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Research Article

Optimal Control of Multiagent Decision-Making Based on Competence Evolution

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We employ the theory of rarefied gas dynamics and optimal control to investigate the kinetic model of decision-making. The novelty of this paper is that we develop a kinetic model that takes into account both the influence of agents' competence and managers' control on decision-making. After each interaction, in addition to the changes in decision directly caused by communication with other agents, the agents' competence evolves and indirectly influences the degree of decision adjustment through the compromise function. By adding a control term to the model, the behavior of the managers who require the group to establish consensus is also described, and the concrete expression of the control term that minimizes the cost function is obtained by model predictive control. The Boltzmann equation is constructed to characterize the evolution of the density distribution of agents, and the main properties are discussed. The corresponding Fokker–Planck equation is derived by utilizing the asymptotic technique. Lastly, the direct simulation of the Monte Carlo method is used to simulate the evolution of decisions. The results indicate that the agents' competence and managers' control facilitate the consistency of collective decisions.

1. Introduction

Kinetic theory was initially proposed by Maxwell [1] and Boltzmann [2] to describe the evolution of rarefied gas in the 19th century. The Boltzmann equation is the fundamental equation of the kinetic theory of gas molecules and plays a crucial role in studying rarefied gas dynamics. In recent years, Boltzmann's analysis has been applied to study socioeconomic phenomena by modeling interacting multiagent systems. The economists [3–7] seek the statistical mechanics of the kinetic models to investigate the formation of heavy tails in the distribution of wealth as predicted by Pareto. The research studies [8–12] apply the kinetic theory to investigate the mechanism of price formation, in which the agent's behavior and preferences are characterized by utility function. Xing et al. [13] develop a bargaining game-based feedback mechanism for dynamic social network group decision-making in which the trust relationship is proposed to reflect the interaction behaviors and generate advice to promote consensus. Ji et al. [14] investigate the overlapping community-driven feedback mechanism to support consensus in social network group decision-making.

As individuals living in a complex and ever-changing modern society, they are often required to communicate with others in various ways and to make decisions based on their knowledge and experience, which decide their competence. Likewise, group managers attempt to guide agents to reach a consensus on a certain event by using rewards or punishments. For instance, since the breakout of the COVID-19 epidemic, the government, as the manager of the country, has attempted to unite the population to reach a consensus about how to combat the epidemic. In this study, we utilize the kinetic model to investigate how agents' competence and managers' control influence the evolution of collective decision-making.

First, because competence is a crucial aspect of decision-making, we employ the theory of rarefied gas dynamics to create a kinetic model of decision-making based on competence.

Kruger and Dunning [15] demonstrate that people with low competence also have low metacognitive abilities, as shown by the fact that they cannot present high levels of performance or correctly perceive their low competence. Low competence people develop an unwarranted sense of overconfidence regarding their competence. In [16–19], they find that incompetent individuals are resistant to reaching a compromise and that competent individuals are more tolerant of it. In the decision model, competence determines the level of compromise, hence impacting the adjustment of decisions.

In recent years, many researchers apply the theory of rarefied gas dynamics to the study of decision-making, which provides the theoretical basis for our paper. Toscani [20] uses the theory of rarefied gas dynamics to develop a kinetic model of decision, in which agents adjust their decisions after binary interactions. The adjustment level is determined by compromise parameters. Toscani [20] proposes a kinetic model of decision as

$$\begin{cases} x^{\diamond} = x + \gamma G(|x|)(x_* - x) + \kappa_1 L(x), \\ x_*^{\diamond} = x_* + \gamma G(|x_*|)(x - x_*) + \kappa_2 L(x_*), \end{cases} \tag{1}$$

where the parameters x and x_* represent the decisions of the two agents before the binary interaction, the parameters $x^{\diamond}$ and $x_*^{\diamond}$ represent the decisions of the two agents after the interaction $(x, x_*, x^{\diamond}, x_*^{\diamond} \in [-1, 1])$, the constant γ represents the compromise coefficient, which belongs to $[0, 1]$, the compromise function $G(\cdot)$ depends on one's own decision, the random variables κ_1 and κ_2 follow the same uniform distribution with mean 0 and variance σ^2, and the function $L(\cdot)$ indicates the local relevance of the diffusion for a given decision.

Later, model (1) is expanded by considering additional factors that impact decisions, including conviction, self-thinking, leadership, external interference, and more complicated interaction networks. Brugna and Toscani [21] show that the agent's conviction is crucial to decision-making and has a resistant impact on compromise. In [22], it is demonstrated that decisions are involved with self-thinking. Pérez-Llanos et al. [23] discuss the influence of attractiveness on decision-making. It shows that the more attractive the other agent's decision is, the easier it is to compromise. Vieira et al. [24] construct a decision model as a function of the parameters that describe conviction, dissent, and independence. In [25–27], the researchers investigate the role of leadership in collective decision-making. Albi et al. [25] and Düring et al. [26] find that while leaders influence followers, followers cannot directly influence leaders. However, the strategy taken by leaders to alter followers' decisions is motivated by their intention to force followers toward a particular decision. Zhao et al. [27] investigate the influence of the leaders on followers under limited confidence and examine the impact of the leaders' quantity and the external environment on collective decision-making. Additionally, Pareschi et al. [28] develop a decision model based on competence to explore the evolutionary mechanism of collective decision-making, which highlights the effect of interaction on competence and proposes that mutual learning can enhance competence. Based on [28], we develop the argument that competence can only be improved by learning when other agents possess competence that we lack. We build a distinct kinetic model of competence, which ultimately results in a distinct kinetic model of decision-making. In order to better understand how competence evolution affects decision-making, the studies in our paper are based on the following hypotheses.

(i) There are a number of N agents in the group, which are identified by two variables (competence and decision), respectively.

(ii) At each time, the agents engage in a random binary interaction, and after each interaction, the agent's competence and decision evolve according to the respective kinetic models.

(iii) The compromise function in the decision-making model depends on the agents' competence.

Second, our work explores the impact of managers' control on the evolution of decisions and develops a kinetic model of decision-making that takes into account the evolution of competence and managers' control. Ideally, the changes in decisions are only derived from the binary interaction. In reality, however, agents are also influenced by various interfering factors, such as being persuaded to vote for a particular candidate in an election or being guided to purchase specific goods or assets through a large number of advertisements. To make the model more convincing, optimal control theory was incorporated into the decision-making model. Bauso et al. [29] utilize mean-field theory to examine the dynamics of decision-making when there are a large number of agents in the group. To investigate the management of emergent behavior in a multiagent system, Caponigro et al. [30] propose the concept of sparse optimization (the sparse control refers to the fact that the policymaker exerts control over a minimal number of agents for a minimal number of times). Tosin and Zanella [31] design feedback control strategies at the level of vehicle-to-vehicle interactions. Preziosi et al. [32] introduce a kinetic model for the study of tumor growth and design feedback control therapies to influence natural tumor growth. In addition, the literature [7, 25, 33, 34] employs the model predictive control (MPC) technique to investigate the decision-making model. Model predictive control is also known as receding horizon strategy or instantaneous control, which is based on the prediction of the model, not limited to the form of the model. The basic idea of MPC is to find the optimal solution on the shorter time interval by decomposing the optimal control problem on the time interval $[0, T]$ into the shorter time optimal control problem [35]. The MPC approach reduces the computational difficulty of the optimal control issue to some amount, even though this optimal solution is not necessarily the global optimum. We use a control term to describe managers' control and subsequently apply the MPC method to establish the concrete formulation of the control term. A kinetic model of decision-making that incorporates the evolution of competence and the managers' control is constructed.

Third, because the Boltzmann equation is the fundamental equation of rarefied gas dynamics, it is typically employed to explain the density distribution of agents. The stationary solution can be obtained for the simple Boltzmann equation [9, 21, 23]. However, when the form is complex, the Boltzmann equation is typically transformed into the Fokker–Planck equation by using the asymptotic method [25–27, 36, 37]. Since the kinetic model built in our paper incorporates the influence of competence and managers, it makes solving the Boltzmann equation very challenging. Thus, the main properties are discussed, and simulation experiments are conducted by using the direct simulation of the Monte Carlo (DSMC) method, which is proposed by Bird [38], to simulate the evolution of decision and describe the density distribution of agents.

The paper is structured as follows: In Section 2, the evolution of competence is modeled by using the theory of rarefied gas dynamics to study the impact of interactions on competence. A Boltzmann equation is constructed to describe the density distribution of competence. The solution of the Boltzmann equation shows that the average competence converges exponentially to a stable value. In Section 3, the situation in which managers' control motivates agents to reach a consensus is investigated. First, we incorporate the kinetic model of competence into the kinetic model of decision-making, where competence influences decision through the compromise function. Then, the manager's behavior is characterized by a control term, and the MPC method is used to derive the concrete expressions for the control term. A kinetic model of decision-making based on agents' competence and managers' control is established. In Section 4, the Boltzmann equation is created to explain the evolution of the density distribution of agents and the main properties are studied. In Section 5, the asymptotic approach is applied to obtain the Fokker–Planck equation corresponding to the Boltzmann equation. In Section 6, the DSMC method is utilized to simulate the evolution of decisions. It shows the solutions of the kinetic decision model at different time steps.

2. Modeling Competence

Before constructing the kinetic model of decision-making based on competence, we develop a kinetic model to examine the evolution of competence in this section.

In the study of competence evolution, Teevan and Birney [39] argue that despite the fact that competence can be acquired genetically and is strongly influenced by parents, it is evident that the primary factor influencing competence is the agent's living background. Bruga and Toscani [21] believe that competence is gained or lost due to two sources: the natural forgetfulness of competence and the improvement derived through learning from the background. Pareschi et al. [28] assert that competence can be gained by learning from others, and the quantity of competence acquired through learning relies on the agent's learning ability. In reality, we find that agents can only acquire competence from others when they possess competence that they lack. When binary interaction occurs, the lower competence agent can improve by learning from the higher one, but the agent with higher competence cannot improve by learning from the lower one. Considering this fact, a different competence dynamic model is developed.

