{p}\mu_i=1$, let $\overline{u}=\sum_{i=1}^{p}\mu_i u_i$, then $\overline{u}\in U$. According to (24), the above equation is transformed to

$$
s\left(\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j+\frac{1}{s}\sum_{j=1}^{q}\eta_j\sum_{r=1}^{q}\lambda_r d_r+\sum_{j=1}^{q}\eta_j d_j-x\right)=A_k x+B_k\overline{u},
\tag{38}
$$

The left of the above equation shows $\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)=\sum_{k=1}^{m}\left(\sum_{i=1}^{p}\mu_i\mu_{ik}\right)w_k$, and the coefficient of the extreme points w_k satisfy

$$
\sum_{k=1}^{m}\sum_{i=1}^{p}\mu_i\mu_{ik}=\sum_{i=1}^{p}\mu_i\sum_{k=1}^{m}\mu_{ik}=\sum_{i=1}^{p}\mu_i=1.
\tag{39}
$$

It implies $\sum_{i=1}^{p}\mu_i(\sum_{k=1}^{m}\mu_{ik}w_k)$ is a convex combination of extreme points of W. On the other hand,

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j+\frac{1}{s}\sum_{j=1}^{q}\eta_j\sum_{r=1}^{q}\lambda_r d_r+\sum_{j=1}^{q}\eta_j d_j, \quad (40)$$

is a nonnegative linear combination of the extreme directions $d_j\,(j=1,\cdots,n)$ of W. Letting

$$\zeta=\sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{m}\mu_{ik}w_k\right)+\sum_{j=1}^{n}\left(\sum_{i=1}^{p}\mu_i\eta_{ij}\right)d_j+\frac{1}{s}\sum_{j=1}^{q}\eta_j\sum_{r=1}^{q}\lambda_r d_r+\sum_{j=1}^{q}\eta_j d_j. \quad (41)$$

Then, $\zeta\in W$, and we have

$$s(\zeta-x)=A_k x+B_k\overline{u}, \quad (42)$$

It shows that for any x, we can obtain $s>0, \zeta\in W, \overline{u}\in U$, such that equation (42) holds, i.e.,

$$A_k x+B_k\overline{u}\in T_W(x). \quad (43)$$

Therefore, the system satisfies the viability condition at x, and by the arbitrariness of x, we know that the switched system (1) is viable on W. This concludes the proof of the theorem.

Theorem 3 has constructed a viability criterion for the switched system on an unbounded polyhedron which expressed by a convex hull of finite number of extreme points and a nonnegative linear combination of finite extreme directions. We have extended and developed the viability criterion. The method we have proposed has three advantages. First, determining the viability for the switched system is transformed into determining the consistency of a system of linear inequalities. It can be implemented in practice easily, and the method is feasible. Second, the method we have proposed only needs to verify the viability condition for some of the extreme points and some of the extreme directions for an unbounded polyhedron. Third, the method has less computational operations in some special cases.

5. Example

In this section, an example is employed to illustrate the effectiveness of the proposed methods.

For the switched system $\dot{x}(t)=A_\sigma x(t)$, where, $x\in \mathrm{R}^3$, $\sigma\in\{1,2\}$,

$$A_1=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},\quad A_2=\begin{bmatrix}1&0&0\\1&1&0\\0&0&-1\end{bmatrix}. \quad (44)$$

The extreme points and extreme directions of W are

$$w_1=\begin{bmatrix}0&0&0\end{bmatrix}^T, w_2=\begin{bmatrix}0&0&1\end{bmatrix}^T,\quad d_1=\begin{bmatrix}1&0&0\end{bmatrix}^T, d_2=\begin{bmatrix}0&1&0\end{bmatrix}^T. \quad (45)$$

In fact, W is an unbounded polyhedron obtained from the intersection of the first quadrant in the space rectangular coordinate system and $x_3=1$.

In what follows, we will determine the viability of the switched system on W. To facilitate presentation, the facet where the x_1x_3-coordinate plane intersects W is denoted as H_1, the x_1x_2-coordinate plane is denoted as H_2, the facet where the x_2x_3-coordinate plane intersects W is denoted as H_3, and the facet where $x_3=1$ intersects W is denoted as H_4. We discuss the viability for each facet, respectively.

