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Research Article

Price Dynamics of a Delay Differential Cobweb Model

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The paper uses a new technique to find a unique solution to a delay differential cobweb model (formulated from a joint supply-demand function of price) with real model parameters via the Lambert W-function without considering any complex branches. The dynamics of the model are demonstrated with simulations and found to complement previous studies using literature values. However, the condition for instability $(\delta/\beta) > 1$ in the previous studies was defied by our model due to the time delay associated with the supply function. The practical application and advantage of this model over the existing models are that the stability of this model is not limited to only the ratio of price elasticity of demand and supply but also the time-delay parameter (i.e., a missing link in the previous models). Our model, on the other hand, loses its stability when the time delay associated with the supply function is fixed at $\tau = 1.8$. Since most of the physical systems, including economical systems, are time-delay inherent and such stability conditionalities should not limit their performance, it is recommended that such systems be modelled using delay differential functions. The novelty of this research is that there has not been a definite general solution to the cobweb model with a time delay whose price dynamics mimic the behaviour of the existing cobweb models in the literature. An illustrative example in a delayed fractional-order differential equation also buttressed the importance of the time delay in the model, aside from the impact of the ratio of the price elasticity of supply and demand.

1. Introduction

The cobweb model is an economic model, which explains the reasons for fluctuations in the prices of commodities in certain markets. The model usually characterizes the equilibrium price in a single market with a time lag between supply and demand functions. It is applied in agricultural markets where there is a time lag between planting and harvesting. Generally, the quantity of produce is determined by the suppliers' future price expectations (i.e., naive price expectations) due to the earlier price trend [1, 2] (see also Ezekiel [3]). Traditionally, the cobweb model is derived from the supply function with a time lag and demand functions of price to determine the current price based on the previous prevailing market price, using the equilibrium market principle in a single market with a single good [2, 4–6]. Let us consider the following piecewise functions of price for supply and demand:

$$\begin{cases} S(p(t)) = \lambda + \delta p(t-\tau), & \delta > 0, \\ D(p(t)) = \alpha - \beta p(t), & \beta > 0, \end{cases} \tag{1}$$

where the demand curve $D(p(t)) = \alpha - \beta p(t)$ gives the total amount of farm produce demanded at each price at a particular time t by the consumers in the market, and the supply curve $S(p(t)) = \lambda + \delta p(t-\tau)$ tells us the total amount farmers are willing to supply at each price. A production time delay τ (i.e., farm produce requires a period τ to grow to maturity and get harvested for the market) is introduced into the supply function of price. Concerning the formation of price expectations, we use the nave price expectation, where the expected price at time t is the market price realized at a time with $t - \tau$. We assume that the constants $\beta > 0$ and $\delta > 0$ (i.e., the price elasticity of demand and supply, respectively), in this case, make the demand curve slope downward and the supply curve slope upward. We also need the constant $\alpha > 0$ to ensure that there is a demand for the farm produce if the price is low enough, and we also assume that $\alpha > \delta$. On the assumption that the relative change in market price at any time t is the balance between demand and supply [2, 5], then we obtain

$$p'(t) = \gamma[D(p(t)) - S(p(t))], \tag{2}$$

where γ is the adjustment coefficient.

As already stated, the inclusion of a time delay in the mathematical formulation is necessary to account for the unknown details underlying the processes from planting to harvest [7]. Thus, we consider a continuous-time price model developed from delay differential equations, which characterizes equilibrium price dynamics in a single commodity market (i.e., a restructured form of demand and supply price functions in [2, 5, 6]). From equation (2), we have the following initial value problem:

$$\begin{cases} p'(t) = \gamma((\alpha - \lambda) - \beta p(t) - \delta p(t - \tau)), & t \in [0, \tau], \\ p(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{3}$$

where $\alpha, \beta, \delta, \gamma \in \mathbb{R}$, with assumption that $\delta > \beta$ and delay $\tau > 0$. The general solution of a mathematical formulation involving supply and demand functions of price has taken different forms depending on the formulation of either the supply function or the demand function or the nature of time around which the system is modelled (i.e., discrete or continuous). If (1) is redefined as

$$\begin{cases} S(p(t)) = \lambda + \delta p(t), & \delta > 0, \\ D(p(t)) = \alpha - \beta p(t + 1), & \beta > 0, \end{cases} \tag{4}$$

then at market clearing equilibrium $(S(p(t)) = D(p(t))$, we obtain the following general solution for an initial value of $p(0) = p_0$:

$$p(t) = p_e + (p_0 - p_e)\left(-\frac{\beta}{\delta}\right)^t, \tag{5}$$

where the initial price $p_0 \in \mathbb{R}$ and the equilibrium price $p_e = (\alpha - \lambda/\delta - \beta)$. This is a discrete-time cobweb model, see [5, 8]. The stability conditionality of the model (5) is described as follows:

(i) That the model (1) exhibits converging oscillations towards the equilibrium price (p_e) if and only if $\delta < \beta$

(ii) That the model (1) exhibits alternating oscillations around the equilibrium price (p_e) if and only if $\delta = \beta$ and

(iii) That the model (1) exhibits explosive oscillations around the equilibrium price (p_e) if and only if $\delta > \beta$ [3, 8].

If we incorporate in (1), a conformable fraction derivative $T_a(p(t))$ with $0 < a \leq 1$ in the demand function and also let $\tau = 0$ in (1), then we have a piece-wise function of the form.

$$\begin{cases} S(p(t)) = \lambda + \delta p(t), & \delta > 0, \\ D(p(t)) = \alpha + \beta(p(t) + T_a(p(t))), & \beta < 0, \\ D(p(t)) = S(p(t)), \end{cases} \tag{6}$$

where a positive β now replaces a negative β in the demand function in equation (1), and the general solution (6) is expressed by

$$p(t) = p_e + (p_0 - p_e)e^{(\delta - \beta/\beta a)(t^a - t_0^a)}, \tag{7}$$

where $p_e = (\alpha - \lambda/\delta - \beta) > 0$. This price function formulation is known as the conformable fractional cobweb model [5]. The stability conditionality of the model (7) is defined by Theorem 3 in [5] as $(\delta/\beta) < 1$ for convergence. By implication, we can say $(\delta/\beta) > 1$ for divergence, as outlined in [3, 8]. If the price expectation in (1) is formed instantaneously or the supply is instantaneous (*i.e.*, $\tau = 0$), then we have an ordinary differential cobweb model [9] whose unique solution is given by

$$p(t) = p_e + (p_0 - p_e)e^{-\gamma(\beta+\delta)t}, \tag{8}$$

where $p_0 \in \mathbb{R}$ and $p_e = (\alpha - \lambda/\beta + \delta)$. Since $(\beta + \delta) > 0$, the equilibrium price has always been locally stable. Thus, the market price will converge to the equilibrium price as $t \longrightarrow \infty$ provided the initial price is close enough to the equilibrium [9]. However, the general solution for (3) cannot be derived with any particular fixed rule as applied to the various models given in literature. Hence the motivation for this study so that we use a new technique to compute the general solution, which complements the previous research. In this study, we apply the Lambert function with a different technique to find only one solution, contrary to the infinite solutions from the Lambert W function, and use only the real branches since our model parameters are all real values. Thus, the key contribution is that there are general solutions to the cobweb model, whether they are discrete-time solutions defined or continuous-time solutions like equations (5), (7) and (8). All these equations converge to the equilibrium price, irrespective of the initial price value set for the model. Also, irrespective of the solution method one uses, the general solution and price dynamics are always alike (unique). However, there is no single defined general solution to the time-delayed cobweb model. The method with the closest general solution has been the Lambert W function, but it has infinite solutions, hence the motivation of the research is to come out with only one general solution defined in the purview of the model parameters (i.e., if they are real or complex values).