Suppose the number of agents is N, the parameter c denotes the competence of the agent, and $c \in C \subset \mathbb{R}^+$ ($\mathbb{R}^+$ is positive real numbers). Before the binary interaction, the competence of the two agents is c and c_*, respectively. After the binary interaction, the competence of the two agents becomes $c^\diamond$ and $c_*^\diamond$. Let the evolution of competence follow the rules:

$$\begin{cases} c^\diamond = (1-\lambda_1(c))c + \lambda_2(c)(c_* - c) + \lambda_3(c)z + \eta c, \\ c_*^\diamond = (1-\lambda_1(c_*))c_* + \lambda_2(c_*)(c - c_*) + \lambda_3(c_*)z + \eta c_*, \end{cases} \tag{2}$$

in which $\lambda_1(\cdot) > 0$ indicates the natural forgetfulness of competence and $\lambda_2(\cdot) > 0$ donates the improvement of competence from mutual learning. We assume that $\lambda_2(\cdot) = \lambda_2 \chi(c_* \geq c)$, where $\chi(\cdot)$ is an indicator function. When other agents have higher competence $(c_* - c \geq 0)$, it will have a positive effect which makes one's competence ungraded. At the same time, when other agents have lower competence $(c_* - c < 0)$, there is no negative effect that makes one's own competence lower. It can be seen that the term $\lambda_2(c)(c_* - c)$ emphasizes that the degree of competence improvement depends on the competence gap between the two agents, which is not investigated in [21]. The variable $\lambda_3(\cdot) > 0$ denotes the tendency to acquire competence from the background. The parameter η is a random variable that indicates the change in competence influenced by unpredictable factors, which obeys a uniform distribution with mean equal to 0 and variance equal to σ_η^2. Let $\lambda_1(\cdot)$, $\lambda_2(\cdot)$, and $\lambda_3(\cdot) \in [\lambda_-, \lambda_+]$, where $0 < \lambda_- < \lambda_+ < 1$. We assume that $\eta \geq -1 + \lambda_+$ and η satisfies a uniform distribution on $[-1 + \lambda_+, 1 - \lambda_+]$. In the following equation, $A(z)$ is the probability distribution of competence in the background which has a bounded mean, i.e.:

$$\int_{\mathbb{R}^+} A(z)dz = 1, \int_{\mathbb{R}^+} zA(z)dz = m_z, \tag{3}$$

where m_z is a constant.

Let $g = g(c, t)$ be the density distribution of agents with competence c. The evolution of $g(c, t)$ with time is expressed in the weak form of the Boltzmann equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \phi(c)g(c,t)dc = (Q(g,g),\phi), \tag{4}$$

in which

$$(Q(g,g),\phi) = \omega\left\langle \int_{\mathbb{R}^+}\int_C \left(\phi\left(c^\diamond\right) - \phi(c)\right)g(c,t)A(z)dcdz \right\rangle. \tag{5}$$

As usual, $\langle\cdot\rangle$ denotes the mathematical expectation on the random variable η. The interaction kernel $\omega > 0$ is related to the frequency of interaction, and the test function $\phi(\cdot)$ satisfies the condition of a smooth function (a smooth function is a function that is continuously derivable to infinite order in its domain of definition). The competence after an interaction is given by equation (2). At any time $t \geq 0$, equation (4) illustrates the evolution of competence from c to $c^\diamond$ through interaction with the background and other agents. The collision operator $(Q(g,g),\phi)$ describes the interaction dynamic of $g(c,t)$.

In order to examine the evolution of average competence, we assume that

$$\int_C g(c,t)dc = 1, \int_C cg(c,t)dc = m_c(t), \tag{6}$$

which is bounded. In (4), we let $\phi(c) = c$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C cg(c,t)dc = \omega\left\langle \int_{\mathbb{R}^+}\int_{C^2} \left(c^\diamond - c\right)g(c,t)g(c_*,t)A(z)dcdc_*dz \right\rangle. \tag{7}$$

According to (6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}m_c(t) = \omega\left\langle \int_{\mathbb{R}^+}\int_{C^2} \left(c^\diamond - c\right)g(c,t)g(c_*,t)A(z)dcdc_*dz \right\rangle. \tag{8}$$

Plugging (2) into (8), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m_c(t) = {} & \omega\int_{\mathbb{R}^+}\int_{C^2} (-(\lambda_1(c) + \lambda_2(c))c + \lambda_2(c)c_*)g(c,t)g(c_*,t)A(z)dcdc_*dz \\ & + \omega\int_{\mathbb{R}^+}\int_{C^2} \lambda_3(c)zg(c,t)g(c_*,t)A(z)dcdc_*dz.\end{aligned} \tag{9}$$

Because of equations (3) and (6), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m_c(t) = {} & \omega\int_{C^2} (-\lambda_1(c)c - \lambda_2(c)c + \lambda_2(c)c_*)g(c,t)g(c_*,t)dcdc_* \\ & + m_z\omega\int_C \lambda_3(c)g(c,t)dc.\end{aligned} \tag{10}$$

When $\lambda_1(\cdot) = \lambda_1, \lambda_2(\cdot) = \lambda_2$, and $\lambda_3(\cdot) = \lambda_3$, where λ_1, λ_2, and λ_3 are all constants on the interval $[\lambda_-, \lambda_+]$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m_c(t) = {} & \omega\int_{C^2} (-\lambda_1 c - \lambda_2 c + \lambda_2 c_*)g(c,t)g(c_*,t)dcdc_* \\ & + m_z\omega\int_C \lambda_3 g(c,t)dc.\end{aligned} \tag{11}$$

Considering equation (3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}m_c(t) = -\omega\lambda_1 m_c(t) + \omega\lambda_3 m_z. \tag{12}$$

The solution of equation (12) is

$$m_c(t) = m_c(0)e^{-\omega\lambda_1 t} + \left(1 - e^{-\omega\lambda_1 t}\right)\frac{m_z\lambda_3}{\lambda_1}, \tag{13}$$

in which $m_c(0)$ denotes the initial value of average competence. As $t \longrightarrow \infty$, the average competence $m_c(t)$ converges to $m_z\lambda_3/\lambda_1$. It is shown that the average competence is reduced by increasing the parameter λ_1 or decreasing the parameter λ_3.

3. Modeling Decision-Making

In [20, 28, 40, 41], the main factors that influence decision-making include the agent's beliefs, self-thinking, degree of compromise, and level of competence, but none of them consider external interference. In reality, the decision-making environment is significantly more complex. In addition to the agent's mutual influence in the group, decisions are influenced by other interfering factors, such as the manager's supervision. Due to the diverse competence and knowledge of agents, reaching a consensus is a difficult process. At this time, managers typically utilize coordination or punishment to urge agents to reach a consensus as soon as possible. Note that managers only have the function of managing the group and formulating the orientation of collective decisions.

In this section, we investigate the evolution of decision-making based on agents' competence and managers' control. When evolution results in the concentration of decisions to a certain point, it indicates that a consensus is reached. The kinetic model of decision-making with a control term is established to describe the influence of managers on decisions. The control term satisfies a cost function with the implication that the difference between all agents' decisions is minimized. We use the MPC method to solve this optimal control problem and obtain a concrete expression to derive a decision model that describes the manager's control and investigate how a consensus is reached.

3.1. Model Predictive Control. To investigate the decision evolution mechanism of N agents, we assume that each agent's competence is $c_i \in C \subset \mathbb{R}^+$ and agent's decision is $x_i \in I = [-1, 1]$ (-1 and 1 represent two extreme opposite decisions), where $i = 1, 2, \ldots, N$. For example, 1 represents investing in a certain stock and -1 represents not investing in a certain stock at all, and the closer it is to 1, the more inclined it is to invest in that stock. The evolution of competence satisfies

$$c_i^{\diamond}(t) = \left(1 - \lambda_1(c_i)\right)c_i + \lambda_2(c_i)\left(c_j - c_i\right) + \lambda_3(c_i)z + \eta c_i. \tag{14}$$

The evolution of decisions satisfies

$$x_i^{\diamond}(t) = x_i(t) + \frac{1}{N}\sum_{j=1}^{N} G_{ij}\left(c_i, c_j, x_i, x_j\right)\left(x_j - x_i\right) + u, x_i(0) = x_{i,0} \geq 0, \tag{15}$$

in which u denotes the control term and $G_{ij} \in [0, 1]$ is the compromise function which implies that the degree of compromise is related not only to the decisions of the interacting agents but also to their competence. The purpose of the control term u is to reduce the difference in decisions between agents. Let the control term satisfy the cost function:

$$u = \operatorname{argmin} \frac{1}{2}\int_0^T \frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{N}\sum_{k=1}^{N}\left(x_j - x_k\right)^2 + \nu u^2\right)ds, u(t) \in \left[u_L, u_R\right]. \tag{16}$$

In (16), T represents the final time of interaction and u_L and $u_R\,(u_R > u_L)$ limit the value of $u(t)$ to ensure that the decision x_i does not exceed the bounded interval $[-1, 1]$ under the influence of the control term. The constant $\nu > 0$ indicates that different dissuasion and disciplinary strategies are applied for different agents.

The extreme value problem (15) and (16) can be regarded as a Mayer problem [42, 43], which can be solved by programming or Pontryagin's maximum principle, but this requires solving the optimization problem in the whole time interval $[0, T]$. In our paper, when N takes larger values, we use the MPC method to solve the optimization problem (15) and (16). The main advantage of the MPC method is reflected in its use of a rolling optimization strategy rather than global optimization, which can compensate for uncertainties due to model mismatch and disturbances timely. The MPC method has better dynamic performance.

Referring to [33], we proceed with the following steps:

(1) The time interval [0, T] is divided into a number of M shorter time intervals with a step length $\Delta t\,(\Delta t > 0)$, and let $t^n = n\Delta t$.

(2) It is assumed that the control term u is piecewise constant on the time interval $[t^n, t^{n+1}]$.

