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Research Article

Oligopoly Modeling between Public and Private Companies with Complementarity

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This study investigated the coexistence of public and private companies using a complementary model to explore mixed oligopoly strategies. Compared to the traditional theory of mixed oligopoly, the main difference of this study is that it revealed that the products produced by both companies are completely complementary. The five findings of the study were as follows: First, under the premise of having one firm classified as a public firm, although it can reach the equilibrium of the maximum solution for social welfare, this causes a loss. Second, more seriously, the private firm would view this as a huge incentive and aggressively pursue to be the price leader, which may result in a greater loss for the public firm. Third, the asymmetry of the model of the complementary mixed oligopoly is of note; that is, when the private firm is in aggressive pursuit to be the price leader, it can elevate its profit margin, but when the public firm is aggressively pursuing be the price leader, this would not result in better profits. Fourth, if the public firm is under budgetary constraints, then the private firm would have no incentive to aggressively pursue being the price leader. Fifth, if the price of the product between the public firm and the private firm is a "strategic substitute," the coexistence of the public firm and the private firm will be better than total privatization.

1. Introduction

This article aims to explore the theoretical model of complementary products of public and private firms when they coexist to analyze the decision-making and influence of a mixed oligopoly. Through the economic model of cooperation between a public and a private company, the final consumption of products is formed together. In this case, the services provided by the public and private firms have two characteristics: the two products are completely complementary and are produced by the public firm and the private firm, respectively. Among them, the goal of the public company is to maximize social welfare, while the private company is engaged in maximizing profit. Under the premise of a mixed oligopoly for perfect substitutes, the public firm can reach the goal of a maximum solution for social welfare under the premise of budgetary balance. In addition, the private firm has no incentive to be the price leader, since price setting in an aggressive manner cannot enhance the profit margin.

The research motivation of this article is that the mixed economy with the coexistence of public firms and private firms began to rise around the world in the 1980s. For example, before the 1980s, the main provider of social care in the UK was the responsibility of the government department, and it emerged in the late 1980s. The "Mixed Economy of Care" has made the role of private firms as service providers extremely important. The abovementioned is the phenomenon of the coexistence of public and private firms, and public and private firms are completely complementary. However, the current theoretical models for the coexistence of public and private firms assume competitive industries. However, so far, there has been no article exploring the theoretical model that public firms and private firms are completely complementary, which forms the research motivation of this study; that is, complete complementarity is the innovation and novelty of this research. This article aims to fill the void of theoretical models for complementarity between public and private firms. Other theoretical articles similar to this article in the existing literature only discuss the competitiveness or substitution between public and private firms. There is no literature that is completely complementary.

In view of the fact that the currently existing literature assumes that public and private firms are competitive or substitutable, there is no one that is similar to the complete complementarity between public and private firms discussed in this study. The aforementioned existing literature is as follows: Chang et al. [1] studied the theoretical model of mixed oligopoly, assuming that a partially privatized public firm and a purely private firm are substitutable, evaluating the welfare level of the two parties when they compete on price and quantity. They found that under the partial privatization of public firms, the social welfare of both firms engaged in quantity competition is higher than that of price competition. Bennett and La Manna [2] also discussed substitutability and competition between the goods of public and private firms. Through the comparison of mixed oligopoly and pure oligopoly, they obtained a finding that has nothing to do with privatization. They believed that, in mixed oligopoly and additional budget constraints, the results of total output, total cost, and social welfare are the same in mixed oligopoly and pure oligopoly. Ghosh and Mitra [3] also studied substitution and competition between public and private firms. Their content mentioned that when public firms adopt partial privatization, the social welfare of Cournot competition will be better than that of Bertrand competition. However, this result is contrary to the findings of Singh and Vives [4] and Ghosh and Mitra [3], which show that if public companies are fully privatized and fully publicized, respectively, the social welfare of Bertrand's competition will be better than Cournot's competition. Mujumdar and Pal [5] studied that public and private firms are substitutable and competitive and explored that in the mixed oligopoly market, there is a public firm that maximizes the social welfare and a private firm that maximizes the profit. If the government taxes, the private firm will not affect the welfare level in a mixed economy, and the privatization of the public firm can increase social welfare and taxation. Matsumura [6] also studied the substitution between public and private firms. The author studied a private firm and a private firm jointly operated by the public and private sectors and found that under certain conditions, neither complete privatization nor complete publicization is the best. George and La Manna [7] also explored the substitutability between public and private firms. They found that under the assumption of asymmetric production costs between public and private firms, the efficiency of the firms' production affects the choice between privatization and publicization. Cremer et al. [8] discussed how the government should use public firms as a policy tool to improve the efficient allocation of resources in partially competitive markets. They assumed that there are several private and public firms engaged in Cournot quantity competition. Based on the assumption that public firms maximize social welfare, they found that if the social optimum is to be achieved, there can only be one public firm in the market. De Fraja and Delbono [9] also explored substitutive competition between public firms and private firms and explored four market structures: public monopoly, mixed oligopoly, pure oligopoly, and public firms, and compared social welfare levels. They found that under the market structure of mixed oligopoly, assuming that one public firm competes with several private firms in Cournot competition, privatization of the public firm can improve social welfare only if there are enough private firms.

In addition, the literature on optimization problems still includes Yadav and Sora [10] and Qin [11]. These two articles are explained as follows: Yadav and Sora [10] use quantitative data to study, that is, they used financial ratios to detect fraud in financial statements. An optimized neural network-based detection of FSF was developed in qualitative data present in financial statements. Qin [11] used the Gaussian random field in the mathematical modeling of new-type detection.

Ramsey [12] emphasized that utilities must at least meet the breakeven and aim to maximize consumer surplus; however, such a theory does exist in real life, and this reasonable profit is normal. Therefore, the first question to be explored in this document was whether the government can achieve the social optimal without losing profit in the public company if one of the companies is retained as a public company.

Cremer et al. [8] established a competitive model of m public firms and n private firms, finding that if only one of them is a public firm, the social optimum can be realized. However, Cremer et al. [8] did not consider the breakeven highlighted by Ramsey pricing, nor discussed the issue of the profit of the public firm. Indeed, if Ramsey pricing and mixed duopoly are both considered, then in a typical hybrid duopoly, if the firms' products are "complete substitutes" and one of the manufacturers is a public company, it can achieve the social optimum without any loss in the public company because if the only public company adopts the average cost price, this forces other companies to adopt the average cost price.

Therefore, this study is different from the traditional literature and aims to explore the mixture of "completely complementary products" between the final products. If a manufacturer retains a public firm, is there still a way to achieve a win-win situation? That is, can we achieve the social optimum and the public firm will not lose profit?