The rest of the paper is structured as follows: Section 2 discusses the existence, stability, and bifurcation of equation (3). The section also considers conditions for the positivity of the solution and the general solution of the equation (3). In Section 3, detailed numerical solutions and stability analysis are done by comparing model (3) to models given in the literature. Section 4 outlines the findings and Section 5 provides conclusions derived from the analyses of the models.

2. Materials and Methods

2.1. Preliminary Analysis. It is widely known that for any delay, there is a differential equation of the form

$$x'(t) = ax(t) + bx(t-\tau), \text{for } t > 0, \tag{9}$$

$$x(t) = \varphi(t), \text{for } t \in [-\tau, 0], \tag{10}$$

and via the Lambert W function, the solution can be written as

$$x(t) = \sum_{i=-\infty}^{\infty} C_i e^{1/\tau} W_i\left(-b\tau e^{a\tau}\right) t, \tag{11}$$

where C_i is determined from the history function, $\varphi(t)$. Every function $W(h)$, such that $W(h)e^{W(h)} = h$, is known as a Lambert function, and it is complex-valued, with a complex argument h, which has an infinite number of branches $W_i(h)$, with $i = -\infty, \ldots -1, 0, 1, \ldots \infty$, including the real branch [10].

On the contrary, this study presents an approach that brings out the real value argument h so that $W(h)e^{W(h)} = h$ is also be considered real and has only a real branch for its solution. This technique is a deduction from [11], and we are using it because the parameters in both the demand and supply functions of price are all real values. The technique provides us with advanced knowledge of the type of W branch to use to save time, taking into account the infinite number of branches.

2.2. Stability Analysis of Equation (3). To find the stability of the differential equation in (3), we linearize it in the neighbourhood of the equilibrium point $p_* = p(t) = p(t-\tau)$ and obtain the following expression:

$$p'(t) + rp(t) + sp(t-\tau) = 0, \tag{12}$$

where $r = \gamma\beta$ and $s = \gamma\delta$. We can observe from equation (12) that $p_* = p(t) = 0$ is the only equilibrium point. If we assume an exponential solution $p(t) = \kappa e^{\mu t}$, where $\kappa \neq 0$, then substituting it into equation (12) yields a characteristic equation (13), from which the stability of equation (12) is determined through the locations of its eigenvalues μ.

$$0 = \mu + r + se^{-\mu\tau}. \tag{13}$$

Now, we seek for a nontrivial solution from equation (12). If we let $\tau = 0$, then for $\mu = -(r+s) < 0$ the steady state p_* is asymptotically stable. On the other hand, when we assume that $\tau > 0$ then there exists $\mu = i\omega$ with $\omega > 0$ such that the real and the imaginary of equation (13) are derived as given by

$$\begin{cases} r + s\cos\omega\tau = 0, \\ \omega - s\sin\omega\tau = 0. \end{cases} \tag{14}$$

Moving the constants and the circular functions to either side of the equations, and then adding the squares of the resulting equations provides the following expression:

$$\omega^2 = (s+r)(s-r). \tag{15}$$

The necessary condition for stability change should be $|r| \leq |s|$. From equation (15), if $s \leq r$, then it contradicts the fact that $\omega > 0$, making the delay parameter harmless. We adopt the theorem from [9].

Theorem 1. *If $s \leq r$, then steady state of equation (12) is asymptotically stable for any positive value of τ.*

On the contrary, if $s > r$, then we can define $\omega > 0$ from equation (15) as follows:

$$\omega = \sqrt{(s+r)(s-r)}. \tag{16}$$

If the results in equation (16) are substituted into the first equation of the piecewise equation (14), then we obtain the threshold value of the delay parameter $\hat{\tau}$ as

$$\hat{\tau} = \frac{\left[\cos^{-1}\left(-(r/s)\right)\right]}{\sqrt{\left(s^2 - r^2\right)}}. \tag{17}$$

2.2.1. Stability Switches and Hopf Bifurcation of (3). In determining the direction of the stability switch, we assume that the roots of equation (13) are a continuous function of the time-delay parameter. Therefore, we take the derivative of equation (13) with respect to the time delay τ and solve for $(\mathrm{d}\mu/\mathrm{d}\tau)$ to arrive at the following results:

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\right)^{-1} = \frac{1 - \tau s e^{-\mu\tau}}{\mu s e^{-\mu\tau}}. \tag{18}$$

Then,

$$\begin{aligned} \left[\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\right]^{-1}_{\mu=i\omega} &= \mathrm{Re}\left(\frac{1 - \tau s e^{-\mu\tau}}{\mu s e^{-\mu\tau}}\right)_{\mu=i\omega}, \\ &= Re\left(\frac{-1}{\mu(\mu + r)}\right)_{\mu=i\omega} \\ \left[\frac{\mathrm{d}(\mathrm{Re}\mu)}{\mathrm{d}\tau}\right]^{-1}_{\mu=i\omega} &= \mathrm{Re}\left(\frac{\omega(\omega + ir)}{\omega^2\left(\omega^2 + r^2\right)}\right) \\ &= \frac{1}{\omega^2 + r^2}. \end{aligned} \tag{19}$$

Therefore,

$$\left[\frac{\mathrm{d}(\mathrm{Re}\mu)}{\mathrm{d}\tau}\right]^{-1}_{\mu=i\omega} > 0. \tag{20}$$

The results in equation (20) indicate that at $i\omega$ on the imaginary axis, all roots of the characteristic equation (13), near the critical value $\hat{\tau}$, will cross from left to right as the delay parameter τ varies continuously from a number less than $\hat{\tau}$ to that greater than $\hat{\tau}$.