(3) The expression of the control term u^n is determined on the time interval $[t^n, t^{n+1}]$ by solving the following optimization problem:

$$\begin{aligned} \dot{x}_i &= \frac{1}{N}\sum_{j=1}^{N} G\left(c_i, c_j, x_i, x_j\right)\left(x_j - x_i\right) + u, x_i\left(t^n\right) = \overline{x}_i, \\ u &= \underset{u\in\mathbb{R}}{\operatorname{argmin}} \frac{1}{2}\int_{t^n}^{t^{n+1}} \frac{1}{N}\sum_{j=1}\left(\frac{1}{N}\sum_{k=1}^{N}\left(x_j - x_k\right)^2 + \nu u^2\right) ds, u \in \left[u_L, u_R\right]. \end{aligned} \tag{17}$$

(4) The control term u^n obtained on the time interval $[t^n, t^{n+1}]$ satisfies the kinetic equation of decision-making:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} G\left(c_i, c_j, x_i, x_j\right)\left(x_j - x_i\right) + u^n, \tag{18}$$

and the new state $\overline{x}_i = x_i\left(t^{n+1}\right)$ is obtained.

(5) The optimization problem (17) is solved again, and u^{n+1} is computed.

(6) The above steps are repeated until $n\Delta t = T$.

With the above steps, the control term u^n can be represented by $\overline{x}_i$ and $x_i\left(t^{n+1}\right)$, where $u_L = -\infty$ and $u_R = +\infty$. The discretization of equation (15) yields

$$x_i^{n+1} = x_i^n + \frac{\Delta t}{N}\sum_{j=1}^{N} G\left(c_i, c_j, x_i, x_j\right)\left(x_j^n - x_i^n\right) + \Delta t u^n. \tag{19}$$

Meanwhile, the discretization of equation (16) takes the form

$$J_{\Delta t}(x, u) = \frac{1}{2N}\sum_{j=1}^{N}\left\{\frac{\Delta t}{N}\sum_{k=1}^{N}\left(x_j^{n+1} - x_k^{n+1}\right)^2 + \nu\left(u^n\right)^2\right\}. \tag{20}$$

In order to minimize the cost function, the necessary condition is that the first-order differentiation of equation (20) for u^n is equal to zero; that is to say,

$$\frac{\mathrm{d}}{\mathrm{d}u^n}\left(\frac{1}{2N}\sum_{j=1}^{N}\left\{\frac{\Delta t}{N}\sum_{k=1}^{N}\left(x_j^{n+1} - x_k^{n+1}\right)^2 + \nu\left(u^n\right)^2\right\}\right) = 0. \tag{21}$$

After calculation, we get

$$\frac{\Delta t}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(x_j^{n+1} - x_k^{n+1}\right)\left(\delta_{i,j} - \delta_{i,k}\right) + \frac{\nu}{N}u^n = 0, \tag{22}$$

where $\delta_{i,j}$ denotes the Kronecker delta. When $i = j$, $\delta_{ij} = 1$, and when $i \neq j$, $\delta_{ij} = 0$. If $j = i = k$, we have $\delta_{i,j} - \delta_{i,k} = 0$, and it indicates that decision makers interact with themselves, which is impossible. If $j \neq i, k \neq i$, we have $\delta_{i,j} - \delta_{i,k} = 0$, and it indicates that no interaction occurs. Only when $j \neq i, k = i$, there is $\delta_{i,j} - \delta_{i,k} = 1$, and it represents that the binary interaction occurs between different decision makers. By solving the above equation, we get

$$\begin{aligned} u^n &= -\frac{2\Delta t}{\nu N}\sum_{j=1}^{N}\left(x_i^{n+1} - x_j^{n+1}\right) \\ &= -\frac{2\Delta t}{\nu}\left(x_i^{n+1} - \overline{x}^{n+1}\right), \end{aligned} \tag{23}$$

where $\overline{x}^{n+1} = \sum_{j=1}^{N} x_j^{n+1}/N$ denotes the average decision of all agents at $t = (n+1)\Delta t$. Combining equations (19) and (23), we obtain

$$u^n = -\frac{2\Delta t}{\nu} \times \left(x_i^n + \frac{\Delta t}{N}\sum_{j=1}^{N} G\left(c_i, c_j, x_i, x_j\right)\left(x_j^n - x_i^n\right) + \Delta t u^n - \overline{x}^{n+1}\right), \tag{24}$$

and

$$u^n = -\frac{2\Delta t}{\nu + 2\Delta t^2} \times \left(x_i^n - \overline{x}_i^{n+1} + \frac{\Delta t}{N} \sum_{j=1}^{N} G\left(x_j^{n+1} - x_j^n\right) \right)$$
$$= -\frac{2\Delta t}{\nu + 2\Delta t^2} \times \left(x_i^n - \overline{x}_i^n + \frac{\Delta t}{N} \sum_{j=1}^{N} G\left(c_i, c_j, x_i, x_j\right)\left(x_j^n - x_i^n\right) - \frac{\Delta t}{N^2} \sum_{k=1}^{N}\sum_{j=1}^{N} G\left(c_k, c_j, x_k, x_j\right)\left(x_j^n - x_k^n\right) \right). \tag{25}$$

Bringing the expression of the control variable u^n into (19), the control equation satisfying the optimality condition can be obtained. To better investigate the role of the control term, some basic assumptions are made next. First, it is assumed that the regularization parameter ν is also scaled such that $\nu = 2\iota\Delta t$. This assumption is based on the idea that the shorter the time interval, the more powerful control is needed to achieve the desired state. Second, the control over the decision must consider the frequency and intensity with which the interaction occurs, and it can be assumed that the value of the parameter ι depends on the compromise function $G(c_i, c_j, x_i, x_j)$. We let

$$\mathscr{C} = \sum_{i=1}^{N}\sum_{j=1}^{N} G\left(c_i, c_j, x_i, x_j\right), \tag{26}$$

where $i \neq j$ and the parameter ι is inversely proportional to $\mathscr{C}$. This assumption ensures that the control term is effective only when interaction occurs.

Bringing $\nu = 2\iota\Delta t$ into equation (25), the concrete expression for the control term u^n is

$$u^n = -\frac{1}{\iota + \Delta t} \times \left(x_i^n - \overline{x}_i^n + \frac{\Delta t}{N} \sum_{j=1}^{N} G\left(c_i, c_j, x_i, x_j\right)\left(x_j^n - x_i^n\right) - \frac{\Delta t}{N^2} \sum_{k=1}^{N}\sum_{j=1}^{N} \left(c_k, c_j, x_k, x_j\right)\left(x_j^n - x_k^n\right) \right). \tag{27}$$

3.2. The Kinetic Model of Decision-Making Based on Competence and Managers' Control. In Section 3.1, the equation for the evolution of decision-making in the presence of N agents under the influence of managers is set, and the expression for the control term is derived by the MPC method. Next, the kinetic model of decision-making is developed.

When the interaction occurs between agents i and j, the binary interaction model of decision-making is set as

$$\begin{cases} x_i^{n+1} = x_i^n + \frac{1}{2}\Delta t G\left(c_i, c_j, x_i, x_j\right)\left(x_j^n - x_i^n\right) + \Delta t u^n + \kappa_1 L\left(c_i, x_i\right), \\ x_j^{n+1} = x_j^n + \frac{1}{2}\Delta t G\left(c_j, c_i, x_j, x_i\right)\left(x_i^n - x_j^n\right) + \Delta t u^n + \kappa_2 L\left(c_j, x_j\right), \end{cases} \tag{28}$$

where $\kappa_i L(\cdot,\cdot)\,(i = 1, 2)$ is the diffusion term, indicating unpredictable factors. The random variables κ_1 and κ_2 obey a uniform distribution with mean 0 and variance σ^2, $L(\cdot,\cdot) \in [0, 1]$.

Since the control term is only related to the agents i and j, by setting $N = 2$, the control terms in (27) become

$$u^n = \frac{1}{\iota + \Delta t} \times \frac{1}{2}\left(1 - \frac{\Delta t}{2}\left(G\left(c_i, c_j, x_i, x_j\right) + G\left(c_j, c_i, x_j, x_i\right)\right) \right)\left(x_j^n - x_i^n\right). \tag{29}$$

Suppose that before the interaction occurs, the competence of the two agents is denoted by c and c_* and the decisions are denoted by x and x_*, respectively. After the binary interaction, the competence of the two agents are denoted by $c^{\diamond}$ and $c_*^{\diamond}$ and the decisions are denoted by $x^{\diamond}$ and $x_*^{\diamond}$, respectively. The variables satisfy $c, c_*, c^{\diamond}, c_*^{\diamond} \in C \subset \mathbb{R}^+$ and $x, x_*, x^{\diamond}, x_*^{\diamond} \in I = [-1, 1]$.

Let $\Delta t/2 = \gamma$ in (28), and the binary interaction model of decision-making becomes

$$\begin{cases} x^{\diamond} = x + \gamma G\left(c, c_*, x, x_*\right)\left(x_* - x\right) + \Delta t u + \kappa_1 L\left(c, x\right), \\ x_*^{\diamond} = x_* + \gamma G\left(c_*, c, x_*, x\right)\left(x - x_*\right) + \Delta t u + \kappa_2 L\left(c_*, x_*\right), \end{cases} \tag{30}$$

where

$$u = \frac{1}{2(\iota + \Delta t)}\left(1 - \gamma G(c_*, c, x_*, x) - \gamma G(c, c_*, x, x_*)\right)(x_* - x). \tag{31}$$

For simplicity, we let

$$\theta = \frac{\Delta t}{2(\iota + \Delta t)}, \theta \in [0,1], \tag{32}$$

in which $\theta = 0$ means no control or interference and $\theta = 1$ means maximum control or interference is generated. Bringing equation (32) into equation (31), we obtain

$$u = \frac{\theta}{\Delta t}\left(1 - \gamma G(c_*, c, x_*, x) - \gamma G(c, c_*, x, x_*)\right)(x_* - x). \tag{33}$$

Plugging (33) into (30), we obtain the kinetic model of decision-making based on agents' competence and managers' control in the form:

$$\begin{cases} x^{\diamond} = x + \left(\gamma G(c, c_*, x, x_*) + \theta\left(1 - \gamma G(c, c_*, x, x_*) - \gamma G(c_*, c, x_*, x)\right)\right) \\ \times(x_* - x) + \kappa_1 L(c, x), \\ x_*^{\diamond} = x_* + \left(\gamma G(c_*, c, x_*, x) + \theta\left(1 - \gamma G(c_*, c, x_*, x) - \gamma G(c, c_*, x, x_*)\right)\right) \\ \times(x - x_*) + \kappa_2 L(c_*, x_*). \end{cases} \tag{34}$$

Comparing (34) with the decision model in [28], it is evident that the evolutionary process of the decision is altered and that the formulation is made more complicated because of the control term. In order to better analyze the effect of managers' control on decision-making, the following discussion is presented.