In the case of the aviation industry, the services provided by the airport and those provided by the airline have two characteristics, they are completely complementary and have the characteristics of mixed operation between the public firm and the private firm, that is, the airport is provided by the government, and the airport services are usually provided by the private firm. Since the 1970s, there has been a new wave of global revolution, known as "regulation reform," and the government has adopted a segmentation strategy to vertically or horizontally divide industries. Taking the power industry as an example, Yarrow [13] pointed out that the privatization of the UK was characterized by dividing the power industry into three parts: power generation, power transmission, as well as power distribution and retail supply. Second, multiple power plants are allowed to produce and compete in power generation. Power transmission refers to the transmission of electricity between various regions of the country and is operated as a monopoly by the company. Regarding the power distribution and retail supply, the country is divided into several regions, with each area operated as a monopoly by one firm. Therefore, in the regulatory reform of the British government, the power industry was divided vertically and horizontally. The vertical division refers to the division of the power industry into power generation, power transmission, power distribution, and retail supply, while the horizontal division refers to the division of power distribution and retail supply into several areas.

If the public firm is divided into several parts and the government privatizes some parts while the other parts are reserved for the public firm, then this reformed industry has the two characteristics that this study wants to explore; that is, complete complementary products are provided by different firms and the public and private firms are mixed.

The mixed duopoly of public and private firms does not necessarily belong to the same level of the industry, as some firms belong to the upstream and some manufacturers belong to the downstream. For example, the aforementioned firms of power generation, power transmission, power distribution, and retail supply have upstream and downstream relationships, and the utility discussed by Chang [14] has the characteristics of upstream and downstream relations. In essence, the relationship between the products of an upstream firm and a downstream firm is necessarily a complete complement, so this study model also covers the mixed duopoly of all upstream and downstream relationships.

Relevant literatures on changes in the welfare of public firm's privatization, such as Fjell and Heywood [15], Fjell and Heywood [16], Myles [17], Poyago-Theotoky [18], Sepahvand [19], Pal and White [20], White [21], and so on, discussed the impact of subsidy policies on output and welfare. All of these articles assumed that there is a competitive industry between public and private firms. Among them, Fjell and Heywood [15] found that the public Stackelberg leader competes with domestic and foreign private firms, and the welfare of the leader is always higher than that of Cournot conjectures. Fjell and Heywood [16] found that the privatization of mixed oligopoly does not have privatization effects when the government uses subsidies to ensure optimal conditions. Myles [17] proved that Poyago-Theotoky's [18] uncorrelated results can be extended from linear-quadratic cases to general demand and cost functions. Sepahvand [19] found that privatization is achievable with appropriate policy imitation in a mixed-market structure. Pal and White [20] found that welfare always increases with privatization if the government uses domestic production subsidies. And if the government uses import tariffs, privatization increases welfare over many parameter spaces. White [21] showed the impact of domestic production subsidies on privatization and efficiency in mixed oligopoly industries. Roland and Sekkat [22], Laffont and Laffont [23], and other literature explore the interaction between ownership transfer and operational efficiency in the process of privatization, while De Fraja [24] and Bös [25] explored the process of privatization and the issues of employers and employees. Vickers and Yarrow [26] argued that in a competitive market where public and private firms coexist, if the public firm is a leader, the social welfare will be higher than the level of decision-making simultaneously between public and private firms. Beato and Mas-Colell [27] reasoned that when public and private firms are in a competitive market, if the public firm is a follower, in some cases its social welfare is greater than that of the public firm as a leader. However, this study compensates for the shortcomings of the existing literature, considering that public and private firms are completely complementary, so the model and the findings of this study will be quite different.

The pricing order of firms is the key to profit, Pal and White [20] pointed out that firms need to make simultaneous or sequential decisions because the order of decisions will make a significant difference in the equilibrium of the firms. However, the order of the decisions can be either an exogenous or an endogenous variable. The difference between the endogenous variable and the exogenous variable lies in the profit comparison of the latter, and both firms decide sequentially, but in fact, the nature of the analysis is the same. The order of decision-making is exogenous, for example, such as Ogawa and Kato [28], the pricing is exogenous to analyze the Bertrand competition between homogeneous products and to compare the equilibriums between the pricing simultaneously and the pricing sequentially. The decision-making sequence is endogenous, as is Tasnádi [29], who analyzed the model of price competition and believed that the order of pricing is selectable. Bárcena-Ruiz [30] analyzed two firms with partial substitution, finding that if the firms are purely duopoly, the follower's profit is greater than the leader's profit and the leader's profit is greater than the profit of the individual firm with simultaneous pricing. Second, if the firms are mixed duopoly, the profit of the private firm as the leader is greater than the profit of the individual firm with simultaneous decisions, and the profit for deciding simultaneously is greater than the profit when the private firm is the follower. Also, when the public firm is the leader, the social welfare is greater than the social welfare for deciding simultaneously, and the social welfare for simultaneous decisions is greater than the social welfare when the public firm is the follower. Bárcena-Ruiz [30] then inferred that when there is a mixed duopoly, both public and private firms will want to be the leader, so the final equilibrium will become two firms pricing simultaneously. However, the biggest difference between this study and the existing literature is that this article explores the mixed duopoly of completely complementary products and that public firms may be subject to self-sufficiency regulation. The conclusions of this study and those of the existing literature will differ significantly.

In this article, in addition to the introduction to this section, the second section details the basic model, and the third section is the equilibrium analysis of the three models. The first model is for the simultaneous pricing of the public firm and the private firm; in the second model, the public firm sets a price first, then the private firm, and in the third model, the private firm sets the prices first, and the public firm sets a price later. The fourth section analyzes the equilibrium of the three models in the self-sufficiency norms of the public firm and compares them with the results obtained in the third section. The fifth section presents the study conclusions.

2. Basic Model

The theoretical analysis of the proposed method uses an optimization problem. The research methods for the optimization problem here include mathematical logic, marginal analysis (or calculus), and the Lagrange method. This study will present the proof process of the research method used in the proposition in the Appendix.

The main research method of this study is to solve the optimization problem. Compared to other relevant kinds of literature using mathematics as a tool at present, it is as follows: Chen et al. [31] used the FO-EEMM research method to carry out the numerical study, respectively, the Fractional-Order (FO) Economic and Environmental Mathematical Models (EEMM), and the correctness of the designed conjugate gradient neural network (SCGNN) using the matching of the implemented and reference solution to verify. Chen et al. [32] discussed the pricing model of Ivancevic options, using two research methods; one is the rational sine-Gordon expansion method and the other is the modified exponential method. Zhang et al. [33] used the research method of the financial risk early warning model in the random effect model and proposed the risk analysis and prediction index system and built the company's financial risk early warning model. Rodríguez et al. [34] used an experimental approach to study the relationship between efficient production cycle design and the environmental sustainability of *Schinus molle* plant metabolites. Through the comparison of the control group and the experimental group, the hypothesis test of the mean difference and Wilcoxon statistics shows that the green business model has a positive relationship with environmental sustainability. Dong [35] used B-theoretical numerical methods applied to differential equation pan-function analysis to analyze and calculate fair value accounting. This article analyzed their stability and convergence. Li et al. [36] used the statistical probability statistical data filtering method and used the descriptive statistical analysis model of big data filtering to improve the detection and filtering capabilities of financial data. Yang [37] used the functional differential equation as a research method and adopted the mathematical analysis method to establish a fair value determination model in order to improve the information quality of financial accounting reports. Ma et al. [38] used the dynamic Copula model to demonstrate the application of the model in financial market risk management and found that the risk of financial markets is contagious.