Furthermore, we partially differentiate equation (17) with respect to the values r and s to determine the critical value of the delay as follows:

$$\frac{\delta\hat{\tau}}{\delta r} = \frac{\left[\left(s^2 - r^2\right)^{1/2} + r\cos^{-1}\left(-(r/s)\right)\right]}{\left(s^2 - r^2\right)^{3/2}} > 0, \tag{21}$$

with

$$\frac{\delta\hat{\tau}}{\delta s} = -\frac{\left[r\left(s^2 - r^2\right)^{1/2} + s^2\cos^{-1}\left(-(r/s)\right)\right]}{s\left(s^2 - r^2\right)^{3/2}} < 0. \tag{22}$$

Therefore, as $(\delta\hat{\tau}/\delta r) > 0$ and $(\delta\hat{\tau}/\delta r) < 0$ then it implies that by increasing the value of r and decreasing the value of s, the stability switching curves are shifted upwards. Thus, the variation of the parameters in the directions indicated has significant effects on stability. Therefore, the condition for transversality is met, and that for Hopf bifurcation is also satisfied at $\tau = \hat{\tau}$. Theorem 2 summaries this assertion as follows:

Theorem 2. *If $r + s < 0$ and $r < s$, then there exists $\hat{\tau} > 0$ such that the steady-state p_* of equation (3) is asymptotically stable for $0 < \tau < \hat{\tau}$, loses stability at $\tau = \hat{\tau}$, and becomes unstable or bifurcates to a limit cycle if $\tau > \hat{\tau}$.*

See full proof in [9, 12].

2.2.2. Existence and Uniqueness Solution of (3). For simplicity, we make $-r$ and $-s$ in equation (12), all positive values, so that it is transformed to

$$p'(t) = r\mathrm{p}(t) + s\mathrm{p}(t - \tau), \text{for } t > 0, \tag{23}$$

$$p(t) = \varphi(t), \text{for } t \in [-\tau, 0], \tag{24}$$

and then we can apply the method of steps to find a nonnegative solution equivalent to equation (12) from equation (23). From the interval $[0, \tau]$, we now generate a nonnegative solution expressed by

$$p(t) = \varphi(0) + \int_0^t [r\mathrm{p}(s) + s\mathrm{p}(s - \tau)]\mathrm{d}s. \tag{25}$$

Since $\varphi(0) \geq 0$, the solution exists and it is unique and nonnegative in the neighbourhood considered. Again in the interval $t \in [(n-1)\tau, n\tau]$, if we let p_n: $[(n-1)\tau, n\tau] \longrightarrow \mathbb{R}^+$ be the solution of equation (23), then for $t \in [n\tau, (n+1)\tau]$ it implies that

$$p(t) = p_n(n\tau) + \int_{n\tau}^t [r\mathrm{p}(s) + s\mathrm{p}(s - \tau)]\mathrm{d}s. \tag{26}$$

Thus, it is observed that for every nonnegative initial function $\varphi(t)$, the solution of equation (23) is defined for $t \geq 0$ in $t \in [n\tau, (n+1)\tau]$ [13].

2.2.3. Existence and Positivity Solution of (3). We now study the conditions, which guarantees nonnegative solutions of equation (23) for every positive initial function $\varphi(t)$. Following the results in Theorem 1.2 [14], the solution of the equation (23) can have negative values for a positive initial condition. This study is very important due to economic reasons. We now transform equation (23) into the following form:

$$p'(t) = \phi\mathrm{p}(t) - \rho\mathrm{p}(t - 1), \text{for } t > 0, \tag{27}$$

$$p(t) = \varphi(t), \text{for } t \in [-1, 0], \tag{28}$$

where $\phi = r\tau$, $\rho = s\tau$ with $t = \bar{t}\tau$ but the bars have been dropped for convenience's sake. We can further transform equation (27) into two composite functions, as expressed by

$$p'(t) = \phi\mathrm{g}(p(t)) - \rho\mathrm{f}(p(t - 1)), \text{for } t > 0, \tag{29}$$

then we consider the following auxiliary problem from the given composite function:

$$p'(t) = \phi\mathrm{g}(p(t)), \text{for } t > 0, \tag{30}$$

$$p(t) = \varphi(0). \tag{31}$$

From the proof of Theorem 1.1 in [13], for every $\varphi(0) \geq 0$, equation (30) has a nonnegative solution. Therefore, we will turn our attention to

$$p'(t) = \phi\mathrm{f}(p(t - 1)), \text{for } t > 0, \tag{32}$$

$$p(t) = \varphi(t), \text{for } t \in [-1, 0], \tag{33}$$

where $\phi \geq 0$ and the f has the following properties:

(i) f: $[0, 1] \longrightarrow [0, 1]$,
(ii) $f \in C^1(\mathbb{R}, \mathbb{R})$,
(iii) $f(0) = f(1) = 0$,
(iv) f is unimodal, i.e., there exists $\overline{p} \in (0, 1)$ with $f(p) < f(\overline{p}) = f_{\max}$, for every $p \neq \overline{p}$, see [14].

Using the method of steps, then for the interval $t \in [n, (n+1)]$, the solution is defined by

$$p(t) = p(n) + \phi\int_n^t \mathrm{p}(s - 1)\mathrm{d}s. \tag{34}$$

Now, we define the auxiliary function such that

$$F(\phi) = 1 + \phi f\left(1 + \phi f_{\max}\right), \tag{35}$$

and let ϕ_0 be the solution to $F(\phi_0) = 0$. Let us consider the following lemma:

Lemma 1. *If $\varphi(t) \in (0, 1]$ and $\phi < \phi_0$ then the corresponding solution to the function (32) is positive.*

Proof. For each $t \geq 0$, if we have $0 \leq p(t) \leq 1$, then $f(p(t-1)) \geq 0$. This implies that the solution is increasing and approaches 1 as $t \longrightarrow +\infty$. Hence, the solution is positive. Also, if there exists $t_0 \geq 0$ such that $p(t_0) = 1$,

$p(t) \in [0,1]$, with $t < t_0$ and $p(t) > 1$, for $t \geq t_0$, then $f(p(t-1)) < 0$, with $t > t_0 + 1$.