For H_1, it can be expressed as

$$H_1=\mathrm{co}\{w_1,w_2\}+\mathrm{cone}\{d_1\}. \quad (46)$$

Since $A_2w_1=\begin{bmatrix}0&0&0\end{bmatrix}^{\mathrm{T}}$, $A_2w_2=\begin{bmatrix}0&0&-1\end{bmatrix}^{\mathrm{T}}$, then

$$A_2w_1\in T_W(w_1), A_2w_2\in T_W(w_2). \quad (47)$$

On the other hand, $A_2d_1=d_1+d_2$, according to Theorem 3, the subsystem A_2 is viable on H_1.

For H_2, it can be expressed as

$$H_2=\mathrm{cone}\{d_1,d_2\}. \quad (48)$$

Since $A_1d_1=d_1$, $A_1d_2=d_1+d_2$, the subsystem A_1 is viable on H_2, and the subsystem A_2 also satisfies the viability condition on H_2.

For H_3, it can be expressed as

$$H_3=\mathrm{co}\{w_1,w_2\}+\mathrm{cone}\{d_2\}. \quad (49)$$

Since $A_2w_1\in T_W(w_1)$, $A_2w_2\in T_W(w_2)$, and $A_2d_2=d_2$, the subsystem A_2 is viable on H_3. However, since

$$A_1w_2=\begin{bmatrix}0&0&1\end{bmatrix}^{\mathrm{T}}\notin T_W(w_2), \quad (50)$$

the subsystem A_1 does not satisfy the viability condition on H_3. Thus, when the state reaches the facet H_3, it is sufficient to switch the system to the subsystem A_2. Similarly, we can calculate and obtain that the subsystem A_2 satisfies the viability condition on H_4. All of these show that the switched system is viable on W. The example implies that the proposed method is feasible and effective. For the case of complex unbounded polyhedron, we can determine it in the same way.

6. Conclusion

We discuss the problem of determining the viability for the switched system on a cone and an unbounded polyhedron. Based on nonsmooth analysis, we have proposed two methods of determining the viability for a cone and an unbounded polyhedron, respectively. We only need to verify the viability condition on the some of the extreme points and extreme directions on the facet of the unbounded polyhedron. These methods presented in the paper are simple and feasible and can be directly used to determine viability. The results are the improvement and development of the viability criterion. There are still several research directions. For instance, determining the viability on the other regions is also important and meaningful. The viability for hybrid systems is also a challenging problem, which leads to strong mathematical difficulties. Finally, viability theory is still not completely explored in practice applications. This deserves more attention and more research activity on the subject in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work under the financial support from the National Natural Science Foundation of China under Grant no. 11171221 (Jianfeng Lv), Research Program of science and technology at Universities of Inner Mongolia Autonomous Region under Grant no. NJZY20098 (Jianfeng Lv), Natural Science Foundation of Inner Mongolia Autonomous Region under Grant nos. 2021LHMS06005 (Jianfeng Lv), 2022MS07010 (Na Zhao), and 2019MS01014 (Jun Tang), and the Fundamental Research Funds for Inner Mongolia University of Science and Technology under Grant nos. 069 (Jianfeng Lv) and 068 (Na Zhao).

References

[1] J. P. Aubin, *Viability Theory*, Springer-Verlag, Berlin, Germany, 2011.

[2] L. Doyen, O. Thebaud, C. Bene et al., "A stochastic viability approach to ecosystem based fisheries management," *Ecological Economics*, vol. 75, pp. 32–42, 2012.

[3] A. Rapaport, J. P. H. Terreaux, and L. Doyen, "Viability analysis for the sustainable management of renewable resources," *Mathematical and Computer Modelling*, vol. 43, no. 5-6, pp. 466–484, 2006.

[4] A. Oubraham and G. Zaccour, "A survey of applications of viability theory to the sustainable exploitation of renewable resources," *Ecological Economics*, vol. 145, pp. 346–367, 2018.

[5] L. Liu, Y. Gao, and Y. Wu, "Speed optimization control for wheeled robot navigation with obstacle avoidance based on viability theory," *Automatika*, vol. 57, no. 2, pp. 428–440, 2016.

[6] L. Liu, Y. Gao, and F. C. Wang, "Road safety analysis for high-speed vehicle in complex environments based on the viability kernel," *IET Intelligent Transport Systems*, vol. 12, no. 6, pp. 495–503, 2018.

[7] M. G. Zarch, V. Puig, J. Poshtan, and M. A. Shoorehdeli, "Fault detection and isolation using viability theory and interval observers," *International Journal of Systems Science*, vol. 49, no. 7, pp. 1445–1462, 2018.