The model assumes that there are two firms, a public firm and a private firm, each of which produces one product, and these two products are completely complementary to each other. Consumers purchase products from both companies and combine them for consumption. The final demand function is set to the open form, with the public firm deciding the price to maximize social welfare, and the private firm setting the price to maximize profit.

It is assumed that product y is a combination of two complete complementary products, x_1 and x_2, also known as components, and component x_i is monopoly produced by firm i $(i = 1, 2)$, where firm 1 is a public firm and firm 2 is a private firm. The unit cost of x_1 is c_1; the unit cost of x_2 is c_2.

Consumers combine one unit of x_1 and x_2 into y for consumption and the production function of y is $y = \min\ \{x_1, x_2\}$. This setting is different from the traditional literature, for example, Dixit [39] and Singh and Vives [4] set x_1 and x_2 products as substitutes, but in this study, it is completely complimentary. Since the product consists of completely complementary products, it is also called a system product. The consumer's utility function $TU(x_1, x_2)$ is a function of y, $TU(x_1, x_2) = U(y) = U\ (\min\ \{x_1, x_2\})$, $U(y) \geq 0$ and $U(0) = 0$, and the consumer cannot choose only x_1 or x_2 to obtain utility (Bárcena-Ruiz [26] follows the demand function of Dixit [39] and Singh and Vives [4] to explore the mixed duopoly of substitute products. The utility function and the demand function of their studies are for substitute goods. However, entirely complementary goods cannot use the utility and demand functions of Dixit [39] and Singh and Vives [4]). Let b_i be the price of a component x_i $(i = 1, 2)$, so the total price of y consumed by a consumer is $b_1 + b_2$. Also, to simplify the representation of the variables, let $p_y = (b_1 - c_1) + (b_2 - c_2) = a_1 + a_2$, where p_y is the "net price" of y or the "unit profit" of y, and $a_1 = b_1 - c_1$ and $a_2 = b_2 - c_2$ are the "net price" or "unit profit" of x_1 and x_2, respectively. In the following, the price is reduced by the unit cost and is called the "net price." The consumer demand function is $\mathrm{D}(y)$ and is defined as a function of the net price of the system product, not a function of the price of the system product, and it meets the demand rule $\mathrm{dD}(p_y)/\mathrm{d}p_y = \mathrm{D}'(p_y)$, and the demand is greater than zero $D(p_y) > 0$, $\forall p_y$.

The profit of firm i is $\Pi^i(a_1, a_2) = a_i\mathrm{D}(a_1 + a_2)$, so the sum of the profits of the two firms is equal to $\Pi^1 + \Pi^2 = p_y\mathrm{D}(p_y)$, and the monopoly net price is defined as $p_y^m = \arg\max_{p_y} p_y D(p_y)$. Second, let $\mathrm{V}(p_y)$ denote the consumer surplus of the consumer when the net price of y is p_y, which is $V(p_y) = \int_{p_y}^{\overline{P}} D(t)\mathrm{d}t$, where $\overline{P}$ is the net reserve price. Social welfare is equal to the sum of firms' profits and the consumer's surplus $W(p_y) = p_y D(p_y) + V(p_y)$.

The following will define the reaction functions of the firms. When the net price a_2 of firm 2 is given, the firm 1 prices a_1 to maximize social welfare, and a_1 is the reaction function expressed as $r_1(a_2)$, that is, $a_1 = r_1(a_2) = \arg\max_{a_1} W(a_1 + a_2)$, where a_2 is given. When the net price a_1 of firm 1 is given, the firm 2 prices a_2 to maximize its profit, and a_2 is the reaction function expressed as $r_2(a_1)$, that is, $a_2 = r_2(a_1) = \arg\max_{a_2} \Pi^2(a_1, a_2)$, where a_1 is given.

3. Model Analysis

First, this section explores the possibility of a first-mover advantage, a second-mover advantage, or a simultaneous mover advantage in price competition; therefore, this study will be divided into three competitions for analysis and comparison. The first one is the model for the simultaneous pricing of the public firm and the private firm; the second one is the model for the public firm to price first, and the private firm prices later; the third one is a model for the private firm to price first, and the public firm prices later. In other words, Model 1: both public and private firms price simultaneously, Model 2: the public firm first sets the price, and Model 3: the private firm first sets the price.

Model 1 is different from the existing literature, as the public firm maximizes social welfare rather than maximizing its profit. This study aims to compare the equilibrium net price with the net price of perfect competition or the net price p_y^m of monopoly. Model 1 is for the pricing of public and private firms simultaneously, and the optimization problem for the public firm is $\max_{a_1} W(p_y) = p_y D(p_y) + V(p_y)$ and the optimization problem for the private firm is $\max_{a_2} \Pi^2(a_1, a_2) = a_2 D(p_y)$. The optimal conditions of the net prices for the components of the public and private firms are as follows:

$$a_1 = r_1(a_2), \tag{1}$$

$$a_2 = r_2(a_1). \tag{2}$$

The following proposition is obtained by solving the two optimal conditions of equations (1) and (2).

Proposition 1. *When the public and private firms price simultaneously, the net price of the public firm is negative $a_1^* < 0$, the net price of the private firm is positive $a_2^* > 0$, and the equilibrium price is equal to the average cost $p_y^* = a_1^* + a_2^* = 0$. At this time, the public firm will lose $\Pi^1 < 0$, while the private firm will have excess profit $\Pi^2 > 0$.*

Proof. See the appendix.

Proposition 1 illustrates an important characteristic, that is, the two products are completely complementary, as long as one firm is a public firm, it can achieve a perfect competitive solution p_y, that is, the best solution for social welfare, but its shortcoming is that the public firm will lose money. In this related case, for example, the tuition fee for basic education is usually lower than the average cost, while the private firm with complementarity with basic education has a higher price than the average cost.