We can observe that for $t > t_0 + 1$, the solution is decreasing; however, it approaches 1, which implies a nonnegative solution. Let t_0 be the first point, such that $p(t_0) = 1$ and $p(t) > 1$, for $t \in (t_0, t_1)$, then on the assumption that there is a sequence (t_n), such that $p(t_n) = 1$, there exists $\bar{t} \in (t_0, t_1)$, where p has a maximum value. We can see that $p(t-1) = 1$ as $p(t) > 0$, for $t \in [-1, \bar{t}]$. Consequently, $\bar{t} = t_0 + 1$ and

$$p(\bar{t}) = 1 + \phi \int_{t_0-1}^{t_0} f(p(s))\mathrm{d}s \leq 1 + \phi f_{\max}. \tag{36}$$

Similarly, if there exists $\bar{t} \in (t_1, t_2)$ within which p has a minimum value, then just as in the previous case of maximal value, we have $\bar{t} = t_1 + 1$ and

$$p(\bar{t}) = 1 + \phi \int_{t_1-1}^{t_1} f(p(s))ds \geq 1 + \phi f\left(1 + \phi f_{\max}\right). \tag{37}$$

Since $p(s) \in [1, 1 + \phi f_{\max}]$, for $s \in [t_1 - 1, t_1]$, the function f decreases from 0 to $f(1 + \phi f_{\max})$ on this interval. We now examine the auxiliary function F. We can see that $F(0) = 1$ and $F(+\infty) < 0$. Therefore, it is apparent that $F'(\phi) = f(1 + \phi f_{\max}) + \phi f_{\max} f'(1 + \phi f_{\max}) < 0$ and there is a unique positive zero solution of the function F. Let $\phi_0 > 0$ denote this point, then for every $\phi < \phi_0$, the solution to equation (32) is positive, because for the next maximal and minimal points, the inequalities (36) and (37) are satisfied [14].

2.2.4. General Solution of Equation (3). If we divide through the differential equation in (3) by $(\beta + \delta)$, then by letting $\bar{r} = \gamma\beta$ and $\bar{s} = \gamma\delta$, we obtain the following differential equation:

$$z'(t) = -\bar{r}\left(p(t) - p_e\right) + -\bar{s}\left(p(t-\tau) - p_e\right), \tag{38}$$

where $p_e = (\alpha - \lambda/\beta + \delta)$ is the equilibrium price. We can further denote $z(t) = p(t) - p_e$ and $z(t-\tau) = p(t-\tau) - p_e$ as deviations from equilibrium price, so that equation (38) is transformed to the form given by

$$z'(t) + \bar{r}z(t) + \bar{s}z(t-\tau) = 0, \tag{39}$$

where $z'(t) = p'(t)$.

This gives us a transcendental equation of the following form if we assume an exponential solution $(z(t) = ce^{\eta}t)$ and then multiply through the resultant equation by $e^{\bar{r}\tau}$.

$$(\eta + \bar{r})e^{\tau(\eta + \bar{r})} = -\bar{s}e^{\bar{r}\tau}. \tag{40}$$

Let $m = \eta + \bar{r}$ and $\sigma = \bar{s}e^{\bar{r}\tau}$, then we have

$$F(m) = me^{m\tau} + \sigma, \tag{41}$$

where the range of values of σ determines the type of solution of the system [11]. To find m from equation (41), we also let $m = x \pm iy$ so that $(x \pm iy)e^{(x \pm iy)\tau} + \sigma = 0$ then we derive the real and imaginary parts of the systems, respectively, as given by

$$x\cos(y\tau) - y\sin(y\tau) = -\sigma e^{-x\tau}, \tag{42}$$

$$y\cos(y\tau) + x\sin(y\tau) = 0. \tag{43}$$

From equation (43), $x = -y\cos(y\tau)$, $y \neq 0$; this gives us

$$\lim_{y \longrightarrow 0} -y\cot(y\tau) = \lim_{y \longrightarrow 0} \frac{-y\tau\cos(y\tau)}{\tau\sin(y\tau)}, \tag{44}$$

$$\lim_{y \longrightarrow 0} -y\cot(y\tau) = \frac{-1}{\tau}. \tag{45}$$

Therefore, as $x \longrightarrow -(1/\tau)$, $y \longrightarrow 0$. Then, from equation (42), we obtain $\sigma = (1/\tau e) > 0$, which means that (41) has exactly one real zero solution [11]. The deduction here implies that we have a Lambert W function with real parameters satisfied by the following function if we multiply through equation (40) by τ

$$\tau(\eta + \bar{r}) = W(\zeta), \tag{46}$$

where $\zeta = \tau\bar{s}e^{\bar{r}\tau}$ and the W satisfies the property $We^{W} = \zeta$, with $W = \tau(\eta + \bar{r})$. Therefore, we find

$$\eta = \frac{1}{\tau}W(\zeta) - \bar{r}, \tag{47}$$

which provides a solution to equation (39) and in turn results in the particular solution of equation (38) in the form given by

$$\begin{aligned} p(t) = {} & p_e + \left(p_0 - p_e\right)e^{(1/\tau W(\zeta) - \bar{r})} \\ & + \left(p_0 - p_e\right)e^{(1/\tau W(\zeta) - \bar{r})(t-\tau)}, \end{aligned} \tag{48}$$

where $\zeta = \tau\bar{s}e^{\bar{r}\tau}$, $\bar{r} = \gamma\beta$, and $\bar{s} = \gamma\delta$. Unlike its counterpart (8) with no time delay and the other models reviewed in the study, the market price dynamics of this model are determined not only by the parameters δ and β but also by the time-delay τ associated with the supply function. The Lambert W branch satisfying the equation (48) is W_0, the principal branch. Hence, one does not need to use other W branches for a solution. This is the knowledge gap this paper sought to establish.

3. Results and Discussion

3.1. Numerical Solution: Price Dynamics of Model (3) and Model (2). In this section, we adopt the same parameter values used in [5] and compare the stability analysis of our model (3) to the stability analyses of the existing price adjustment models ((4), (6), and (8) reviewed in this study). In all the models, except (8), the convergence of the market price was dependent on the condition that $(\delta/\beta) < 1$, which conforms to the condition of convergence in the classical cobweb model Ezekiel in [3]. We now use the following parameters $\alpha = 0.8$, $\beta = 0.4$, $\lambda = a_1 = 1$, $\delta = 0.2$, and $p_e = 3.0$, with $p_0 = 5.0$. All the numerical simulations will be done using MATLAB software.

It is observed in Figure 1 that, for the same parameter values except for the time-delay parameter of $\tau = 1$, the

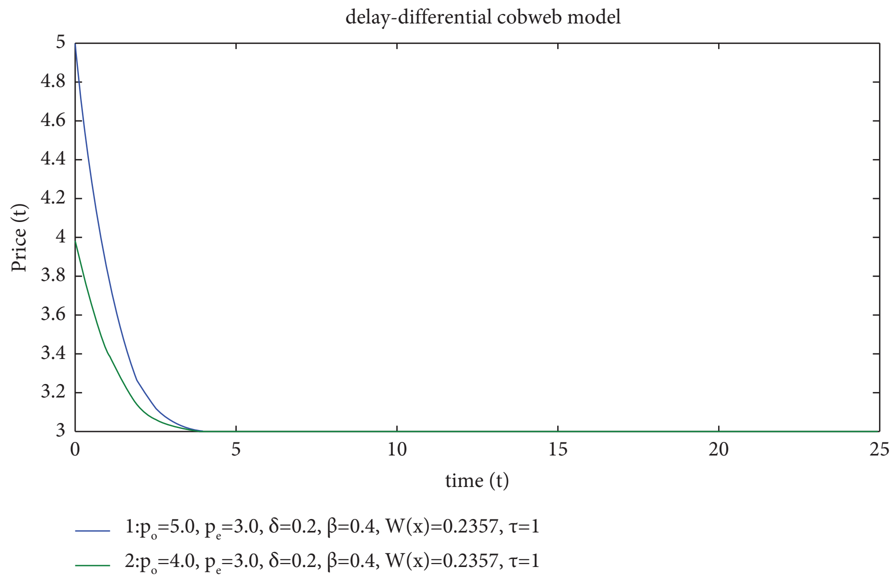

FIGURE 1: Price dynamics of model (3) for different values of p_0.

market price converges to the equilibrium price set for the system. The dynamics of model (49) complement the assertions of the papers in [3, 5, 8, 15, 16], including models (4) and (8).