First, we let

$$\begin{cases} \gamma G(c, c_*, x, x_*) + \theta\left(1 - \gamma G(c, c_*, x, x_*) - \gamma G(c_*, c, x_*, x)\right) = \widetilde{G}(c, c_*, x, x_*), \\ \gamma G(c_*, c, x_*, x) + \theta\left(1 - \gamma G(c_*, c, x_*, x) - \gamma G(c, c_*, x, x_*)\right) = \widetilde{G}(c_*, c, x_*, x). \end{cases} \tag{35}$$

The model of decision making influenced by managers can be further transformed into

$$\begin{cases} x^{\diamond} = x + \widetilde{G}(c, c_*, x, x_*)(x_* - x) + \kappa_1 L(x, x_*), \\ x_*^{\diamond} = x_* + \widetilde{G}(c_*, c, x_*, x)(x - x_*) + \kappa_2 L(x_*, x). \end{cases} \tag{36}$$

Notably, the compromise function satisfies

$$0 \le G(c, c_*, x, x_*), G(c_*, c, x_*, x) \le 1, \tag{37}$$

and $\theta \in [0,1]$ and $\gamma \in [0,1]$. From (35), we have

$$1 - \theta \le \widetilde{G}(c, c_*, x, x_*), \widetilde{G}(c_*, c, x_*, x) \le 1. \tag{38}$$

The function $\widetilde{G}(c, c_*, x, x_*), \widetilde{G}(c_*, c, x_*, x)$ contains the variable θ. According to (33), $\widetilde{G}(c, c_*, x, x_*), \widetilde{G}(c_*, c, x_*, x)$ represent not only the meaning of compromise but also the managers' control.

Next, without considering the diffusion term, we obtain

$$\begin{aligned} x_*^{\diamond} + x^{\diamond} &= x_* + x + \gamma\left(G(c, c_{,*}, x, x_*) - G(c_*, c, x_*, x)\right)(x_* - x), \\ x_*^{\diamond} - x^{\diamond} &= (1 - 2\theta)\left(1 - \gamma\left(G(c, c_*, x, x_*) + G(c_*, c, x_*, x)\right)\right)(x_* - x). \end{aligned} \tag{39}$$

In particular, when $G(c, c_*, x, x_*) = G(c_*, c, x_*, x)$, we have $x_*^{\diamond} + x^{\diamond} = x_* + x$. It implies that the interaction does not change the average decision; thus, we have $\overline{x}^{n+1} = \overline{x}$. Since the compromise function $G(\cdot,\cdot,\cdot,\cdot) \in [0,1]$ and $0 \le G(c, c_*, x, x_*) + G(c_*, c, x_*, x) \le 2$, we let $\theta \in [0, 1/2]$ and $\gamma \in [0, 1/2]$ to obtain

$$\left|x_*^{\diamond}-x^{\diamond}\right|=\left|(1-2\theta)\left(1-\gamma\left(G\left(c,c_*,x,x_*\right)+G\left(c_*,c,x_*,x\right)\right)\right)\left(x_*-x\right)\right| \\ \leq\left|x_*-x\right|, \tag{40}$$

which shows that the difference in decisions between the two agents does not increase after interaction.

Some constraints need to be set to ensure that the decision $x^{\diamond}$ after each interaction will not exceed the finite interval $[-1,1]$. Theorem 1 gives a sufficient condition for this problem.

Theorem 1. *It is assumed that* $0<G\left(c,c_*,x,x_*\right)\leq 1$ *and*

$$\theta\geq\frac{(1-\gamma p)}{(1-2\gamma p)},\left|\kappa_i\right|<\Lambda\left(1-\gamma G\left(c,c_*,x,x_*\right)-\theta\Gamma\right),i=1,2, \tag{41}$$

in which $p=\min_{x,x_*\in I}\{G(c,c_*,x,x_*)\}>0$, $\Lambda=\min_{x,x_*\in I}\{(1-x)/L(c,x),L(c,x)\neq 0\}>0$. *Then, according to the rules of the decision model (34), the postinteraction decision* $x^{\diamond}$ *is contained in the limited interval* $[-1,1]$.

Proof. The proof process is divided into two steps. In the first step, the diffusion term is not considered. We let

$$\Gamma=1-\gamma G\left(c,c_*,x,x_*\right)-\gamma G\left(c_*,c,x_*,x\right). \tag{42}$$

According to the rules identified in (34), the post-interaction decision is

$$x^{\diamond}=\left(1-\gamma G\left(c,c_*,x,x_*\right)-\theta\Gamma\right)x+\left(\gamma G\left(c,c_*,x,x_*\right)+\theta\Gamma\right)x_*, \tag{43}$$

in which

$$0\leq\gamma G\left(c,c_*,x,x_*\right)+\theta\Gamma\leq 1. \tag{44}$$

By calculation, we get

$$\frac{-\gamma G\left(c,c_*,x,x_*\right)}{\theta}\leq\Gamma\leq\frac{1-\gamma G\left(c,c_*,x,x_*\right)}{\theta}. \tag{45}$$

By definition, the compromise function $0<G(\cdot,\cdot,\cdot,\cdot)=p\leq 1$. Because $\theta\geq(1-\gamma p)/(1-2\gamma p)$, we have

$$1-2\gamma\leq\Gamma\leq 1-2\gamma p. \tag{46}$$

In the second step, the diffusion term is considered. According to the model (34), we obtain

$$x^{\diamond}=\left(1-\gamma G\left(c,c_*,x,x_*\right)-\theta\Gamma\right)x+\left(\gamma G\left(c,c_*,x,x_*\right)+\theta\Gamma\right)x_*+\kappa_1 L(c,x). \tag{47}$$

Since

$$\kappa_1\leq\Lambda\left(1-\gamma G\left(c,c_*,x,x_*\right)-\theta\Gamma\right), \tag{48}$$

we have

$$x^{\diamond}\leq\left(1-\gamma G\left(c,c_*,x,x_*\right)-\theta\Gamma\right)x+\gamma G\left(c,c_*,x,x_*\right)+\theta\Gamma+\kappa_1 L(c,x) \\ \leq 1. \tag{49}$$

Similarly, we can obtain $-1\leq x^{\diamond}$. The same results are readily obtained for the post-interaction decision $x_*^{\diamond}\in[-1,1]$. □

4. Main Properties of the Boltzmann Description for Decision

4.1. The Boltzmann Equation. Let the function $f=f(c,x,t)$ represent the density distribution of agents with competence c and decision x at time $t\geq 0$. If decisions are defined in the subinterval $I_1\subset I$, then the integral,

$$\int_{C\times I_1} f(c,x,t)dcdx, \tag{50}$$

denotes the distribution of agents whose decisions belong to the subinterval I_1 at time $t\geq 0$. Also, we have

$$\int_{C\times I} f(c,x,t)dcdx=1. \tag{51}$$

The evolution of the density function $f(c,x,t)$ over time depends on the interactions that occur. In this process, there is not only a change in the competence c but also an adjustment of the decision x. The level of decision adjustment is determined by the degree of compromise.

The function $\phi(c,x)$ is a smooth function, which is a test function in the Boltzmann equation. The symbol $\langle\cdot\rangle$ denotes the mathematical expectation on the random variable κ_1 or κ_2. Then, the variation of the density distribution is described by the weak form of the Boltzmann equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I\times C}\phi(c,x)f(c,x,t)dcdx = \frac{\omega}{2}\int_{\mathbb{R}^+}\int_{C^2\times I^2}\Big(\phi\big(c^\diamond,x^\diamond\big)+\phi\big(c_*^\diamond,x_*^\diamond\big)-\phi(c,x)\Big\rangle\\ \big\langle-\phi(c_*,x_*)\big)f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz,\end{aligned}\tag{52}$$

where the parameter ω is a scaling constant that depends on the frequency of interactions. The relevant properties of the decision under the influence of the manager are discussed by assuming $\phi(c,x)=\phi(x)=1,x,x^2$, respectively.

4.2. Main Properties. In equation (52), suppose $\phi(c,x)=\phi(x)=1$, then

$$\frac{\partial}{\partial t}\int_{I\times C}f(c,x,t)dcdx=0,\tag{53}$$

which indicates that the total number of agents is constant.

The average decision of all agents is expressed as

$$Y(t)=\int_{I\times C}xf(c,x)dcdx.\tag{54}$$

According to the decision model (34) and the Boltzmann equation (52), when $\phi(c,x)=x$, the evolution in the average decision of all agents over time occurs as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I\times C}xg(c,x,t)dcdx &= \frac{\omega}{2}\int_{\mathbb{R}^+}\int_{I^2\times C^2}\big(x_*^\diamond+x^\diamond-x_*-x\big)\Big\rangle\\ &\quad\big\langle\times f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz\\ &=\frac{\omega\gamma}{2}\int_{I^2\times C^2}\big(G(c,c_*,x,x_*)-G(c_*,c,x_*,x)\big)(x_*-x)\\ &\quad\times f(c,x,t)f(c_*,x_*,t)dcdc_*dx\,dx_*,\end{aligned}\tag{55}$$

namely,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I\times C}xg(c,x,t)dcdx = \omega\gamma\int_{I^2\times C^2}\big(G(c,c_*,x,x_*)-G(c_*,c,x_*,x)\big)x_*\\ \times f(c,x,t)f(c_*,x_*,t)dcdc_*dx\,dx_*.\end{aligned}\tag{56}$$

Since $0\le G(\cdot,\cdot,\cdot,\cdot)\le 1$ and $|G(c,c_*,x,x_*)-G(c_*,c,x_*,x)|\le 1$, we have

$$-\omega\gamma Y(t)\le\frac{d}{dt}Y(t)\le\omega\gamma Y(t).\tag{57}$$

From (57), we get

$$Y(0)e^{-\omega\gamma t}\le Y(t)\le Y(0)e^{\omega\gamma t},\tag{58}$$

where $Y(0)$ is the initial value of the average decision. When $G(c,c_*,x,x_*)=G(c_*,c,x_*,x)$, it is calculated that

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t)=0.\tag{59}$$

Therefore, when $t>0$, the average decision $Y(t)=Y(0)$, which indicates that managers' control does not change the average decision of agents. However, the control influences the internal structure of decisions and eliminates the decision gap between agents, which helps reach a consensus.