If the two products are "substitutive" when the private firm raises the price of x_2, the demand for x_2 will decrease, but it will not necessarily reduce the demand for $x_1 + x_2$, it still needs to be determined by the substitutivity between x_1 and x_2. The model in this study is "completely complementary" between the two products. When the private firm raises the price of x_2, the demand for x_2 may decrease, but x_2 must be combined with x_1 to consume and consumers care for the total price of the whole group of $x_1 + x_2$, so when the private firm has excess profit, this causes loss in the public firm because it aims to maximize social welfare.

In Model 2, the public firm first sets the price, then the private firm sets the price, so the optimization of public and private firms is analyzed by the order of the game, which is equation (3) of firm 1 and equation (4) of firm 2.

$$\max_{a_1} \Pi^1(a_1, r_2(a_1)) + \Pi^2(a_1, r_2(a_1)) + V(a_1 + r_2(a_1)) \text{ and} \tag{3}$$

$$\max_{a_2} \Pi^2(a_1, a_2), \text{given } a_1. \tag{4}$$

In Model 3, the private firm first sets the price, then the public firm sets the price, so the optimization of private and public firms is analyzed by the order of the game, which is equation (5) of firm 2 and equation (6) of firm 1.

$$\max_{a_2} \Pi^2(r_1(a_2), a_2) \text{ and} \tag{5}$$

$$\max_{a_1} \Pi^1(a_1, a_2) + \Pi^2(a_1, a_2) + V(a_1 + a_2), \text{given } a_2. \tag{6}$$

Model 2, Model 3, and Model 1 are compared to obtain the following proposition of price and profit in equilibrium: □

Proposition 2

(1) When the public firm first sets the price, the equilibrium prices $(\hat{a}_1, \hat{a}_2)$ are equivalent to the equilibrium prices at the time of simultaneous pricing of both firms.

(2) If the private firm first sets the price, then the equilibrium price $(\tilde{a}_1, \tilde{a}_2)$ of the net price of the final product is $\tilde{p}_y = \tilde{a}_1 + \tilde{a}_2$. The profit of the private firm is higher than that of the public firm first pricing or both firms pricing simultaneously, $\Pi^2(\tilde{a}_1, \tilde{a}_2) > \Pi^2(\hat{a}_1, \hat{a}_2)$. The profit of the public firm is lower than that of the public firm first pricing or both firms pricing simultaneously, $\Pi^1(\tilde{a}_1, \tilde{a}_2) < \Pi^1(\hat{a}_1, \hat{a}_2)$.

(3) Regardless of whether the public or private firm first sets the price or the two firms set the price at the same time, the public firm will lose money, and the private firm will have excess profit in equilibrium, the sum of both profits will be zero, and the net price of the final product will satisfy the optimal condition of social welfare.

Proof. See the appendix.

The first item in Proposition 2 finds that if the public firm is the leader of pricing, the equilibrium result is the same as the equilibrium at the same time. The economic meaning of the first item is that even if the public company can preempt pricing, it cannot reduce the loss. When the firm is a leader, it is equivalent to taking the "reaction function" of the other firm as a set of opportunities and selecting the most favorable decision from the set of opportunities. If the equilibrium is best when the public firm is pricing simultaneously with the private firm, it will also be selected when the public firm is the leader. Proposition 1 is equivalent to convey the following message, that is, when the public firm and the other firm simultaneously price, they can achieve social optimization.

The second item in Proposition 2 finds that if a private firm preempts the price leader, it can increase its profit, but the loss of the public firm will also increase because when the private firm is price leader, it will take advantage of the public firm, that is, the private firm will set a higher price because the public firm will be forced to cut its price to optimize social welfare, with the sum of the prices of the two complementary products equal to the price of a competitive solution.

Also, if the order of the pricing is for the endogenous decision of the firms, the private firm will rush to set the price first. However, if the public firm lets the private firm preemptively price, the government will suffer serious losses; therefore, it is a very important policy for the public firm to play the role of a leader or to price at the same time. But how do you make the government a price leader? In addition to legal and historical factors, a viable method is to price as the leader and let the private firm believe that the public firm will not change the price. For example, legislation can be used to make the price of a public firm rigid and, at this time, the public firm has credibility.

If the public firm is still unable to prevent the private firm from becoming a leader, the following method will be used to avoid being taken advantage of: the law expressly stipulates that the public firm cannot lose money. If the government specifies this regulation, then the private firm will not be able to take advantage of the public firm even if it preempts the price. This policy, intuitively speaking, will prevent the private firm from making the public firm a cash cow; therefore, in the fourth section, this study will assume that the public firm is faced with legal regulation that at least earns a normal profit and analyzes the response of the public and private firms. □

4. Analysis of the Self-Sufficiency Public Firm

As mentioned above, if the public firm only maximizes social welfare, it will generate losses and become a burden on public finances; so for the public firm to survive, it must be satisfied with the condition of self-sufficiency to maximize social welfare. The regulations governing certain utilities are self-sufficient norms; therefore, under the basic structure of the model, this section only adds another consideration, that is, the public firm must at least earn normal profit for analysis, which is expressed as follows:

$$\Pi^1(a_1, a_2) \geq 0. \quad (7)$$

The public firm maximizes social welfare, but it is subjected to equation (7), so the objective function of the private firm is the same as that of the previous section, to maximize profit; therefore, the response function of the private firm is the same as that of the third section as follows:

$$r_1(a_2) = \arg\max_{a_1} W(a_1 + a_2), \text{s.t. } \Pi^1(a_1, a_2) \geq 0, \text{given } a_2. \quad (8)$$

After adding this restriction to the self-sufficiency of the public firm, the optimal problems for the two firms are as follows: (1) In Model 1, the net price of the public firm and the private firm in equilibrium, respectively, satisfies the optimal conditions equation (8) $a_1 = r_1(a_2)$ and equation (2) $a_2 = r_2(a_1)$. (2) In Model 2, according to the order of the game, the optimization problem of the public firm is equation (3) with restriction in equation (7), and the optimization problem of the private firm is equation (4). (3) In Model 3, according to the order of the game, the optimization problem of the private firm is equation (5), and the optimization problem of the public firm is equation (6) with restriction in equation (7), that is, the price decision of the public firm meets the optimal condition equation (8).

From the third section, it is found that if the private firm is a price leader, it will gain the greater benefit; however, if the public firm needs to satisfy self-sufficiency, does the private firm still have the advantage of the first mover?

Proposition 3. *When the public company is regulated by self-sufficiency, regardless of the order of the firm, in Model 1, Model 2, or Model 3, the net price of the equilibrium product of the system product y equilibrium is the monopoly pricing* p_y^m *and*

(1) *The net price of the public company is zero, that is, the price is equal to the unit cost*
(2) *The equilibrium pricing of the private firm is the monopoly price of the system product, namely,* $a_2 = p_y^m$.

Proof. See the appendix.