3.2. Numerical Solution: Price Dynamics of Model (3) for $(\delta/\beta) > 1$. In the case of the condition $(\delta/\beta) > 1$, the two models reviewed, models (4) and (6), will have the market price diverged from the equilibrium price fixed for the price adjustment model, with the exception of models (3) and (8). These models (3) and (8) do not have the same defined conditions for stability as the others. Even though the price of an agricultural commodity is affected by the ratio of the price elasticity of supply and demand, the time to supply the commodity cannot be factored out but that will be reserved for future research. The time delay is a paramount factor missing in the other models. We now attest the condition for $(\delta/\beta) > 1$.

In Figure 2, we interchanged the values for δ and β and had the condition for convergence now $(\delta/\beta) = (0.4/0.2) = 2 > 1$. However, instead of divergence of the system, it is converging. We observe that if $(\delta/\beta) = \epsilon$, then the time delay associated with the supply is multiplied by the inverse of that proportion $(i.e. (1/\epsilon)\tau)$ and we will achieve stability. We can loosely infer that the supply of commodities is also affected by time delay aside from the ratio of price elasticity of supply and demand parameters.

3.3. Numerical Solution: Limitation of Model (3). In this section, we study conditions under which model (3) loses its stability concerning parameters used in the numerical simulation. We have already seen that $(\delta/\beta) > 1$ does not have much impact on the stability of the model, but the time-delay parameter τ does, so we should try values greater than one.

In Figure 3, we maintained the literature values and varied the time-delay parameters to assess their impact on the stability of the model (3). We increased the delay value to $\tau = 1.5$, and for the two initial prices, we had all prices converged to the equilibrium price.

For $\tau = 1.8$, the display in Figure 4 conforms to the behaviour of model (23) in [5] with $cT_a(p(t))$ incorporated in the supply function of price with c assigned a negative value. However, the highest time-delay parameter permissible for the supply function is to be $\tau = 1.7$. Above this limit value, model (3) becomes unstable; the market price diverges from the equilibrium price set for the system.

3.4. Illustrative Application of the Technique. In this section, we provide an illustrative example of the technique proposed in the study applied to a common ordinary differential model, transform it into a delay differential equation with integer-order and fractional-order, and then discuss the dynamics of the solutions using numerical simulations. Thus, we will consider the general solution technique of the following differential equation (49) and then apply the technique proposed in the study to both the delay differential equation and the fractional-order differential equation.

$$x'(t) = ax(t); x(0) = 1. \tag{49}$$

The variable $x(t)$ is farm produce supplied at a unit time t. The general solution of equation (49) is given by

$$x(t) = e^{(at)}, \tag{50}$$

where stability is guaranteed if $a \leq 0$. When a delay parameter is incorporated in (49), such that we have $x'(t) = ax(t-\tau)$, then with the proposed technique, we can express the general solution as

$$x(t) = e^{1/\tau (W(a\tau)t)}, \tag{51}$$

whose stability condition is such that $(1/\tau)(W(a\tau)t) \leq 0$ for all real values of τ, a, and t.

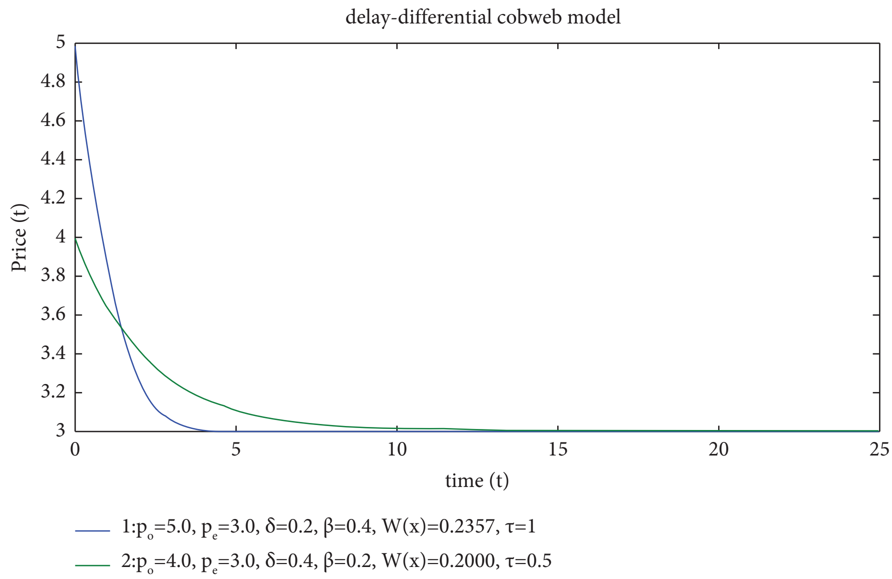

Figure 2: Price dynamics of model (3) for different values of p_0 and $(\delta/\beta) > 1$.

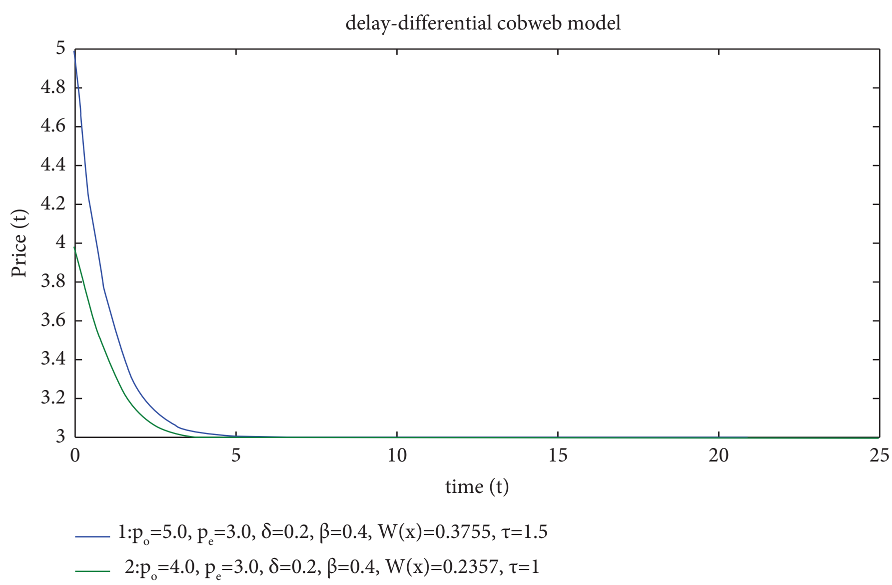

Figure 3: Price dynamics of model (3) for different values of p_0 and τ.