We let $\phi(c,x)=x^2$ in (52) to obtain the second-order moments:

$$W(t)=\int_{I\times C}x^2f(c,x,t)dcdx.\tag{60}$$

From (34) and (52), the change in second-order moments with time is

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t)=\frac{\omega}{2}\left\langle \begin{array}{l}\int_{\mathbb{R}^+}\int_{I^2\times C^2}\left(\left(x_*^{\diamond}\right)^2+\left(x^{\diamond 2}\right)-x_*^2-x^2\right)\\ \times f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz,\end{array}\right\rangle \tag{61}$$

where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W(t)=&\frac{\omega}{2}\int_{I^2\times C^2}((\gamma G(c,c_*,x,x_*)+\theta(1-\gamma G(c,c_*,x,x_*)\\&-\gamma G(c,c_*,x,x_*)))^2+(\gamma G(c_*,c,x_*,x)+\theta(1-\gamma G(c,c_*,x,x_*)\\&-\gamma G(c_*,c,x_*,x)))^2)(x-x_*)^2f(c,x,t)f(c_*,x_*,t)dcdc_*dx\,dx_*\\&-\omega\int_{I^2\times C^2}(x(\gamma G(c,c_*,x,x_*)+\theta(1-\gamma G(c,c_*,x,x_*)\\&-\gamma G(c_*,c,x_*,x)))-x_*(\gamma G(c_*,c,x_*,x)+\theta(1-\gamma G(c,c_*,x,x_*)\\&-\gamma G(c_*,c,x_*,x))))(x-x_*)f(c,x,t)f(c_*,x_*,t)dcdc_*dx\,dx_*.\end{aligned} \tag{62}$$

We let $G(c,c_*,x,x_*)=G(c_*,c,x_*,x)=p\in[0,1]$,and equation (62) is simplified to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W(t)=\omega\int_{I^2\times C^2}&(\gamma p+\theta(1-2\gamma p))(\gamma p+\theta(1-2\gamma p)-1)(x-x_*)^2\\&\times f(c,x,t)f(c_*,x_*,t)dcdc_*dx\,dx_*.\end{aligned} \tag{63}$$

Since the average decision $Y(t)=Y(0)$, it is further calculated that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W(t)=&2\omega(\gamma p+\theta(1-2\gamma p))(\gamma p+\theta(1-2\gamma p)-1)W(t)\\&-2\omega(\gamma p+\theta(1-2\gamma p))(\gamma p+\theta(1-2\gamma p)-1)Y(0)^2.\end{aligned} \tag{64}$$

The solution of equation (64) is

$$\begin{aligned}W(t)=&W(0)\exp\{2\omega(\gamma p+\theta(1-2\gamma p))(\gamma p+\theta(1-2\gamma p)-1)t\}\\&+(1-\exp\{2\omega(\gamma p+\theta(1-2\gamma p))(\gamma p+\theta(1-2\gamma p)-1)t\})Y(0)^2,\end{aligned} \tag{65}$$

in which $W(0)$ is the initial value of the second-order moments. Since $\omega(\gamma p+\theta(1-2\gamma p))>0$ and $\gamma p+\theta(1-2\gamma p)-1<0$, we obtain that $W(t)$ converges exponentially to $Y(0)^2$ when $t\longrightarrow\infty$. Thus, the equation,

$$\int_{C\times I}(x-Y(0))^2f(c,x,t)dcdx=W(t)-Y(0)^2, \tag{66}$$

converges toward 0 in the limit $t\longrightarrow\infty$. It is concluded that, under the same assumption, the steady-state solution of the decision model is a Dirac δ function centered on $Y(0)$, indicating that the managers' control drives the agents to achieve decision consensus.

5. Fokker–Planck Description

5.1. The Quasi-Invariant Decision Limit. We assume that $\epsilon > 0$ is a scaling parameter, and the individual parameters in the kinetic model of competence (2) are scaled by taking the form of

$$\lambda_1 = \epsilon\varrho_1, \lambda_2 = \epsilon\varrho_2, \lambda_3 = \epsilon\varrho_3, \sigma_\eta^2 = \epsilon\varsigma_\eta^2, \omega = \frac{1}{b\epsilon}. \tag{67}$$

After scaling, the competence model becomes

$$\begin{cases} c^\diamond = (1 - \epsilon\varrho_1(c))c + \epsilon\varrho_2(c)(c_* - c) + \epsilon\varrho_3(c)z + \eta_\epsilon c, \\ c_*^\diamond = (1 - \epsilon\varrho_1(c_*))c_* + \epsilon\varrho_2(c_*)(c - c_*) + \epsilon\varrho_3(c_*)z + \eta_\epsilon c_*, \end{cases} \tag{68}$$

where η_ϵ obeys a uniform distribution with mean 0 and variance $\epsilon\varsigma_\eta^2$. Applying the same method, the parameters in the decision model (34) are scaled by taking the form of

$$\gamma = \epsilon, \omega = \frac{1}{b\epsilon}, \sigma^2 = \epsilon\varsigma^2, \nu = \epsilon\rho. \tag{69}$$

Correspondingly, the parameters in (34) become

$$\theta = \frac{4\epsilon}{\rho + 8\epsilon}. \tag{70}$$

The decision model (34) is transformed to

$$\begin{cases} x^\diamond = x + \left(\epsilon G(c, c_*, x, x_*) + \dfrac{4\epsilon}{\rho + 8\epsilon}(1 - \epsilon G(c, c_*, x, x_*) - \epsilon G(c_*, c, x_*, x))\right) \\ \times (x_* - x) + \kappa_{1\epsilon} L(c, x), \\ x_*^\diamond = x_* + \left(\epsilon G(c_*, c, x_*, x) + \dfrac{4\epsilon}{\rho + 8\epsilon}(1 - \epsilon G(c_*, c, x_*, x) - \epsilon G(c, c_*, x, x_*))\right) \\ \times (x - x_*) + \kappa_{2\epsilon} L(c_*, x_*), \end{cases} \tag{71}$$

where $\kappa_{1\epsilon}$ and $\kappa_{2\epsilon}$ obey a uniform distribution with mean 0 and variance $\epsilon\varsigma^2$. The scaled Boltzmann equation is

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{I\times C} \phi(c, x) f(c, x, t)\,dc\,dx \\ &= \frac{1}{2b\epsilon}\left\langle \begin{array}{c} \displaystyle\int_{\mathbb{R}^+}\int_{C^2\times I^2} \left(\phi(c^\diamond, x^\diamond) + \phi(c_*^\diamond, x_*^\diamond) - \phi(c, x) - \phi(c_*, x_*)\right) \\ \times f(c, x, t) f(c_*, x_*, t) A(z)\,dc\,dc_*\,dx\,dx_*\,dz \end{array} \right\rangle. \end{aligned} \tag{72}$$

Thus, after scaling, the evolution of the average decision over time is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{I\times C} xg(c,x,t)dcdx = \frac{1}{b}\int_{I^2\times C^2}(G(c,c_*,x,x_*) - G(c_*,c,x_*,x))x_* \times f(c,x,t)f(c_*,x_*,t)dcdc_*dx\,dx_*. \tag{73}$$

When the function $G(c,c_*,x,x_*) = G(c_*,c,x_*,x)$, there is $dY(t)/dt = 0$. Hence, when $t > 0$, we have $Y(t) = Y(0)$, indicating that scaling does not change the convergence value of the average decision. Although for each interaction, the variation of the decision is very small, and the same law of the average decision can still be obtained as long as a sufficient number of interactions are performed. We call this property the quasi-invariant decision limit.

5.2. Fokker–Planck Equation. Based on the scaled competence model (68) and the scaled decision model (71), we calculate the following expectation values:

$$\begin{aligned}
\left\langle c^{\diamond} - c\right\rangle &= -\epsilon\left(\varrho_1(c)c + \varrho_2(c)(c_* - c) + \varrho_3(c)z\right) = \epsilon K(c,c_*,z),\\
\left\langle x^{\diamond} - x\right\rangle &= \epsilon\left(G(c,c_*,x,x_*) + \frac{4}{\rho + 8\epsilon}\left(1 - \epsilon G(c,c_*,x,x_*) - \epsilon G(c_*,c,x_*,x)\right)\right)\\
&\quad\times(x_* - x)\\
&= \epsilon H(c,c_*,x,x_*),\\
\left\langle\left(c^{\diamond} - c\right)^2\right\rangle &= \epsilon^2K^2(c,c_*,z) + \epsilon\varsigma_\eta^2c^2,\\
\left\langle\left(x^{\diamond} - x\right)^2\right\rangle &= \epsilon^2H^2(c,c_*,x,x_*) + \epsilon\varsigma^2L^2(c,x),\\
\left\langle\left(c^{\diamond} - c\right)\left(x^{\diamond} - x\right)\right\rangle &= \epsilon^2K(c,c_*,z)H(c,c_*,x,x_*).
\end{aligned} \tag{74}$$