Proposition 3 finds that regardless of the order of pricing, the public company will adopt unit cost pricing, while the private companies adopt monopoly pricing of the system product to make an exclusive profit because the public firm is restricted to $\Pi^1(a_1, a_2) \geq 0$, hence cannot reduce the price to subsidize consumers. Proposition 3 finds that when the public firm is regulated by self-sufficiency, the private firm cannot take advantage of the public firm even if it rushes to set the price first.

The conclusions of Proposition 1–3 obtained in this article are closely related to the setting of the complete complementary model. If the public firm is allowed to lose money (Proposition 1 and 2), it will adjust the pricing of the system product y to the unit cost to maximize social welfare. Therefore, in the case of complete complementarity, the price of the public firm will be lower than the cost to reduce the price of the system product. If it is considered that the public firm must at least earn a normal profit, it will be priced at the unit cost. At this time, the net price of the public firm is $a_1 = 0$, which does not affect the price of the system product y; therefore, the private firm will be able to adopt a monopoly price. Where $(a_1, a_2) = (0, p_y^m)$, and p_y^m is greater than 0. That is to say, the pricing of the public firm is set at the unit cost, but the target of the unit cost will change according to different situations, which will affect the profits of the public and the private firms.

The pros and cons of mixed duopoly and pure duopoly are compared in Proposition 4. A mixed duopoly will focus on the complementarity between the public firm and the private firm, and the public firm will be regulated by self-sufficiency; pure duopoly refers to the situation where both firms are private firms. □

Proposition 4. *When the public firm in the mixed duopoly is subjected to the norm of self-sufficiency, and the private firm in the pure duopoly satisfies the strategic substitute, that is, when the profit function satisfies the condition* $\Pi_{ij}^{i} = D'(p_y) + a_i D''(p_y)$, *where* $i, j = 1, 2$ *and* $i \neq j$, *then regardless of the order of pricing between the firms (Model 1, Model 2, or Model 3), the total profit, consumer utility, and social welfare of the public firm and private firm coexist (semiprivatization) and are all higher than that if the private firm and the private firm coexist (total privatization).*

Proof. See the appendix.

Proposition 4 explores the importance and necessity of the existence of the public firm in a complete complementary market structure because semiprivatization is superior to total privatization if it is based on "social welfare" and semiprivatization is greater than total privatization in terms of "total profit of two firms" or "consumer utility." Another noteworthy aspect of Proposition 4 is that this study found that in the complete complementary mixed duopoly model, the balance of profit and loss of the public firm is equivalent to removing the "horizontal externality" emphasized by Tirole [40] because in the case of "total privatization," if the private firm raises the price separately, it only needs to bear some of the unfavorable effects of the price increase, so p_y will be higher than the monopoly net price of the monopoly, which is partly affected by the unfavorable factor of price increase. This finding can be explained by the "horizontal externality" highlighted by Tirole [40], which explores the optimal pricing of a multi-product monopolist and finds that if these products are complementary, then the profit margins of these products (the difference between the price and marginal cost divided by price) will be lower than if these products are independent (there is neither substitution nor complementarity between products). On the contrary, if these products are substitutable, the profit margins of these products will be higher than if these products are independent, which will explain the above phenomenon with "horizontal externality." Similarly, this study explores the complementarity of products. In this complimentary situation, if the price of a certain product is increased, the demand for the other product will be reduced. It is also a negative externality for the "provider" of the other complementary product; therefore, if these products are only provided by a single firm, this firm is internalized based on externality and will be less willing to raise the price. From the perspective of horizontal externality, it can be understood why p_y is higher than the monopoly net price in the case of "total privatization"; because the company that provides the component at this time will not consider the negative externality caused by the price increase for the other company of the component.

However, in the mixed duopoly of the public firm and the private firm, the restriction on the norm of self-sufficiency of the public firm, so that the private firm must bear the unfavorable factor of price increases alone (The change in profit caused by the unit price increase of the private enterprise is $D(p_y) + p_y D'(p_y)$, where $D(p_y)$ is the direct effect, where $p_y D'(p_y)$ is the indirect effect, and this indirect effect is the negative effect of the decrease in demand caused by the price increase), so the private firm dare not set the price too high. In the terminology of Tirole [40], the fact that the public firm earns at least a normal profit is equivalent to having the advantage of removing "horizontal externality." Therefore, semiprivatization is better than full privatization in terms of the sum of profits of the two firms, the utility of consumers and social welfare. If the demand function satisfies the condition of Proposition 4, then the existence of semiprivatization is justified and important, and the government should not privatize the remaining public firm.

The main contribution of this article in this field is the analysis of mixed oligopoly with complete complementarity, which is a groundbreaking research that is not found in the existing literature. Research findings on the four propositions of mixed oligopoly are as follows. First, if the public firm and the private firm set prices at the same time, the public firm will suffer losses. Second, if the public firm and the private firm do not set prices at the same time, the private firm will preemptively become the price leader, causing the public firm to suffer even more losses. Third, when self-sufficiency is given to the public firm, the private firm will have no incentive to become the price leader. Fourth, mixed oligopoly is better than full privatization in the case of strategic substitution in pricing.

5. Conclusions

This study proves that, for complementary public and private products, the optimal price of the public firm can achieve social welfare under first-best, and if the public firm is limited by the balance of profit and loss, it will enable the private firm to price its product at a monopoly price.

In a completely competitive market, if the public firm is limited by the balance of profit and loss, the optimal social welfare equilibrium will be reached, and at this time, the private firm has no incentive to lead the pricing. Compared with the traditional literature of completely substitute and mixed duopoly, the main difference is that the production between firms has complete complementarity, thus the equilibrium results are much different. This study revealed five important findings. First, in the mixed duopoly market structure, the public firm will make a loss, but it can achieve the optimization of social welfare. Second, the private firm will have strong incentives to preempt the leader in price, causing more serious losses in the public firm. Third, there is a notable asymmetry, that is, the private firm preemptive price leader can increase profit, but the public firm cannot reduce loss if it preempts the price leader. Fourth, if the public firm is limited by the norm of profit and loss balance, then the private firm will not be preemptive of the price leader, because even if the price is preemptive, it will not be able to increase profit; therefore, if the public firm has a norm of profit and loss balance, it will generate a strategic consideration and successfully prevent the private firm from the strategic behavior of preemptive pricing. Fifth, when the public firm is regulated by the balance of profit and loss, and if the pricing satisfies "strategic substitution," then regardless of the order of pricing of public and private firms, semiprivatization is better than total privatization. These findings highlight the importance and necessity of the existence of a public company, providing the government with an important reference for formulating industrial policies.