Similarly, we can change both the left and the right-hand sides of (49) to the form $(d^r x(t)/dt^r) = ax(t-\tau)$ known as the delayed fractional order differential equation (DFODE) and apply the Laplace transform to derive its characteristic equation and in turn obtain the following general expression via the Lambert function:

$$x(t) = e^{r/\tau\left(W\left(\tau/ra^{1/r}\right)t\right)}, \tag{52}$$

where r is a fractional-order (noninteger), with τ, a, and t as real values.

3.5. Numerical Solution: From the Illustrative Application. In this section, we consider the case that $a \geq 0$, which makes equation (50) unstable and uses the same value in equations (51) and (52) to ascertain the impact of a in relation to time-delay τ and the fractional-order r on the stability of the systems. We have already observed that model (3) defines the stability conditionality defined by $(\delta/\beta) > 1$ due to the presence of the time-delay parameter, and we want to find out if the technique is applicable to the application examples specified in this section.

For $a = 0.85 > 0$ and the starting function value of $x = 0.02$, the display in Figure 5 conforms to the behaviour of every exponential function having $e^a > 0$ and occupying a large space located at $(x(t), t) = (0.7, 500)$ as the functional line is extended to 0.7 from 0.2.

For $a = 0.85 > 0$ and $\tau = 1.0$, the display in Figure 6 shows a quick movement down the functional line, then halfway in between 0.02 and 0, diverged to the point

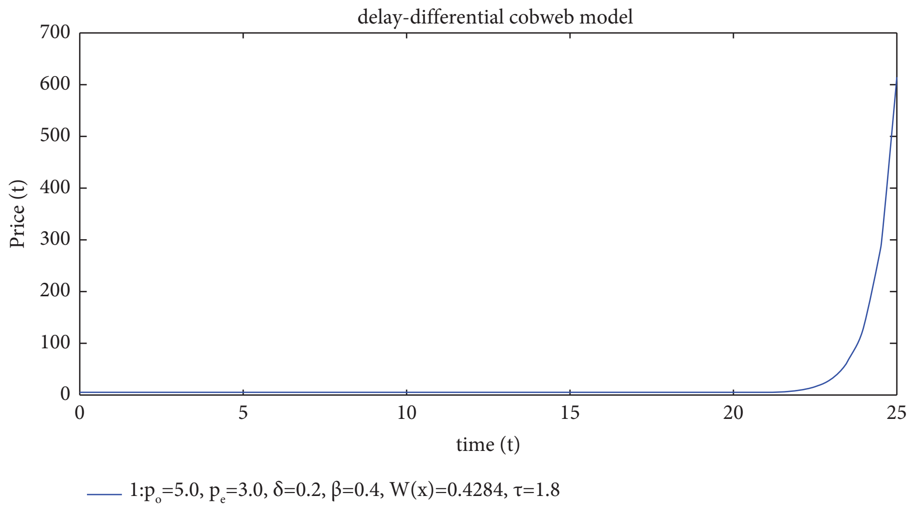

FIGURE 4: Price dynamics of model (3) for $(\delta/\beta) < 1$ and highest τ value.

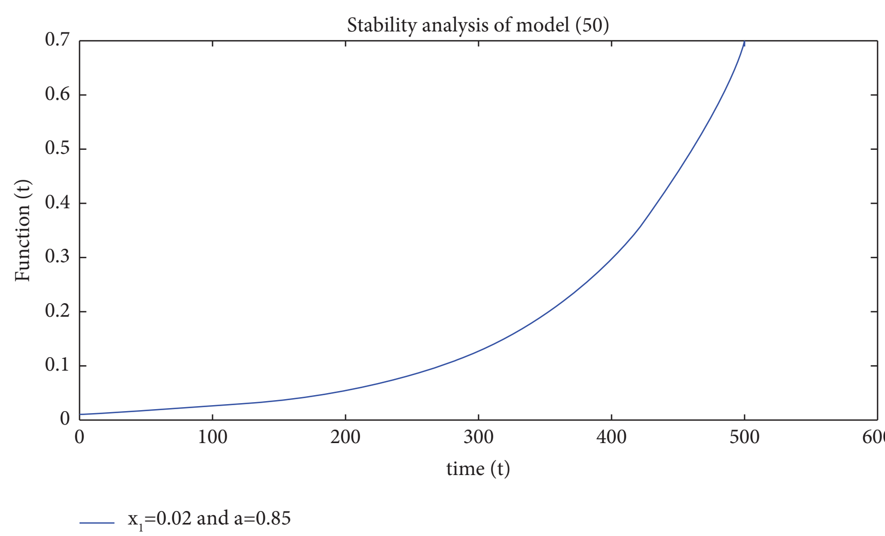

FIGURE 5: Supply dynamics of model (50) for $a = 0.85 > 0$.

$(x(t), t) = (0.13, 500)$ that extended the functional line of the system but was less than that of the model (50).

For $a = 0.85 > 0$, $\tau = 1.0$, and the fractional-order of $r = 1/3$, the display in Figure 7 indicates a very long and slow movement along the functional line up to the point 0.01 and then at time $t = 0$ diverged to increase the functional line to 0.038 (close to the next point in Figure 6, after the starting values 0.02) and occupy space defined by $(x(t), t) = (0.038, 500)$, a very small space compared to the display of the model (50) and (51). This confirms that with a delayed fractional-order differential equation, the instability of the model is drastically reduced.

Figure 8 indicates that for $a = 0.85 > 0$ and $\tau = 0.5$, a very long and slow movement is observed this time along the functional line up to the point 0.01 and then at time $t = 0$ diverged to increase the functional line to 0.022 to occupy a space smaller than that in Figure 6, when we had $\tau = 1$.

For $a = 0.85 > 0$, $\tau = 0.5$, and the fractional-order of $r = 1/5$, Figure 9 shows similar characteristics as observed in Figure 7 and occupies a smaller space as defined by the next point in Figure 8, after the starting point 0.02. Thus, the smaller the fractional-order, the more drastic the instability of the function becomes.