Since the function $\phi(\cdot,\cdot)$ is a smooth function that is continuously derivable of infinite order in its domain of definition, through Taylor expansion at the point (c,x), we obtain

$$\begin{aligned}
\left\langle\phi\left(c^{\diamond},x^{\diamond}\right)\right\rangle &= \epsilon K(c,c_*,z)\frac{\partial\phi}{\partial c}(c,x) + \epsilon H(c,c_*,x,x_*)\frac{\partial\phi}{\partial x}(c,x)\\
&\quad+\frac{1}{2}\Bigg(\left(\epsilon^2K^2(c,c_*,z) + \epsilon\varsigma_\eta^2c^2\right)\frac{\partial^2\phi}{\partial c^2}(c,x) + \Big(\epsilon^2H(c,c_*,x,x_*)\\
&\quad+\epsilon\varsigma^2L^2(c,x)\Big)\frac{\partial^2\phi}{\partial x^2} + \epsilon^2K(c,c_*,z)H(c,c_*,x,x_*)\frac{\partial^2\phi}{\partial c\partial x}(c,x)\Bigg)\\
&\quad+R_\epsilon(c,c_*,x,x_*),
\end{aligned} \tag{75}$$

where $R_\epsilon(c,c_*,x,x_*)$ is the higher order term of the Taylor expansion. Therefore, the Boltzmann equation (72) becomes

$$
\begin{aligned}
&\frac{d}{dt}\int_{I\times C}\phi(c,x)f(c,x,t)dcdx \\
&= \frac{1}{b\epsilon}\int_{\mathbb{R}^+}\int_{I^2\times C^2}\left(\epsilon K(c,c_*,z)\frac{\partial\phi}{\partial c}(c,x)+\epsilon H(c,c_*,x,x_*)\frac{\partial\phi}{\partial x}(c,x)\right) \\
&\quad\times f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz \\
&\quad+\frac{1}{b\epsilon}\int_{\mathbb{R}^+}\int_{I^2\times C^2}\left(\frac{\epsilon\varsigma_\eta^2}{2}c^2\frac{\partial^2\phi}{\partial c^2}(c,x)+\frac{\epsilon\varsigma^2}{2}L^2(c,x)\frac{\partial^2\phi}{\partial x^2}\right) \\
&\quad\times f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz \\
&\quad+r(\epsilon)+O(\epsilon),
\end{aligned}
\tag{76}
$$

in which

$$
\begin{aligned}
r(\epsilon) &= \frac{1}{2b\epsilon}\int_{\mathbb{R}^+}\int_{I^2\times C^2}\left[\epsilon^2\left(K^2(c,c_*,z)\frac{\partial^2\phi}{\partial c^2}(c,x)+H^2(c,c_*,x,x_*)\frac{\partial^2\phi}{\partial x^2}(c,x)\right.\right. \\
&\quad\left.\left.+K(c,c_*,z)H(c,c_*,x,x_*)\frac{\partial^2\phi}{\partial c\partial x}(c,x)\right)+R_\epsilon(c,c_*,x,x_*)\right] \\
&\quad\times f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz.
\end{aligned}
\tag{77}
$$

Similar to the literature [9, 20], when $\epsilon \longrightarrow 0$, $r(\epsilon)$ converges to 0. Thus, equation (76) becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{I\times C}\phi(c,x)f(c,x,t)dcdx &= \frac{1}{b}\int_{\mathbb{R}^+}\int_{I^2\times C^2}K(c,c_*,z)\frac{\partial\phi}{\partial c}(c,x) \\
&\quad\times f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz \\
&\quad+\frac{1}{b}\int_{\mathbb{R}^+}\int_{I^2\times C^2}H(c,c_*,x,x_*)\frac{\partial\phi}{\partial x}(c,x) \\
&\quad\times f(c,x,t)f(c_*,x_*,t)A(z)dcdc_*dx\,dx_*dz \\
&\quad+\frac{\varsigma_\eta^2}{2b}\int_{I\times C}c^2\frac{\partial^2\phi}{\partial c^2}(c,x)f(c,x,t)dcdx \\
&\quad+\frac{\varsigma^2}{2b}\int_{I\times C}L^2(c,x)\frac{\partial^2\phi}{\partial x^2}(c,x)f(c,x,t)dcdx.
\end{aligned}
\tag{78}
$$

By using integration back by part, we obtain the corresponding Fokker–Planck equation:

$$\frac{\partial}{\partial t} f(c,x,t) = -\frac{\partial}{\partial c}\frac{1}{b}\mathcal{K}[f](c,t)f(c,x,t) - \frac{\partial}{\partial x}\frac{1}{b}\mathcal{H}[f](c,x,t)f(c,x,t) + \frac{\varsigma_\eta^2}{2b}\frac{\partial^2}{\partial c^2}\left(c^2 f(c,x,t)\right) + \frac{\varsigma^2}{2b}\left(L^2(c,x)f(c,x,t)\right), \tag{79}$$

where

$$\begin{aligned}\mathcal{K}[f](c,t) &= \int_{\mathbb{R}^+}\int_{I\times C} K(c,c_*,z)f(c_*,x_*,t)A(z)dc_*dx_*dz \\ &= -(\varrho_1+\varrho_2)c + \varrho_2 m_c + \varrho_3 m_z, \\ \mathcal{H}[f](c,x,t) &= \int_{I\times C} H(c,c_*,x,x_*)f(c_*,x_*,t)dc_*dx_* \\ &= \int_{C\times I}\Bigg(G(c,c_*,x,x_*) + \frac{4}{\rho+8\epsilon}(1-\epsilon G(c,c_*,x,x_*) \\ &\quad -\epsilon G(c_*,c,x_*,x)))(x_*-x)f(c_*,x_*,t)dc_*dx_*.\end{aligned} \tag{80}$$

Considering the complicated structure of the Fokker–Planck equation in this article, it is difficult to calculate the solution. To further analyze the evolution of decision-making, simulation experiments will be conducted by applying the DSMC approach.

6. Simulation Experiments

The DSMC method is commonly employed in the study of rarefied gas dynamics. The key notion of the simulation is to decouple molecular mobility and collision within a given time step. In this section, the DSMC approach is utilized to simulate the decision model. We employ simulated molecules to represent agents and directly simulate the evolution of competence and decisions which are caused by binary interactions. The images of the density distribution of agents are the kinetic solutions at different time steps, respectively. To demonstrate the evolution of decision-making more clearly, we assume that the distribution of agents obeys the Gaussian distribution at the initial time. In the graph, the horizontal axis represents decision and the vertical axis represents competence. The color changes from blue to yellow, indicating an increasing concentration of agents. After the interaction, the movement of the yellow area indicates a shift in the decision consensus, while the shrinking of the yellow area suggests a higher consistency of collective decision. If not specified, the number of agents in the simulation is $N = 10^4$. The evolution of competence is conducted according to the competence model (2), and the parameters take the values $\lambda_1 = \lambda_2 = \lambda_3 = 0.01$, $z = 0.01$, $\sigma_\eta^2 = 0.01$, and $m_z = 1$.

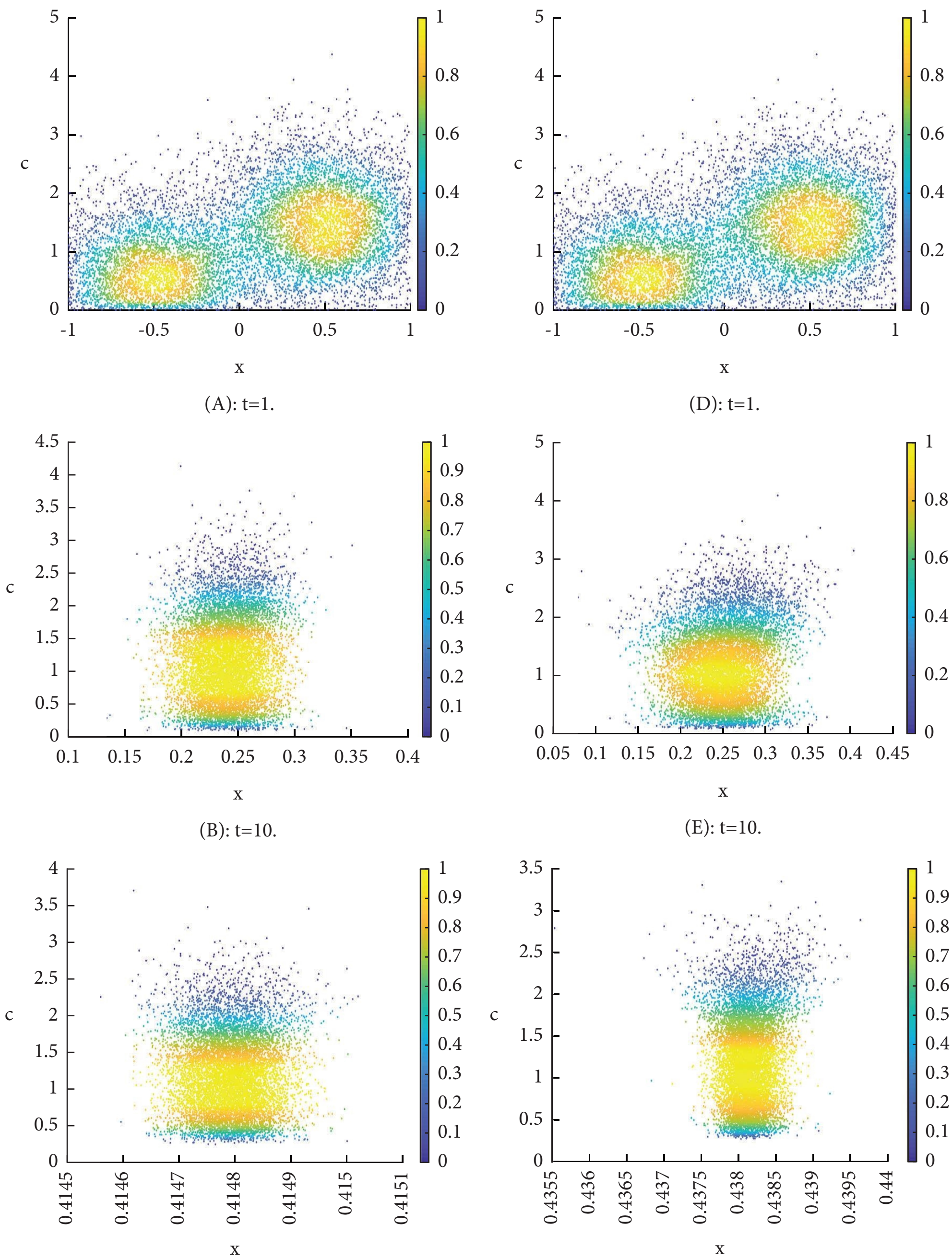
6.1. A Consensus Is Required. The evolution of decision-making is conducted according to the decision model (34), in which the manager's objective is to make agents reach a consensus. We simulate the effect of different control forces on the evolution of decision-making.