Appendix

Proof of Proposition 1. Let (a_1^*, a_2^*) be expressed as the equilibrium net price of model 1. The optimization problem for firm 1 is $\max_{a_1} W(p_y) = p_y D(p_y) + V(p_y) = p_y D(p_y) + \int_{p_y}^{\overline{p}} D(t)\, dt$, and the first-order necessary condition is $D(p_y) + p_y D'(p_y) - D(p_y) = (a_1 + a_2)D'(p_y) = 0$, which is due to chain rule. The optimization problem for firm 2 is $\max_{a_2} \Pi^2(a_1, a_2) = a_2 D(p_y)$, and the first-order necessary condition is $a_2 D'(p_y) + D(p_y) = 0$. From the above two necessary conditions, $D'(p_y) < 0$ and $D(p_y) > 0$; therefore, the optimal solution $a_1^* + a_2^* = 0$, where $a_1^* < 0 < a_2^*$, and $\Pi^1 = a_1^* D(p_y) < 0$ and $\Pi^2 = a_2^* D(p_y) > 0$ are obtained. Q.E.D. □

Proof of Proposition 2. Let $(\widehat{a}_1, \widehat{a}_2)$ and $(\widetilde{a}_1, \widetilde{a}_2)$ be expressed as the equilibrium net price of Model 2 and Model 3, respectively.

(1) In Model 1, it is known from the Proof 1 that the first-order necessary condition for firm 1 to optimize social welfare is $a_1 + a_2 = 0$. The first-order necessary condition for optimization in equation (4) of Model 2 is $a_2 \mathrm{D}'(p_y) + \mathrm{D}(p_y) = 0$, thereby obtaining the response function $r_2(a_1)$ of firm 2, and since the first-order condition is the same as the first-order condition of Model 1, the reaction function and Model 1 is the same. The first-order necessary condition of equation (3) is $(1 + r_2'(a_1))(a_1 + r_2(a_1))D'(p_y) = 0$, and because $D'(p_y) < 0$ and $1 + r_2'(a_1) > 0$, $a_1 + r_2(a_1) = 0$ is obtained. The reason for the above $1 + r_2'(a_1) > 0$ is because the condition $a_2 D'(p_y) + D(p_y) = 0$ can obtain the reaction function $r_2(a_1)$ of firm 2, and its slope is $r_2'(a_1) = -(D'(p_y) + a_2 D''(p_y))/(2D'(p_y) + a_2 D''(p_y))$, so $1 + r_2'(a_1) = \mathrm{D}'(p_y)/(2D'(p_y) + a_2 D''(p_y)) > 0$, because the numerator $D'(p_y) < 0$ and the denominator $2D'(p_y) + a_2 D''(p_y)$ is the second-order sufficient condition for optimization, so it is less than zero. Therefore, the equilibrium prices of Model 1 and Model 2 are equal.

(2) We can derive $r_1(a_2) + a_2 = 0$ by using the first-order necessary condition of the equation (6). From this condition, the optimization problem of firm 2 can be written as $\max_{a_2} a_2 D(0)$, so $\widetilde{a}_2 = \arg\max_{a_2} \Pi^2(r_1(a_2), a_2) = \infty$; therefore, $\widetilde{a}_1 = r_1(\infty) = -\infty$. The final net price is $\widetilde{p}_y = \widetilde{a}_1 + \widetilde{a}_2$, the net price of the private firm is $\widetilde{a}_2 > \widehat{a}_2$, and the net price of the public firm is $\widetilde{a}_1 < \widehat{a}_1$; therefore, $\Pi^2(\widetilde{a}_1, \widetilde{a}_2) > \Pi^2(\widehat{a}_1, \widehat{a}_2)$ and $\Pi^1(\widetilde{a}_1, \widetilde{a}_2) < \Pi^1(\widehat{a}_1, \widehat{a}_2)$.

(3) It is known from the above two proofs that the equilibrium net prices of Model 1, 2, and 3 are all $a_1 + a_2 = 0$, that is, $a_1^* + a_2^* = \widehat{a}_1 + \widehat{a}_2 = \widetilde{a}_1 + \widetilde{a}_2 = 0$. In Model 1 and Model 2, the first-order necessary condition for the optimization of firm 2 is $a_2 D'(p_y) + D(p_y) = 0$, and $D(p_y) > 0$ and $D'(p_y) < 0$, so $a_2 > 0$; in Model 3, the profit of firm 1 is negative, the profit of firm 2 is positive, so the sum of the profit of these two firms is zero. The optimization of social welfare is $\max_{p_y} W(p_y)$, and the first-order condition is $p_y = 0$. Since the equilibrium of the net price of y is zero in the three models, the equilibrium of Models 1, 2, and 3 meets the condition for the optimization of social welfare. Q.E.D. □

Proof of Proposition 3

(1) In Model 1, the Lagrange function of the optimization problem of firm 1 is $\mathscr{L}(\lambda, a_1) = \Pi^1(a_1, a_2) + \Pi^2(a_1, a_2) + \mathrm{V}(a_1 + a_2) + \lambda\Pi^1(a_1, a_2)$, then the first-order necessary condition for the optimization of firm 1 is $[(\lambda + 1)a_1 + a_2]D'(p_y) + \lambda D(p_y) = 0$, and the first-order condition for the optimization problem of firm 2 is $a_2 D'(p_y) + D(p_y) = 0$, where $D(p_y) > 0$ is known, and $D'(p_y) < 0$. The following will use the contradiction proof method: if $\lambda = 0$, then $a_1 + a_2 = 0$ is derived from the first-order condition of the abovementioned firm 1, and $a_2 > 0$ is derived from the first-order condition of firm. 2. Therefore, $a_1 < 0$, then $\Pi^1(a_1, a_2) < 0$, but this result contradicts the non-negative profit of firm 1, so $\lambda > 0$ is true. According to the Kuhn-Tucker theorem, the profit of firm 1 is the identity equation, that is, $a_1^* = 0$. Replace this condition with the optimization problem $\max_{a_2} a_2 D(a_1 + a_2)$ of firm 2 to obtain $\max_{a_1 + a_2} (a_1 + a_2)\mathrm{D}(a_1 + a_2)$, where $a_1 = 0$. Therefore, firm 2 will take monopoly net pricing p_y^m.