This assertion will be further investigated in comparison with other models [17] in future research, where the bifurcation properties of the fractional-order delay differential cobweb model will be discussed as shown in the related articles [17–21] and then assessed for the practical implication of the bifurcation of price in the market.

4. The Practical Application of the Proposed Technique

The change in the price of a commodity, be it farm produce or a product manufactured by an industry, depends on a certain unit of time it takes to be made available at the market. If it takes a short time to grow and be harvested for the market or manufactured by industry, then the price will

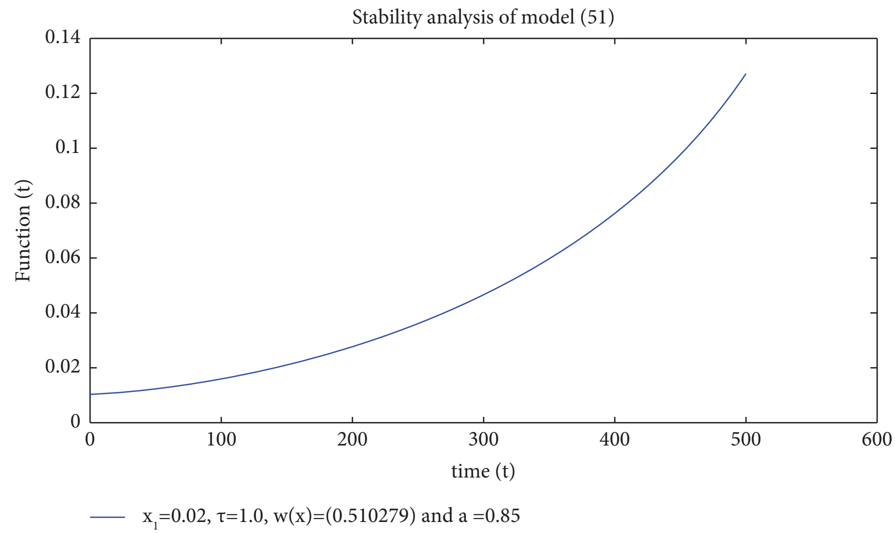

Figure 6: Supply dynamics of model (51) for $\tau = 1$.

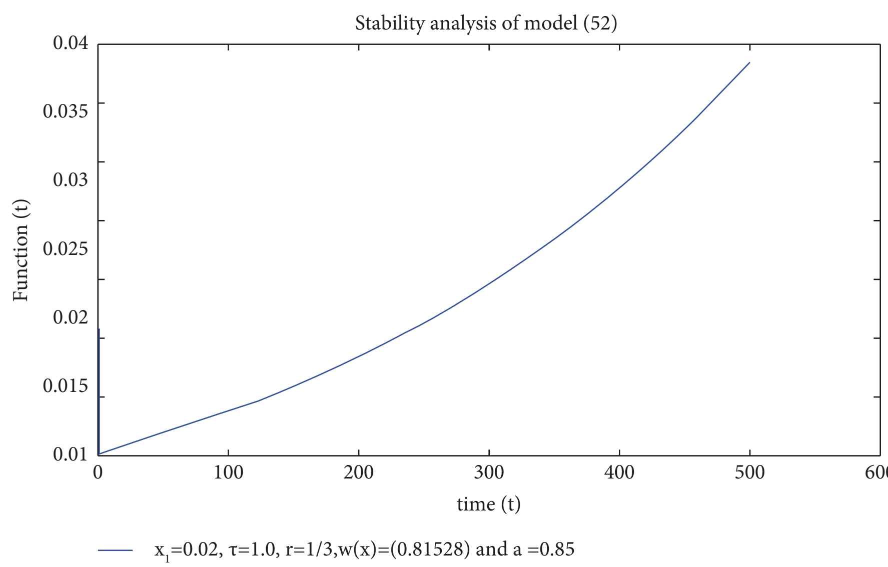

Figure 7: Supply dynamics of the model (52) for $r = (1/3)$.

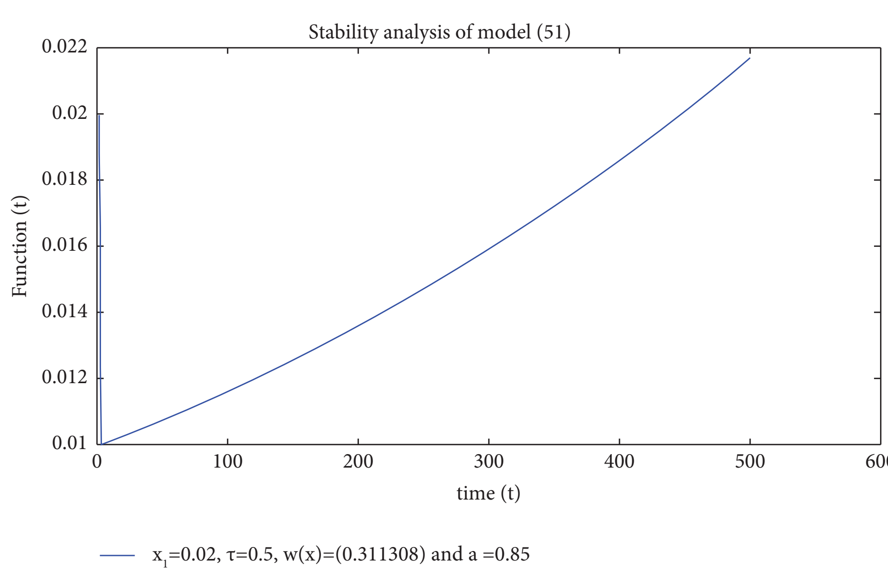

Figure 8: Supply dynamics of model (51) for $\tau = 0.5$.

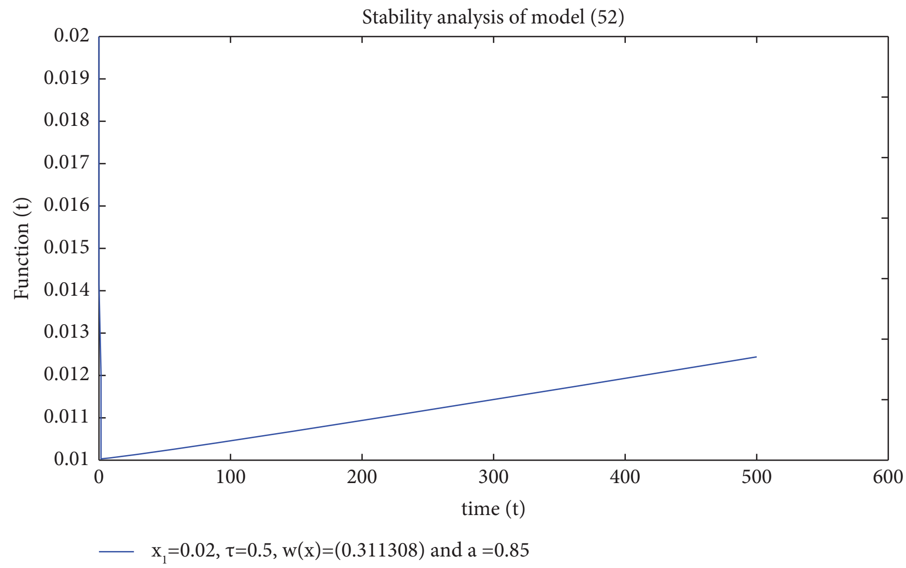

Figure 9: Supply dynamics of model (52) for $r = (1/5)$.

not escalate, irrespective of the demand level of the commodity at the market. We will witness the reverse if the time for harvest or manufacturing is long. Since the time delay in the price adjustment models accounts for the time details underlying the processes from planting to harvest, the time delay in the model is very important, aside from the impact of the ratio of the price elasticity of supply and demand.