[8] N. Zhao, X. Liu, J. F. Lv, and J. L. Yang, "Viability discrimination of a class of control systems on a nonsmooth region," *Discrete Dynamics in Nature and Society*, vol. 2014, Article ID 127185, 6 pages, 2014.

[9] Y. Gao, "Viability criteria for differential inclusions," *Journal of Systems Science and Complexity*, vol. 24, no. 5, pp. 825–834, 2011.

[10] J. F. Lv, Y. Gao, and N. Zhao, "Viability criteria for a switched system on bounded polyhedron," *Asian Journal of Control*, vol. 20, no. 6, pp. 2380–2387, 2018.

[11] N. Bonneuil, "Computing the viability kernel in large state dimension," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1444–1454, 2006.

[12] P. Saint-Pierre, "Approximation of the viability kernel," *Applied Mathematics and Optimization*, vol. 29, no. 2, pp. 187–209, 1994.

[13] S. Kaynama, J. Maidens, and M. Oishi, "Computing the viability kernel using maximal reachable sets," *International Conference on Hybrid Systems: Computation and Control, ACM*, pp. 55–64, 2012.

[14] J. F. Lv and Y. Gao, "The computation of the viability kernel for switched systems," *Advances in Difference Equations*, vol. 2018, no. 1, p. 297, 2018.

[15] J. N. Maidens, S. Kaynama, I. M. Mitchell, M. M. Oishi, and G. A. Dumont, "Lagrangian methods for approximating the viability kernel in high-dimensional systems," *Automatica*, vol. 49, no. 7, pp. 2017–2029, 2013.

[16] Z. Wang, R. M. Jungers, and C. J. Ong, "Computation of invariant sets via immersion for discrete-time nonlinear systems," *Automatica*, vol. 147, Article ID 110686, 2023.

[17] Y. Gao, "Determining viability of polytopic set for a linear control system," *Control and Decision*, vol. 31, no. 9, pp. 1720–1722, 2016.

[18] H. C. Wu, "Continuous-time linear programming problems revisited: a perturbation approach," *Optimization*, vol. 62, no. 1, pp. 33–70, 2013.

[19] N. T. Vinh, D. S. Kim, N. N. Tam, and N. Yen, "Duality gap function in infinite dimensional linear programming," *Journal of Mathematical Analysis and Applications*, vol. 437, no. 1, pp. 1–15, 2016.

[20] R. T. Rockafellar, "Conjugate duality and optimization," in *Proceedings of the 16th in Conference Board of Math. Sciences Series*, SIAM Publications, Laramie, Wyo, August 1974.

[21] Z. Chen and Y. Gao, "Determining the viable unbounded polyhedron under linear control systems," *Asian Journal of Control*, vol. 16, no. 5, pp. 1561–1567, 2014.

[22] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[23] I. M. Mitchell, A. Bayen, and C. Tomlin, "A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games," *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 947–957, 2005.

[24] A. A. Neznakhin, "Construction of the viability kernel for a generalized dynamical system," *Journal of Applied Mathematics and Mechanics*, vol. 70, no. 5, pp. 706–714, 2006.

[25] A. A. Neznakhin and V. N. Ushakov, "Construction of the viability kernel for a non-linear controlled system with a target set," *Journal of Applied Mathematics and Mechanics*, vol. 69, no. 6, pp. 875–884, 2005.

[26] G. Deffuant, L. Chapel, and S. Martin, "Approximating viability kernels with support vector machines," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 933–937, 2007.

[27] Y. Gao, J. Lygeros, and M. Quincampoix, "On the reachability problem for uncertain hybrid systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1572–1586, 2007.

[28] Y. Gao ∗, J. Lygeros, M. Quincampoix, and N. Seube, "On the control of uncertain impulsive systems: approximate stabilization and controlled invariance," *International Journal of Control*, vol. 77, no. 16, pp. 1393–1407, 2004.

[29] H. Haimovich and M. M. Seron, "Componentwise ultimate bound and invariant set computation for switched linear systems," *Automatica*, vol. 46, no. 11, pp. 1897–1901, 2010.

[30] J. F. Lv, Y. Gao, and N. Zhao, "The viability of switched nonlinear systems with piecewise smooth Lyapunov functions," *Journal of Industrial and Management Optimization*, vol. 17, no. 4, pp. 1825–1843, 2021.

[31] Q. L. Wei and H. Yan, "A method of transferring polyhedron between the intersection-form and the sum-form," *Computers & Mathematics with Applications*, vol. 41, no. 10-11, pp. 1327–1342, 2001.