(2) In Model 2, the Lagrange function optimized by firm 1 is $\mathscr{L}(\lambda, a_1) = \Pi^1(a_1, r_2(a_1)) + \Pi^2(a_1, r_2(a_1)) + \mathrm{V}(a_1 + r_2(a_1)) + \lambda\Pi^1(a_1, r_2(a_1))$, then the first-order necessary condition for the optimization problem of firm 1 is $(1 + r_2'(a_1))[(\lambda + 1)a_1 + r_2(a_1)]D'(p_y) + \lambda D(p_y) = 0$, and the first-order necessary condition for the optimization problem of firm 2 is $r_2(a_1)D'(p_y) + D(p_y) = 0$, where $D(p_y) > 0$ is known, and $D'(p_y) < 0$. The following will use the method of contradiction: if $\lambda = 0$, then $1 + r_2'(a_1) = D'(p_y)/(2D'(p_y) + a_2 D''(p_y)) > 0$ has been obtained from the first item of Proposition 2's proof, then $a_1 + r_2(a_1) = 0$ is derived from the first-order condition of the abovementioned firm 1, and $r_2(a_1) > 0$ is derived from the first-order condition of the firm 2. Therefore, $a_1 < 0$, then $\Pi^1(a_1, a_2) < 0$,

but this result contradicts that the profit of firm 1 is not negative, so $\lambda > 0$ is true. According to the Kuhn-Tucker theorem, the profit of firm 1 is the identity equation, that is, $\hat{a}_1 = 0$, and this condition is replaced by the optimization problem of firm 2 equation (4) $\max_{a_2} \Pi^2(a_1, a_2)$, given a_1, so $max_{a_1+a_2}$ $(a_1 + a_2)D(a_1 + a_2)$, where $a_1 = 0$. Therefore, firm 2 will take monopoly net pricing p_y^m.

(3) In Model 3, the Lagrange function optimized by firm 1 is $\mathscr{L}(\lambda, a_1) = \Pi^1(a_1, a_2) + \Pi^2(a_1, a_2) + V(a_1 + a_2) + \lambda \Pi^1(a_1, a_2)$, then the first-order necessary condition for the optimization problem of firm 1 is $[(\lambda + 1)a_1 + a_2]D'(p_y) + \lambda D(p_y) = 0$, the first-order condition for firm 2 optimization problem is $(1 + r_1'(a_2))a_2 D'(p_y) + D(p_y) = 0$, and the response function $r_1(a_2)$ is obtained by the first-order necessity condition of firm 1, and its slope is $r_1'(a_2) = -\{(1+\lambda)D'(p_y)\ [(1+\lambda)a_1 + a_2]\ D''(p_y)\}/\{(1+2\lambda) D'(p_y)[(1+\lambda)a_1 + a_2]D''(p_y)\}$. The following will use the contradiction proof method: if the net price of firm 1 is greater than zero $r_1(a_2)$, in other words, $\Pi^1(r_1(a_2), a_2) > 0$, $\lambda = 0$ according to the Kuhn-Tucker theorem, and substitute $r_1'(a_2)$ above, so $r_1'(a_2) = -1$ is obtained, and this condition is substituted into the first-order condition of the above firm 2 to obtain $D(p_y) = 0$, but this result contradicts $D(p_y) > 0$; therefore, $r_1(a_2) = 0$ is true, and this condition is brought back to equation (5) $\max_{a_2} \Pi^2(r_1(a_2), a_2)$ to obtain $\max_{r_1(a_2)+a_2}(r_1(a_2) + a_2)D(r_1(a_2) + a_2)$, where $r_1(a_2) = 0$, that is, firm 2 will take a monopoly net price p_y^m. Q.E.D. □

Proof of Proposition 4. In the market structure of both firms being private, the reaction function of this proposition is redefined, so that the response function of firm i is $r_i(a_j) = \arg\max_{a_i} \Pi^i(a_i, a_j)$, given a_j, $i, j = 1, 2$, and $i \neq j$. Therefore, the response function $r_i(a_j)$ of firm i differentiates at the net price a_j of firm j to obtain $r_i'(a_j) = -\Pi^i_{ij}/\Pi^i_{ii} = -D'\ [(p_y) + a_i D''(p_y)]/[2D'(p_y) + a_i D''(p_y)]$. In the above equation, since it is necessary to satisfy the optimal second-order sufficient condition, $\Pi^i_{ii} = 2D'(p_y) + a_i D''(p_y) \leq 0$ and $1 + r_i'(a_j) > 0$ are also satisfied. If the numerator of the above equation satisfies the condition $D'(p_y) + a_i D''(p_y) < 0$, then the slope of the above reaction function $r_i'(a_j) < 0$. The first item below demonstrates the characteristics of the equilibrium net price of system products when both private firms are priced at the same time; the second item proves the characteristics of the equilibrium net price of system products when two private firms are priced sequentially; the third item proves the comparison of the net price of the mixed duopoly and pure duopoly in equilibrium; and the fourth item demonstrates the comparison between the mixed duopoly and pure duopoly related to the sum of profit, consumers' utility, and social welfare.

(1) It shows that in the market structure of both private firms when the two firms "make decisions at the same time," the equilibrium net price p_y^{**} of the system product is greater than the monopoly net price p_y^m, which is $p_y^{**} = a_1^{**} + a_2^{**} > p_y^m$: (1) We adopted the method of contradiction, that is, first assume that the condition of $a_1^{**} + a_2^{**} < p_y^m$ is satisfied, and prove that this is a contradiction. Due to the symmetry of the model, $a_1^{**} = a_2^{**}$, so $p_y^m/2 + a_2^{**} = a_1^{**} + p_y^m/2 < p_y^m$ can be obtained and because it meets the law of demand, so $p_y^m D(p_y^m) < (p_y^m/2)D(p_y^m/2 + a_2^{**}) + (p_y^m/2)D(a_1^{**} + p_y^m/2)$. The following $(p_y^m/2)\, D(p_y^m/2 + a_2^{**}) + (p_y^m/2)D\ (a_1^{**} + p_y^m/2) \leq \max_{a_1} \Pi^1(a_1, a_2^{**}) + \max_{a_2} \Pi^2(a_1^{**}, a_2) = \Pi^1(a_1^{**}, a_2^{**}) + \Pi^2(a_1^{**}, a_2^{**}) = (a_1^{**} + a_2^{**})\, D(a_1^{**}, a_2^{**})$ is obtained by the principle of optimization. Rearranging the two main results mentioned above resulted in $p_y^m D(p_y^m) < \Pi^1(a_1^{**}, a_2^{**}) + \Pi^2\ (a_1^{**}, a_2^{**}) = (a_1^{**} + a_2^{**})\ D(a_1^{**}, a_2^{**})$; however, this result violates the logic of maximizing profit by the monopoly net price p_y^m. Therefore, the above assumption $a_1^{**} + a_2^{**} < p_y^m$ is contradictory, so $a_1^{**} + a_2^{**} \geq p_y^m$ must be true. (2) We adopted the method of contradiction, that is, first assume that the condition of $a_1^{**} + a_2^{**} = p_y^m$ is satisfied and prove its contradiction. In a pure duopoly market structure, when two firms make decisions at the same time, two equations are added according to the optimized first-order necessary condition, and $2D(p_y^m) + p_y^m D'(p_y^m) = 0$ is obtained by considering the above assumption. However, monopoly net pricing p_y^m satisfies the first-order necessary condition to be $D(p_y^m) + p_y^m D'(p_y^m) = 0$. Therefore, if $D(p_y^m) = 0$ is obtained from the above two equations, it conflicts with $D(p_y^m) > 0$, so the assumption of $a_1^{**} + a_2^{**} = p_y^m$ is not true. (3) The results of items (1) and (2) above were combined to prove that $p_y^{**} = a_1^{**} + a_2^{**} > p_y^m$.