5. Conclusion

The paper studied the stability analysis of the delay differential cobweb model formulated from a joint supply-demand function of price. We derived a particular solution to the model through the Lambert W function using a new technique that identifies the nature of the zero solution from the corresponding transcendental equation. As a result of the technique, we had the advanced knowledge that our model would have its solution in the principal branch W_0 of the Lambert function because it satisfied the real parameters in our model. This is contrary to the general solution that is obtained from the Lambert W function [10]. The model adopted the literature values from the previous research, and the price dynamics were found to complement the previous studies. However, the condition for instability $(\delta/\beta) > 1$ in the previous studies was defied by our model since it was still stable. The reason for this stability is found to be associated with a time delay in the supply function. Thus, the practical application and advantage of this model over the existing models are that the stability of this model is not limited to only the ratio of price elasticity of demand and supply but also the time-delay parameter (i.e., a missing link in the previous models). Our model (3) also has a limitation. The model diverged from the equilibrium price defined in the system when we applied the time-delay parameter $\tau = 1.8$ and beyond, and the behaviour confirmed the assertion from the previous studies. Since most of the physical systems, including economical systems, are time-delay-inherent and the stability conditionalities in the previous models should not limit their performance, it is recommended that such systems be modelled using delay differential functions. Beyond the general solution of the time-delay cobweb is the complex dynamics of price in a chaotic state when the bifurcation properties of the model are studied. Therefore, future research will consider modelling the price adjustment model with a time delay, addressing bifurcation properties and the practical implications of the bifurcations on the price in the market using the fractional-order delay [22] differential technique.

Data Availability

No data are available. Literature values were used.

Disclosure

This research was performed as part of the employment requirements by the employers, University of Cape Coast, Ghana, Kwame Nkrumah University of Science and Technology, Ghana, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Ghana, and Sunyani Technical University, Ghana, for the promotion of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Brianzoni, C. Mammana, E. Michetti, and F. Zirilli, "A stochastic cobweb dynamical model," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 219653, pp. 1–18, 2008.

[2] L. Gori, L. Guerrini, and M. Sodini, "Hopf bifurcation and stability crossing curves in a cobweb model with

heterogeneous producer and time delay," *Nonlinear Analysis: Hybrid Systems*, vol. 18, pp. 117–133, 2015.
[3] M. Ezekiel, "The cobweb theorem," *Quarterly Journal of Economics*, vol. 52, no. 2, pp. 255–280, 1938.
[4] W. Nicholson, *Microeconomic Theory: Basic Principles and Extensions*, Dryden Press, Hinsdale, IL, USA, 1978.
[5] M. Bohner and V. F. Hatipoglu, "Cobweb model with conformable fractional derivatives," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 18, pp. 9010–9017, 2018.
[6] C. H. Hommes, "Dynamics of the Cobweb model with adaptive expectations and nonlinear supply and demand," *Journal of Economic Behavior & Organization*, vol. 24, no. 3, pp. 315–335, 1994.
[7] K. L. Cooke and Z. Grossman, "Discrete delay, distributed delay and stability switches," *Journal of Mathematical Analysis and Applications*, vol. 86, no. 2, pp. 592–627, 1982.
[8] D. Dufresne and F. Vzquez-Abad, "Cobweb theorems with production lags and price forecasting," *Economics*, vol. 2013, Article ID 2035306, 2013.
[9] A. Matsumoto and F. Szidarovszky, "The asymptotic behavior in a nonlinear cobweb model with time delays," *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 312574, pp. 1–14, 2015.
[10] F. M. Asl and A. G. Ulsoy, "Analysis of a system of linear delay differential equations," *Journal of Dynamic Systems, Measurement, and Control*, vol. 125, no. 2, pp. 215–223, 2003.
[11] C. E. Falbo, "Analytic and numerical solutions to the delay differential equation y't=αyt-δ," revised," in *Proceedings of the Joint Meeting of the Northern and Southern California Sections of the MAA*, San Luis Obispo, CA, May 1995.
[12] S. Hal, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer Science and Business Media, Berline, Germany, 2010.
[13] U. Forys, "Delayed equations in applications," 2015, http://www.czm.mif.pg.gda.pl/wp-content/uploads/fam/publ/Forys.pdf.
[14] M. Bodnar, "The nonnegativity of solutions of delay differential equations," *Applied Mathematics Letters*, vol. 13, no. 6, pp. 91–95, 2000.
[15] U. Foryś, "Global stability for a class of delay differential equations," *Applied Mathematics Letters*, vol. 17, no. 5, pp. 581–584, 2004.
[16] M. Bohner and V. F. Hatipoglu, "Dynamic cobweb models with conformable fractional derivatives," *Nonlinear Analysis: Hybrid Systems*, vol. 32, pp. 157–167, 2019.
[17] C. Xu, D. Mu, Z. Liu et al., "Comparative exploration on bifurcation behavior for integer-order and fractional-order delayed BAM neural networks," *Nonlinear Analysis Modelling and Control*, vol. 27, no. 6, pp. 1–24, 2022.
[18] C. Huang, J. Wang, X. Chen, and J. Cao, "Bifurcations in a fractional-order BAM neural network with four different delays," *Neural Networks*, vol. 141, pp. 344–354, 2021.
[19] C. Huang, H. Liu, X. Shi et al., "Bifurcations in a fractional-order neural network with multiple leakage delays," *Neural Networks*, vol. 131, pp. 115–126, 2020.
[20] C. Xu, M. Liao, P. Li, Y. Guo, and Z. Liu, "Bifurcation properties for fractional order delayed BAM neural networks," *Cognitive Computation*, vol. 13, no. 2, pp. 322–356, 2021.
[21] C. Xu, D. Mu, Z. Liu, Y. Pang, M. Liao, and C. Aouiti, "New insight into bifurcation of fractional-order 4D neural networks incorporating two different time delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 118, Article ID 107043, 2023.