6.1.1. Test 1. The Effect of θ Taking on Decisions When the Aim Is Reaching a Consensus. For simplicity, we assume that the compromise functions satisfy

$$\begin{cases} G(c,c_*,x,x_*) = G(c) = (1+c)^{-2}, \\ G(c_*,c,x_*,x) = G(c_*) = (1+c_*)^{-2}, \end{cases} \tag{81}$$

which are decreasing functions in $c \in (0,+\infty]$. According to the compromise function, low competence decision makers are often more likely to compromise due to their lack of knowledge and cognition. However, low competence decision makers are not useless. As the interaction progresses, the abilities of low competence individuals are also improved and lead to greater confidence which results in a decrease in compromise. Therefore, the consensus is not solely determined by high competence decision makers. In the decision model (34), the parameters take the values $\gamma = 1$ and $\sigma^2 = 0.01$ and the diffusion term $L(c,x) = 1 - x^2$. The larger the value of θ, the greater the control force. By setting $\theta = 1/3$ (on the left side of Figure 1) and $\theta = 1/5$ (on the right side of Figure 1), we investigate the different effects of various control forces on the evolution of decision-making under the same remaining conditions.

In Figure 1, at the initial time $t = 1$, there are two subgroups with distinct decisions, each including $N = 10^4/2$ agents. In the low competence subgroup, the average competence equals 0.5 and the average decision equals −0.5. In the high competence subgroup, the average competence equals 1.5 and the average decision equals 0.5. This indicates that agents with different competence make different decisions. The simulation results are consistent with the

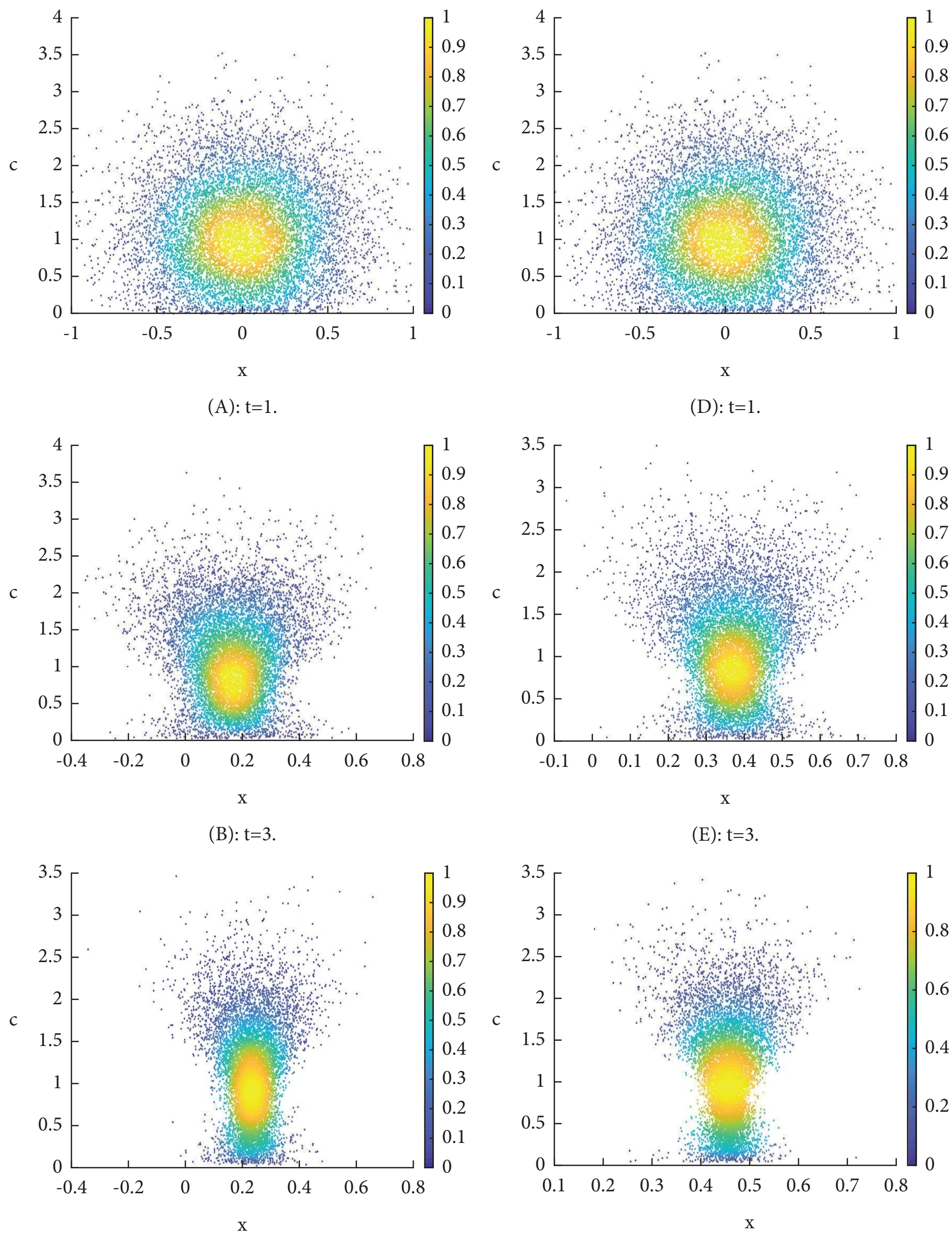


(A): t=1.
(D): t=1.
(B): t=10.
(E): t=10.
(C): t=30.
(F): t=30.
(a)
(b)

Figure 1: Kinetic solution at different time steps when $\theta = 1/3$ (a) and $\theta = 1/5$ (b).

conclusion of equation (13). That is to say, after $t = 30$ interactions, the average competence of the whole group equals m_z and the competence gap between agents is gradually reduced due to the natural forgetfulness of competence, mutual learning, and improvement from the background.

Besides, we find that the consistency of collective decisions is improved. At the initial time, there exist two opposite subgroup decisions. After $t = 10$ binary interactions, the consensus is obtained. Comparing Figures 1(C) and 1(F), when $\theta = 1/3$, agents concentrate in the interval (0.4147, 0.4149), and when $\theta = 1/5$, the agents concentrate in the interval (0.4375, 0.4385). It demonstrates that the greater the managers' control, the higher the concentration of decisions. Meanwhile, due to the narrowing of the competence gap, it is easier to have similar cognition about a certain event. It is concluded that competence evolution and managers' control facilitate the consistency of collective decisions.



(A): t=1. (D): t=1.

(B): t=3. (E): t=3.

(C): t=5. (F): t=5.

(a) (b)

Figure 2: Kinetic solution at different time steps when $x_d = 0.3$ (a) and $x_d = 0.5$ (b).

6.2. Reaching a Specific Consensus. In fact, there exists a rare circumstance in which managers require agents to attain a particular consensus x_d. It is essential to highlight that the article's consensus x_d is not necessarily the ultimate decision, as x_d simply indicates the subjective intents of management. Thus, the control term satisfies the cost function:

$$u = \operatorname{argmin} \int_0^T \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2} \left(x_j - x_d \right)^2 + \frac{\nu}{2} u^2 \right) ds, u(t) \in [u_L, u_R], \tag{82}$$

which is a special form of the cost function (16). We still use the model predictive control (MPC) method to obtain the concrete expression of the control term. After calculation, we obtain

$$\begin{aligned} u^n = &-\frac{\Delta t}{2\left(\nu + \Delta t^2\right)} \left(\left(x_j^n - x_d \right) + \left(x_i^n - x_d \right) \right) \\ &- \frac{\Delta t^2}{4\left(\nu + \Delta t^2\right)} \left(G\left(c_i, c_j, x_i, x_j\right) - G\left(c_j, c_i, x_j, x_i\right) \right) \left(x_j^n - x_i^n \right). \end{aligned} \tag{83}$$