(2) It shows that in the market structure of both private firms when the two firms "have a series of decisions," the equilibrium net price of system products is p_y^{ss}, which is not less than the system's equilibrium net price p_y^{**} when the two companies "make decisions at the same time," which is $p_y^{ss} \geq p_y^{**}$. The following does not lose generality; let firm 1 be the leader in price setting and let firm 2 be the follower in price setting. The same reasoning can also be applied to the model where firm 2 sets the price first, and firm 1 sets the price later. The symbols a_L^{ss} and a_F^{ss} are defined to represent the equilibrium net prices of firms 1 and 2, respectively. The following is to prove $p_y^{ss} = a_L^{ss} + a_F^{ss} \geq p_y^{**}$: (1) We first prove that the condition of $r_1(a_F^{ss}) \leq a_L^{ss}$ is true and adopt the method of contradiction, first assume that the condition of $r_1(a_F^{ss}) > a_L^{ss}$ is true and prove that this assumption is contradictory. Since the slope of the response function of firm 2 obtained above is negative, $r_2(r_1(a_F^{ss})) < r_2(a_L^{ss}) = a_F^{ss}$; therefore, the following equation $\Pi^1(a_L^{ss}, a_F^{ss}) \leq_1 \Pi^1(r_1(a_F^{ss}), a_F^{ss}) \leq_2 \Pi^1(r_1(a_F^{ss}), r_2(r_1(a_F^{ss})))$ is obtained.

Inequality 1 in the above equation is true due to the characteristics of the response function, and the reason why inequality 2 is true because of the abovementioned $r_2(r_1(a_F^{ss})) < a_F^{ss}$, and the demand function $D(a_1 + a_2)$ or $\Pi^1(a_1 + a_2)$ is a decreasing function of a_2. We obtained $\Pi^1(a_L^{ss}, a_F^{ss}) < \Pi^1(r_1(a_F^{ss}), r_2(r_1(a_F^{ss})))$ which contradicts the definition of equilibrium solution as (a_L^{ss}, a_F^{ss}), so $r_1(a_F^{ss}) \le a_L^{ss}$ holds. (2) Since the price leader (firm 1) can choose the optimal strategy, it can also take the nonoptimal strategy to choose the equilibrium price a_1^{**} of firm 1 when the two firms "make decisions at the same time." The price response function of firm 2 is a_2^{**} at this time; therefore, the equilibrium profit of firm 1 when the two firms "making decisions sequentially" will be greater than or equal to the equilibrium profit of the two firms "making decisions simultaneously," that is, $\Pi^1(a_L^{ss}, a_F^{ss}) \ge \Pi^1(a_1^{**}, a_2^{**})$. (3) To prove that $a_2^{**} \ge a_F^{ss}$ is true, adopt the method of contradiction and first assume that the condition of $a_2^{**} < a_F^{ss}$ is true and prove that this is a contradiction. The inequalities $\Pi^1(a_1^{**}, a_2^{**}) \ge \Pi^1(r_1(a_F^{ss}), a_2^{**}) > \Pi^1(r_1(a_F^{ss}), a_F^{ss}) \ge \Pi^1(a_L^{ss}, a_F^{ss})$ are obtained by the inference of the characteristics of the response function and the demand function, but the result $\Pi^1(a_1^{**}, a_2^{**}) > \Pi^1(a_L^{ss}, a_F^{ss})$ conflicts with the abovementioned result item (2) $\Pi^1(a_L^{ss}, a_F^{ss}) \ge \Pi^1(a_1^{**}, a_2^{**})$; therefore, $a_2^{**} \ge a_F^{ss}$ is true. (4) We obtain $r_1(a_F^{ss}) \ge r_1(a_2^{**}) = a_1^{**}$ from the result $a_2^{**} \ge a_F^{ss}$ of the above item (3) and the negative slope of the reaction function $r_1(a_2)$ and obtained the result $r_1(a_F^{ss}) \le a_L^{ss}$ from the above item (1), so we infer that $a_1^{**} \le a_L^{ss}$. (5) From the proof in Proposition 4 above, the slope of the response function of firm 2 is negative $r_i'(a_j) < 0$ and its absolute value is less than 1, that is, $1 + r_i'(a_j) > 0$, therefore, $a_1 + r_2(a_1)$ is a monotonically increasing function of a_1, and from the above, the result $a_1^{**} \le a_L^{ss}$ obtained in item (4), we obtained $p_y^{ss} = a_L^{ss} + r_2(a_L^{ss}) \ge a_1^{**} + r_2(a_1^{**}) = p_y^{**}$.

(3) The first item above resulted in $p_y^{**} > p_y^m$, and the second item resulted in $p_y^{ss} \ge p_y^{**}$, so $p_y^{ss} \ge p_y^{**} > p_y^m$, therefore, under the market structure of two private firms, regardless of order of the pricing between the two firms, the net price p_y of the system product must be greater than the monopoly net price p_y^m. It is also known from Proposition 3 that in the mixed duopoly market structure where public and private firms coexist, the equilibrium net price is equal to the monopoly net price regardless of the order of the two firms' pricing. Therefore, in terms of the equilibrium net price p_y of system product, the "pure duopoly" pricing $p_y = p_y^{ss}$ or $p_y = p_y^{**}$ of the two private firms coexisting is greater than the "mixed duopoly" pricing $p_y = p_y^m$ of the public and private firms coexisting.

(4) From the above item (3), the result $p_y^{ss} \ge p_y^{**} > p_y^m$ is known. The following will prove the ranking of total profit, consumer utility, and social welfare: (1) in terms of total profits of the two firms, under the norm that the public firm is self-sufficient, the coexistence of the public firm and the private firm will lead to monopoly profit, that is, maximize the profit of system product y. The total profit of the coexistence of both firms is greater than or equal to the profit of both private firms. (2) In terms of consumer utility, $V'(p_y) = -D(p_y) < 0$ can be found from Roy's identity. Consumer utility is a monotonically decreasing function of price; therefore, consumer utility when coexisting with public and private firms is better than that when both are private firms. (3) In terms of social welfare, $W'(p_y) = p_y D'(p_y) < 0$ is derived from the derivative of social welfare with p_y. Social welfare is a monotonically decreasing function of price; therefore, the social welfare when public firms and private firms coexist is greater than that when both are private firms. Q.E.D. □

Data Availability

No data were used in this study.

Conflicts of Interest

The author declares no conflicts of interest.

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