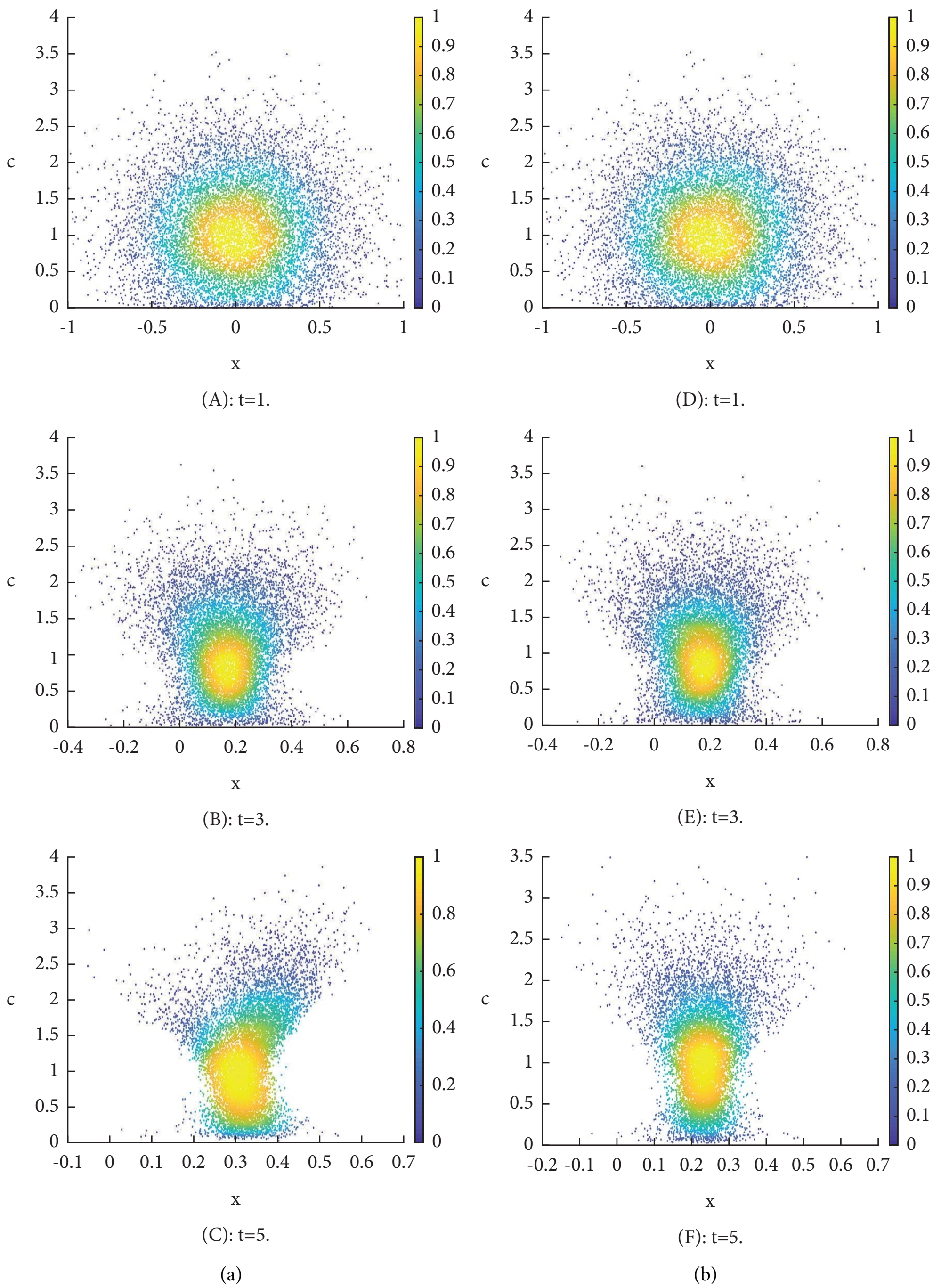
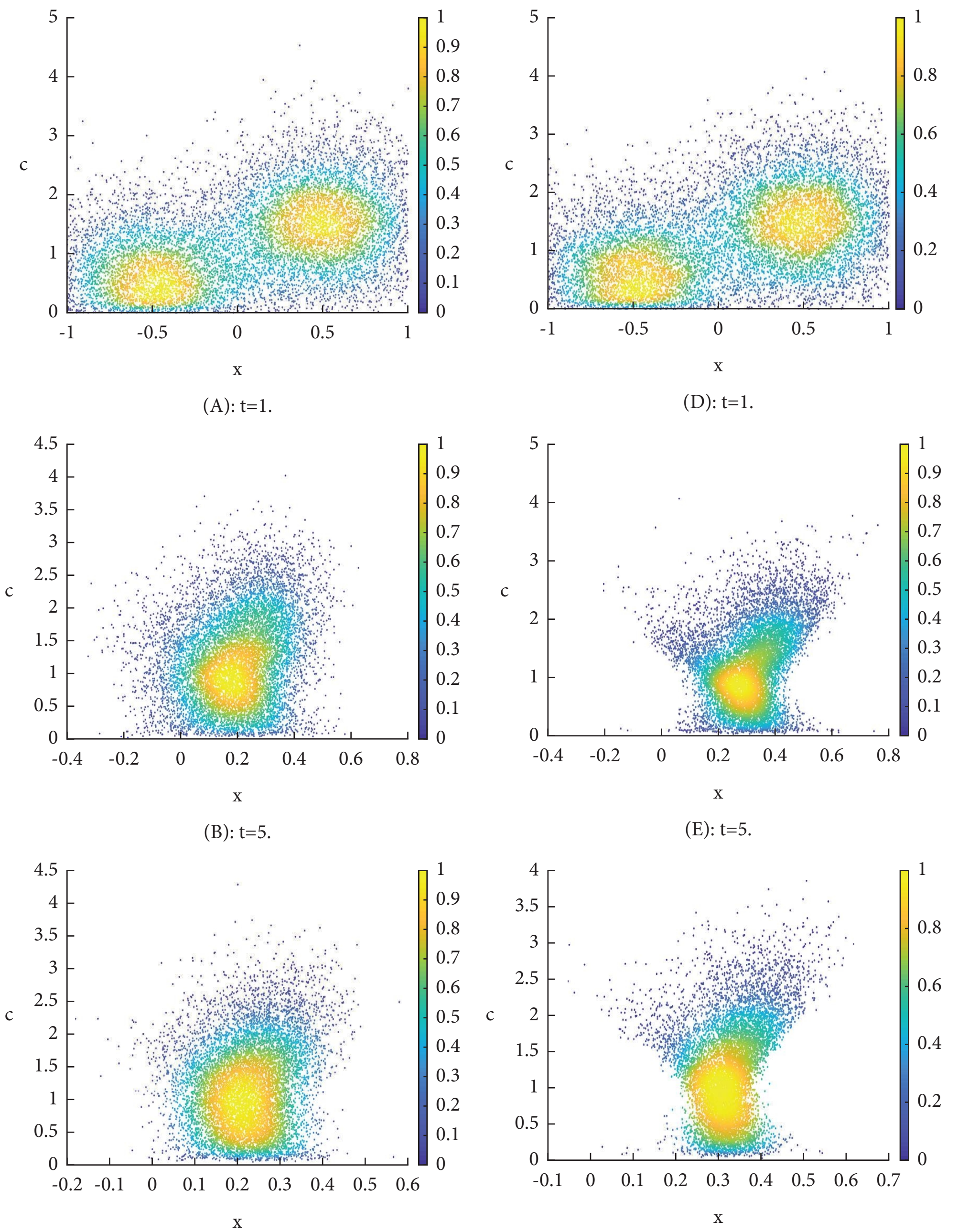


Figure 3: Kinetic solution at different time steps when θ takes different values.



(A): t=1. (D): t=1.

(B): t=5. (E): t=5.

(C): t=7. (F): t=7.

(a) (b)

Figure 4: Comparison of decision models (34) and (84).

Therefore, the decision model becomes

$$\begin{cases} x^{\diamond} = x + \gamma G(c, c_*, x, x_*)(x_* - x) - \dfrac{\theta}{2}((x_* - x_d) + (x - x_d)) \\ -\dfrac{\gamma\theta}{2}(G(c, c_*, x, x_*) - G(c_*, c, x_*, x))(x - x_*) + \kappa_1 L(c, x), \\ x_*^{\diamond} = x_* + \gamma G(c_*, c, x_*, x)(x - x_*) - \dfrac{\theta}{2}((x - x_d) + (x_* - x_d)) \\ -\dfrac{\gamma\theta}{2}((G(c_*, c, x_*, x) - G(c, c_*, x, x_*))(x_* - x)) + \kappa_2 L(c_* x_*), \end{cases} \tag{84}$$

where $\theta = 4\gamma^2/(\nu + 4\gamma^2)$ and $\theta \in [0, 1]$.

6.2.1. Test 2. Impact of x_d Taking on Decisions. In Figure 2, the parameters of the decision model (84) take the values $\gamma = 1$, $\theta = 1/3$, and $\sigma^2 = 0.01$ and the diffusion term $L(c, x) = 1 - x^2$. At the time $t = 1$, the mean competence is equal to 1 and the mean value of the decision is 0. In the left graph, the expected consensus of managers $x_d = 0.3$, and in the right graph, the expected consensus of managers $x_d = 0.5$. After $t = 5$ interactions, the decision is centered in the interval $[0.2, 0.3]$ when $x_d = 0.3$ and the interval $[0.4, 0.5]$ when $x_d = 0.5$. This indicates that the manager's control makes the decisions close to the target consensus x_d.

6.2.2. Test 3. Impact of θ Taking on Decisions As Obtaining a Particular Consensus Is Required. Figure 3 illustrates the influence of managers' control on decisions by comparing the magnitude of the parameter θ. In the decision model (84), we let $\gamma = 1$ and $x_d = 0.3$. At the initial time, the mean value of competence is 1 and the mean decision is 0. In the left graph, the parameter $\theta = 1/3$, and in the left graph, the parameter $\theta = 1/5$. When the managers' control force $\theta = 1/3$, the decisions after interaction are mainly concentrated in the interval (0.25, 0.35). When the managers' control force $\theta = 1/5$, the decisions after interaction are mainly concentrated in the interval (0.2, 0.3). It demonstrates that the stronger the manager's influence, the more decisions move toward the expected consensus.

6.2.3. Test 4. Comparison of Decision Models (34) and (84). In Figure 4, the left graph simulates decision model (34), where manager's goal is to urge the agents to reach a consensus; the right graph simulates decision model (84), where the managers require the agents to reach a particular consensus x_d. We let the parameters in decision model (34) take the values $\gamma = 1$ and $\theta = 1/5$, and the parameters in decision model (84) take the values $\gamma = 1$, $x_d = 0.3$, and $\theta = 1/5$. After the interaction, in the left graph, the actual consensus obtained is centered around 0.2, and in the right graph, the actual consensus obtained is centered around 0.3. This shows that if managers require a specific consensus $xd = 0.3$, the control will drive agents to move closer to 0.3.

7. Conclusion

In collective decision-making, agents are influenced by their competence and managers' control. In order to combat natural disasters, for instance, the government unites the populace in order to formulate a concerted counterstrategy, which is necessary for the country's stability. While the previous research on decision-making has focused on the influence of a single factor, such as beliefs, biases, and knowledge, we investigate the influence of agents' competence and managers' control simultaneously, which is the novelty of this paper. First, we develop a kinetic model of decision based on competence in which competence evolves in interaction. Therefore, we incorporate the model of competence into the model of decision. Second, our model explores the impact of managers' control on decision-making. A control term is used to describe a situation in which managers, through supervision and coordination, urge agents to achieve a consensus. The control term satisfies a cost function requiring that the decision variance between agents be minimized. Using the MPC approach, we construct a concrete expression for the control term and derive a kinetic model of decision-making (34) that includes agents' competence and managers' control. Third, the Boltzmann equation is established to describe the evolution of the density distribution of agents, and the main properties are discussed. It is mathematically proved that while the average decision of all agents remains constant, the managers' control will drive the distribution of agents to become increasingly concentrated. Fourth, the DSMC method is used to simulate the evolution of decisions based on agents' competence and managers' control. It demonstrates that the greater the influence of the managers' control, the more concentrated and consistent the decision distribution. The simulation also shows that the interaction narrows the variance of competence between agents, which helps agents easily achieve a consensus. The limitation of this paper is that the stationary solution of the Boltzmann equation is not obtained. In the future, with the improvement in mathematics, it is believed that the stationary solution of the model can be calculated. Besides, only the binary interaction case is considered, and the interaction process is linear. Inspired by references [13, 14], we can try to add a trust index to the decision interaction model or consider the evolution of group decision-making in more complex interaction networks.

Data Availability

No data were involved in this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Chen is mainly responsible for the theoretical derivation part of the article, and the rest of the authors contribute equally to the simulation part. Hongjing Chen and Zhi Huang contributed equally to this work.

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